GROWTH DIAGRAMS, DOMINO INSERTION AND SIGN-IMBALANCE

THOMAS LAM

Abstract. We study some properties of domino insertion, focusing on aspects related to Fomin's growth diagrams [3, 4]. We give a self-contained proof of the semistandard domino-Schensted correspondence given by Shimozono and White [21], bypassing the connections with mixed insertion entirely. The correspondence is extended to the case of a nonempty 2-core and we give two dual domino-Schensted correspondences. We use our results to settle Stanley's '$2^{n/2}$' conjecture on sign-imbalance [25] and to generalise the domino generating series of Kirillov, Lascoux, Leclerc and Thibon [9] .

1. Introduction

Recently in [21] Shimozono and White described a semistandard generalisation of Barbasch and Vogan's domino insertion [1], relating domino insertion to Haiman's mixed insertion [8]. This semistandard domino Schensted algorithm establishes a bijection between colored biwords and pairs of semistandard domino tableaux of the same shape. That such a bijection exists can already be seen by combining Littlewood's p-quotient construction [17] with the usual Robinson-Schensted-Knuth algorithm. Shimozono and White's key observation is that Barbasch-Vogan domino insertion has a color-to-spin property. This property appears to have been used earlier by Kirillov, Lascoux, Leclerc and Thibon [9] for some special colored involutions.

Earlier, van Leeuwen [15] had described domino insertion in terms of Fomin's growth diagrams. He connected Barbasch and Vogan's left-right insertion description [1] with Garfinkle's traditional bumping description [6]. He also defines insertion in the presence of a 2-core.

Our first aim in this paper is to give a self contained proof of the semistandard domino-Schensted correspondence, using elementary growth diagram calculations to prove all the main properties of the bijection which we also extend to the nonempty 2-core case. Thus our approach allows us to avoid mention of mixed insertion completely. We also describe two dual domino-Schensted bijections. These are bijections between multiplicity-free sets of biletters and pairs of semistandard domino tableaux which have conjugate shapes. All three bijections are essentially the same on the set of hyperoctahedral permutations. In fact we will make clear that the most important difference is that different notions of 'standardisation' of a set or multiset of biletters are used. Finally, we perform a detailed analysis of symmetric growth diagrams for domino insertion.

Date: August 26, 2003; revised February 18, 2004; March 29, 2004.

The study of growth diagrams leads us to a number of applications. These include a number of enumerative results for domino tableaux, an application to sign-imbalance, and a collection of product expansions for generating series of domino functions.

The sign $sign(T)$ of a standard Young tableau T is the sign (as a permutation) of its reading word obtained by reading the tableau from left to right along the rows, starting from the top row. The sign imbalance of a shape λ is defined as

$$\sum_{SYT\,T\,:\,sh(T)=\lambda} sign(T).$$

That sign-imbalance is related to domino tableaux has been made explicit in work of White [27] and Stanley [25]. In particular, White gives a formula for the sign of the Young tableau $T(D)$ associated to a domino tableau D:

$$sign(T) = (-1)^{ev(D)}$$

where $ev(D)$ is the number of vertical dominoes in even columns of D. Domino tableaux are in bijection with hyperoctahedral involutions and we prove that in fact $ev(D)$ is equal to the number of barred two-cycles of π, where $D = P_d(\pi)$ is the insertion tableau of π. This allows us to prove Stanley's conjecture on sign-imbalance, our Theorem 23, which is a 4-parameter generalisation of the following elegant result:

$$\sum_{SYT\,T:sh(T)\vdash m} sign(T) = 2^{\lfloor m/2 \rfloor}.$$

Recently, another proof of Stanley's conjecture has appeared which uses the usual Schensted correspondence, due to Astrid Reifegerste [19] and Jonas Sjöstrand [23].

Carré and Leclerc [2] and Kirillov, Lascoux, Leclerc and Thibon [9] have studied certain generating functions $H_\lambda(X;q)$ for domino tableaux which we loosely call *domino functions*. More general domino functions $G_\lambda(X;q)$ were developed also in [11], where they were connected with the Fock space representation of $U_q(\hat{\mathfrak{sl}}_2)$. These are defined as

$$G_\lambda(X;q) = \sum_D q^{sp(D)} x^D$$

where the sum is over all semistandard domino tableaux of shape λ. The H_λ are defined by $H_\lambda(X;q) = G_{2\lambda}(X;q)$. Product expansions of the sums $\sum H_\lambda(X;q)$ and $\sum H_{\lambda\vee\lambda}(X;q)$ were given in [9].

By studying colored involutions we give a product expansion for a 3-parameter generalisation of the sum $\sum_\lambda G_\lambda(X;q)$. When the parameters of this sum is specialised, we obtain both of the product expansions of [9].

A generalisation of the functions $G_\lambda(X;q)$ from dominoes to p-ribbons is given by Lascoux, Leclerc and Thibon in [11] and the connection with representation theory further explored in [13, 14, 12]. The study of ribbon tableaux appears to be even more interesting, though considerably harder, than that of domino tableaux, inspiring much recent work. A Schensted-correspondence for ribbon tableaux has been given by van Leeuwen [16], though the correspondence cannot be described in terms of insertion. Descriptions of the spin of a ribbon tableau in terms of the p-quotient have been given by Schilling, Shimozono and White [20], and also by Haglund, Haiman, Loehr, Remmel and Ulyanov [7]. In [10], we study these ribbon functions in analogy with Schur functions, by proving ribbon Cauchy, Pieri and Murnaghan-Nakayama formulae.

We now briefly describe the organisation of this paper. In Section 2 we give some notation and definitions for domino tableaux and multisets of biletters. We also give a description of domino insertion bumping in an informal manner, following mostly [21]. In Section 3, we introduce and study growth diagrams. This is followed by a proof of the semistandard domino-Schensted correspondence and a description of the dual domino-Schensted correspondences. The section ends with a study of symmetric growth diagrams and some enumerative results. In Section 4, we apply the results of Section 3 to sign-imbalance. In Section 5, we combine the results of Section 3 with a study of the standardisation of colored involutions. These lead to a number of product expansions for generating series of domino functions. In Section 6, we give some final remarks concerning possible generalisations to longer ribbons.

Acknowledgements I am indebted to my advisor, Richard Stanley, for introducing the subjects of domino tableaux and sign-imbalance to me and for suggesting his conjecture for study. My work on generating series of domino functions was inspired by the sum $\sum(-1)^{ev(\lambda)}G_\lambda(X;-1)$, where $ev(\lambda)$ is the maximum number of vertical dominoes in even columns in some domino tiling of λ, which he suggested to me. I would also like to thank Marc van Leeuwen, and two anonymous referees who made many helpful comments on an earlier version of this paper.

2. Preliminaries

2.1. **Domino Tableaux.** We will let $[n] = \{1, 2, \ldots, n\}$ throughout.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{l(\lambda)} > 0)$ be a partition of n. We will often not distinguish between a partition λ and its diagram (often called $D(\lambda)$) but the meaning will always be clear from the context. The partition $\lambda \cup \mu$ is obtained by taking the union of the parts of λ and μ (and reordering to form a partition). We denote by $\tilde{\lambda}$ and $(\lambda^{(0)}, \lambda^{(1)})$ the 2-core and 2-quotient of λ respectively (see [18]). Every 2-core has the shape of a staircase $\delta_r = (r, r-1, \ldots, 0)$ for some integer $r \geq 0$. As usual, when λ and μ are partitions satisfying $\mu \subset \lambda$ we will use λ/μ to denote the shape corresponding to the set-difference of the diagrams of λ and μ.

We denote the set of partitions by $\mathcal{P}$ and the set of partitions with 2-core δ_r by $\mathcal{P}_r$. The set of all partitions λ satisfying the two conditions:

$$\tilde{\lambda} = \delta_r$$
$$|\lambda| = \delta_r + 2n$$

will be denoted $\mathcal{P}_r(n)$. Note that $\mathcal{P} = \cup_{r,n}\mathcal{P}_r(n)$.

A (standard) domino tableau (SDT) D of shape λ consists of a tiling of the shape $\lambda/\tilde{\lambda}$ by dominoes and a filling of each domino with an integer in $[n]$ so that the numbers are increasing when read along either the rows or columns. Here, n is $\frac{1}{2}|\lambda/\tilde{\lambda}|$. A domino is any 2×1 or 1×2 shape, or equivalently, two adjacent squares sharing a common edge. The *value* of a domino is the number written inside it. We will write dom_i to indicate the domino with the value i inside. We will also write $sh(D) = \lambda$. An alternative way of describing a standard domino tableau of shape λ is by a sequence of partitions $\left\{\tilde{\lambda} = \lambda^0 \subset \lambda^1 \subset \ldots \subset \lambda^n = \lambda\right\}$, where $sh(dom_i) = \lambda_i/\lambda_{i-1}$.

A semistandard domino tableau (SSDT) D of shape λ consists of a tiling of the shape $\lambda/\tilde{\lambda}$ by dominoes and a filling of each domino with an integer, so that the numbers are non-decreasing when read along the rows and increasing when read along the columns. The *weight* of such a tableau D is the composition $wt(D) = (\mu_1, \mu_2, \ldots)$ where there are μ_i occurrences of i's in D. Let $v(D)$ be the number of vertical dominoes in a domino tableau D. The spin $sp(D)$, is defined as $v(D)/2$. The standardisation of a semistandard domino tableau D of weight μ is a standard domino tableau D^{st} obtained from D by replacing the dominoes containing 1's with $1, 2, \ldots, \mu_1$ from left to right, the dominoes containing 2's by $\mu_1+1, \mu_1+2, \ldots, \mu_1+\mu_2$, and so on.

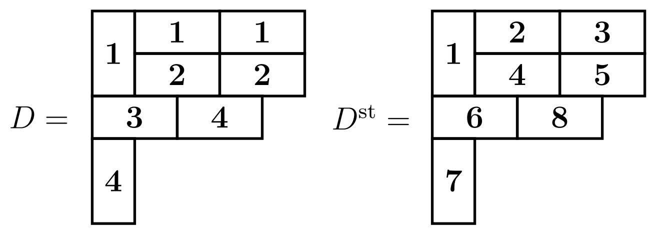

FIGURE 1. A domino tableau D with shape $(5, 5, 4, 1, 1)$ and weight $(3, 2, 1, 2)$ and its standardisation D^{st}.

More general skew (semi)standard domino tableaux are defined in a similar manner.

We should remark that Littlewood's 2-quotient map [17] gives a bijection between semistandard domino tableaux of shape λ and pairs of semistandard Young tableaux of shapes $\lambda^{(0)}$ and $\lambda^{(1)}$.

2.2. **Biletters and Colored Words.** The definitions in this section are essentially those of [21] except that we will consider multisets of biletters instead of colored words, and our definitions of inverse and standardisation will emphasise this point of view.

A *letter* will be an integer with possibly a bar over it. If x and y are letters, we will say $x < y$ if

(1) $x < y$ as integers and both are unbarred,
(2) $x > y$ as integers and both are barred, or
(3) x is barred and y is unbarred.

A *colored word* is a word made of letters. A colored word $\mathbf{w}$ is a *colored permutation* if each integer of $[n]$ is used exactly once, for some n. Such a word will also be called a hyperoctahedral permutation or a signed permutation. The set (in fact group) of all such words will be denoted B_n. The weight of a word is defined in the usual way, with the bars ignored. The operation *ev* removes the bars from a colored word. Thus if $\mathbf{w} = (2\overline{3}\overline{1})$ then $\mathbf{w}^{ev} = (231)$.

A *biletter* $\mathbf{l}$ is an ordered pair of letters, denoted $\binom{x}{y}$ such that x is unbarred and y may be barred or not. The inverse $\mathbf{l}^{\text{inv}} = \binom{y}{x}$ of $\mathbf{l} = \binom{x}{y}$ is the biletter obtained from $\mathbf{l}$ by swapping the pair of integers preserving the barred-ness of the lower letter.

There is a (total) ordering $<$ on biletters defined by $\binom{x}{y} < \binom{k}{l}$ if

(1) $x < k$, or
(2) $x = k$ and $y < l$ as letters.

Let $\mathbf{m}$ be a multiset of biletters. The *length* of $\mathbf{m}$ is simply its size as a multiset. The *top* or *upper weight* of $\mathbf{m}$ is the weight (in the usual sense) of the multiset of top letters, and similarly for the *bottom* or *lower weight*. The *inverse* $\mathbf{m}^{\text{inv}}$ of $\mathbf{m}$ is the multiset $\left\{\binom{x}{y}^{\text{inv}} \mid \binom{x}{y} \in \mathbf{m}\right\}$.

The *total color* of a multiset of biletters $\mathbf{m}$ or a colored word $\mathbf{w}$, denoted $tc(\mathbf{m})$ or $tc(\mathbf{w})$, is the number of barred letters in the multiset or word.

Standardisation st is defined as follows for a multiset of biletters $\mathbf{m}$. It will send a multiset of biletters to a hyperoctahedral permutation $\mathbf{m}^{\text{st}} = \pi = \pi_1\pi_2\cdots\pi_n$ where n is the size of $\mathbf{m}$. We set $\pi_i = j$ if the i^{th} smallest biletter $\mathbf{l} = \binom{x}{y}$ in $\mathbf{m}$ under $<$ becomes the j^{th} smallest biletter in $\mathbf{m}^{\text{inv}}$ under $<$. We then make j barred if and only if y is barred. Ties are broken as follows. Suppose $\mathbf{l} = \binom{x}{y}$ occurs k times in $\mathbf{m}$. Let the first occurrence of $\mathbf{l}$ in $\mathbf{m}$, ordered by $<$, be its i-th letter, and let the first occurrence of $\mathbf{l}^{\text{inv}}$ in $\mathbf{m}^{\text{inv}}$ be its j-th letter. Then

(1) If y is unbarred, we set $\pi_i = j$, $\pi_{i+1} = j+1, \ldots \pi_{i+k-1} = j+k-1$.
(2) If y is barred, we set $\pi_i = \overline{j+k-1}$, $\pi_{i+1} = \overline{j+k-2}, \ldots \pi_{i+k-1} = \overline{j}$.

It is immediate from the definitions that standardisation and inverse commute.

One may check that these definitions agree with those of [21] by identifying a multiset of biletters with a colored biword by ordering the biletters canonically via $<$. We also note the following property of standardisation which we will need later. If in $\mathbf{m}$, ordered by $<$, the biletters with a fixed number y as lower letter (barred or not) occur at indices $i_1 < \ldots < i_l$, then in $\pi = \mathbf{m}^{\text{st}}$ one has $\pi_{i_1} < \cdots < \pi_{i_l}$ as letters.

For example, let $\mathbf{m}$ be the multiset of biletters

$$\mathbf{m} = \left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}3\\\overline{1}\end{pmatrix}, \begin{pmatrix}3\\\overline{1}\end{pmatrix} \right\}.$$

Then $\mathbf{m}$ has top weight $(2,2,1)$ and bottom weight $(2,1,1,1)$. Its inverse $\mathbf{m}^{\text{inv}}$ is given by

$$\mathbf{m}^{\text{inv}} = \left\{ \begin{pmatrix}2\\\overline{1}\end{pmatrix}, \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}1\\\overline{3}\end{pmatrix}, \begin{pmatrix}1\\\overline{3}\end{pmatrix} \right\}.$$

Its standardisation $\mathbf{m}^{\text{st}}$ is

$$\mathbf{m}^{\text{st}} = \overline{3}45\overline{2}\overline{1}.$$

Lemma 1. *A multiset of biletters* $\mathbf{m}$ *is uniquely determined by its standardisation* $\mathbf{m}^{\text{st}}$ *and its top and bottom weights.*

Proof. This is easy to check directly from the definitions, but can also be derived from results in [21]. □

We will occassionally identify a colored word $\mathbf{w}$ or a hyperoctahedral permutation π with a multiset of biletters obtained by filling the top row with $\{1, 2, \ldots, n\}$ from left to right, and splitting into biletters. We note that under this identification, the inverse for multisets of biletters is compatible with the usual inverse of B_n.

2.3. **Domino insertion.** The normal Robinson-Schensted algorithm gives a bijection between permutations of S_n and pairs (P, Q) of standard Young tableaux (SYT) of size n and the same shape. A semistandard generalisation of this was given by Knuth. This is a bijection between certain matrices with non-negative integer entries (or alternatively multisets of unbarred biletters) and pairs of semistandard Young Tableaux of the same shape. We refer the reader to [5, 24] for further details. Henceforth, familiarity with usual Robinson-Schensted insertion will be assumed.

In this section we describe the corresponding bijection for domino tableaux in a traditional insertion 'bumping' procedure. We will follow the description given by Shimozono and White [21] for the rest of this section where more details may be found. As the whole theory will be developed completely from the growth diagram point of view in Section 3, we will not be completely formal. The reader is referred to [6], [21], [15] for full details.

Let D be a domino tableau with $sh(D) = \lambda$, no values repeated, and i a value which does not occur in D. We will describe how to insert both a vertical and horizontal domino with value i into D. Let $A \subset D$ be the sub-domino tableau containing values less than i. If λ has a 2-core $\lambda = \tilde{\lambda}$, then we will always assume that $\tilde{\lambda} \subset sh(A)$. We set B to be the domino tableau containing A and an additional vertical domino in the first column or an additional horizontal domino in the first row labelled i. Let $C = D/D'$ be the skew domino tableau containing values greater than i. Now we recursively define a bumping procedure as follows.

Let (B, C) be a pair of domino tableau (with no values repeated) overlapping in at most a domino. The combined shape of B and C must be a valid skew shape and the values of C larger than those of B. Let λ be the shape of B and j be the largest value of C respectively. Denote the corresponding domino by dom_j. We now distinguish four cases:

(1) If $\lambda \cap dom_j = \emptyset$ do not touch, then we set $B' = B \cup dom_j$ and $C' = C - dom_j$.
(2) If $\lambda \cap dom_j = (k, l)$ is exactly one square, then we add a domino containing j to B to obtain a tableau B' which has shape $\lambda \cup dom_j \cup (k + 1, l + 1)$. We set $C' = C - dom_j$.
(3) If $\lambda \cap dom_j = dom_j$ and dom_j is horizontal, then we 'bump' the domino dom_j to the next row, by setting B' to be the union of B with an additional (horizontal) domino with value j one row below that of dom_j. We set $C' = C - dom_j$.
(4) If $\lambda \cap dom_j = dom_j$ and dom_j is vertical, then we 'bump' the domino dom_j to the next column, by setting B' to be the union of B with an additional (vertical) domino with value j one column to the right of dom_j. We set $C' = C - dom_j$.

This procedure is repeated with (B, C) replaced by (B', C') until the (skew) domino tableau C becomes empty.

The resulting B tableau will be denoted by $D \leftarrow i$ for the insertion of a horizontal domino and $D \leftarrow \bar{i}$ for a vertical domino.

Let $\mathbf{w} = w_1 w_2 \cdots w_n$ be a colored permutation and δ_r be a 2-core assumed to be fixed throughout. Then the insertion tableau $P_d^r(\mathbf{w})$ is defined as $((\ldots((\delta_r \leftarrow w_1) \leftarrow w_2)\cdots) \leftarrow w_n)$. The sequence of shapes obtained in the process defines another standard domino tableau called the recording tableau $Q_d^r(\mathbf{w})$.

As an example, the domino tableau $P_d^0(\bar{3}42\bar{1})$ is constructed as in Figure 2.

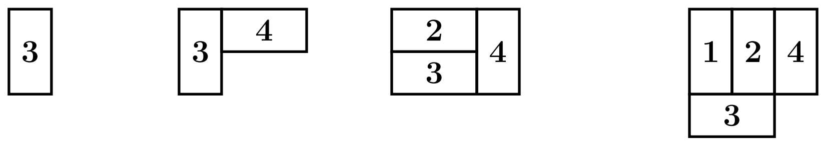

FIGURE 2. Insertion of $\mathbf{w} = \bar{3}42\bar{1}$ into $\emptyset$.

The following theorem will be proven in Section 3.

Theorem 2. *Fix $r \geq 0$. The above algorithm defines a bijection between signed permutations $\pi \in B_n$ and pairs of standard domino tableaux (P, Q) of the same shape $\lambda \in \mathcal{P}_r(n)$. This bijection satisfies the equality*

$$tc(\pi) = sp(P_d(\pi)) + sp(Q_d(\pi)).$$

The insertion algorithm is due to Barbasch and Vogan [1] in a different form, and can be implemented by performing multiple (usual) Schensted algorithms. For example, to calculate the tableau $P_d^0(\bar{3}42\bar{1})$ of Figure 2, we would start with the word $[1, -2, -4, 3, -3, 4, 2, -1]$. One computes the shapes of the different tableaux obtained by Schensted insertion, applied to the successive words obtained by erasing $(4, -4)$, $(3, -3)$, $\ldots$ in succession. This sequence of shapes differs by single dominoes, and give the domino tableau $P_d^0(\bar{3}42\bar{1})$. The insertion described here in terms of bumping is essentially that of Garfinkle [6]. Van Leeuwen [15] proves that the Barbasch-Vogan algorithm is the same as the bumping description, and also shows that the bijection holds in the presence of a 2-core. That this algorithm sends total color to the sum of spins seems to have been first used by Kirillov, Lascoux, Leclerc and Thibon in [9] for certain hyperoctahedral involutions, though no details or proofs are present. More recently, the color-to-spin property is made explicit by Shimozono and White in [21].

Shimozono and White [21] only prove the color-to-spin property in the absence of a 2-core. However, the color-to-spin property is proven by studying the spin change for all the 'bumps' in the insertion and these are unaffected by the presence of a 2-core. Thus the generalisation of the domino insertion bijection to the 2-core case is immediate. Shimozono and White also give a semistandard generalisation of this bijection which is the case $r = 0$ of the following theorem. Their theory of domino insertion is developed in conjunction with other combinatorial algorithms including Haiman's mixed insertion and left-right insertion.

Theorem 3. *Fix a 2-core δ_r. There is a bijection between mutlisets of biletters* $\mathbf{m}$ *of length n and pairs $(P_d^r(\mathbf{m}), Q_d^r(\mathbf{m}))$ of semistandard domino tableaux with the same shape $\lambda \in \mathcal{P}_r(n)$ with the following properties:*

(1) *The bijection has the color-to-spin property:*
$$tc(\mathbf{m}) = sp(P_d^r(\mathbf{m})) + sp(Q_d^r(\mathbf{m})).$$
(2) *The weight of $P_d^r(\mathbf{m})$ is the upper weight of* $\mathbf{m}$*. The weight of $Q_d^r(\mathbf{m})$ is the lower weight of* $\mathbf{m}$*.*
(3) *The bijection commutes with standardisation in the following sense:*
$$P_d^r(\mathbf{m})^{\mathrm{st}} = P_d^r(\mathbf{m}^{\mathrm{st}}).$$
$$Q_d^r(\mathbf{m})^{\mathrm{st}} = Q_d^r(\mathbf{m}^{\mathrm{st}}).$$

The proof of this will be left until the next section, where we give an alternative description of domino insertion in terms of growth diagrams.

3. Growth Diagrams and Domino Insertion

3.1. **Properties of Growth Diagrams.** The insertion algorithm of subsection 2.3 can also be phrased in terms of Fomin's growth diagrams [3, 4] (also known as the poset-theoretic description, or language of shapes). This was first made explicit by van Leeuwen [15]. We will show how growth diagrams are relevant to the semistandard generalisation of domino insertion of [21]. Thus our aim will be to give a short, stand-alone proof of Theorem 3 using elementary considerations of growth diagrams only, bypassing the connection with mixed insertion used by Shimozono and White. Thus their lemma [21, Lemma 33] is replaced by our Lemma 9. The use of growth diagrams makes the generalisation to the case of nonempty 2-core immediate. In fact one could use growth diagrams to define the entire correspondence and develop the theory beginning from that.

Let $M(i,j)$ be a $n \times n$ matrix taking values from $\{0, 1, -1\}$ thought of as the matrix representing a hyperoctahedral permutation. Thus it has one non-zero value in each row or column. We will take the row and column indices to lie in $[n]$.

The growth diagram (of $M(i,j)$) is an array of partitions $\lambda_{(i,j)}$ for $1 \leq i,j \leq n+1$. Two 'adjacent' partitions $\lambda_{(i,j)}$ and $\lambda_{(i+1,j)}$ or $\lambda_{(i,j)}$ and $\lambda_{(i,j+1)}$ are either identical or differ by exactly one domino. Initially, all the partitions $\lambda_{(1,j)}$ and $\lambda_{(i,1)}$ are set to the same partition μ. For our purposes this will usually be a partition satisfying $\mu = \tilde{\mu}$. The remainder of the growth diagram will be determined from μ and the data $M(i,j)$ according to the following local rules.

Let $\lambda = \lambda_{(i,j)}$, $\mu = \lambda_{(i+1,j)}$, $\nu = \lambda_{(i,j+1)}$, $\rho = \lambda_{(i+1,j+1)}$ be the corners of a 'square'. Assume (inductively) that λ, μ and ν are known. Then ρ is determined as follows:

(1) If $M(i,j) = 1$ then it must be the case that $\lambda = \mu = \nu$. Obtain ρ from λ by adding two to the first row.
(2) If $M(i,j) = -1$ then it must be the case that $\lambda = \mu = \nu$. Obtain ρ from λ by adding two to the first column.
(3) If $M(i,j) = 0$ and $\lambda = \mu$ or $\lambda = \nu$ (or both) then ρ is set to the largest of the three partitions.

(4) Otherwise $M(i,j) = 0$ and ν and μ differ from λ by dominoes γ and γ'. If γ and γ' do not intersect then ρ is set to be the union $\lambda \cup \gamma \cup \gamma'$. If $\gamma \cap \gamma'$ is a single square (k,l), then ρ is the union of $\lambda \cup \gamma \cup \gamma' \cup (k+1, l+1)$. If $\gamma = \gamma'$ is a vertical domino then ρ is obtained from $\lambda \cup \gamma$ by adding two to the column immediately to the right of γ. If $\gamma = \gamma'$ is a horizontal domino then ρ is obtained from $\lambda \cup \gamma$ by adding two to the row immediately below γ.

We will call these rules the *local rules* of the growth diagram.

Proposition 4. *The above algorithm is well defined. The growth diagram models the insertion of the colored permutation π corresponding to $M(i,j)$ into a 2-core δ_r (in fact more generally any initial partition) .*

The partition $\lambda_{(i,j)}$ is the shape of the tableau obtained after the first i insertions and restricted to values less than j. Thus $\left\{\lambda_{(n+1,j)} : j \in [n+1]\right\}$ is a chain of partitions determining $P_d^r(\pi)$ and $\left\{\lambda_{(i,n+1)} : i \in [n+1]\right\}$ is a chain of partitions determining $Q_d^r(\pi)$.

Proof. This is proven via induction, by comparing domino insertion with the local rules of the growth diagram. The details can be found in [15]. □

For example, Figure 3 is the growth diagram corresponding to the insertion procedure of Figure 2.

Lemma 5. *The local rules of a growth diagram are reversible in the following sense. Let $\lambda = \lambda_{(i,j)}$, $\mu = \lambda_{(i+1,j)}$, $\nu = \lambda_{(i,j+1)}$, $\rho = \lambda_{(i+1,j+1)}$ be the corners of a 'square' of the growth diagram. Then ρ, μ and ν determine λ and $M(i,j)$.*

Proof. This is a simple verification of the local rules. □

Note, that there can be two legitimate standard domino tableaux corresponding to $\left\{\lambda_{(i,n+1)} : i \in [n+1]\right\}$ and $\left\{\lambda_{(n+1,j)} : j \in [n+1]\right\}$ which do not give a growth diagram corresponding to an insertion procedure. For example if $\lambda_{(1,2)} = (2) = \lambda_{(2,1)}$ and $\lambda_{(2,2)} = (2,2)$ then $\lambda_{(1,1)}$ must be $\emptyset$. This is not a valid growth diagram for insertion as $\lambda_{(1,1)} \neq \lambda_{(2,1)}$.

Lemma 6. *The correspondence*

$$\pi \longrightarrow (P_d^r(\pi), Q_d^r(\pi))$$

is a bijection between $\pi \in B_n$ and pairs of standard domino tableaux of the same shape $\lambda \in \mathcal{P}_r(n)$.

Proof. The previous Lemma implies that this correspondence is injective. As no dominoes can be removed from δ_r, the 'initial' row and column of the growth diagram ($\lambda_{(1,j)}$ and $\lambda_{(i,1)}$) will consist completely of partitions equal to δ_r. Thus setting $\lambda_{(i,n+1)} : i \in [n+1]$ and $\lambda_{(n+1,j)} : j \in [n+1]$ to two tableaux of the shape $\lambda \in \mathcal{P}_r(n)$ will give a growth diagram corresponding to the insertion of some hyperoctahedral permutation π. □

Lemma 7. *Let π be a hyperoctahedral permutation. Domino insertion possesses the symmetry property*

$$P_d^r(\pi) = Q_d^r(\pi^{\text{inv}}).$$

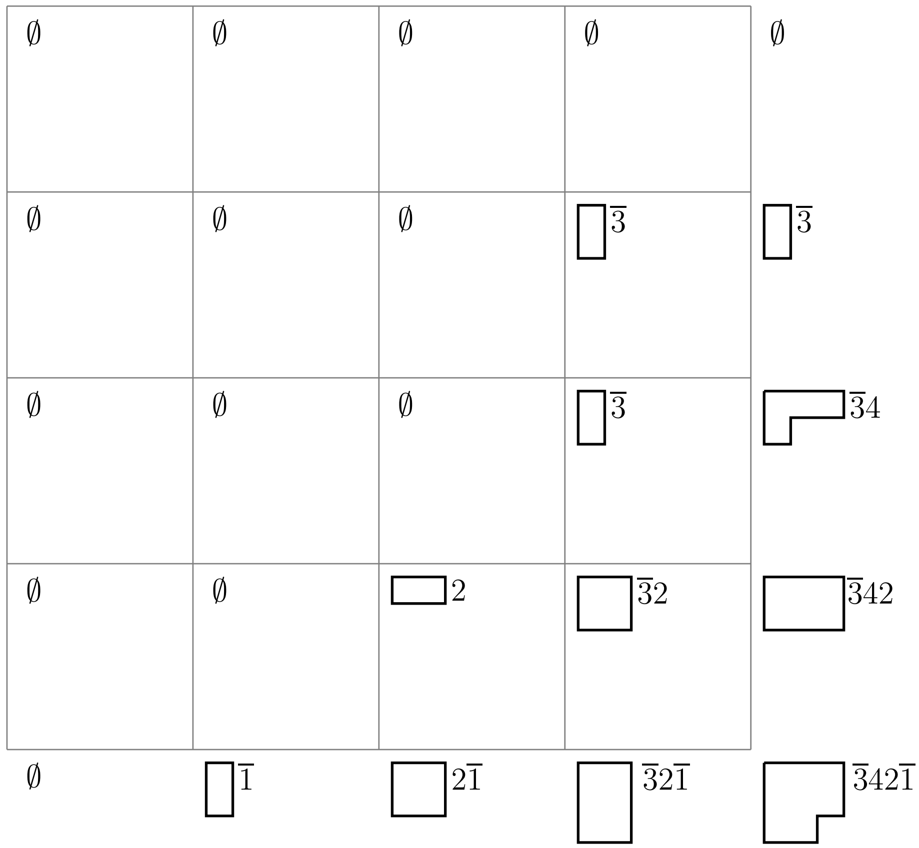

FIGURE 3. Growth diagram for the insertion of $\mathbf{w} = \overline{3}42\overline{1}$ into $\emptyset$. The colored word whose insertion tableau corresponds to each shape is written next to the shape.

Proof. This is a consequence of the fact that the growth diagram local rules are symmetric. □

Lemma 8. *Domino insertion for hyperoctahedral permutations π possesses the color-to-spin property:*

$$tc(\pi) = sp(P_d^r(\pi)) + sp(Q_d^r(\pi)).$$

Proof. Let $\lambda = \lambda_{(i,j)}$, $\mu = \lambda_{(i+1,j)}$, $\nu = \lambda_{(i,j+1)}$, $\rho = \lambda_{(i+1,j+1)}$ be the corners of a square of the growth diagram. Then the Lemma follows from the observation that

$$sp(\rho/\mu) + sp(\rho/\nu) = sp(\mu/\lambda) + sp(\nu/\lambda) + \begin{cases} 1 & \text{if } M(i,j) = -1 \\ 0 & \text{otherwise.} \end{cases}$$

This can be checked by considering the local rules case by case. □

Lemma 9. *Let $\pi = \pi_1 \cdots \pi_n$ be a colored permutation. Then $\pi_i < \pi_{i+1}$ if and only if dom_i lies to the left of dom_{i+1} in $Q_d^r(\pi)$.*

Proof. The main idea is to analyze a 1×2 rectangle of the growth diagram. Let $\lambda_0 = \lambda_{(i,j)}$, $\lambda_1 = \lambda_{(i+1,j)}$, $\lambda_2 = \lambda_{(i+2,j)}$, $\mu_0 = \lambda_{(i,j+1)}$, $\mu_1 = \lambda_{(i+1,j+1)}$ and $\mu_2 = \lambda_{(i+2,j+1)}$

be the corners of a 1×2 rectangle of the growth diagram. We will call the two squares of the 1×2 rectangle the first and second squares. We further assume that $M(i,j) = M(i+1,j) = 0$.

Now suppose that $\alpha_0 = \lambda_1/\lambda_0$ and $\alpha_1 = \lambda_2/\lambda_1$ are both dominoes so that α_0 lies to the left of α_1. Then it is easy to check that $\beta_0 = \mu_1/\mu_0$ and $\beta_1 = \mu_2/\mu_1$ are both dominoes since $M(i,j) = M(i+1,j) = 0$. We claim that in fact β_0 lies to the left of β_1. If $\lambda_0 = \mu_0$ this is trivial and most of the cases of the local rules are a simple verification.

The only interesting case is when $\lambda_1 = \mu_0$ and α_0 is a vertical domino. In this case, β_0 has moved to the right when compared to α_0. The key observation is that β_0 is placed in the column immediately to the right of α_0, so it is either still to the left of α_1 or it overlaps α_1. When overlap occurs, β_1 will be moved further to the right and β_0 will remain to the left of β_1. This proves our claim.

To show (one direction of) our lemma, we just need to check, case by case, that the initial condition (α_0 lying to the left of α_1) holds for $j = \max(\pi_i^{ev}, \pi_{i+1}^{ev}) + 1$. As adding a new domino to the first column will be furthest to the left, and adding a new domino to the first row will be the furthest right this is a simple verification. The claim implies inductively that the same will continue to hold when we get to $\lambda_{(i,n+1)}$, $\lambda_{(i+1,n+1)}$ and $\lambda_{(i+2,n+1)}$, which give exactly dom_i and dom_{i+1} of $Q_d^r(\pi)$.

The other direction of the lemma is proven in exactly the same way, or one could replace 'left' by 'above' and 'row' by 'column'. □

Lemma 10. *Let $\pi = \pi_1 \cdots \pi_n$ be a colored permutation. Then $\left(\pi^{\mathrm{inv}}\right)_i < \left(\pi^{\mathrm{inv}}\right)_{i+1}$ if and only if dom_i lies to the left of dom_{i+1} in $P_d^r(\pi)$.*

Proof. This is a consequence of Lemma 9 and Lemma 7. □

We are now ready to prove the semistandard domino-Schensted correspondence.

Proof of Theorem 3. That the correspondence exists for hyperoctahedral permutations is Lemma 6. Then the color-to-spin property follows from Lemma 8.

For the semistandard case, fix two weights μ and λ and let these be the upper and lower weights of a multiset of biletters $\mathbf{m}$. We define $P_d^r(\mathbf{m})$ by requiring it to have weight λ and satisfy $P_d^r(\mathbf{m})^{\mathrm{st}} = P_d^r(\mathbf{m}^{\mathrm{st}})$. We now show that such a (semistandard) tableau exists. Let $\pi = \mathbf{m}^{\mathrm{st}}$. Suppose dom_i lies to the right of dom_{i+1} in $P_d^r(\pi)$ and j and k satisfy $\pi_i^{\mathrm{inv}} = j$ and $\pi_{i+1}^{\mathrm{inv}} = k$. Then by Lemma 10, $j > k$ as letters. This means that the i^{th} smallest biletter of $\mathbf{m}^{\mathrm{inv}}$ has a different top letter to the $(i+1)^{th}$ smallest biletter $\mathbf{m}^{\mathrm{inv}}$ by the definition of standardisation. So we are never in the situation where we need to relabel dom_i and dom_{i+1} with the same integer. Such a tableau is unique since standardisation is injective for tableaux when the weight is fixed.

We then define $Q_d^r(\mathbf{m})$ by

$$Q_d^r(\mathbf{m}) = P_d^r(\mathbf{m}^{\mathrm{inv}}).$$

It is clear that these definitions commute with standardisation.

Since standardisation is injective (for both multisets of biletters and tableaux) when the weights μ and λ are fixed, this proves that the correspondence

$$\mathbf{m} \to (P_d^r(\mathbf{m}), Q_d^r(\mathbf{m}))$$

is injective for multisets of biletters with fixed weights for the top and bottom rows. The color-to-spin property is also a consequence of the standardisation procedure, as

$$tc(\mathbf{m}) = tc(\mathbf{m}^{\mathrm{st}}) = sp(P_d^r(\mathbf{m}^{\mathrm{st}})) + sp(Q_d^r(\mathbf{m}^{\mathrm{st}})) = sp(P_d^r(\mathbf{m})) + sp(Q_d^r(\mathbf{m})).$$

Finally, one can show that correspondence is a surjection as follows. Suppose we are given a pair (P, Q) of semistandard domino tableaux of shape $sh(P) = sh(Q) \in \mathcal{P}_r(n)$ such that $wt(P) = \lambda$ and $wt(Q) = \mu$. Let π be the signed permutation corresponding to $(P^{\mathrm{st}}, Q^{\mathrm{st}})$ by Lemma 6, and let $f, g\colon [n] \to \mathbf{Z}$ be the maps assigning to an entry of P^{st} (respectively of Q^{st}) the corresponding entry of P (respectively of Q); f and g are weakly increasing. We claim that the multiset $\mathbf{m}$ of biletters, obtained by considering π as a (multi)set of biletters (adding a top row $1, \ldots, n$) and then replacing each pair of integers $\binom{x}{y}$ by $\binom{g(x)}{f(y)}$ while preserving the barred-ness of the bottom letter, corresponds to (P, Q). Since the top and bottom weights are correct, this amounts to showing $\mathbf{m}^{\mathrm{st}} = \pi$. The 'if' part of Lemma 9 implies that if q is any entry of Q, and $\{a, a+1, a+2, \ldots, b\} = g^{-1}(q)$, then $\pi_a < \pi_{a+1} < \cdots < \pi_b$, which means that the sequence of biletters of $\mathbf{m}$ obtained from $\pi_a, \ldots, \pi_b$ are in weakly increasing order. Since among biletters from distinct such sequences order is also preserved due to the top letter, we see that each biletter $\binom{i}{\pi_i}$ of π gives rise to the i^{th} smallest biletter of $\mathbf{m}$; since π^{inv} corresponds to $(Q^{\mathrm{st}}, P^{\mathrm{st}})$, similar reasoning shows that the biletter $\binom{j}{\pi_j^{\mathrm{inv}}}$ of π^{inv} gives rise to the j^{th} smallest biletter of $\mathbf{m}^{\mathrm{inv}}$. One checks that the definition of standardisation (in particular the rule for breaking ties) now ensures that $\mathbf{m}^{\mathrm{st}} = \pi$.

This completes the proof. □

An alternative way of proving the surjectiveness of the correspondence is by enumerating both multisets of biletters and pairs of tableaux of the same shape. Littlewood's 2-quotient map will accomplish the latter.

For the case $r = 0$, it is easy to see that the definition used in the proof agrees with that of Shimozono and White [21].

Corollary 11. *The semistandard domino correspondence possesses the symmetry property:*

$$P_d^r(\mathbf{m}) = Q_d^r(\mathbf{m}^{\mathrm{inv}}).$$

Proof. This is a consequence of the definition used in the proof. □

3.2. **Dual domino-Schensted correspondence.** In this section we give a description of two dual domino-Schensted correspondences α and β. They are bijections between (multiplicity-free) sets of biletters and pairs of tableaux of the same shape, one of which is semistandard and the other is column-semistandard. The definitions of the three domino-Schensted correspondences differ only in the order on biletters used to define standardisation.

For a description of the dual RSK correspondence for Young tableaux see [24].

A domino tableau D is *column-semistandard* if its transpose is semistandard.

We define a new order $<_d$ on biletters as follows. The biletter $\binom{x}{y} <_d \binom{k}{l}$ if

(1) $x < k$, or
(2) $x = k$ and $y > l$.

Now we define two new kinds of standardisation st_α and st_β. Let $\mathbf{m}$ be a set of biletters of size n. We define $\mathbf{m}^{\mathrm{st}_\alpha} = \pi_1\pi_2\cdots\pi_n$ as follows. We set $\pi_i = j$ if the i^{th} largest biletter of $\mathbf{m}$ under $<_d$ becomes the j^{th} largest biletter of $\mathbf{m}^{\mathrm{inv}}$ under $<$ when we take inverses. Since $\mathbf{m}$ is multiplicity free, we do not need to worry about ties. Similarly we define $\mathbf{m}^{\mathrm{st}_\beta}$ by using $<$ as the order for $\mathbf{m}$ and $<_d$ as the order for $\mathbf{m}^{\mathrm{inv}}$. In both cases the barred-ness of individual biletters is preserved as for the original standardisation.

The inverse $\mathbf{m}^{\mathrm{inv}}$ of a set of biletters is defined as for multisets of biletters.

We may now define the two dual domino-Schensted correspondences α and β. Let $\mathbf{m}$ be a multiplicity-free set of biletters. We define $P^r_\alpha(\mathbf{m})$ to be the unique semistandard tableau which satisfies $P^r_\alpha(\mathbf{m})^{\mathrm{st}} = P^r_d(\mathbf{m}^{\mathrm{st}_\alpha})$ and the usual equality of weights. We define $Q^r_\alpha(\mathbf{m})$ to be the unique column-semistandard tableau satisfying $Q^r_\alpha(\mathbf{m})^{\mathrm{st}} = Q^r_d(\mathbf{m}^{\mathrm{st}_\alpha})$.

We define the correspondence β in the same way, replacing st_α by st_β, and requiring that $P^r_\beta(\mathbf{m})$ be column-semistandard and $Q^r_\beta(\mathbf{m})$ be semistandard. That α and β are unique and well-defined is part of Theorem 12.

Note that both correspondences agree with the usual domino correspondence when applied to hyperoctahedral permutations.

Theorem 12. *Let $r \geq 0$ be fixed. The map α*

$$\alpha : \mathbf{m} \to (P^r_\alpha(\mathbf{m}), Q^r_\alpha(\mathbf{m}))$$

is a weight preserving bijection between multiplicity-free sets of biletters $\mathbf{m}$ of length n and pairs of tableaux (P, Q) of the same shape $\lambda \in \mathcal{P}_r(n)$ such that P is semistandard and Q is column-semistandard.

The map β

$$\beta : \mathbf{m} \to (P^r_\beta(\mathbf{m}), Q^r_\beta(\mathbf{m}))$$

is a weight preserving bijection between multiplicity-free sets of biletters $\mathbf{m}$ of length n and pairs of tableaux (P, Q) of the same shape $\lambda \in \mathcal{P}_r(n)$ such that P is column-semistandard and Q is semistandard.

These maps satisfy the following properties:

(1) *They commute with standardisation (by definition). Thus*

$$(P^r_\alpha(\mathbf{m})^{\mathrm{st}}, Q^r_\alpha(\mathbf{m})^{\mathrm{st}}) = (P^r_d(\mathbf{m}^{\mathrm{st}_\alpha}), Q^r_d(\mathbf{m}^{\mathrm{st}_\beta}))$$

and similarly for β.

(2) *The maps α and β are related by*

$$(Q^r_\alpha(\mathbf{m}), P^r_\alpha(\mathbf{m})) = (P^r_\beta(\mathbf{m}^{\mathrm{inv}}), Q^r_\beta(\mathbf{m}^{\mathrm{inv}})).$$

(3) *Both maps have the color-to-spin property.*

Proof. The proof is analogous to that of Theorem 3, requiring use of Lemmas 9 and 10. □

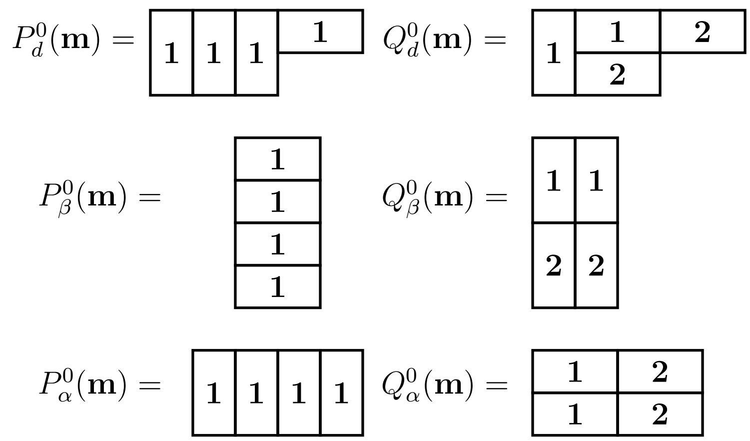

FIGURE 4. The three domino-Schensted correspondences for $\mathbf{m} = \left\{\binom{1}{1}, \binom{1}{1}, \binom{2}{1}, \binom{2}{1}\right\}$.

As an example we have calculated the insertion and recording tableaux for all three correspondences in Figure 4 for the set of biletters $\mathbf{m} = \left\{\binom{1}{1}, \binom{1}{1}, \binom{2}{1}, \binom{2}{1}\right\}$.

3.3. **Statistics on Domino Tableaux.** In this subsection we will introduce and study a number of statistics on partitions and domino tableaux. Let λ be a partition with 2-core $\tilde{\lambda}$. Let $o(\lambda)$ be the number of odd rows of λ . Thus $o(\lambda')$ is the number of odd columns. Let

$$d(\lambda) = \sum_{i=1}^{l(\lambda)/2} \left\lfloor \frac{\lambda_{2i}}{2} \right\rfloor.$$

Note that $d(\lambda) = d(\lambda')$ (see for example [25]). Also let

$$v(\lambda) = \sum_{i=1}^{l(\lambda')} \left\lfloor \frac{\lambda'_i}{2} \right\rfloor = \sum_{i=1}^{l(\lambda)/2} \lambda_{2i}.$$

Now let D be a domino tableau of shape λ. As before $v(D)$ is the number of vertical dominoes in D and $sp(D) = v(D)/2$. Let $ov(D)$ and $ev(D)$ be the number of vertical dominoes in odd and even columns respectively. Thus $sp(D) = (ov(D) + ev(D))/2$. Let $mspin(\lambda)$ be the maximum spin over all domino tableaux of shape λ. Similarly, let $ov(\lambda)$ be the maximum of $ov(D)$ over all domino tableau of shape λ. Define $ev(\lambda)$ similarly. The cospin of a domino tableau D is $cosp(D) = mspin(\lambda) - sp(D)$ (and is always an integer).

The following lemma is a strengthening of a lemma in [27].

Lemma 13. *Let D be a domino tableau of shape λ with 2-core $\tilde{\lambda}$. Then*

$$ov(D) - ev(D) = \frac{o(\lambda) - o(\tilde{\lambda})}{2}. \tag{1}$$

Proof. We proceed by induction on the size of λ, while keeping $\tilde{\lambda}$ fixed. When D has shape $\tilde{\lambda}$ then both sides are 0. Now let D have shape λ and suppose the Lemma is true for all shapes μ that can be obtained from λ by removing a domino. Let γ be the domino with the largest value in D. Removing γ from D gives a domino tableau D' for which (1) holds. If γ is a horizontal domino then neither side changes. If γ is a vertical domino in an odd row then both sides decrease by 1 (changing from D to D'). If γ is a vertical domino in an even row then both sides increase by 1. □

Note that this implies that a domino tableau D which has the maximum spin (amongst all domino tableaux of shape λ) will also have the most number of odd vertical and even vertical dominoes. Thus for example, $mspin(\lambda) = ev(\lambda) + ov(\lambda)$.

3.4. **Symmetric Growth Diagrams.** We now specialise to the case where the matrix $M_\pi(i,j)$ corresponds to a hyperoctahedral involution π. Thus $M_\pi(i,j)$ is symmetric and π satisfies $\pi^2 = 1$ in the group B_n. The hyperoctahedral involution π will consist of a number of fixed points, barred fixed points, two-cycles and barred two-cycles. For example, let $\pi = (1\overline{6}\overline{3}54\overline{2}\overline{7})$. Then π has one fixed point, two barred fixed points, one two-cycle and one barred two-cycle.

In this case we obtain the following proposition, part of which was first observed by van Leeuwen [15, p.26].

Proposition 14. *Let $\pi \in B_n$ be a hyperoctahedral involution. Suppose π has θ fixed points, ϑ barred fixed points, ι two-cycles and κ barred two-cycles. Fix a 2-core δ_r. Let the insertion tableau $P_d^r(\pi) = Q_d^r(\pi)$ of π into δ_r have shape $\lambda = sh(P_d^r(\pi))$ (which satisfies $\tilde{\lambda} = \delta_r$). Then*

$$\begin{aligned} sp(P_d^r(\pi)) &= \frac{\vartheta}{2} + \kappa \\ \frac{o(\lambda) - o(\delta_r)}{2} &= \vartheta \\ \frac{o(\lambda') - o(\delta_r)}{2} &= \theta \\ d(\lambda) - d(\delta_r) &= \iota + \kappa. \end{aligned}$$

Proof. Since $P_d^r(\pi) = Q_d^r(\pi)$ for a hyperoctahedral involution by Lemma 7, the first equation is a consequence of the color-to-spin property of Theorem 2. For the other statements, note that the symmetry of $M_\pi(i,j)$ and of the local rules of the growth diagram imply that the growth diagram $\lambda_{(i,j)}$ itself is symmetric. We focus our attention on the partitions $\lambda_{(i,i)}$. If $M_\pi(i,i) = 1$ then $\lambda_{(i+1,i+1)}$ has two boxes added to its first row, and so $o(\lambda'_{(i+1,i+1)}) = o(\lambda'_{(i,i)}) + 2$. Similarly, if $M_\pi(i,i) = -1$ then $o(\lambda_{(i+1,i+1)}) = o(\lambda_{(i,i)}) + 2$. In both cases $d(\lambda_{(i,i)}) = d(\lambda_{(i+1,i+1)})$.

If $M_\pi(i,i) = 0$ and $\lambda_{(i+1,i)} = \lambda_{(i,i)} = \lambda_{(i,i+1)}$ then $\lambda_{(i,i)} = \lambda_{(i+1,i+1)}$. The only remaining case is if $\lambda_{(i+1,i)}$ differs from $\lambda_{(i,i)}$ by a domino, in which case $\lambda_{(i,i+1)} = \lambda_{(i+1,i)}$ as well. This implies that $\lambda_{(i+1,i+1)}$ differs from $\lambda_{(i,i)}$ by two dominoes, that are either in two adjacent columns or in two adjacent rows. Regardless, the number of odd columns and rows is unchanged while $d(\lambda_{(i+1,i+1)}) = d(\lambda_{(i,i)}) + 1$. □

Corollary 15. *Let $D = P_d(\pi)$ correspond to a hyperoctahedral involution π with ϑ barred fixed points and κ barred two-cycles. Then*

$$ev(D) = \kappa.$$
$$ov(D) = \vartheta + \kappa.$$

Proof. As before, let π have ϑ barred fixed points. Then by Proposition 14,

$$ev(D) + ov(D) = 2sp(D) = \vartheta + 2\kappa.$$

Combining Lemma 13 with Proposition 14 again we have,

$$ov(D) - ev(D) = \frac{o(\lambda) - o(\tilde{\lambda})}{2} = \vartheta.$$

Subtracting the two equations and dividing by two, we obtain the first result. Summing the two equations give the second result. □

The significance of this Corollary will become apparent in Section 4.

3.5. **Some Enumeration for Domino Tableaux.** Let f^λ be the number of SYT of shape λ. The Robinson-Schensted algorithm for standard Young tableaux (SYT) leads to a number of enumerative results including the following well known result.

Proposition 16. *Let $n \geq 1$. Then*

$$\sum_{\lambda \vdash n} (f^\lambda)^2 = n!. \tag{2}$$

$$\sum_{\lambda \vdash n} f^\lambda = t(n). \tag{3}$$

We can easily generalise these to domino tableaux. Define

$$f_2^\lambda(q) = \sum_D q^{sp(D)}$$

where the sum is over all standard domino tableaux D of shape λ. It is unlikely that a 'hook-length' formula holds for $f_2^\lambda(q)$. Note that $f_2^\lambda(q)$ depends on more than just the 2-quotient $(\lambda^{(0)}, \lambda^{(1)})$ of λ. For example, $(3,1,1)$ and $(2,2)$ have the same 2-quotient but $f_2^{(3,1,1)}(q) = 2q^{1/2}$ and $f_2^{(2,2)}(q) = 1 + q$. A cospin version of $f_2^\lambda(q)$ for more general ribbon tableaux was studied by Schilling, Shimozono and White in [20].

We have the following analogue of (2):

Proposition 17. *Let $n \geq 1$ and $r \geq 0$ be fixed. Then*

$$\sum_\lambda \left(f_2^\lambda(q)\right)^2 = (1+q)^n n!$$

where the sum is over all partitions $\lambda \in \mathcal{P}_r(n)$.

Proof. This is an immediate consequence of the bijection in Theorem 2. □

The $q = -1$ specialisation of Proposition 17 has an interpretation in terms of sign-imbalance (see Corollary 24).

Now define $h_r(n)$ as follows:

$$h_r(n) = \sum_{\lambda \in \mathcal{P}_r(n)} a^{(o(\lambda)-o(\delta_r))/2} b^{(o(\lambda')-o(\delta_r))/2} c^{d(\lambda)-d(\delta_r)} f_2^{\lambda}(q).$$

When $a = b = c = q = 1$, this is the number of hyperoctahedral involutions in B_n and thus a domino analogue of $t(n)$.

Proposition 18. *The function $h(n) = h_r(n)$ does not depend on r. It satisfies the recursion*

$$h(n+1) = (b + aq^{1/2})h(n) + nc(1+q)h(n-1).$$

The exponential generating function defined as

$$E_h = \sum h(n)\frac{t^n}{n!}$$

is given by the formula

$$E_h = exp\left((b + aq^{1/2})t + c(1+q)\frac{t^2}{2}\right).$$

Proof. That $h_r(n)$ does not depend on r follows from the fact that the tableau being enumerated are in bijection with hyperoctahedral involutions. Furthermore, the bijection preserves the appropriate weighting according to Proposition 14. Thus we are in fact enumerating hyperoctahedral involutions.

The recursion for $h(n)$ is immediate from the construction of a hyperoctahedral involution from barred and non-barred fixed points and two-cycles.

For the exponential generating function, we can use the exponential formula (see [24, Corollary 5.1.6]). Thus we think of a hyperoctahedral involution as a partition of $[n]$ into one and two element subsets. The one element subsets can be given a weight of b or $aq^{1/2}$ while the two element subsets can be given a weight of c or cq. □

We remark that the usual exponential generating function for the number of involutions in S_n is $exp(t + t^2/2)$. Setting $a = b = q^{1/2} = c = 1$ in Proposition 18, we confirm that the generating function for the number of hyperoctahedral involutions is its square.

4. Sign-Imbalance and Stanley's Conjecture

Sign imbalance can be defined for posets in general, but we will only concern ourselves with the posets arising from partitions.

Let T be a standard Young tableau. Its reading word $reading(T)$, for our purposes, will be obtained by reading the first row from left to right, then the second row, and so on. We set $sign(T) = sign(reading(T))$ where $reading(T)$ is treated as a permutation.

Let λ be a partition. Then we set

$$I_\lambda = \sum_T sign(T)$$

where the sum is over all standard Young tableaux T of shape λ. We say I_λ is the *sign-imbalance* of λ.

It is not difficult to see that I_λ is related to domino tableaux. Suppose λ has no 2-core, then define an involution on standard Young tableaux of shape λ by swapping $2i-1$ with $2i$ for the smallest possible value of i where this is possible. If no such swap is possible the tableau is fixed by the involution.

The fixed points correspond exactly to the standard domino tableau of shape λ. We obtain a standard Young tableau $T(D)$ from a standard domino tableau D, by filling the domino with a 1 with the values 1 and 2, the domino with a 2, with the values 3 and 4, and so on.

When λ has 2-core δ_1 (a single box) then we use an involution which swaps $2i$ with $2i+1$ for the smallest value of i where it is possible. Again, the fixed points are the standard domino tableau of shape λ.

It is easy to see that these involutions are sign-reversing on tableaux which are not fixed points and thus we obtain the following proposition.

Proposition 19. *Let $r \in \{0,1\}$, $n \geq 1$ and $\lambda \in \mathcal{P}_r(n)$. Then*

$$I_\lambda = \sum_{sh(D)=\lambda} sign(D)$$

where the sum is over standard domino tableaux of shape λ and the sign of a domino tableau D is the sign of the corresponding standard Young tableau $T(D)$. If

For other values of r, the same involutions (chosen based on he parity of $|\lambda|$) give the following result.

Proposition 20. *Let λ have 2-core δ_r for $r > 1$, then*

$$I_\lambda = 0.$$

There is another natural involution on standard Young tableaux whose fixed points can be identified with standard domino tableaux. This is Schützenberger's involution S, also known as evacuation. The fixed points of this involution are exactly in correspondence with the domino tableau of shape λ satisfying $\tilde{\lambda} = \delta_r$ for $r \in \{0,1\}$ (see [15]). For a fixed shape λ, Stanley [25] has shown that S is either always parity-reversing or parity-preserving.

By analysing the positions of horizontal and vertical dominoes in a standard domino tableau, White [27] proves the following proposition. We give a short proof of this, which was suggested by the referee.

Proposition 21. *Let D be a domino tableau of shape λ which has 2-core $\emptyset$ or δ_1. Then*

$$sign(D) = (-1)^{ev(D)}.$$

Proof. We begin with a standard Young tableau T of shape λ whose reading word is the identity permutation. Keeping the other values in reading order, we now move the two largest values (say i and $i+1$) to the location of the domino γ with the largest value in D. If γ is horizontal, then i and $i+1$ will both pass the same set of smaller values, so the sign of T does not change. If γ is vertical, then one checks that the sign changes if and only if γ is in an even column. Now we move the next largest domino into position, and so on, the analysis being identical. □

White has also given an explicit formula (in terms of shifted tableaux) for the sign-imbalance of partitions which have 'near-rectangular' shape.

Combining Proposition 21 with Corollary 15 we obtain the following theorem.

Theorem 22. *Fix $r \in \{0, 1\}$. Let π be a hyperoctahedral involution. Then the sign of its insertion tableau sign$(P_d^r(\pi))$ is equal to the number of barred 2-cycles.*

Proof. This follows immediately from Corollary 15 and Proposition 21. □

We can now prove the following conjecture of Stanley [25], known as the '$2^{\lfloor n/2 \rfloor}$' conjecture.

Theorem 23. *Let $m \geq 1$ be an integer. Then*

$$\sum_{\lambda \vdash m} x^{v(\lambda)} y^{v(\lambda')} q^{d(\lambda)} t^{d(\lambda')} I_\lambda = (x + y)^{\lfloor m/2 \rfloor}.$$

Note that $d(\lambda) = d(\lambda')$ so that one of q and t is not needed.

Proof. Since $I_\lambda = 0$ for λ with a 2-core larger than δ_1, we may assume the sum is over $\lambda \in \mathcal{P}_r(n)$, for the unique $r \in \{0, 1\}$ and n satisfying $2n + r = m$. Note that $o(\delta_1) = o(\delta_1') = 1$ and $d(\delta_1) = 0$.

The standard domino tableau of such shape correspond exactly to hyperoctahedral involutions $\pi \in B_n$. We define an involution α on all such π by turning the two-cycle (i, j) with the smallest value of i from barred to non-barred or vice versa, if such an i exists. By Theorem 22, α is sign-reversing for domino tableaux which are not fixed points. Furthermore, by Proposition 14, all of the statistics $o(\lambda) - r$, $o(\lambda') - r$ and $d(\lambda)$ remain fixed by α.

The fixed points of α are exactly the hyperoctahedral involutions without two-cycles. Hence we obtain, using Proposition 14

$$\sum a^{(o(\lambda)-r)/2} b^{(o(\lambda')-r)/2} c^{d(\lambda)} I_\lambda = (a + b)^n.$$

To change this into the form of Stanley's conjecture, observe that $2v(\lambda) + o(\lambda) = m = 2n + r$ implying that $(o(\lambda) - r)/2 = n - v(\lambda)$ and similarly for $v(\lambda')$ and $o(\lambda')$. Now substitute this and also $x = 1/a$ and $y = 1/b$. Finally multiply both sides by $(xy)^n$. □

Theorem 23 is compatible with the involution on B_n which changes barred letters to non-barred letters and vice-versa. This operation preserves hyperoctahedral involutions, and transposes the corresponding insertion tableau.

Note that the fixed points of α in the proof are exactly the domino tableaux which are hook shaped: each such tableau D with $v(D)$ vertical dominoes and $h(D)$ horizontal dominoes contributes a term $x^{v(D)} y^{h(D)}$. That these give the right hand side of the conjecture was shown by Stanley [25]. When we set $x = y = q = 1$ we obtain the following signed analogue of (3):

$$\sum_{SYT\,T} sign(T) = 2^{\lfloor n/2 \rfloor}$$

where the sum is over all standard Young tableaux T of size n.

As a corollary of Proposition 17 we also obtain Theorem 3.2(b) and the $t = 1$ case of Conjecture 3.3(b) of [25].

Corollary 24. *Let $n \geq 1$ be a positive integer. Then*

$$\sum_{\lambda \vdash n} (-1)^{v(\lambda)} \left(I_\lambda\right)^2 = 0.$$

Proof. Let D be a standard domino tableau of shape $\lambda \vdash n$. By Proposition 21, $sign(D) = (-1)^{ev(D)}$. Now, $sp(D) = (ev(D) + ov(D))/2$ and by Lemma 13 $ov(D) - ev(D) = (o(\lambda) - o(\tilde{\lambda}))/2$ giving

$$sp(D) = ev(D) + \frac{o(\lambda) - o(\tilde{\lambda})}{4}.$$

Thus $sign(D) = (-1)^{sp(D)}(-1)^{(o(\lambda)-o(\tilde{\lambda}))/4}$. (This may involve -1 to the power of a half integer, which we can consider to be some fixed square root of -1.) Thus summing over all standard domino tableaux of shape λ we get

$$f^\lambda(-1) = (-1)^{(o(\lambda)-o(\tilde{\lambda}))/4} I_\lambda. \tag{4}$$

Now we note that when $\tilde{\lambda} = \delta_r$ for $r \in \{0, 1\}$, we have $v(\lambda) \equiv o(\lambda) - o(\tilde{\lambda})/2 \mod 2$ which can easily be estalbished by induction. Squaring (4), and summing over $\lambda \vdash n$ we obtain

$$\sum_{\lambda \vdash n} \left(f^\lambda(-1)\right)^2 = \sum_{\lambda \vdash n} (-1)^{v(\lambda)} \left(I_\lambda\right)^2,$$

using Proposition 20. Thus the Corollary follows from setting $q = -1$ in Proposition 17. ☐

Similar results were also obtained by Reifegerste [19] and Sjöstrand [23].

5. Domino Generating Functions

Let Λ denote the ring of symmetric functions in a set of variables $X = (x_1, x_2, \ldots)$ taking coefficients in $\mathbb{C}[q^{1/2}]$ (though the coefficient field will not affect the results). Its completion, $\tilde{\Lambda}$ includes symmetric power series of unbounded degree (though the coefficient of a monomial m_λ will always be well defined).

Carré and Leclerc [2] have defined symmetric functions $H_\lambda(X; q)$ via semistandard domino tableaux, in the same way that Schur functions arise from semistandard Young tableaux. Slightly more general functions $G_\lambda(X; q)$ were used in [11] and the two are connected via $H_\lambda(X; q) = G_{2\lambda}(X; q)$. In fact, [11] defines these functions much more generally for p-ribbon tableaux.

Let λ be a partition. Define

$$G_\lambda = \sum_D q^{sp(D)} x^{wt(D)}$$

where the sum is over all semistandard domino tableaux of shape λ and $x^\mu := x_1^{\mu_1} x_2^{\mu_2} \dots$ for a partition μ. There is a cospin version of this function which we will not need. In the notation of [11], our G_λ would be denoted $G_{\lambda/\tilde{\lambda}}$.

That the G_λ are symmetric functions is a consequence of a combinatorial interpretation of their expansion into Schur functions given by Carré and Leclerc. We will call the G_λ *domino functions*. Theorem 3 leads immediately to the following *domino Cauchy identity*.

Proposition 25. *Fix $r \geq 0$. Then*

$$\sum_{\lambda \in \mathcal{P}_r} G_\lambda(X; q) G_\lambda(Y; q) = \frac{1}{\prod_{i,j}(1 - x_i y_j)(1 - q x_i y_j)}.$$

The dual domino-Schensted correspondence of Theorem 12 leads to the following *dual domino Cauchy identity.*

Proposition 26. *Fix $r \geq 0$. Then*

$$\sum_{\lambda \in \mathcal{P}_r} q^{|\lambda/\delta_r|/2} G_\lambda(X; q) G_{\lambda'}(Y; q^{-1}) = \prod_{i,j}(1 + x_i y_j)(1 + q x_i y_j).$$

Proof. This follows from the fact that column-semistandard domino tableaux D are in bijection with semistandard domino tableaux D' of the conjugate shape with spin given by

$$sp(D') = \frac{m}{2} - sp(D),$$

where m is the number of dominoes in the tableau. $\square$

These results are generalised to p-ribbons for any p in [10], using algebraic methods.

In [9], Kirillov, Lascoux, Leclerc and Thibon give two product expansions for certain sums of the G_λ. These will be seen as specialisations of our Theorem 28. As the paper [9] contains no proofs, our theorem can be considered both as a proof and as a generalisation.

We will call a multiset of biletters $\mathbf{m}$ a colored involution if $\mathbf{m} = \mathbf{m}^{\text{inv}}$. We begin by studying closely the effect of standardisation on a colored involution.

Every such colored involution is given by the number of fixed points $\binom{i}{i}$, barred fixed points $\binom{i}{\bar{i}}$, two-cycles $\binom{i}{j}\dots\binom{j}{i}$ and barred two-cycles $\binom{i}{\bar{j}}\dots\binom{j}{\bar{i}}$. Let there be a_i, b_i, c_{ij} and d_{ij} of these respectively. Thus $c_{ij} = c_{ji}$ and $d_{ij} = d_{ji}$.

Lemma 27. *Let $\mathbf{m}$ be a colored involution. Then its standardisation $\mathbf{m}^{\text{st}}$ is a hyperoctahedral involution with θ fixed points, ϑ barred fixed points, ι two-cycles and κ barred two-cycles, where:*

$$\theta = \sum_i a_i,$$

$$\vartheta = \sum_i b_i - 2\sum_i \left\lfloor \frac{b_i}{2} \right\rfloor$$
$$\iota = \sum_{i<j} c_{ij}$$
$$\kappa = \sum_{i<j} d_{ij} + \sum_i \left\lfloor \frac{b_i}{2} \right\rfloor .$$

In other words, the only change that occurs is that of barred fixed points becoming barred two-cycles.

Proof. It is clear that $\mathbf{m}^{\mathrm{st}}$ is a hyperoctahedral involution.

Fix an integer i. Then in the multiset of biletters $\mathbf{m}$, then there are exactly

$$A = \sum_{j<i}(a_j + b_j + c_{jk} + d_{jk}) + b_i + \sum_k d_{ik} + \sum_{k<i} c_{ki}$$

biletters smaller than the fixed points of the form $\binom{i}{i}$. Exactly the same formula holds for these fixed points in $\mathbf{m}^{\mathrm{inv}}$. So by the description of how to break ties when standardising, we see that all these biletters become fixed points.

Now consider barred fixed points $\binom{i}{\bar{i}}$. There are

$$A = \sum_{j<i}(a_j + b_j + c_{jk} + d_{jk}) + \sum_{k>i} d_{ik}$$

smaller biletters. Again the same formula holds in $\mathbf{m}^{\mathrm{inv}}$. However, because of the special way in which ties are broken in the presence of a bar, the numbers assigned for the upper letters will be the reverse of the numbers assigned to the lower letters. So all but at most one of these will change from fixed points into two-cycles.

Now consider what happens to the collection of biletters of the form $\binom{i}{j}$ and $i \neq j$. We need only show that these all become two-cycles when $\mathbf{m}$ is standardised. Since $\mathbf{m}^{\mathrm{st}}$ is an involution we only need to check that these biletters do not become fixed points. Such a biletter has between

$$A = \sum_{l<i}(a_l + b_l + c_{lk} + d_{lk}) + b_i + \sum_k d_{ik} + \sum_{k<j} c_{ki}$$

and

$$B = \sum_{l<i}(a_l + b_l + c_{lk} + d_{lk}) + b_i + \sum_k d_{ik} + \sum_{k<j} c_{ki} + c_{ij} - 1$$

smaller biletters. After taking the inverse, exactly the same formula holds with i swapped with j. We see that the top and bottom letters will never get the same number via standardisation (in fact if $i < j$ then i will become a smaller number than what j becomes).

Exactly the same analysis holds for a biletter of the form $\binom{i}{\bar{j}}$ and $i \neq j$. $\square$

As an example, let $\mathbf{m}$ be the colored involution

$$\mathbf{m} = \left\{ \binom{1}{\bar{3}}, \binom{1}{3}, \binom{2}{\bar{2}}, \binom{2}{\bar{2}}, \binom{2}{\bar{2}}, \binom{3}{\bar{1}}, \binom{3}{1}, \binom{4}{5}, \binom{5}{4} \right\}.$$

with 3 barred fixed points, 2 two-cycles and 1 barred two-cycle. Then its standardisation

$$\mathbf{m}^{\mathrm{st}} = \overline{6}7\overline{5431}298 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \overline{6} & 7 & \overline{5} & \overline{4} & \overline{3} & \overline{1} & 2 & 9 & 8 \end{pmatrix}$$

has 1 barred fixed point, 2 two-cycle and 2 barred two-cycles.

Theorem 28. *Let $r \geq 0$ be fixed. Let $S(X; a, b, c, q) \in \tilde{\Lambda}[[a, b, c]]$ be the symmetric power series*

$$S(X; a, b, c, q^{1/2}) = \sum_{\lambda \in \mathcal{P}_r} a^{(o(\lambda)-o(\delta_r))/2} b^{(o(\lambda')-o(\delta_r))/2} c^{d(\lambda)-d(\delta_r)} G_\lambda(X; q).$$

Then $S(X; a, b, c, q^{1/2})$ does not depend on r and has a product formula given by

$$\frac{\prod_i (1 + a q^{1/2} x_i)}{\prod_i (1 - b x_i) \prod_i (1 - c q x_i^2) \prod_{i<j} (1 - c x_i x_j) \prod_{i<j} (1 - c q x_i x_j)}.$$

Proof. Semistandard domino tableaux are in one-to-one correspondence with colored involutions by Theorem 3 and Corollary 11. If $\mathbf{m}$ is a colored involution then the shape and spin of $P_d^r(\mathbf{m})$ is that of $P_d^r(\mathbf{m}^{\mathrm{st}})$ and thus we may use Proposition 14 and Lemma 27 to calculate the contributions each colored involution makes to the weights $o(\lambda)$, $o(\lambda')$, $d(\lambda)$ and $sp(P_d^r(\mathbf{m}))$.

Such colored involutions consist of a number of fixed points $\binom{i}{i}$ corresponding to the product $\prod_i 1/(1 - b x_i)$. The barred fixed points $\binom{i}{\bar{i}}$ correspond to the product $\prod_i (1 + a q^{1/2} x_i)/(1 - c q x_i^2)$ since according to Lemma 27 all but at most one of the barred fixed points of each weight will pair to become a two-cycle upon standardisation. The two-cycles correspond to $\prod_{i<j} 1/(1 - c x_i x_j)$ and the barred two-cycles correspond to $\prod_{i<j} 1/(1 - c q x_i x_j)$. □

There are a number of interesting specialisations. We will set $r = 0$ for the next few examples.

(1) When $a = b = c = q^{1/2} = 1$, we obtain the square of a well known identity:

$$\left(\sum_{\lambda \in \mathcal{P}} s_\lambda(X) \right)^2 = \left(\frac{1}{\prod_i (1 - x_i) \prod_{i<j} (1 - x_i x_j)} \right)^2.$$

(2) Substituting $q^{1/2} = 0$ and using the fact that $G_\lambda(X; 0) = s_\mu(X)$ for λ which satisfy $\lambda = 2\mu$ (see [2]), while $G_\lambda(X; 0) = 0$ for other $\lambda \in \mathcal{P}_0$, we get

$$\sum_{\lambda \in \mathcal{P}} b^{o(\lambda)} c^{v(\lambda)} s_\lambda(X) = \frac{1}{\prod_i (1 - b x_i) \prod_{i<j} (1 - c x_i x_j)}.$$

This is another well known identity which can be proved using growth diagrams for normal RSK.

(3) The case $b = c = 1$ and $a = 0$ picks out the G_λ of the form $G_{2\mu} = H_\mu$ and we obtain the first formula of [9]:

$$\sum_\lambda H_\lambda(X; q) = \frac{1}{\prod_i (1 - x_i) \prod_{i<j} (1 - x_i x_j) \prod_{i \leq j} (1 - q x_i x_j)}.$$

(4) The case $a = b = 0$ and $c = 1$ picks out the partitions of the form $2\lambda \vee 2\lambda$ giving us the second formula of [9]:

$$\sum_{\lambda} H_{\lambda\vee\lambda}(X;q) = \frac{1}{\prod_{i<j}(1-x_ix_j)\prod_{i\leq j}(1-qx_ix_j)}.$$

(5) The case $a = c = 1$ and $b = 0$ picks out the G_λ of the form $G_{\mu\vee\mu}$ and we obtain:

$$\sum_{\lambda} G_{\lambda\vee\lambda}(X;q) = \frac{\prod_i(1+q^{1/2}x_i)}{\prod_i(1-qx_i^2)\prod_{i<j}(1-x_ix_j)\prod_{i<j}(1-qx_ix_j)}.$$

Note that while $\sum G_\lambda$ over $\lambda \in \mathcal{P}_r(n)$ does not depend on r, the individual G_λ can differ greatly. In particular, two partitions λ and μ with the same 2-quotient but with $\tilde{\lambda} \neq \tilde{\mu}$ may not have the same G function. For example, $G_{(2,2)} = qs_2 + s_{1,1}$ while $G_{(3,1,1)} = q^{1/2}(s_2 + s_{1,1})$. Both $(2,2)$ and $(3,1,1)$ have 2-quotient $\{(1),(1)\}$.

6. Ribbon Tableaux

In this last section we make a few remarks concerning which results might be generalised to ribbon tableaux. We refer the reader to [11] for the important definitions.

Shimozono and White [22] also give a spin-preserving insertion algorithm for standard ribbon tableaux. Subsequently, van Leeuwen [16] has found a full spin-preserving Knuth-correspondence for ribbon tableaux. Focusing on the standard correspondence only, we get a spin-preserving bijection between pairs of standard ribbon tableaux and permutations π of the wreath product $C_p \S S_n$. Again the involutions are in bijection with standard ribbon tableaux and thus we obtain a p-ribbon analogue of Proposition 18 with an identical proof.

Proposition 29. *Let $h(n)$ be the polynomial in q defined as*

$$h(n) = \sum_T q^{sp(T)}$$

where the sum is over all standard ribbon tableaux of size n (and fixed p-core). Then $h(n)$ satisfies the recurrence

$$h(n+1) = (1 + q^{1/2} + \ldots + q^{(p-1)/2})h(n) + n(1 + q + \ldots + q^{p-1})h(n-1)$$

and has exponential generating function

$$E_h(t) = exp\left((1 + q^{1/2} + \ldots + q^{(p-1)/2})t + (1 + q + \ldots + q^{p-1})\frac{t^2}{2}\right).$$

The statistics $o(\lambda)$ and $d(\lambda)$ are no longer suitable for longer ribbons. It seems likely that the statistic

$$o_k(\lambda) = \#\{i : \lambda_i \equiv k \mod p\}$$

may be interesting, but we have been unable to find any applications.

Possibly more promising is the following potential generalisation. The sums over standard Young tableaux of size n

$$\begin{aligned}\sum_T 1 &= t(n) \\ \sum_T sign(T) &= 2^{\lfloor n/2 \rfloor}\end{aligned}$$

suggest that we might consider the sum

$$\sum_T \chi(reading(T))$$

for some other character χ of S_n. If this were to be related to p-ribbon tableaux and the wreath product $C_p \S S_n$ then χ should take p^{th} roots of unity as its values. One possibility is the (virtual) character which on the conjugacy class of cycle type λ takes the value

$$\chi(C_\lambda) = \omega^{\lambda - l(\lambda)}$$

for some p^{th} root of unity ω.

References

[1] D. Barbasch, D. Vogan, Primitive ideals and orbital integrals on complex classical groups, *Math. Ann.*, **259** (1982), pp. 153-199.

[2] C. Carré and B. Leclerc, Splitting the square of a Schur function into its symmetric and antisymmetric parts, *J. Alg. Combin.*, **4** (1995), pp.201-231.

[3] S. Fomin, Duality of graded graphs, *J. Algebraic Combin.*, **3** (1994), pp. 357-404.

[4] S. Fomin, Schensted algorithms for dual graded graphs, *J. Algebraic Combin.*, **4** (1995), pp. 5-45.

[5] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts **35**, Cambridge University Press (1997).

[6] D. Garfinkle, On the classification of primitive ideals for complex classical Lie algebras II, *Compositio Math.*, **81** (1992) pp. 307-336.

[7] J. Haglund, M. Haiman, N. Loehr, J.B. Remmel, A. Ulyanov, A combinatorial formula for the character of the diagonal coinvariants, preprint, 2003, math.CO/0310424.

[8] M. Haiman, On mixed insertion, symmetry, and shifted Young tableaux, *J. Combin. Theory Ser. A* **50** (1989), pp. 709-727.

[9] A.N. Kirillov, A. Lascoux, B. Leclerc, and J.-Y. Thibon, Séries génératrices pour les tableaux de dominos, *C. R. Acad. Sci. Paris*, Série I **318** (1994), pp.395-400.

[10] T. Lam, Ribbon tableaux and the Heisenberg algebra, preprint, 2003; math.QA/0310250.

[11] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Ribbon tableaux, Hall-Littlewood symmetric functions, quantum affine algebras, and unipotent varieties, *J. Math. Phys.*. **38**(3) (1997), pp. 1041-1068.

[12] B. Leclerc Symmetric functions and the Fock space representation of $U_q(\widehat{\mathfrak{sl}}_n)$, *Lectures given at the Isaac Newton Institute* (2001).

[13] B. Leclerc, J.-Y. Thibon, Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials; *Combinatorial Methods in Representation Theory*, Advanced Studies in Pure Mathematics 28, (2000), 155-220.

[14] B. Leclerc, J.-Y. Thibon, Canonical bases of q-deformed Fock spaces, *Int. Math. Res. Notices*, **9** (1996), 447-456.

[15] M. vanLeeuwen, The Robinson-Schensted and Schutzenberger algorithms, an elementary approach, The Foata Festscrhift, *Electron. J. Combin.* **3(2)** (1996), Research Paper 15.

[16] M. vanLeeuwen, Spin-preserving Knuth correspondences for ribbon tableaux, preprint, 2003; math.CO/0312020.

[17] D.E. Littlewood, Modular representations of symmetric groups, *Proc. Royal Soc. London Ser. A*, **209** (1951), pp. 333-353.

[18] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, 1995.

[19] A. Reifegerste, Permutation sign under the Robinson-Schensted-Knuth correspondence, preprint, 2003; math.CO/0309266.

[20] A. Schilling, M. Shimozono and D.E. White, Branching formula for q-Littlewood-Richardson coefficients, *Advances in Applied Mathematics*, **30** (2003), pp. 258-272.

[21] M. Shimozono and D.E. White, A color-to-spin domino Schensted algorithm, *Electron. J. Combinatorics*, **8** (2001).

[22] M. Shimozono and D.E. White, Color-to-spin ribbon Schensted algorithms, *Discrete Math.*, **246** (2002), pp. 295-316.

[23] J. Sjöstrand, On the imbalance of partition shapes, preprint, 2003; math.CO/0309231.

[24] R. Stanley, *Enumerative Combinatorics, Vol 2*, Cambridge, 1999.

[25] R. Stanley, Some Remarks on Sign-Balanced and Maj-Balanced Posets, preprint, 2002; math.CO/0211113.

[26] D. Stanton and D. White, A Schensted algorithm for rim-hook tableaux, *J. Combin. Theory Ser. A.*, **40** (1985), pp. 211-247.

[27] D. White, Sign-balanced posets, *J. Combinatorial Theory Ser. A.*, to appear.

Department of Mathematics, M.I.T., Cambridge, MA 02139

E-mail address: thomasl@math.mit.edu