(1) If $p$ is prime and $G$ is a finite abelian group of order divisible by $p$, then show that $G$ contains an element of order $p$ via the following approach: induct on the order of $G$, using the fact that by inductive hypothesis the result is true for $G/H$ whenever $H$ is a nontrivial normal subgroup of $G$.

Note: the “trivial subgroup” of a group $G$ is the subgroup generated by (and consisting solely of) the identity element of $G$.

(2) Let $p$ be prime, and let $G$ be a finite group of order divisible by $p$. Show that $G$ contains an element of order $p$ via the following approach:

(a) show that the set

$$S := \{(x_1, x_2, \ldots, x_p) : x_i \in G, x_1x_2\cdots x_p = 1\}$$

has size divisible by $p$

(b) show that the subset

$$T := \{(x_1, x_2, \ldots, x_p) \in S : \text{there exist } i, j \text{ with } x_i \neq x_j\}$$

has size divisible by $p$

(c) show that the complement $S \setminus T$ is nonempty

(d) conclude that $G$ contains an element of order $p$.

(3) Show that if $n$ is a positive integer which is neither 1 nor prime, then there exists a finite group $G$ of order divisible by $n$ which does not contains an element of order $n$.

(4) Show that if $H$ is a subgroup of $G$ then for each $g \in G$ the set $gHg^{-1}$ is a subgroup of $G$. Give a “nice” description of the set of all $g \in G$ for which $gHg^{-1} = H$. Show that this set is a subgroup of $G$.

(5) Show that if $H, J$ are normal subgroups of a group $G$ such that $H \cap J = 1$, then the group generated by $H$ and $J$ is isomorphic to the direct product $H \times J$.

You may use Prop. 2.11.4 of Artin, although it would be better to try to prove this on your own before looking up that result.

(6) Let $G$ be a finite group and let $N$ be a nontrivial normal subgroup of $G$ which does not properly contain another nontrivial normal subgroup of $G$. Show that there is a simple group $L$ and a positive integer $k$ for which $N$ is isomorphic to the direct product $L \times L \times \cdots \times L$ of $k$ copies of $L$.

(7) Determine the group of all automorphisms of $\mathbb{Z}/n\mathbb{Z}$, for each positive integer $n$.

(Partial credit for doing this just when $n$ is prime.)