(1) Let $G$ be the group of rotational symmetries of a cube. Explicitly, the elements of $G$ are rotations of $\mathbb{R}^3$ about a line such that the image of the cube under the rotation occupies the same position in $\mathbb{R}^3$ as the cube did originally; here the line we’re rotating around can be different for different elements of $G$. Show that $G$ acts on the set of four main diagonals of the cube, and that this action has trivial kernel and induces an isomorphism $G \cong S_4$.

(2) For any finite group $G$, consider a chain of subgroups $G = N_0 > N_1 > N_2 > \cdots > N_k = 1$, where if $1 \leq i \leq k$ then $N_i$ is a maximal proper normal subgroup of $N_{i-1}$ (i.e., a normal subgroup of $N_{i-1}$ which doesn’t equal $N_{i-1}$, and for which there is no normal subgroup of $N_{i-1}$ lying strictly between $N_i$ and $N_{i-1}$). Show that each quotient group $N_{i-1}/N_i$ is simple. Further, if $G = M_0 > M_1 > M_2 > \cdots > M_\ell = 1$ is another chain of subgroups where $M_j$ is a maximal proper normal subgroup of $M_{j-1}$, then show that $k = \ell$ and that the sequence of quotient groups $N_0/N_1, N_1/N_2, \ldots, N_{k-1}/N_k$ is a permutation of the sequence $M_0/M_1, M_1/M_2, \ldots, M_{\ell-1}/M_\ell$, in the sense that there is some $\sigma \in S_k$ for which $N_{i-1}/N_i \cong M_{\sigma(i)-1}/M_{\sigma(i)}$ for all $i$.

(3) For each $k$ with $1 \leq k \leq 4$, and each $n \geq k$, how many orbits does $S_n$ have on the set of ordered $k$-tuples of (not necessarily distinct) elements of $\{1, 2, \ldots, n\}$, via the action $g \star (a_1, \ldots, a_k) = (g(a_1), \ldots, g(a_k))$?

(4) Let $\mathbb{Z}^2$ act on $\mathbb{R}^2$ via $(a, b) \star (x, y) = (a + x, b + y)$. Give a geometric/topological description for the “shape” of the set of all orbits, in which each orbit is viewed as a point. (You don’t need to be rigorous in this problem.)

(5) For any prime $p$, describe all groups of order $p^2$ (up to isomorphism) (Hint: first show that every such group is abelian.)

(6) Let $N$ be a normal subgroup of a group $G$ with $[G : N] = n$. Show that $g^n \in N$ for each $g \in G$. Is the same conclusion true without requiring $N$ to be normal?

(7) If $N$ is a subgroup of a group $G$, then show that $N$ is normal in $G$ if and only if $N$ is a union of conjugacy classes of $G$.

(8) Give a representative for each conjugacy class of $A_5$, and determine the size of the conjugacy class. Show that $A_5$ is simple (i.e., it has no nontrivial proper normal subgroups). Deduce that if $H$ is a proper subgroup of $A_5$ then $[A_5 : H] \geq 5$. 