(1) For any action of a finite group $G$ on a finite set $S$, describe the character of the corresponding permutation representation in terms of data about $G$ and $S$. Then compute the character of the regular representation of a finite group $G$ (meaning the permutation representation associated to the action of $G$ on itself by left multiplication).

(2) Let $V$ and $W$ be finite-dimensional complex representations of a finite group $G$. Show that there are induced $G$-representations on the dual space $V^*$ (i.e., the set of $\mathbb{C}$-linear maps $V \to \mathbb{C}$), the space $\text{Hom}(V, W)$ of $\mathbb{C}$-linear maps $V \to W$, and the tensor product $V \otimes W$ (which may be viewed as all $\mathbb{C}$-linear combinations of formal symbols $v \otimes w$ with $v \in V$ and $w \in W$, subject to the relations $(\lambda v_1 + v_2) \otimes w = \lambda (v_1 \otimes w) + v_2 \otimes w$ and $v \otimes (\lambda w_1 + w_2) = \lambda (v \otimes w_1) + v \otimes w_2$ for $\lambda \in \mathbb{C}$). Writing $\chi_M$ for the character of a $G$-representation on $M$, show:

(a) $\chi_{V \oplus W} = \chi_V + \chi_W$
(b) $\chi_{\text{Hom}(V, W)} = \overline{\chi_V} \cdot \chi_W$
(c) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
(d) $\chi_{V^*} = \overline{\chi_V}$.

(3) Compute the characters of the irreducible complex representations of $S_3$ (note: one can avoid computation by using the above problems). Check that these characters are orthonormal under the inner product defined at the end of Wednesday’s class.

(4) Show that the character $\chi$ of a finite-dimensional complex representation of a finite group $G$ is constant on conjugacy classes of $G$, in the sense that $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$. Assuming the last theorem stated in class on Wednesday, deduce:

(a) The number of irreducible (finite-dimensional) representations of $G$ is at most the number of conjugacy classes of $G$.
(We will show next week that equality holds.)
(b) If $V$ and $W$ are finite-dimensional representations of $G$ with $V$ irreducible, then the number of copies of $V$ in an expression of $W$ as the sum of irreducible representations is $(\chi_V, \chi_W)$.
(c) Two finite-dimensional representations of $G$ have the same character if and only if they are isomorphic representations.
(d) A finite-dimensional representation $V$ of $G$ is irreducible if and only if $(\chi_V, \chi_V) = 1$.

(5) Assuming the last theorem stated in class on Wednesday, show that if $G$ is a finite group then the regular representation $V$ of $G$ (over $\mathbb{C}$) decomposes (as a $G$-representation) as $V \cong W_1^{\dim W_1} \oplus \ldots W_r^{\dim W_r}$ where the $W_i$’s comprise all the different irreducible $G$-representations (up to isomorphism). Deduce that $|G| = \sum_{i=1}^r (\dim W_i)^2$. 