1. We can write every positive integer $n$ in exactly one way as $2^a \cdot \prod_{j=1}^r p_j^{e_j} \cdot \prod_{\ell=1}^s q_{\ell}^{f_{\ell}}$ where $a, r, s \in \mathbb{N}_0$, the $p_j$'s are primes congruent to 1 mod 4 with $p_1 < p_2 < \cdots < p_r$, and the $q_{\ell}$'s are primes congruent to 3 mod 4 with $q_1 < q_2 < \cdots < q_s$. For each $n \in \mathbb{N}$, give a formula for the number of pairs $(x, y)$ of nonnegative integers such that $x^2 + y^2 = n$ and $0 \leq x \leq y$.

**Hint:** The formula should only involve the values of $a$, the $e_j$'s, and the $f_{\ell}$'s. Use unique factorization in $\mathbb{Z}[i]$. Also use the fact proved in class, that every prime number congruent to 3 mod 4 is a prime in $\mathbb{Z}[i]$, and every prime number $p$ which is not congruent to 3 mod 4 can be written as $p = a^2 + b^2$ with $a, b \in \mathbb{N}$, in which case $a + bi$ and $a - bi$ are primes in $\mathbb{Z}[i]$. Moreover, every prime in $\mathbb{Z}[i]$ is a unit times one of the primes named in the previous sentence. Also use that if $x \in \mathbb{Z}[i]$ and $\bar{x}$ is its complex conjugate then $N(x) := x\bar{x}$ is a nonnegative integer such that $N(x) = 0$ precisely when $x = 0$, and $N(x) = 1$ precisely when $x$ is a unit, and also $N(xy) = N(x) \cdot N(y)$. The last assertion follows from the easy-to-verify fact that $|xy| = \bar{x} \cdot \bar{y}$.

2. Describe the prime elements in $\mathbb{Z}[\sqrt{-2}]$, and then use this description to give a formula for the number of ways to express a positive integer $n$ as $x^2 + 2y^2$ with $x, y \in \mathbb{N}_0$. You may use without proof the fact that if $p$ is an odd prime then $-2$ is a square in $(\mathbb{Z}/p\mathbb{Z})^*$ if and only if $p$ is congruent to 1 or 3 mod 8 (I will prove this fact in a piazza post).

3. Problems 5.1, 5.3, 3.6, 4.4, 4.10(a,e,j,k) from chapter 12 of Artin.