(1) For $n \in \mathbb{N}$, define $\phi(n)$ to be the cardinality of $(\mathbb{Z}/n\mathbb{Z})^*$, i.e., the number of integers $k$ with $1 \leq k \leq n$ for which $\gcd(k, n) = 1$. For each prime power $q$ and each positive integer $d$ dividing $q - 1$, express the number of order-$d$ elements of $\mathbb{F}_q^*$ as a value of $\phi$. Deduce from this a positive lower bound on the number of monic irreducible degree-$n$ polynomials in $\mathbb{F}_q[x]$ (express the lower bound in terms of a value of $\phi$).

(2) Determine all primitive elements for the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$.

(3) Let $q$ be a prime power with $q \equiv 1 \pmod{4}$, and let $f(X)$ and $g(X)$ be distinct monic irreducible polynomials in $\mathbb{F}_q[X]$. Show that the image of $f(X)$ in $\mathbb{F}_q[X]/(g(X))$ is a square if and only if the image of $g(X)$ in $\mathbb{F}_q[X]/(f(X))$ is a square.

(I will post hints on piazza.)

(4) Problems 7.1–7.4, 7.6, 7.11 from chapter 16 of Artin (in 7.1, assume that $a, b, ab$ are all nonsquares in $F$).

(5) Let $N/K$ be a Galois extension, and let $L$ be a field with $K \subseteq L \subseteq N$. Let $H$ be the set of all elements $h \in \text{Gal}(N/K)$ such that $h(L) = L$. Show that $H$ is the normalizer of $\text{Gal}(N/L)$ in $\text{Gal}(N/K)$.

Note that the condition $h(L) = L$ says $h$ preserves $L$ as a set, which is a different assertion than saying that $h$ fixes every element of $L$. That is, it says $h$ fixes $L$ setwise but not necessarily pointwise.

(6) Let $N$ be a field and let $G$ be a finite group of automorphisms of $N$. Writing $N^G := \{n \in N : g(n) = n \forall g \in G\}$, show that $N/N^G$ is Galois with Galois group $G$.

(7) Fill in the missing details in the following sketch of a proof that $\mathbb{C}$ is algebraically closed. Note that the only non-algebraic ingredient is the intermediate value theorem on $\mathbb{R}$.

Let $M/\mathbb{C}$ be any finite extension, and let $N$ be the normal closure of $M/\mathbb{R}$. Show (easily) that $N/\mathbb{R}$ is Galois. Let $H$ be a Sylow 2-subgroup of $G := \text{Gal}(N/\mathbb{R})$, and put $L := N^H$. Show that $[L : \mathbb{R}]$ is odd. Then use the intermediate value theorem to show that $[L : \mathbb{R}]$ cannot be greater than 1. Conclude that $G = H$, so that $[N : \mathbb{R}]$ is a power of 2. Now $N/\mathbb{C}$ is Galois with Galois group being a 2-group. Deduce that if $N \neq \mathbb{C}$ then there is a field $K$ with $\mathbb{C} < K \leq N$ and $[K : \mathbb{C}] = 2$. Then obtain a contradiction by showing directly that there is no degree-2 extension $K/\mathbb{C}$.