

ON THE STRUCTURE OF POLYNOMIAL MAPPINGS MODULO AN ODD PRIME POWER

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ABSTRACT. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, let n be a positive integer, and p an odd prime. Then the mapping $x \mapsto f(x)$ sends $\mathbb{Z}/p^n\mathbb{Z}$ into $\mathbb{Z}/p^n\mathbb{Z}$. We study the topological structure of this mapping.

1. INTRODUCTION

Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients, let n be a positive integer, and let p be an odd prime. Then the mapping $x \mapsto f(x)$ sends $\mathbb{Z}/p^n\mathbb{Z}$ into $\mathbb{Z}/p^n\mathbb{Z}$. We shall study the structure of this mapping. Since the mapping $f \pmod{p^n}$ must project to a well-defined mapping $f \pmod{p^{n-1}}$, only a certain class of mappings on $\mathbb{Z}/p^n\mathbb{Z}$ can arise from polynomials. But there are many more restrictions on which mappings can occur than just the above observation—in Section 3 we show that there is a certain linearity causing one such restriction. In later sections we take advantage of this linearity to derive numerous results about the cycles of $f \pmod{p^n}$. Our results give an algorithm which, for almost any given polynomial f , finds the lengths of the cycles of $f \pmod{p^n}$ for all n , usually very quickly. Our results also indicate how to construct a polynomial with any (possible) desired cycle structure mod p^n . Our methods also apply in much more general situations¹; we will briefly discuss this in Section 11.

2. NOTATION

Henceforth, p will denote a fixed odd prime, $f(x) \in \mathbb{Z}[x]$ a fixed polynomial, and n a positive integer. We denote by f_n the mapping $\mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ which sends $x \mapsto f(x) \pmod{p^n}$. We let $\sigma =$

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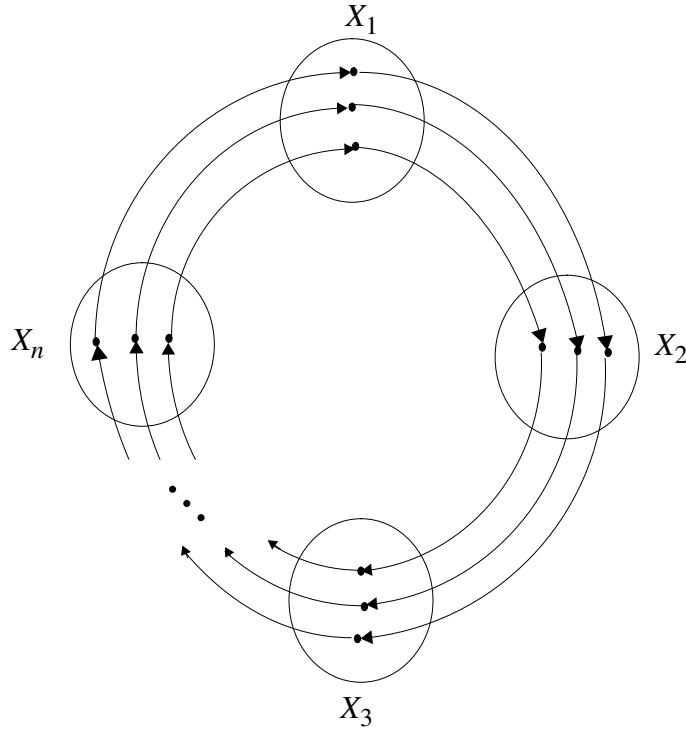
This paper is unchanged from the version circulated in August 1994. See the second author's Ph.D. thesis (Berkeley, 1996) for further results and for comments relating this paper to the mathematical literature.

¹For instance, we can allow our polynomials to have coefficients in the p -adic integers \mathbb{Z}_p , and in fact every argument we make will be unchanged if we replace every symbol \mathbb{Z} by the symbol \mathbb{Z}_p .

(x_1, \dots, x_k) be a cycle of f_n of length k ; that is, $f_n(x_1) = x_2$, $f_n(x_i) = x_{i+1}$, and $f_n(x_k) = x_1$. (We view the x_i as integers lying in the appropriate classes (mod p^n)). Finally, $g = f^k$ is the k^{th} iterate of f .

3. CYCLE LIFTING

In this section we examine the structure of f_{n+1} on the set of points of $\mathbb{Z}/p^{n+1}\mathbb{Z}$ which are congruent mod p^n to elements of σ . Let X_i be the preimage of x_i under the projection $\mathbb{Z}/p^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$; thus, $|X_i| = p$, and by the definition of σ , $f_{n+1}(X_i) \subseteq X_{i+1}$. For $g = f^k$, the k^{th} iterate of f , we have $g_{n+1}(X_1) \subseteq X_1$.



Now, let $X = X_1 \cup X_2 \cup \dots \cup X_k$; then $f_{n+1}(X) \subseteq X$, and any cycle of f_{n+1} in X must have length divisible by k . We call these cycles the *lifts* of σ . When we divide the lengths of these lifts by k , we get the cycle lengths of g_{n+1} in X_1 .

We can define a bijection between X_1 and $\mathbb{Z}/p\mathbb{Z}$ by the rule

$$x_1 + p^n t \longleftrightarrow t.$$

By Taylor's theorem for polynomials,²

$$\begin{aligned} g(x_1 + p^n t) &\equiv g(x_1) + p^n t g'(x_1) \pmod{p^{2n}} \\ &\equiv x_1 + p^n \left(\frac{g(x_1) - x_1}{p^n} \right) + p^n g'(x_1) t \pmod{p^{2n}} \\ &\equiv x_1 + p^n b_n + p^n a_n t \pmod{p^{2n}} \end{aligned}$$

where we define $a_n = g'(x_1)$ and $b_n = (g(x_1) - x_1)/p^n$. (Note that a_n and b_n are defined over \mathbb{Z} .) Thus, if we define the map $\Phi : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ to be induced by restricting g_{n+1} to X_1 and applying the above bijection, then $\Phi(t) = b_n + a_n t$. The linearity of this map is the key to what follows.

Note that:

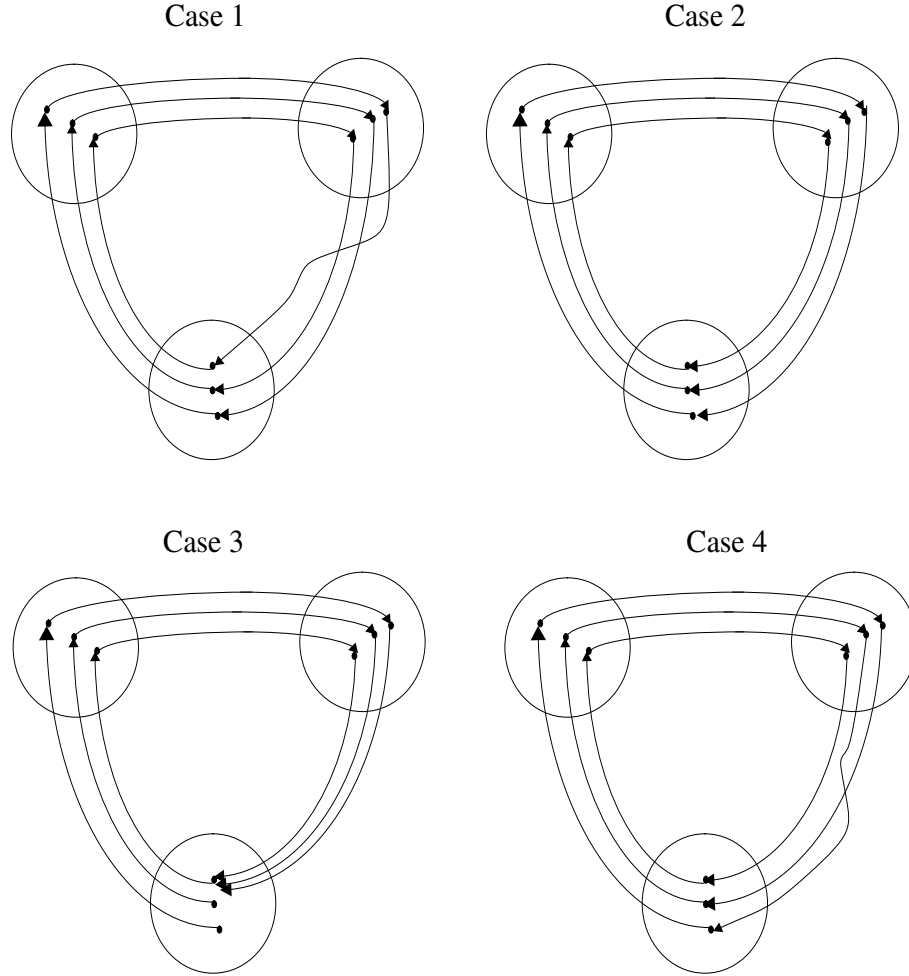
- (1) If $a_n \equiv 1 \pmod{p}$ and $b_n \not\equiv 0 \pmod{p}$, then Φ consists of a single cycle of length p , so that f_{n+1} restricted to X consists of a single cycle of length pk . In this case we say that σ *grows*.
- (2) If $a_n \equiv 1 \pmod{p}$ and $b_n \equiv 0 \pmod{p}$, then Φ is the identity, so f_{n+1} restricted to X consists of p cycles, each of length k . In this case we say that σ *splits*.
- (3) If $a_n \equiv 0 \pmod{p}$, then Φ is constant, so f_{n+1} on X contains one k -cycle, and the remaining points of X are mapped into this cycle by f^k . In this case we say that σ *grows tails*.
- (4) If $a_n \not\equiv 0, 1 \pmod{p}$, then Φ is a permutation, and Φ^ℓ , the ℓ^{th} iterate of Φ , sends

$$\begin{aligned} t &\rightarrow (b_n + a_n b_n + a_n^2 b_n + \cdots + a_n^{\ell-1} b_n) + a_n^\ell t \\ &= b_n (a_n^\ell - 1) / (a_n - 1) + a_n^\ell t, \end{aligned}$$

so $\Phi^\ell(t) - t = (t + b_n/(a_n - 1))(a_n^\ell - 1)$. Thus, Φ has a single fixed point, namely $t = -b_n/(a_n - 1)$, and the remaining points of X_1 lie on cycles of length d , where d is the order of a_n in $(\mathbb{Z}/p\mathbb{Z})^*$. Thus, f_{n+1} restricted to X contains one k -cycle and $(p-1)/d$ cycles of length kd . In this case we say that σ *partially splits*.

The above results already rule out many potential ways for σ to lift. For instance, the lifts of σ can only have two distinct lengths. If two lengths do occur, then one equals the length of σ and occurs only once; if only one length occurs, it is either the length of σ or p times this length.

²This says that $g(x+y) = \sum_{i=0}^{\infty} y^i \frac{g^{(i)}(x)}{i!}$; note that the sum is finite, since all terms with $i > \text{degree}(g)$ vanish, and also note that $\frac{g^{(i)}(x)}{i!}$ is an integer.



Before proceeding any further, we comment on the definitions of a_n and b_n . Our definition of a_n and b_n depends on three things: the cycle σ , the choice of x_1 from the elements of σ , and the integer x_1 chosen to represent the congruence class $x_1 \pmod{p^n}$. However, to some extent a_n and b_n are independent of these last two choices. First,

$$a_n = (f^k)'(x_1) = \prod_{i=0}^{k-1} f'(f^i(x_1)) \equiv \prod_{i=0}^{k-1} f'(x_i) \pmod{p^n},$$

so the class of $a_n \pmod{p^n}$ does not depend on the choices. Secondly,

$$g(x_1 + p^n z) - (x_1 + p^n z) \equiv p^n b_n + p^n z(a_n - 1) \pmod{p^{2n}},$$

so replacing x_1 by $x_1 + p^n z$ has the effect of replacing b_n with $b_n + z(a_n - 1) \pmod{p^n}$. Thus, for $A = \min\{\text{ord}_p(a_n - 1), n\}$, the choice of the integer x_1 from the congruence class $x_1 \pmod{p^n}$ does not affect $b_n \pmod{p^A}$. Finally,

$$g(f(x_1)) - f(x_1) = f(x_1 + p^n b_n) - f(x_1) \equiv p^n b_n f'(x_1) \pmod{p^{2n}},$$

and since $p \nmid f'(x_1)$, $\min\{\text{ord}_p(b_n), n\}$ is independent of the choice of which a particular element of σ is called x_1 .

4. RELATIONSHIPS BETWEEN a 'S AND b 'S

Let $\tilde{\sigma} = (\tilde{x}_1, \dots, \tilde{x}_{rk})$ be a lift of σ to an rk -cycle of f_{n+1} . We will show that the manner in which σ lifted restricts how $\tilde{\sigma}$ can lift. We may assume that $\tilde{x}_1 \equiv x_1 \pmod{p^n}$, and (as before, viewing \tilde{x}_1 as an integer) we write $\tilde{x}_1 = x_1 + p^n t$. Then

$$\begin{aligned} a_{n+1} &= (g^r)'(x_1 + p^n t) \equiv (g^r)'(x_1) = \prod_{i=0}^{r-1} g'(g^i(x_1)) \\ &\equiv g'(x_1)^r = a_n^r \pmod{p^n}. \end{aligned}$$

Now we apply this calculation:

- (1) If σ splits or grows, then $a_{n+1} \equiv 1^r \equiv 1 \pmod{p}$, so $\tilde{\sigma}$ either splits or grows.
- (2) If σ partially splits, then its k -cycle lift also partially splits (with the same d , since $a_{n+1} \equiv a_n \pmod{p}$ and so the order of a_{n+1} in $(\mathbb{Z}/p\mathbb{Z})^*$ is the same as the order of a_n in $(\mathbb{Z}/p\mathbb{Z})^*$), and its kd -cycle lifts either split or grow (since $a_{n+1} \equiv a_n^d \equiv 1 \pmod{p}$).
- (3) If σ grows tails, then the single k -cycle lift $\tilde{\sigma}$ also has $a_{n+1} \equiv a_n \equiv 0 \pmod{p}$, so it grows tails as well.

We will need another basic calculation. As before,

$$g^r(x_1 + p^n t) \equiv x_1 + p^n (t a_n^r + b_n(1 + a_n + \dots + a_n^{r-1})) \pmod{p^{2n}},$$

so

$$\begin{aligned} p^{n+1} b_{n+1} &= g^r(x_1 + p^n t) - (x_1 + p^n t) \\ &\equiv p^n (t(a_n^r - 1) + b_n(1 + a_n + \dots + a_n^{r-1})) \pmod{p^{2n}} \end{aligned}$$

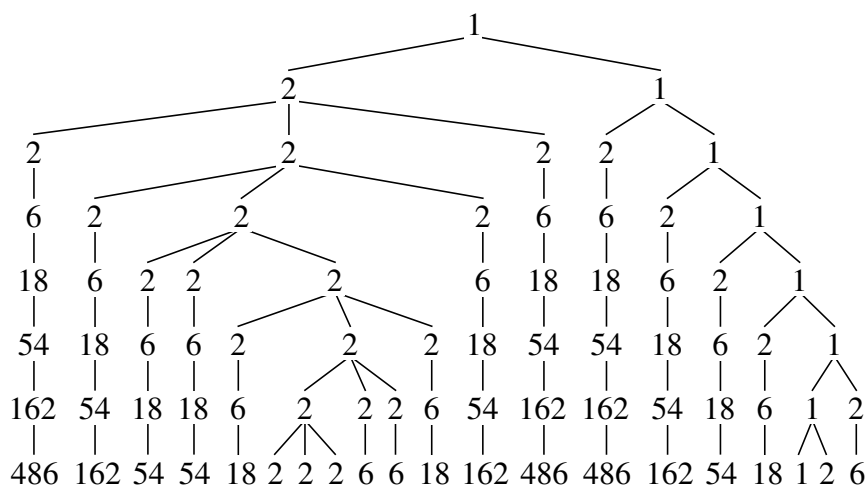
and therefore

$$p b_{n+1} \equiv t(a_n^r - 1) + b_n(1 + a_n + \dots + a_n^{r-1}) \pmod{p^n}.$$

5. OUTLINE OF GOALS

Now that we have established the basic setup, we briefly pause to discuss the general questions we are studying. We have seen that the cycle structure of f_n greatly depends on that of f_{n-1} . Thus, it will be possible to obtain results which apply to the structure of f_n for all n . More precisely, we study an infinite tree which contains a node for each cycle of f_n , for every $n \geq 0$, and where each node is labeled with the length of the corresponding cycle. The tree is defined as follows: at the top level, level 0, is a single node labeled with 1, the length of the single cycle of f_0 . At each lower level, level n , there is a node for each cycle of f_n , labeled with its length, and a node at level $n + 1$ is a child of a node at level n if it is a lift of the corresponding cycle.

Here is an example of such a tree, for a polynomial with $p = 3$:



We would like to do the following:

- (1) Describe all trees that can occur.
- (2) Give a method for constructing a polynomial having a prescribed tree.
- (3) Give a method for determining the tree of a given polynomial.

We will derive a number of results of the form: if a certain (finite) part of the tree has a certain form, then this constrains the behavior of another (possibly infinite) part of the tree. For example, we will show that whenever a cycle for some f_n (with $n \geq 2$) grows, then its lift grows, and the lift of that lift grows, and so on.

Results of this form severely restrict the class of trees which can occur. They are also useful for determining the tree of a given polynomial; in fact, except for a certain pathological class of functions, we will see that the first n levels of the tree, for some n , will determine the entire tree. We have looked at thousands of random polynomials of small degree, and in every case the first nine levels were sufficient; usually five were enough, and it seemed that fewer levels were needed for larger p . The pathological cases appear to be quite rare, since none arose randomly. However, we do not believe that, in the non-pathological cases, there is a bound on the number of levels of the tree needed to determine the entire tree; large numbers of levels should sometimes be necessary, but only very rarely.

Finally, these results help us construct polynomials having prescribed trees. As long as the tree is determined by its first n levels, we need only find a polynomial whose tree has those first n levels; i.e., a polynomial having a certain structure mod p^n .

We will generally not study cycles which grow tails, except in Section 6.4. This case is easy to identify and distinguish, because cycles which grow tails will only occur in subtrees rooted at cycles mod p which grow tails.

6. CYCLE STRUCTURES

6.1. If σ grows. Suppose σ grows. We showed above that $\tilde{\sigma}$ either splits or grows. From $a_n \equiv 1 \pmod{p}$, it follows that $a_n^p \equiv 1 \pmod{p^2}$, because

$$\frac{a_n^p - 1}{a_n - 1} = a_n^{p-1} + \cdots + a_n + 1 \equiv 1 + \cdots + 1 + 1 \equiv 0 \pmod{p}.$$

So, for $n \geq 2$,

$$pb_{n+1} \equiv b_n(1 + a_n + \cdots + a_n^{p-1}) \pmod{p^2}.$$

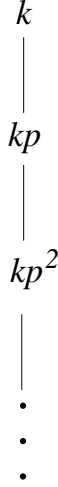
If $a_n = 1$, then $1 + a_n + \cdots + a_n^{p-1} = p$. Otherwise, let $a_n = 1 + p^\gamma \delta$, where $p \nmid \delta$ and $\gamma \geq 1$. Then

$$\begin{aligned} 1 + a_n + \cdots + a_n^{p-1} &= \frac{a_n^p - 1}{a_n - 1} = \frac{\binom{p}{1}p^\gamma \delta + \binom{p}{2}p^{2\gamma} \delta^2 + \cdots}{p^\gamma \delta} \\ &= \binom{p}{1} + \binom{p}{2}p^\gamma \delta + \cdots \\ &\equiv p \pmod{p^2}. \end{aligned}$$

Thus, in either case $pb_{n+1} \equiv pb_n \pmod{p^2}$. So, $b_{n+1} \not\equiv 0 \pmod{p}$, so $\tilde{\sigma}$ grows.

Here we have shown that, for $n \geq 2$, whenever σ grows, its lift $\tilde{\sigma}$ also grows; it follows that the lift of $\tilde{\sigma}$ also grows, and so on. In Appendix B we will show that, for $p > 3$, this result holds for $n = 1$ as well.

In this case, the subtree rooted at σ has the following structure:



6.2. If σ partially splits. Suppose that $\tilde{\sigma}$ is a kd -cycle lift of σ , where $1 < d < p$; thus, d is the order of a_n in $(\mathbb{Z}/p\mathbb{Z})^*$. Let y be any element of $\tilde{\sigma}$, and let $h = f^{kd}$. In Corollary 3 of Section 9 we show that

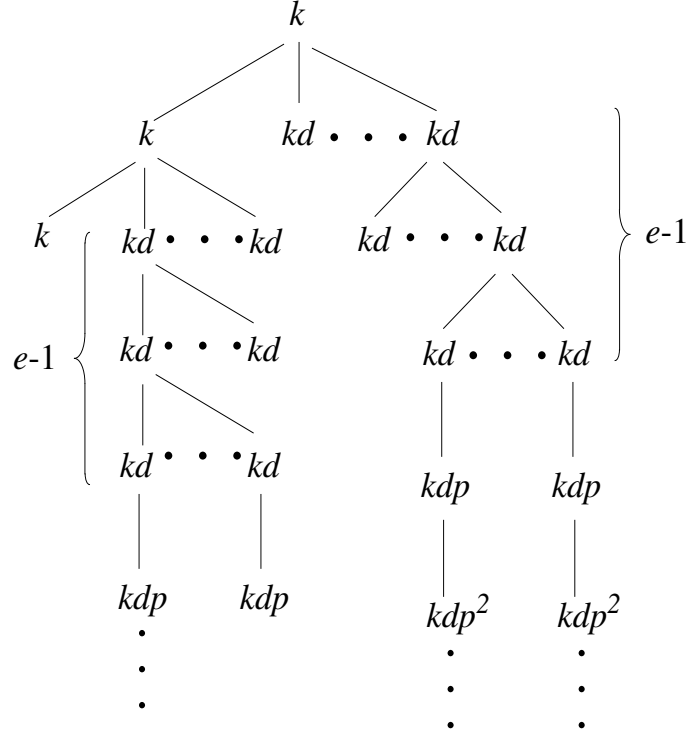
$$\min\{\text{ord}_p(h(y) - y) - n, nd\} = \min\{\text{ord}_p(a_{n+1} - 1), nd\}.$$

In this section we note some implications of this result:

- (1) If $e = \text{ord}_p(a_{n+1} - 1) < nd$, then $h(y) \equiv y \pmod{p}^{n+m}$ for $m \leq e$, but $h(y) \not\equiv y \pmod{p}^{n+e+1}$, so $\tilde{\sigma}$ splits $(e - 1)$ times, and then the descendants of $\tilde{\sigma}$ at level $n + e$ grow.
- (2) If $e = \text{ord}_p(a_{n+1} - 1) \geq nd$, then $\tilde{\sigma}$ splits $(nd - 1)$ times, but we do not know what happens to its descendants at level $n + nd$.

Note that, if $e < n$, then every kd -cycle lift of σ has the same e (since $a_{n+1} \equiv a_n^d \pmod{p^n}$), so they all behave the same way.

In case 1, the subtree rooted at σ has the following structure:



6.3. **If σ splits.** Suppose that σ splits. Let x be an element of σ , and let $y = x + p^n z$ be an element of $\tilde{\sigma}$. Recall that $a_n = g'(x) \equiv 1 \pmod{p}$ and $b_n = (g(x) - x)/p^n \equiv 0 \pmod{p}$. Then

$$\begin{aligned} p^{n+1}b_{n+1} &= g(y) - y = g(x + p^n z) - (x + p^n z) \\ &\equiv g(x) - x + p^n z(g'(x) - 1) + p^{2n} z^2 g''(x)/2 \pmod{p^{3n}} \\ &\equiv p^n b_n + p^n z(a_n - 1) + p^{2n} z^2 g''(x)/2 \pmod{p^{3n}} \end{aligned}$$

so

$$pb_{n+1} \equiv b_n + z(a_n - 1) + p^n z^2 g''(x)/2 \pmod{p^{2n}}.$$

Similarly,

$$a_{n+1} = g'(y) = g'(x + p^n z) \equiv a_n + p^n z g''(x) \pmod{p^{2n}}.$$

Combining these two expressions gives

$$\begin{aligned} \frac{z}{2}(a_{n+1} + a_n - 2) &\equiv \frac{z}{2}(2a_n + p^n z g''(x) - 2) \pmod{p^{2n}} \\ &\equiv pb_{n+1} - b_n \pmod{p^{2n}}. \end{aligned}$$

Now we apply this result. Let $A = \text{ord}_p(a_n - 1)$ and $B = \text{ord}_p(b_n)$. We know that $A, B \geq 1$. Since $a_{n+1} \equiv a_n \pmod{p^n}$, we have $\text{ord}_p(a_{n+1} - 1) = A$ if $A < n$, and $\text{ord}_p(a_{n+1} - 1) \geq n$ if $A \geq n$. Now,

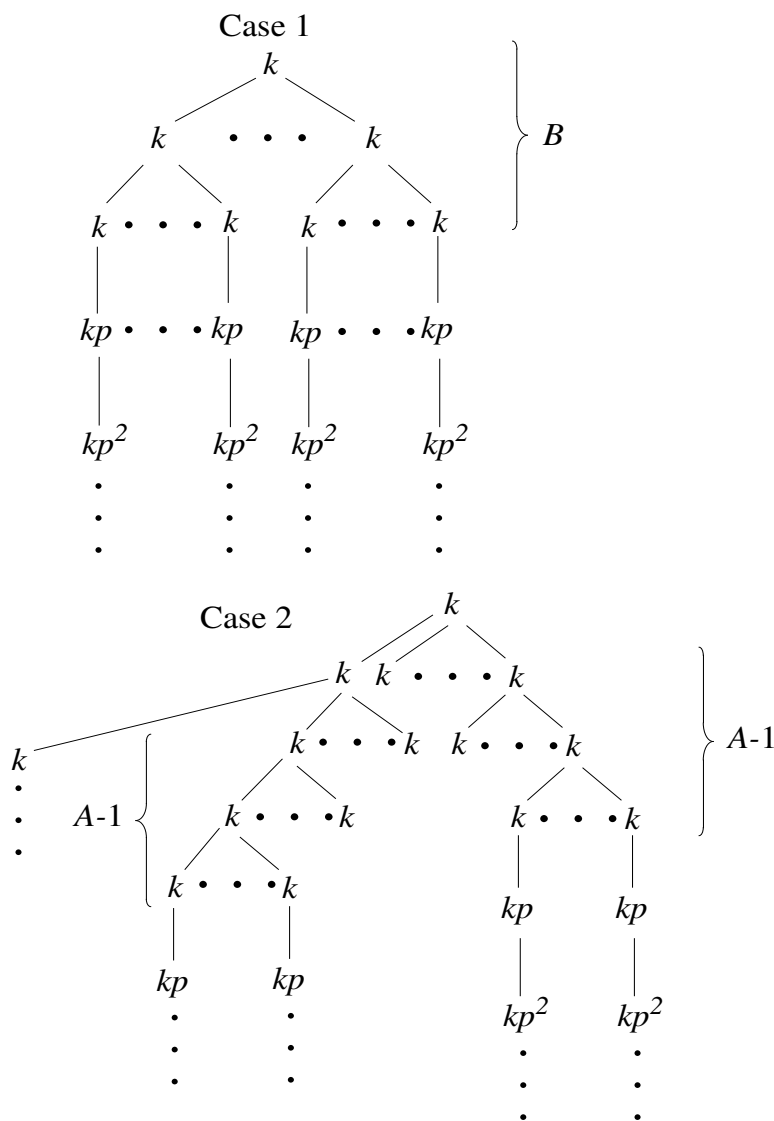
- (1) If $B < A$ and $B < n$, then $\text{ord}_p(b_{n+1}) = B - 1$.
- (2) If $A \leq B$ and $A < n$, then $b_n + z(a_n - 1) \equiv pb_{n+1} \pmod{p^n}$. There is a unique $z \pmod{p}$ for which $b_n + z(a_n - 1) \equiv 0 \pmod{p^{A+1}}$, so that $\text{ord}_p(b_{n+1}) \geq A$. For all other $z \pmod{p}$, $b_n + z(a_n - 1)$ is divisible by p^A but not by p^{A+1} , so that $\text{ord}_p(b_{n+1}) = A - 1$.
- (3) If $A, B \geq n$ then p^n divides $(a_{n+1} - 1)$ and p^{n-1} divides b_{n+1} .

Interpreting these results in terms of the tree structure, we see that:

- If $B < A$ and $B < n$, then every lift of σ splits $(B - 1)$ times, and then grows.
- If $A \leq B$ and $A < n$, then every lift of σ , except for one, splits $(A - 1)$ times, and then grows. The single exceptional lift behaves precisely the same way as does σ .
- If $A, B \geq n$ then every lift of σ splits $n - 1$ times, but we do not know what happens to their descendants at level $2n$.

Note that, in addition to using the above results by computing A and B to predict the structure of the tree, we can use the results by observing the tree to determine which case we are in.

In cases 1 and 2, the subtrees rooted at σ have the following structures:



6.4. **Tails.** A *tail* of f_n is a sequence of elements of $\mathbb{Z}/p^n\mathbb{Z}$ of the form y_1, y_2, \dots, y_ℓ , where $y_{j+1} = f_n(y_j)$, and none of the y_j is in the image of f_n^m , for m sufficiently large. All of the points of $\mathbb{Z}/p^n\mathbb{Z}$ either lie on tails of f_n or in cycles of f_n .

Suppose (x_1, x_2, \dots, x_k) is a cycle of f_1 . If $f'(x_i) \not\equiv 0 \pmod{p}$, then f_n maps $\{x \mid x \equiv x_i \pmod{p}\}$ to $\{x \mid x \equiv x_{i+1} \pmod{p}\}$ bijectively. This follows by induction on n . Let $x \in \mathbb{Z}/p^{n-1}\mathbb{Z}$ be congruent to $x_i \pmod{p}$. Then the p elements of $\mathbb{Z}/p^n\mathbb{Z}$ that are congruent to x

$(\text{mod } p^{n-1})$ map bijectively to the p elements of $\mathbb{Z}/p^n\mathbb{Z}$ that are congruent to $f(x) \pmod{p^{n-1}}$, since

$$f(x + p^{n-1}y) \equiv f(x) + p^{n-1}yf'(x) \equiv f(x) + p^{n-1}yf'(x_i) \pmod{p^n}.$$

Thus, if $f'(x_i) \not\equiv 0 \pmod{p}$ for $i = 1, \dots, k$, then all of the elements of $\mathbb{Z}/p^n\mathbb{Z}$ which are congruent to $x_1, \dots, x_k \pmod{p}$ lie on cycles of f_n .

However, if $f'(x_i) \equiv 0 \pmod{p}$, then f_n maps the p elements of $\mathbb{Z}/p^n\mathbb{Z}$ that are congruent to $x \pmod{p^{n-1}}$ all to the same element of $\mathbb{Z}/p^n\mathbb{Z}$, by the above computation. Thus, the elements of $\mathbb{Z}/p^n\mathbb{Z}$ which are congruent to $x_1, \dots, x_k \pmod{p}$ contain only a single cycle of length k , and the remaining points lie on tails of f_n . If y_1, \dots, y_ℓ is such a tail, with $y_j \equiv x_i \pmod{p}$, then $y_{j+1} \pmod{p^2}$ must be on the cycle of f_2 . Similarly $y_{j+k+1} \pmod{p^3}$ must be on the cycle of f_3 , and so on. By induction, $y_{j+(n-2)k+1}$ must be on the cycle of f_n . Thus, for such a cycle of f_n , the maximum length of a tail leading to that cycle is $p + (n-2)k$.

Of course, the tails of f_n form trees, with every tail eventually leading to a cycle, but possibly first joining another tail. The above result gives a bound on how long it takes for all the tails to coalesce into the cycle.

If $f'(x_i) \equiv 0 \pmod{p}$, but $f''(x_i) \not\equiv 0 \pmod{p}$, then we can describe precisely how f_n maps $\{x \mid x \equiv x_i \pmod{p}\}$ into $\{x \mid x \equiv x_{i+1} \pmod{p}\}$. The preimages of the points in the image of f_n have sizes p^j or $2p^j$; precisely, for $1 \leq j < n/2$ there are $p^{n-2j-1}(p-1)/2$ preimages of size $2p^j$, and there is a single preimage of size $p^{\lfloor n/2 \rfloor}$. The proof is similar to many we have already presented.

7. PERIODIC ORBITS OF f

In this section we describe the possible lengths of periodic orbits in the p -adic integers \mathbb{Z}_p for a polynomial $f(x) \in \mathbb{Z}[x]$.³ Such an orbit corresponds to a sequence of cycles of f_n , for $n = 1, \dots, \infty$, where each cycle is a lift of its predecessor. The lengths of the cycles are bounded, and length of the orbit is the lim sup of the lengths of the cycles. All the relevant properties of \mathbb{Z}_p are presented in Appendix A. We use the term ‘periodic orbit’ for \mathbb{Z}_p , while we reserve the term ‘cycle’ for $\mathbb{Z}/p^n\mathbb{Z}$.

Let $x \in \mathbb{Z}_p$ lie in a periodic orbit of f of length c . For each n , let c_n be the length of the cycle σ_n of f_n containing $x \pmod{p^n}$. Thus, $c_1 \leq c_2 \leq \dots = c$. Clearly $c_1 \leq p$. If σ_1 grows tails, then each $c_n = c_1$, so $c = c_1$. Otherwise, whenever some σ_n either splits or grows, all further σ_{n+i} either split or grow, so c_{n+i+1}/c_{n+i} is always either 1 or p ,

³Or, more generally, a polynomial in $\mathbb{Z}_p[x]$.

thus c/c_n is a power of p . If σ_1 partially splits, then its lifts of length dc_1 either split or grow, and its lift of length c_1 partially splits just as does σ_1 .

Thus, there are three possibilities:

- (1) $c = c_1$, if either σ_1 grows tails or every σ_{i+1} is the single c_i -cycle lift of σ_i , which partially splits.
- (2) c/c_1 is a power of p , if σ_1 splits or grows.
- (3) c/dc_1 is a power of p , if σ_1 partially splits but some σ_{i+1} is a dc_i -cycle lift of σ_i (which partially splits).

We showed in Section 6.1 that, for $n \geq 2$, whenever σ grows, its lift also grows; then that cycle's lift grows, and so on. So, under the hypothesis that $c = \limsup_{i \rightarrow \infty} c_i$ is finite, we can never have $c_{n+1} = pc_n$ for $n \geq 2$. Thus, $c = c_1$ or $c = dc_1$ or $c = c_2 = pc_1$.

For $p > 3$, we show in Appendix B that the case $c = c_2 = pc_1$ never occurs.

In summary, any periodic orbit of f in \mathbb{Z}_p has length at most p^2 , and this length is the product of a positive integer not exceeding p and a divisor of $p - 1$ (except if $p = 3$, in which case length 9 can occur). Note that this upper bound on the lengths of periodic orbits implies, if f is not linear, an upper bound on the number of periodic points. For, any element of an orbit of length c must be a root of the polynomial $f^c(z) - z$, which has only finitely many roots.

8. LIFTS OF A PERIODIC ORBIT

In this section we describe the behavior of cycles which separate from a periodic orbit of f at some stage. Precisely, let $\alpha \in \mathbb{Z}_p$ be an element of a periodic orbit of f of length k , so that $g(\alpha) = \alpha$ for $g = f^k$, and assume that $g'(\alpha) \not\equiv 0 \pmod{p}$. Let $c_n = k$ (i.e., $\alpha \pmod{p^n}$ is in a k -cycle of f_n , not a shorter cycle), and let $y \in \mathbb{Z}_p$ have $n = \text{ord}_p(y - \alpha)$. Then $\alpha \pmod{p^{n+1}}$ and $y \pmod{p^{n+1}}$ lie in different cycles of f_{n+1} . We will say that the cycle containing y separates from α at level $n + 1$.

Let d be the order of $g'(\alpha) \pmod{p}$. Then we know that $y \pmod{p^{n+1}}$ is in a kd -cycle of f_{n+1} . Let $h = g^d = f^{kd}$. Then $h(\alpha) = \alpha$ and $h'(\alpha) \equiv 1 \pmod{p}$.

Suppose that $h'(\alpha) = g'(\alpha)^d \not\equiv 1$. Let $m = \text{ord}_p(h'(\alpha) - 1)$. Then

$$\begin{aligned} h(y) - y &= h(\alpha + (y - \alpha)) - \alpha - (y - \alpha) \\ &\equiv h(\alpha) + (y - \alpha)h'(\alpha) - \alpha - (y - \alpha) \pmod{p^{2n}} \\ &\equiv (y - \alpha)(h'(\alpha) - 1) \pmod{p^{2n}}. \end{aligned}$$

Thus, if $n > m$, $\text{ord}_p(h(y) - y) = n + m$.⁴ In this case, $y \pmod{p^{n+m}}$ is a fixed point of h_{n+m} , but $y \pmod{p^{n+m+1}}$ is not a fixed point of h_{n+m+1} . So $y \pmod{p^{n+m}}$ lies in a kd -cycle of f_{n+m} , but $y \pmod{p^{n+m+1}}$ lies in a longer cycle of f_{n+m+1} . It follows that the kd -cycle of f_{n+m} which contains $y \pmod{p^{n+m}}$, and all of its descendants, must grow.

Note that in the above case there will always be some n such that the behavior of the infinite subtree consisting of cycles which separate from α at levels greater than n is determined by the finite tree up to level n . For there will be some cycle which separates from α at level $n + 1$, and then splits m times where $m < n$. By the above argument, all cycles which separate from α at higher levels will behave the same way.

In fact, by observing only a finite part of the tree, we can determine that we are in that case. If we have a cycle which partially splits, then we know that it has a lift which partially splits, and so on, so each point on that cycle corresponds to a periodic element α . Then, if one of the other cycles which is a lift of that cycle behaves as above (splits $m < n$ times, then grows), then we know that we are in the above case and all cycles which separate from α at higher levels will behave the same way.

If we have a periodic element α which is on a cycle which splits completely, and it has a lift which splits $m < n$ times and then grows, then simply by observing that feature of the tree, by the results of Section 6.3 we must be in case 2 of that section, and so we know that there is a periodic point α on the cycle, and the above results apply.

However, when $h'(\alpha) = 1$, it does not seem to be true that by observing a finite part of the tree we can predict all subsequent behavior, nor can we determine that $h'(\alpha) = 1$ by observing only a finite portion of the tree.

Suppose that $h'(\alpha) = 1$. If f is not linear, then h' is not constant, so there is an integer $\ell \geq 2$ for which $h^{(\ell)}(\alpha) \neq 0$ while $0 = h^{(2)}(\alpha) = \dots = h^{(\ell-1)}(\alpha)$.⁵ Let $m = \text{ord}_p(h^{(\ell)}(\alpha)/\ell!)$. Then

$$\begin{aligned} h(y) - y &= h(\alpha + (y - \alpha)) - \alpha - (y - \alpha) \\ &= h(\alpha) + (y - \alpha)h'(\alpha) + \dots - \alpha - (y - \alpha) \\ &= (y - \alpha)^\ell h^{(\ell)}(\alpha)/\ell! + (y - \alpha)^{\ell+1} h^{(\ell+1)}(\alpha)/(\ell + 1)! + \dots \\ &\equiv (y - \alpha)^\ell h^{(\ell)}(\alpha)/\ell! \pmod{p^{n(\ell+1)}}. \end{aligned}$$

⁴In the next section, we will show that the same conclusion holds if $n > m/d$.

⁵In the next section, we will show that $\ell > d$.

Thus, if $n > m$, $\text{ord}_p(h(y) - y) = n\ell + m$. In this case, the image of $y \pmod{p^{n\ell+m}}$ is in a kd -cycle of $f_{n\ell+m}$ which grows and all of whose descendants grow. Thus, the lifts of α which separate from it at any stage $n + 1$, where $n > m$, will split $n(\ell - 1) + (m - 1)$ times and then grow. (And, since this is greater than or equal to n , the above results for the case $h'(\alpha) \neq 1$ never apply.) The lifts of α which separate from it at stage $n + 1$, where $n \leq m$, will split at least $n\ell - 1$ times, but we do not know whether they then grow.

We consider the case $h'(\alpha) = 1$ to be pathological; it did not arise in any of thousands of random examples we studied. We can construct an example, though: let $p = 3$ and $f(x) = x + 3x^2$. Then $f(0) = 0$, so take $\alpha = 0$. Since $f(3^n\beta) \equiv 3^n\beta \pmod{3^{2n+1}}$ for any n, β , the cycles which separate from 0 at level $n + 1$ split n times and then grow.

9. IMPROVING THE BOUNDS

In the previous section we described the dynamics of f sufficiently close to a periodic orbit; in this section we will show that the same results hold in a somewhat larger neighborhood of the periodic orbit. We will prove the following result:

Proposition 1. *If $f(x) \in \mathbb{Z}_p[x]$ has a periodic orbit of length k containing $\alpha \in \mathbb{Z}_p$, $d > 1$ is the order of $(f^k)'(\alpha) \pmod{p}$, and $h = f^{kd}$, then each of $h^{(2)}(\alpha), \dots, h^{(d)}(\alpha)$ is divisible by $(h'(\alpha) - 1)$, in \mathbb{Z}_p .*

Our interest is in the following two corollaries:

Corollary 1. *Under the hypotheses of Proposition 1, if $h'(\alpha) = 1$ then $h^{(2)}(\alpha) = \dots = h^{(d)}(\alpha) = 0$.*

Corollary 2. *Under the hypotheses of Proposition 1, if $m = \text{ord}_p(h'(\alpha) - 1)$ and $y \in \mathbb{Z}_p$ has $n = \text{ord}_p(y - \alpha)$, then*

$$h(y) - y \equiv (y - \alpha)(h'(\alpha) - 1) \pmod{p^{\min\{n(d+1), 2n+m\}}}.$$

Proof. Observe that

$$h(y) - y = h(\alpha) - \alpha + (y - \alpha)(h'(\alpha) - 1) + (y - \alpha)^2 h''(\alpha)/2! + \dots,$$

where $h(\alpha) = \alpha$ and p^{ni+m} divides $(y - \alpha)^i h^{(i)}(\alpha)/i!$ for $2 \leq i \leq d$. \square

The following corollary was used in Section 6.2:

Corollary 3. *Under the hypotheses of Corollary 2, if we define $a_{n+1} = h'(y)$, then*

$$\min\{\text{ord}_p(h(y) - y) - n, nd\} = \min\{\text{ord}_p(a_{n+1} - 1), nd\}.$$

Proof. We have

$$a_{n+1} = h'(y) = h'(\alpha) + (y - \alpha)h''(\alpha) + \dots$$

But $(y - \alpha)^{i-1}h^{(i)}(\alpha)$ is divisible by $p^{m+n(i-1)}$ for $2 \leq i \leq d$, so

$$a_{n+1} \equiv h'(\alpha) \pmod{p^{\min\{n+m, nd\}}}.$$

Thus, $\min\{\text{ord}_p(a_{n+1} - 1), n + m, nd\} = \min\{m, n + m, nd\}$. From Corollary 2,

$$\begin{aligned} \min\{\text{ord}_p(h(y) - y) - n, n + m, nd\} &= \min\{m, n + m, nd\} \\ &= \min\{\text{ord}_p(a_{n+1} - 1), n + m, nd\}. \end{aligned}$$

But $n + m > \text{ord}_p(a_{n+1} - 1)$, so the minimum of the right-hand side is less than $n + m$, so the minimum of the left-hand side is also less than $n + m$. \square

Proof of Proposition 1. First, we will translate the periodic orbit so that it passes through 0; this simplifies the algebra in our proof. Let $T : x \mapsto x + \alpha$. Then the function $\hat{f} = T^{-1}fT$ also has a periodic orbit of length k , namely $(0, f(\alpha) - \alpha, f^2(\alpha) - \alpha, \dots, f^{k-1}(\alpha) - \alpha)$. But $\hat{f}(x) = f(x + \alpha) - \alpha$, so $\hat{f}'(x) = f'(x + \alpha)$, and similarly for higher derivatives; likewise, any iterate $\hat{f}^\ell = T^{-1}f^\ell T$, so $(\hat{f}^\ell)^{(t)}(x) = (f^\ell)^{(t)}(x + \alpha)$. So computations assuming that $\alpha = 0$ will also hold for arbitrary α .

Now, for $g = f^k$ we have $g(x) = g'(0)x + O(x^2) \in \mathbb{Z}_p[x]$ and $h(x) = g^d(x) = g'(0)^d x + h_2 x^2 + h_3 x^3 + \dots + h_d x^d + O(x^{d+1})$, where $O(x^j)$ denotes a polynomial in x in which every term has degree at least j . Since d is the order of $g'(0) \pmod{p}$, each of $g'(0) - 1, \dots, g'(0)^{d-1} - 1$ is coprime to p . Thus, for any $\ell \leq m = \text{ord}_p(g'(0)^d - 1)$, we can project to $R = \mathbb{Z}_p/p^\ell \mathbb{Z}_p$ and apply the following lemma, which implies that each of h_2, \dots, h_d is divisible by p^ℓ , completing the proof of the Proposition. \square

Lemma 1. *For any commutative ring R and any primitive d^{th} root of unity $\zeta \in R$ such that none of $\zeta - 1, \zeta^2 - 1, \dots, \zeta^{d-1} - 1$ is a zero-divisor, let $g(x) = \zeta x + O(x^2) \in R[x]$ and $h(x) = g^d(x) = x + ax^i + O(x^{i+1})$, where $a \neq 0$ is the first nonzero coefficient of $h(x)$ of degree greater than 1. Then $i \equiv 1 \pmod{d}$, and in particular $i \geq d + 1$.*

Proof. Write $g(x) = g_1 x + g_2 x^2 + \dots$. Then the compositions

$$h \circ g = g_1 x + \dots + g_{i-1} x^{i-1} + (g_i + a\zeta^i) x^i + O(x^{i+1})$$

and

$$g \circ h = g_1 x + \dots + g_{i-1} x^{i-1} + (g_i + \zeta a) x^i + O(x^{i+1}).$$

Since $g \circ h = h \circ g$, the coefficients of x^i are equal, so $a\zeta^i = a\zeta$, so $a\zeta(\zeta^{i-1} - 1) = 0$. Since ζ is a root of unity, it is not a zero-divisor. Therefore if $i \not\equiv 1 \pmod{d}$, then $\zeta^{i-1} - 1$ would be a zero-divisor, contradicting the hypothesis. \square

10. ODDS AND ENDS

10.1. Analyzing polynomials. For a given polynomial, our results generally allow us to find the cycle structure of f_n rather quickly. We can compute the first few levels of the tree directly, and then our results will usually imply the structure of the entire tree. We have done this for thousands of randomly selected polynomials, for small primes p ; in theory one should be able to construct polynomials which will take us arbitrarily long to analyze, but these polynomials seem to be extremely rare. Also, the numbers a_n and b_n are sometimes useful for determining the structure of the remainder of the tree.

The tree shown in Section 5 is a typical example. For this tree, levels 0–3 suffice to determine the structure of the entire tree. Once we observe a cycle (mod p^2) which splits, for which one lift grows and another splits, we know that the lift which splits will behave in the same way. Also, for the 1-cycle (mod p) which partially splits, since its 2-cycle lift splits 0 times before growing, and $0 < kr - 1 = 1$, this behavior must persist.

Conversely, our results also allow us to construct polynomials with desired cycle structures (mod p^n). For instance, we can construct polynomials having periodic orbits (in \mathbb{Z}_p) of length kr , for any $1 \leq k \leq p$ and any r dividing $p - 1$.

10.2. Polynomial with 3-adic 9-cycle. We mention the polynomial $f(x) = 2 + x + 3x^2 + x^3 + 3x^4 + 2x^5$, which has a 3-adic 9-cycle, since it has a 9-cycle (mod 81) for which $\text{ord}_p(a_4 - 1) = 3$ and $\text{ord}_p(b_4) = 4$, namely the cycle containing 0 (mod 81).

A p -adic cycle of length p^2 is impossible for $p > 3$, by the results of Appendix B.

10.3. Permutation polynomials and single-cycle polynomials. We give a straightforward method for determining whether a given polynomial $f(x) \in \mathbb{Z}[x]$ induces a permutation $f_n : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$. We claim that, for $n \geq 2$, f_n is a permutation if and only if f_1 is a permutation and $f'(x)$ has no roots in $\mathbb{Z}/p\mathbb{Z}$; it follows that, for any $n \geq 2$, f_n is a permutation if and only if f_2 is a permutation.

We prove the claimed result by induction. Certainly, if f_n is a permutation, then f_{n-1} is a permutation, which implies that f_1 is a permutation. Now, given that f_{n-1} is a permutation, f_n will be a permutation if and only if, for each integer x , the numbers $f(x), f(x+p^{n-1}), \dots, f(x+p^{n-1}(p-1))$ are all distinct (mod p^n); but $f(x+p^{n-1}t) \equiv f(x) + p^{n-1}tf'(x) \pmod{p^n}$, so f_n is a permutation if and only if f' has no roots in $\mathbb{Z}/p\mathbb{Z}$. This completes the proof.

We can also give a simple criterion for when f_n is a single cycle of length p^n . For $p > 3$, for any $n \geq 2$, f_n is a p^n -cycle if and only if f_2 is a p^2 -cycle. For $p = 3$, for any $n \geq 3$, f_n is a 3^n -cycle if and only if f_3 is a 3^3 -cycle.

11. FURTHER GENERALITY

There are more general situations in which our arguments, perhaps with slight modifications, will apply. They include various combinations of the following:

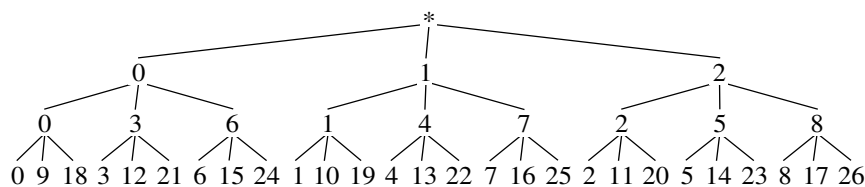
- First of all, we can replace \mathbb{Z} by \mathbb{Z}_p in all of our arguments.
- Most of the results which we have derived for polynomials also hold for rational functions whose denominators have no roots in $\mathbb{Z}/p\mathbb{Z}$; we will show this in Appendix C.
- More generally, we can consider rational functions over \mathbb{Q}_p having “good reduction” (mod p) at all points of a cycle in $\mathbb{P}^1(\mathbb{Q}_p)$.
- All of the above proofs work just as well for power series over \mathbb{Z}_p , with one caveat: if the power series only converges on $p\mathbb{Z}_p$, then we must only consider elements of $p\mathbb{Z}_p$, and in particular $f(0)$ must be divisible by p .
- The above arguments apply, in modified form, if we replace \mathbb{Z}_p by the valuation ring of any finite extension of \mathbb{Q}_p .
- For polynomials with coefficients in a number field, we can pick a good prime of the number field (almost any would do) and apply the results for the valuation ring of the completion of the field at that prime, to give bounds on the cycle lengths.
- Our basic approach yields interesting results for polynomial mappings from \mathbb{Z}^n to \mathbb{Z}^n (thanks to Greg Kuperberg for pointing this out).

We have studied all of the above, and we have numerous partial results; we hope eventually to write a comprehensive paper covering at least the above situations.

APPENDIX A. A QUICK INTRODUCTION TO p -ADICS.

For the reader's convenience, we set forth the basic properties of p -adic integers which we use in Sections 7, 8, & 9. The p -adic integers are the projective limit of the rings $\mathbb{Z}/p^n\mathbb{Z}$. Explicitly, an element of \mathbb{Z}_p is a sequence (x_1, x_2, \dots) , where $x_n \in \mathbb{Z}/p^n\mathbb{Z}$, such that $x_m \equiv x_n \pmod{p^n}$ for all $m > n$. Addition and multiplication are defined component-by-component, which makes \mathbb{Z}_p into a ring. Note that \mathbb{Z}_p contains \mathbb{Z} , since any nonnegative integer n is represented by (n, n, n, \dots) . Also note that \mathbb{Z}_p is a domain, namely, $a \cdot b = 0$ only happens when a or b is zero. It makes sense to reduce elements of \mathbb{Z}_p modulo p^n , in the usual ring-theoretic way or just by extracting the n^{th} component; the ring $\mathbb{Z}_p/p^n\mathbb{Z}_p$ is canonically isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$. Finally, the invertible elements of \mathbb{Z}_p are precisely the elements not divisible by p (*i.e.*, the elements for which $x_1 \not\equiv 0 \pmod{p}$), for one can write down the inverse of such an element component-by-component, in much the same way as one multiplies in \mathbb{Z}_p .

The p -adic integers can be visualized as an infinite tree much like the ones we have described above. Construct the infinite p -ary tree, analogous to the infinite binary tree for $p = 2$, viewing the nodes on the n^{th} level as classes $(\text{mod } p^n)$, labeled in such a way that a node $(\text{mod } p^n)$ is connected by an edge to the p nodes $(\text{mod } p^{n+1})$ which are congruent to the first class $(\text{mod } p^n)$. The first few levels of the tree for $p = 3$ are pictured below:



Then the p -adic integer (x_1, x_2, \dots) corresponds to the infinite path down the tree which passes through each node $x_n \pmod{p^n}$. \mathbb{Z}_p is the set of all such infinite paths in the tree. This interpretation makes it easy to see when two p -adic integers are congruent $(\text{mod } p^n)$: if and only if their paths coincide for the first n levels. If $\text{ord}_p(x - y) = n$, then the paths x and y coincide at the first n levels, and separate at level $n + 1$.

APPENDIX B. CYCLE GROWTH

We prove that, for $p > 3$, if a cycle of f_1 grows, then its lift also grows. From previous results we then know that the next lift grows, and so on. We also describe when this fails for $p = 3$.

Say our polynomial f has a k -cycle mod p which contains x . Let $g = f^k$, which has x as a fixed point mod p , but for which x is in a p -cycle mod p^2 . We must show that x is in a p^2k -cycle of f_3 , or equivalently that x is in a p^2 -cycle of g_3 . Let $a = a_1 = g'(x)$, $b = b_1 = (g(x) - x)/p$, and $c = g''(x)/2$. Then $a \equiv 1 \pmod{p}$ and $b \not\equiv 0 \pmod{p}$.

First, we show that, for each $i \geq 1$,

$$g^i(x) \equiv x + pb \sum_{j=0}^{i-1} a^j + p^2 cb^2 \sum_{j=0}^{i-2} a^{i-2-j} (1 + a + \cdots + a^j)^2 \pmod{p^3}.$$

For, this is true for $i = 1$, and inductively

$$\begin{aligned} g^{i+1}(x) &\equiv g \left(x + p \left(b \sum_{j=0}^{i-1} a^j + pcb^2 \sum_{j=0}^{i-2} a^{i-2-j} (1 + a + \cdots + a^j)^2 \right) \right) \pmod{p^3} \\ &\equiv g(x) + pb \sum_{j=1}^i a^j + p^2 cb^2 \sum_{j=0}^{i-2} a^{(i+1)-2-j} (1 + a + \cdots + a^j)^2 \\ &\quad + p^2 cb^2 \left(\sum_{j=0}^{i-1} a^j \right)^2 \pmod{p^3} \\ &= x + pb \sum_{j=0}^i a^j + p^2 cb^2 \sum_{j=0}^{i-1} a^{(i+1)-2-j} (1 + a + \cdots + a^j)^2, \end{aligned}$$

which completes the induction.

Now,

$$\begin{aligned} g^p(x) &\equiv x + pb \sum_{j=0}^{p-1} a^j + p^2 cb^2 \sum_{j=0}^{p-2} a^{p-2-j} \left(\sum_{\ell=0}^j a^\ell \right)^2 \pmod{p^3} \\ &\equiv x + pb \sum_{j=0}^{p-1} a^j + p^2 cb^2 \sum_{j=0}^{p-2} (j+1)^2 \pmod{p^3} \quad (\text{since } a \equiv 1 \pmod{p}) \\ &= x + pb \sum_{j=0}^{p-1} a^j + p^2 cb^2 \frac{(p-1)(p)(2p-1)}{6} \end{aligned}$$

and, for $p > 3$, the last term is $0 \pmod{p^3}$, so

$$g^p(x) \equiv x + pb \sum_{j=0}^{p-1} a^j \pmod{p^3}.$$

But, as shown in Section 6.1, $\sum_{j=0}^{p-1} a^j$ is not divisible by p^2 . Thus, $g^p(x) \not\equiv x \pmod{p^3}$, so the p -cycle of g_2 which includes x does not split, hence it grows.

Using the above methods, we can describe when a cycle $\pmod{3}$ will grow and then split. For $p = 3$ we have

$$\begin{aligned} g^p(x) &\equiv x + pb \sum_{j=0}^{p-1} a^j + p^2 cb^2 \frac{(p-1)(p)(2p-1)}{6} \pmod{p^3} \\ &\equiv x + pb \sum_{j=0}^{p-1} a^j - p^2 c \pmod{p^3} \\ &\equiv x + p^2 b - p^2 c \pmod{p^3}, \end{aligned}$$

so a cycle $\pmod{3}$ which grows will then split if and only if $b \equiv c \pmod{3}$.

APPENDIX C. RATIONAL FUNCTIONS

Let $h = f/g$ be a ratio of polynomials $f, g \in \mathbb{Z}[x]$ such that g takes values coprime to p on any cycle being considered; in particular, this condition certainly holds if g has no roots in the field $\mathbb{Z}/p\mathbb{Z}$. We will show that the results we have derived for polynomials over \mathbb{Z} also hold for h . We do this by constructing a sequence of polynomials $h_n \in \mathbb{Z}[x]$ such that h and h_n agree (mod p^n) on the cycles being considered, and the a_i 's and b_i 's for the various h_n are compatible.

Precisely, put

$$h_n(x) = f(x) \cdot g(x)^{\phi(p^{2n})-1},$$

where ϕ is the Euler quotient function; then $h_n(x) \in \mathbb{Z}[x]$. For any x such that $p \nmid g(x)$, $h_n(x) \equiv h(x) \pmod{p^{2n}}$. Let $a_{i,n}$ and $b_{i,n}$ be the values of a_i and b_i for the polynomial h_n , for $i \leq n$, and say that x is in a cycle of $h_n \pmod{p^i}$ of length $\alpha_{i,n}$. Then

$$b_{i,n} = \frac{h_n^{\alpha_{i,n}}(x) - x}{p^i} \equiv \frac{h^{\alpha_{i,n}}(x) - x}{p^i} \pmod{p^n}.$$

Next,

$$\begin{aligned} h'_n(x) &= f'(x) \cdot g(x)^{\phi(p^{2n})-1} + f(x) \cdot (\phi(p^{2n}) - 1) \cdot g(x)^{\phi(p^{2n})-2} \cdot g'(x) \\ &\equiv \frac{f'(x)}{g(x)} + f(x) \cdot (p^{2n} - p^{2n-1} - 1) \cdot \frac{g'(x)}{g(x)^2} \pmod{p^{2n}} \\ &\equiv \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \pmod{p^{2n-1}} \\ &= h'(x), \end{aligned}$$

so

$$\begin{aligned} a_{i,n} &= (h_n^{\alpha_{i,n}})'(x) = \prod_{\ell=0}^{\alpha_{i,n}-1} h'_n(h_n^\ell(x)) \equiv \prod_{\ell=0}^{\alpha_{i,n}-1} h'(h^\ell(x)) \pmod{p^{2n-1}} \\ &= (h^{\alpha_{i,n}})'(x). \end{aligned}$$

Now, for any $n > i$, h agrees with $h_n \pmod{p^i}$; thus, each $\alpha_{i,n}$ equals the length of the cycle of $h \pmod{p^i}$ containing x . Hence, the classes $a_{i,n} \pmod{p^i}$ and $b_{i,n} \pmod{p^i}$ are independent of n . This shows the

compatibility of the h_n ; thus, because our earlier results apply to each h_n , they apply as well to the function h .

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