Computing Twisted Kazhdan-Lusztig-Vogan polynomials

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Some terminology

Kazhdan-Lusztig polynomials: (KL 1979) Hecke algebras of Coxeter groups

Kazhdan-Lusztig conjectures: these polynomials give change of basis matrices in category $\mathcal{O}$ (KL 1979, proved by Beilinson/Bernstein, Brylinski/Kashiwara 1980)

Kazhdan-Lusztig-Vogan polynomials: change of basis matrices for $(\mathfrak{g}, K)$-modules, also known as Kazhdan-Lusztig polynomials, (Lusztig-Vogan, 1980s). $(G, \theta) \rightarrow K = G^\theta \rightarrow (\mathfrak{g}, K)$. Related to representations of the Hecke algebra of W.

Twisted Kazhdan-Lusztig-Vogan polynomials: introduced by Lusztig-Vogan 2013. (G, θ, δ), $\theta^2 = \delta^2 = 1$, $\theta\delta = \delta\theta \rightarrow (\mathfrak{g}, K)$ modules with an action of δ.Related to representations of the "folded" Hecke algebra. These arise from the study of the unitary dual.

Today: Discuss twisted KLV polynomials; their role in the unitary dual, and what is needed to compute them.

The Kazhdan-Lusztig-Vogan picture

Fix an infinitesimal character

(finite) Parameter set: $\{\gamma \mid \gamma \in S\}$

$\gamma \to J(\gamma)$: an irreducible $(\mathfrak{g}, K)$-module

$\gamma \to I(\gamma)$: a standard $(\mathfrak{g}, K)$-module (induced from limit of discrete series)

$\mathcal{M} =$ Grothendieck group of admissible $(\mathfrak{g}, K)$-modules with given infinitesimal character

$$= \mathbb{Z}\langle J(\gamma) \mid \gamma \in S\rangle$$

$$= \mathbb{Z}\langle I(\gamma) \mid \gamma \in S\rangle$$

Change of basis matrices:

$$I(\gamma) = \sum_{\delta \in S} m(\delta, \gamma) J(\gamma)$$
$$J(\gamma) = \sum_{\delta \in S} M(\delta, \gamma) I(\gamma)$$

$m(\delta, \gamma) \in \mathbb{N}$ (multiplicities)

$M(\delta, \gamma) \in \mathbb{Z}$ (character formula)

The Kazhdan-Lusztig-Vogan polynomials

$\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, $\mathcal{M} \to \mathcal{M} = \mathcal{M} \otimes \mathcal{A}$

Theorem

(Lusztig/Vogan 1983) Define KLV polynomials (next slide)

$$P(\gamma, \delta) \in \mathbb{Z}[q]$$

then

$$M(\gamma, \delta) = (-1)^{\ell(\gamma) - \ell(\delta)} P(\gamma, \delta)(1)$$

$P(\gamma, \delta)$ are related to singularities of closures of K orbits on G/B:

$$P(\gamma, \delta) = \sum_i (\text{dim.stalk of intersection cohomology sheaf}) q^i$$

The set S

For later use...

$G = G(\mathbb{C}), K = G^\theta \to G^\vee$, and an inner class of choices of $\theta^\vee, K^\vee$

$$S = \{(x, y) \mid x \in K\backslash G/B, \quad y \in K^\vee\backslash G^\vee/B^\vee, -\theta_x^t = \theta^y\}$$

(this set (a "block") works fine; to get $\mathcal{M}$ take the union of these as you vary $K^\vee$)

Key property of S: symmetric in G and $G^\vee$ (Vogan duality)

Computing the KLV polynomials:

There is a natural representation of $\mathcal{H}(W)$ on
$\mathcal{M} = \mathcal{A}\langle I(\gamma) \mid \gamma \in S\rangle$,

a duality operation $\mathcal{D}$ on $\mathcal{M}$

such that $P(\, , \,)$ is a change of basis matrix, computed entirely from the data $(\mathcal{H}, \mathcal{M}, D)$, given by a set of recurrence relations which determine them uniquely.

Basic principle: everything you want to know about $(\mathfrak{g}, K)$-modules can be computed from the KLV polynomials.

Character theory, parabolic induction, K-types and multiplicities, etc.

K-types:

$$\mathrm{mult}_K(\mu, J(\gamma)) = \sum_\delta M(\delta,\gamma)\mathrm{mult}_K(\mu, I(\delta))$$

$\mathrm{mult}_K(\mu, I(\delta))$ is "known" (Blattner formula, induced character formula)

Question: can we understand Hermitian forms from this point of view?

Problem: Until now everything was complex/holomorphic: G is complex, $\mathfrak{g} = Lie(G)$, θ is a holomorphic involution of G, and $K = G^\theta$ is a complex group$(\mathfrak{g}, K)$-modules.

Now: need to use the antiholomorphic involution σ definining the real form $G(\mathbb{R}) \subset G$

Invariant Hermitian form on (V, π):

$$\langle \pi(X)v, w\rangle + \langle v, \pi(X)w\rangle = 0 \quad (X \in \mathfrak{g}_0)$$

or equivalently in terms of $\mathfrak{g}$:

$$\langle \pi(X)v, w\rangle + \langle v, \pi(\sigma(X))w\rangle = 0 \quad (X \in \mathfrak{g})$$

Basic Idea

Suppose $J = J(\gamma)$ admits an invariant Hermitian form $\langle , \rangle_J$. Then

$$(?) \qquad (J(\gamma), \langle , \rangle_J) = \sum_\delta M_h(\delta,\gamma)(I(\delta), \langle , \rangle_{I(\delta)})$$

where $M_h(\delta,\gamma) \in \mathbb{Z}[s]$ ($s^2 = 1$), $(a + bs)(I(\delta, \langle , \rangle_{I(\delta)})$ means $I(\delta, \langle , \rangle_{I(\delta)})$ occurs a times with $+\langle , \rangle_{I(\delta)}$ and b times with $-\langle , \rangle_{I(\delta)}$.

This ideal picture has two serious flaws:
(1) $I(\delta)$ may have no invariant form (even if J does)

(2) The invariant form on $I(\delta)$ (or $J(\delta)$) is not canonical: there is no way to prefer $+\langle , \rangle$ over $-\langle , \rangle$ (already occurs for the odd principal series of $SL(2,\mathbb{R})$).

Any formula will depend on these choices, and can't possibly be related to anything geometric.

The c-invariant form

Solution:

Replace $\langle \, , \, \rangle$ with the c-invariant form.

c-invariant form

Fix σ_c a compact real form of G and $\mathfrak{g}$. The c-invariant form satisfies:

$$\langle \pi(X)v, w\rangle_c + \langle v, \pi(\sigma_c(X))w\rangle_c = 0 \quad (X \in \mathfrak{g})$$

Assume "real" infinitesimal character (standard reduction to this case)

The c-invariant form

Theorem (Adams/Trapa/van Leeuwen/Vogan)

Every irreducible $(\mathfrak{g}, K)$ modules admits a c-invariant Hermitian form. It can be normalized to be positive definite on all lowest K-types, and is then unique up to positive real scalar.
Then $M_c(\delta, \gamma)$ is well defined, and

$$M_c(\delta, \gamma) = P_h(\delta, \gamma)(1)$$

where

$$P_h(\delta, \gamma)(q) = \iota P(\delta, \gamma)(sq)$$

($\iota = 1$ or s is an elementary factor)

$$(J(\gamma), \langle , \rangle_J) = \sum_{\delta} M_c(\delta, \gamma)(I(\delta), \langle , \rangle_{I(\delta)})$$

Digression on p-adic groups

The c-invariant is a natural object (more natural than a Hermitian form).

Question: Is there an analogue in the p-adic case?

Ciubotaru, Barbasch: in the context of the graded affine Hecke algebra

Definition: $(\pi, V) \to (\pi^{ch}, V^h)$

$$V^h = \{f : V \to \mathbb{C} \mid f(\lambda v) = \overline{\lambda} f(v)\}$$

$$\pi^{ch}(X)(f)(v) = -f(\pi(\sigma_c(X)v))$$

Easy: $\pi \simeq \pi^{ch}$ if and only if π admits a c-invariant Hermitian form.

Consider the involution $\pi \to \pi^{ch}$

Look at the dual side: $G^\vee$ is a complex group.
$\pi : W_{\mathbb{R}} \to^L G \rightsquigarrow \Pi(\phi)$:

Lemma

The c-Hermitian dual is given by:

$$\Pi(\phi)^{ch} = \Pi(\sigma_s \circ \phi)$$

where σ_s is complex conjugating defining the split form of $G^\vee$.

In the p-adic case, $G^\vee$ is still a complex group. So:

Question: What does $\phi \to \sigma_s \circ \phi$ give in the p-adic case?

Equal rank case

Definition:

$(\mathfrak{g}, K)$ is of equal rank if θ is an inner automorphism, equivalently $G(\mathbb{R})$ has discrete series representations, equivalently the inner class of G contains the compact real form.

Theorem (Adams/Trapa/van Leeuwen/Vogan)

If $(\mathfrak{g}, K)$ is equal rank, there is an elementary formula relating $\langle , \rangle$ and $\langle , \rangle_c$. Consequently, there is an explicit algorithm to compute whether a given irreducible $(\mathfrak{g}, K)$-module is unitary.

Formula: $\mu \in \widehat{K}$ -on the μ-isotypic component:

$$\langle v, w \rangle = \zeta \mu(z_K) \langle v, w \rangle_c$$

where $z_K \in Z(K)$, ζ is independent of μ, and $(\zeta \mu(z_k))^2 = 1$.

Unequal Rank Case

Where are we?

We can compute $\langle , \rangle_c$ in terms of the usual KLV polynomials $P(\gamma,\delta)(sq)$

Problem:

This does NOT determine $\langle , \rangle$: some information is missing.

Set ${}^\theta K = \langle K, \theta\rangle$ and consider $(\mathfrak{g}, {}^\theta K)$ modules.

$\leftrightsquigarrow$ representations of ${}^\theta G(\mathbb{R}) = \langle G(\mathbb{R}), \theta\rangle$

Unequal Rank Case

Key fact: $\pi^h \simeq \pi^\theta$

So:
π admits an invariant Hermitian form if and only if $\pi^\theta \simeq \pi$

Proposition

An irreducible $(\mathfrak{g}, K)$-module π admits an invariant Hermitian form if and only if π extends to an irreducible $(\mathfrak{g}, {}^\theta K)$-module $\hat{\pi}$. If this holds there are exactly two such extensions, differing by tensoring with the sgn representation.

Key Problem

There is no canonical way (in general) to choose between these two extensions.

Digression

Consider irreducible finite dimensional representations of $O(n)$ (disconnected).

Fix an irreducible representation π of $SO(n)$, fixed by the extra element.
If n is odd, $\pi_\pm : \pi_\epsilon(-I) = \epsilon$

If n is even? $\pi_\epsilon(\mathrm{diag}(1, \dots, 1, -1)) = \epsilon$? On the highest weight space? What is $(1, 1, \dots, -1)$?

Conclusion: no canonical way to choose π_+.

Lusztig-Vogan

Let δ be the distinguished automorphism inner to θ (preserving a splitting datum)

δ is the Cartan involution of the most compact form in the inner class; $\delta = 1$ if and only if (G, K) is of equal rank

δ defines a "folding" of the Dynkin diagram, and a "folded" Hecke algebra $\widehat{\mathcal{H}}/\mathcal{A}$

Recall: $\mathcal{M} = \mathcal{A}\langle J(\gamma) \mid \gamma \in S\rangle$, Grothendieck group of $(\mathfrak{g}, K)$-modules, carries an action of $\mathcal{H} = \mathcal{H}(W)$.

Definition

$\widehat{\mathcal{M}}$ is the corresponding object for $(\mathfrak{g}, {}^{\theta}K)$-modules, modulo the relation

$$\widehat{\pi} \otimes \mathrm{sgn} \equiv -\widehat{\pi}$$

Basis: parametrized by $\{\pi \text{ irreducible } \mid \pi^{\theta} \simeq \pi\}$

Parameter set: S^{θ}

Twisted KLV polynomials

Theorem (Lusztig/Vogan 2013)

There is a natural action of $\widehat{\mathcal{H}}$ on $\widehat{\mathcal{M}}$, equipped with a duality map $\widehat{\mathcal{D}}$.

This defines the twisted KLV polynomials by the standard procedure applied to $(\widehat{\mathcal{H}}, \widehat{\mathcal{M}}, \widehat{\mathcal{D}})$. These are $\{\widehat{P}(\gamma, \delta) \mid \gamma, \delta \in S^\theta\}$.

Furthermore

$$\widehat{P}(\gamma, \delta) = \sum_i (\text{trace of } \theta \text{ acting on the stalk} \\ \text{of an intersection homology sheaf}) q^i$$

Formulas for the Hecke algebra action

1C+: $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma}$

1C-: $T_{w_\kappa}(a_\gamma) = (u-1)a_\gamma + ua_{w_\kappa \times \gamma}$

1i1: $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma} + a_{\gamma^\kappa}$

1i2f: $T_{w_\kappa}(a_\gamma) = a_\gamma + (a_{\gamma_1^\kappa} + a_{\gamma_2^\kappa})$

1i2s: $T_{w_\kappa}(a_\gamma) = -a_\gamma$

1ic: $T_{w_\kappa}(a_\gamma) = ua_\gamma$

1r1f: $T_{w_\kappa}(a_\gamma) = (u-2)a_\gamma + (u-1)(a_{\gamma_\kappa^1} + a_{\gamma_\kappa^2})$

1r1s: $T_{w_\kappa}(a_\gamma) = ua_\gamma$

1r2: $T_{w_\kappa}(a_\gamma) = (u-1)a_\gamma - a_{w_\kappa \times \gamma} + (u-1)a_{\gamma_\kappa}$

1rn: $T_{w_\kappa}(a_\gamma) = -a_\gamma$

2C+: $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma}$

2C-: $T_{w_\kappa}(a_\gamma) = (u^2-1)a_\gamma + u^2 a_{w_\kappa \times \gamma}$

2Ci: $T_{w_\kappa}(a_\gamma) = ua_\gamma + (u+1)a_{\gamma^\kappa}$

2Cr: $T_{w_\kappa}(a_\gamma) = (u^2-u-1)a_\gamma + (u^2-u)a_{\gamma_\kappa}$

2i11: $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma} + a_{\gamma^\kappa}$

Formulas for the Hecke algebra action

2i12: $T_{w_\kappa}(a_\gamma) = a_\gamma + \sum \epsilon(\lambda,\gamma) a_\lambda$

2i22: $T_{w_\kappa}(a_\gamma) = a_\gamma + (a_{\gamma_1^\kappa} + a_{\gamma_2^\kappa})$

2r22: $T_{w_\kappa}(a_\gamma) = (u^2-1)a_\gamma - a_{w_\kappa \times \gamma} + (u^2-1)a_{\gamma_\kappa}$

2r21: $T_{w_\kappa}(a_\gamma) = (u^2-2)a_\gamma + (u^2-1)\sum \epsilon(\gamma,\lambda) a_\lambda$

2r11: $T_{w_\kappa}(a_\gamma) = (u^2-2)a_\gamma + (u^2-1)(a_{\gamma_\kappa^1} + a_{\gamma_\kappa^2})$

2rn: $T_{w_\kappa}(a_\gamma) = -a_\gamma$

2ic: $T_{w_\kappa}(a_\gamma) = u^2 a_\gamma$

3C+: $T_{w_\kappa}(a_\gamma) = w_\kappa \times a_\gamma$

3C-: $T_{w_\kappa}(a_\gamma) = (u^3-1)a_\gamma + u^3(a_{w_\kappa \times a_\gamma})$

3Ci: $T_{w_\kappa}(a_\gamma) = ua_\gamma + (u+1)a_{\gamma^\kappa}$

3Cr: $T_{w_\kappa}(a_\gamma) = (u^3-u-1)a_\gamma + (u^3-u)a_{\gamma_\kappa}$

3i: $T_{w_\kappa}(a_\gamma) = ua_\gamma + (u+1)a_{\gamma^\kappa}$

3r: $T_{w_\kappa}(a_\gamma) = (u^3-u-1)a_\gamma + (u^3-u)a_{\gamma_\kappa}$

3rn: $T_{w_\kappa}(a_\gamma) = -a_\gamma$

3ic: $T_{w_\kappa}(a_\gamma) = u^3 a_\gamma$

Formulas for the Hecke algebra action

For example:

`1i1:` $T_{w_\kappa}(a_\gamma) = a_{w_\kappa \times \gamma} + a_{\gamma^\kappa}$:

$\gamma \in S^\theta$, α is an imaginary, noncompact simple root,$\delta\alpha = \alpha$, $\kappa = \alpha$, $w_\kappa = s_\alpha$

$\gamma \to$ *two* representations $\widehat{\pi}_\clubsuit(\gamma), \widehat{\pi}_\spadesuit(\gamma) = \widehat{\pi}_\clubsuit \otimes \mathrm{sgn}$

No preference. In $\widehat{\mathcal{M}}$ $\widehat{\pi}_\clubsuit \otimes \mathrm{sgn} = -\widehat{\pi}_\clubsuit$, so:

`1i1:` $T_{s_\alpha}(\widehat{\pi}(\gamma)) = \pm\widehat{\pi}(s_\alpha \times \gamma) \pm \widehat{\pi}(\gamma^\alpha)$

where the signs depend on the choices.

Obviously: Given $\gamma \in S^\theta$: we can *choose* $\widehat{\pi}(\gamma')$, so that, for example

`1i1:` $T_{s_\alpha}(\widehat{\pi}(\gamma)) = \widehat{\pi}(s_\alpha \times \gamma) + \widehat{\pi}(\gamma^\alpha)$

Formulas for the Hecke algebra action

Lusztig-Vogan: For each line in the table, the given formula holds for some choice of $\widehat{\pi}(\gamma)$.

Question

1. Is it possible to choose the $\widehat{\pi}(\gamma)$ consistently?
2. If so, is this the right thing to do?

Answer: (1) No in general (so (2) is moot).

Recall: $S = \{(x, y) \in K\backslash G/B \times K^\vee\backslash G^\vee/B^\vee \ (-\theta_x^t = \theta_y)$

Need a natural set of *extended parameters* generalizing S.
Desideratum: the data should be symmetric in $G, G^\vee$, reflecting some (unknown) Vogan duality for $(\mathfrak{g}, {}^\theta K)$ modules.

Extended Parameters

Definition

An extended parameter is:

$$(\gamma, g, x, \lambda, \tau, y, l, t)$$

where

1. $\gamma \in X^* \otimes \mathbb{Q}$ (infinitesimal character)
2. $g \in X_* \otimes \mathbb{Q}$ (infinitesimal cocharacter)
3. $x \in K\backslash G/B$
4. $y \in K^\vee\backslash G^\vee/B^\vee$
5. $\lambda, \tau \in X^*$
6. $l, t \in X_*$

satisfying various conditions, the new ones are:

1. $(1-\theta_x)\lambda = (1-\theta_x)(\gamma-\rho)$
2. $(1-\theta_y^\vee)l = (1-\theta_y^\vee)(g-\rho^\vee)$
3. $(1-\delta)\lambda = (1-\theta_x)\tau$
4. $(1-\delta^t)\lambda = (1-\theta_y^\vee)t$

Extended Parameters

There is a natural map

$$E = (\gamma, g, x, \lambda, \tau, y, l, t) \to \widehat{\pi}(E)$$

Other ingredients:

1. A natural map $E = (\gamma, g, x, \lambda, \tau, y, l, t) \to \widehat{\pi}(E)$ $(\mathfrak{g}, {}^{\theta}K)$-module
2. Cayley transforms of extended parameters
3. Cross action of extended parameters
4. Definition of equivalence of extended parameters

Note: There is a lot of flexibility in this construction. An essential tool in choosing the right one is the requirement that some version of Vogan duality hold.

Conclusion

Theorem (Adams/Vogan)

This can be carried out, in such a way as to make the formulas for the Hecke algebra action hold on the level of extended parameters. The resulting recurrence relations for the twisted KLV polynomials may then be solved.

Currently being implemented in the `atlas` software.