A computation with Bernstein projectors of depth 0 for SL(2)

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1 Introduction

Computation based on a conversation with Roger Howe (Aug 2013).

- The computation is elementary but tedious. The end result is an interesting spectral expansion of $\delta_{1_{\mathrm{SL}(2)}}$:
$$\delta_{1_{\mathrm{SL}(2)}} = e_0 + e_{\frac{1}{2}} + e_1 + \cdots$$
into invariant orthogonal idempotent distributions e_k belonging to the Bernstein center, with the very important properties:

· e_k is related to representations of depth k.

· e_k has support in the topologically unipotent set of elements of SL(2). In particular, the distribution $E_k = e_k \circ \exp$ is an invariant distribution on the Lie algebra supported on the topologically nilpotent set. I will mention geometrically what I think is the Fourier Transform $FT(E_k)$.

- The computation, in particular, relies on the SL(2) discrete series character table computed by Sally-Shalika in 1968.

2 Notation

- F a p-adic field (characteristic 0) with residue field $\mathbb{F}_q = \mathcal{R}_F/\mathcal{P}_F$.
- $\mathbb{G}/F$ a connected reductive algebraic group, $\mathrm{Lie}(\mathbb{G})$ The Lie-algebra of $\mathbb{G}$, $\quad G := \mathbb{G}(F)$, $\mathfrak{g} = \mathrm{Lie}(\mathbb{G})(F)$.

  Fix Haar measures μ_G on G and $\mu_{\mathfrak{g}}$ on $\mathfrak{g}$.
- $C_c^\infty(G)$, $C_c^\infty(\mathfrak{g})$ the vectors spaces of locally constant compactly supported functions.
- ψ a non-trivial character of F. Will assume conductor is $\mathcal{P}_F$.

3 Review of Bernstein Center distributions, Components and Projectors.

For next few sections G is a general connected reductive p-adic group. Suppose $D \in \mathrm{Hom}_{\mathbb{C}}(C_c^\infty(G), \mathbb{C})$, i.e., a distribution and $f \in C_c^\infty(G)$. We have the convolution $D \star f$:

$$(D \star f)(x) \ := \ D(\lambda_x(C(f))) \ \text{ where } \begin{cases} (\lambda_x(h))(y) \ := \ h(x^{-1}y) \text{ is left translation} \\ C(h)(y) := h(y^{-1}) \text{ is inversion} \end{cases}$$

D is called left-essentially compact if:

$$\forall f \in C_c^\infty(G), \quad \text{we have} \quad D \star f \in C_c^\infty(G).$$

Similar notion of right-essentially compact. An essentially compact distribution is, by definition, one which is both left and right essentially compact.

Bernstein center (geometric version):

$$\mathcal{Z}(G) \ : = \ \text{algebra of essentially compact } G\text{-invariant distributions}$$

For any $z \in \mathcal{Z}(G)$, and any smooth representation π, there is a canonical way to define $\pi(z) \in \mathrm{End}_{\mathbb{C}}(V_\pi)$.

Remark: Explicit examples of Bernstein center distributions are rather sparse:

- The delta function δ_z of a central element $z \in G$, e.g., δ_{1_G}.
- When G is semisimple, and π is an irreducible cuspidal representation, then the character $f \longrightarrow \mathrm{trace}(\pi(f))$ is a Bernstein center distribution.
- (Bernstein's example) When $G = \mathrm{SL}(n)$, and ψ is a non-trivial additive character, then the distribution which is integration against the function

$$\psi \circ \mathrm{trace}$$

  is in $\mathcal{Z}(G)$.
- When G is quasi-split, M-Tadić produced some Bernstein center distributions as linear combinations of orbital integrals of split elements.

Spectral realization of Bernstein Center

Consider pairs $[M, \sigma]$ where M is a Levi subgroup of G and σ is a (equivalent class of) cuspidal representation of M.

- The group G acts by the Adjoint map on the collection of pairs, and the resulting set of orbits is the space $\Omega(G)$ of infinitesimal characters. There is an map from the smooth dual:

$$\widehat{G} \xrightarrow{\text{Inf}} \Omega(G)$$

- For a fixed Levi M, the unramified characters $\Psi(M)$ of M act on pairs with $[M, \sigma]$ by twisting σ. Then

$$\Omega([M, \sigma]) \;=\; \mathrm{Inf}(\; \Psi(M)\,[M, \sigma]\;) \;,$$

  is a Bernstein component. It and therefore $\Omega(G)$ too is a complex variety.

- If $z \in \mathcal{Z}(G)$, and π and π' are two irreducible smooth representations with $\mathrm{Inf}(\pi) = \mathrm{Inf}(\pi')$, then $\pi(z) = \pi'(z)$. In particular, each $z \in \mathcal{Z}(G)$ defines a function $z_{\Omega(G)}$ on $\Omega(G)$.

- **Spectral expansion characterization of** $z \in \mathcal{Z}(G)$.

(i) $z_{\Omega(G)}$ is a regular function on each component Ω, and

$$z(f) \;=\; \int_{\widehat{G}_{\mathrm{temp}}} z_{\Omega(G)}(\pi)\, \Theta_\pi\,(\,f\,)\; d\mu(\pi)$$

(ii) Conversely, given a system of regular functions on the Bernstein components Ω, then the above integral gives a distribution in $\mathcal{Z}(G)$.

Bernstein Projectors

Recall the abstract Plancherel formula

$$\delta_1(f) \;=\; \int_{\widehat{G}_{\mathrm{temp}}} \Theta_\pi\,(\,f\,)\;d\mu(\pi)\;.$$

Suppose Ω is a Bernstein component. The distribution

$$e_\Omega(\,f\,) \;:=\; \int_{\widehat{G}_{\mathrm{temp}}\,\cap\,\mathrm{Inf}^{-1}(\Omega)} \Theta_\pi\,(\,f\,)\;d\mu(\pi)\;.$$

is an idemponent in $\mathcal{Z}(G)$. The e_Ω's are called the Bernstein component projectors. We have:

$$\delta_1 \;=\; \sum_\Omega e_\Omega\;, \quad \text{and}$$

$$C_c^\infty(G) \;=\; \bigoplus_\Omega e_\Omega \star C_c^\infty(G) \star e_\Omega$$

is a decomposition of the (non-unital) algebra $C_c^\infty(G)$ into ideals.

4 Depths of representations and components.

Suppose $\pi \in \widehat{G}$.

Work of M-Prasad defines a rational non-negative number (depth) $\rho(\pi)$ attached to π.

Furthermore, if $\mathrm{Inf}(\pi)$, $\mathrm{Inf}(\pi')$, belong to the same component then $\rho(\pi) = \rho(\pi')$, so one can define the depth of a component Ω.

For a given depth d there are only finitely many components Ω with depth d.

Set

$$e_d := \sum_{\rho(\Omega)=d} e_\Omega .$$

I'll state the interesting outcome of a computation for the very special case of SL(2) and $d = 0$ when the residual characteristic of F is odd. For SL(2):

1. M-Tadić explicitly computed the projectors e_Ω for principal series components in 2001. Marko and I did the initial work during a one month NSF international collaboration visit (July 2000) here in Chicago.

2. For a cuspidal component, i.e., representation π, the projector is given as:

$$e_\pi \;=\; d_\pi \, \Theta_\pi \qquad (\Theta_\pi \text{ is the character}) \;.$$

   Sally and Shalika computed these characters in 1968.

5 Topologically nilpotent and unipotent and compact sets.

Back to general setting. A failing of the p-adic situation is:

$$\exp \;:\; \mathfrak{g} \longrightarrow G$$

is not always defined.

An element $\gamma \in G$ is called *topologically unipotent* if for any F-rational representation $\tau \;:\; G \longrightarrow \mathrm{GL}(V)$, the characteristic polynomial charpoly$(\tau(\gamma), x)$ satisfies:

$$\mathrm{charpoly}(\tau(\gamma), x) \;\in\; \mathcal{R}_F[x]\,, \quad \text{and} \quad \equiv\; (x-1)^{\dim(V)} \bmod \mathcal{P}_F\;.$$

Similarly $\gamma \in \mathfrak{g}$ is *topologically nilpotent* if:

$$\mathrm{charpoly}(\tau(\gamma), x) \;\in\; \mathcal{R}_F[x]\,, \quad \text{and} \quad \equiv\; x^{\dim(V)} \bmod \mathcal{P}_F\;.$$

Let $\mathcal{N}_{\mathrm{top}}$ and $\mathcal{U}_{\mathrm{top}}$ denote respectively the sets of topologically nilpotent and topologically unipotent. Then under suitable conditions,

$$\exp \text{ is a } G\text{-equivariant bijection of } \mathcal{N}_{\mathrm{top}} \text{ with } \;\mathcal{U}_{\mathrm{top}}\;.$$

In particular, functions/distributions on G supported on $\mathcal{U}_{\mathrm{top}}$ can be pulled back to $\mathcal{N}_{\mathrm{top}}$.

An element $\gamma \in G$ is *compact* if it lies in some compact subgroup. This is equivalent to:

$$\forall\ \tau\ :\quad \mathrm{charpoly}(\tau(\gamma), x)\ \in\ \mathcal{R}_F[x]\ .$$

Set

$$\mathcal{C}\ :=\ \text{set of compact elements.}$$

Obviously,

$$\mathcal{U}_{\mathrm{top}}\ \subset\ \mathcal{C}$$

Result of Dat in general setting:

$$\forall\ \Omega\ :\quad \mathrm{supp}(e_\Omega)\ \subset\ \mathcal{C}.$$

When G is semi-simple, a cuspidal representation π gives a singleton Bernstein component Ω, and $e_\Omega = d_\pi \theta_\pi$. In this situation, Deligne was the first to note θ_π has support in $\mathcal{C}$.

6 Statement of a $\mathrm{SL}(2)$ calculation result.

For $G = \mathrm{SL}(2)(F)$, the idempotent distribution

$$e_0 := \sum_{\rho(\Omega)=0} e_\Omega$$

has support in $\mathcal{U}_{\mathrm{top}}$.

Remark: The components for $\mathrm{SL}(2)$ are either principal series or cuspidal. Neither of the two idempotents

$$e_{0,\mathrm{PS}} := \sum_{\substack{\Omega\ \mathrm{PS} \\ \rho(\Omega)=0}} e_\Omega \quad , \quad e_{0,\mathrm{cusp}} := \sum_{\substack{\Omega\ \mathrm{cuspidal} \\ \rho(\Omega)=0}} e_\Omega$$

has support in $\mathcal{U}_{\mathrm{top}}$. More generally, no linear combinations of just PS (or just cusp) idempotents has support in $\mathcal{U}_{\mathrm{top}}$.

Some elementary remarks on elements in $\mathcal{U}_{\text{top}}$ and $\mathcal{C}$ for SL(2).

1. If $\gamma \in \mathcal{U}_{\text{top}}$ is not unipotent, then it is semi-simple (either split or elliptic) with eigenvalues α, α^{-1} which are principal units ($|\alpha - 1|_E < 1$). (E is F or relevant quadratic extension.)

2. We note

$$\mathcal{C} \;=\; \mathcal{U}_{\text{top}} \;\coprod\; -\mathrm{I}_{2\times 2}\mathcal{U}_{\text{top}} \;\coprod\; \textsc{rest}$$

   $\textsc{rest}$ is the set of strongly regular (compact) elements:

$$\textsc{rest} \;=\{\, \gamma \in \mathcal{C} \mid \text{the eigenvalues of } \gamma \text{ modulo } \mathcal{P} \text{ are distinct } \,\}$$

7 PS projectors for $\mathrm{SL}(2)$.

The PS components of depth zero are parameterized by characters pairs $\{\chi, \chi^{-1}\}$ of $\mathcal{R}_F^\times/(1+\mathcal{P}_F)$. The Bernstein projectors are given by the following table: For $y \in \mathcal{C}_{\mathrm{reg}}$,

Regular PS

$$e_{\Omega(\{\chi,\chi^{-1}\})}(y) = \begin{cases} (q+1)\,\dfrac{\chi(\alpha)+\chi(\alpha^{-1})}{|\,\alpha-\alpha^{-1}\,|_F}\,, & y \text{ split with eigenvalues } \alpha,\ \alpha^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Sgn PS

$$e_{\Omega(\mathrm{sgn})}(y) = \begin{cases} (q+1)\,\dfrac{\mathrm{sgn}(\alpha)}{|\,\alpha-\alpha^{-1}\,|_F}\,, & y \text{ split with eigenvalues } \alpha,\ \alpha^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Unramified PS (Iwahori fixed vectors)

$$e_{\Omega}(y) = \begin{cases} \dfrac{2q}{|\,\alpha-\alpha^{-1}\,|_F} - (q-1)\,, & y \text{ split with eigenvalues } \alpha,\ \alpha^{-1} \\ -(q-1) & y \text{ elliptic} \end{cases}$$

8 Cuspidal projectors for $\mathrm{SL}(2)$.

We give a description of the cuspidal depth zero representations. We recall there are two conjugacy classes of maximal compact subgroups in $\mathrm{SL}(2)$ with representatives:

$$\begin{aligned} K &= \mathrm{SL}(2)(\mathcal{R}_F)\ , \\ K' &= v\ K\ v^{-1}\ , \quad \text{where} \quad v := \begin{bmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{bmatrix}\ . \end{aligned} \tag{8.1}$$

Let K_1 and K_1' be the 1st congruence subgroups of K and K' respectively. The quotients K/K_1 and K'/K_1' are naturally isomorphic to $\mathrm{SL}(2)(\mathbb{F}_q)$.

- Any irreducible cuspidal depth zero representation π is (compactly) induced uniquely from either K or K' of a cuspidal reprentation inflated from $\mathrm{SL}(2)(\mathbb{F}_q)$.

- Recall $\mathrm{SL}(2)(\mathbb{F}_q)$ has:
    1. $(q-1)/2$ cuspidal representations of degree $(q-1)$.
    2. a pair of cuspidal representations of degree $(q-1)/2$.

  They are associated to pairs of regular characters, and the sgn character of the elliptic torus.

  1. If σ is a cuspidal representation of degree $(q-1)$, then:

     $$\pi \;:=\; \mathrm{Ind}_K^G \,\sigma \circ \inf \quad , \qquad \pi' \;:=\; \mathrm{Ind}_{K'}^G \,\sigma \circ \inf$$

     have characters Θ_π and $\Theta_{\pi'}$ conjugate under the adjoint action of the element v. They form a 2 element L-packet.
  2. The two degree $(q-1)/2$ cuspidal representations, when inflated to K and K', and then induced to G, give a 4 element L-packet.

The next slide is the Sally-Shalika tables of 'unramified' discrete series. The case of the level $h = 1$ corresponds to depth 0.

TABLE 2. *Unramified discrete series.*

Representation $\Pi(\Phi,\psi,V)$, $\quad V = k(\sqrt{\epsilon})$, $\quad \mathrm{cond}\ \psi = C_\epsilon^{(h)}$, $\quad h \geq 1$, $\quad \psi^2 \not\equiv 1$.

Representative		$\Theta_\Pi(g)$
$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$	$\lambda \notin \pm U_h$	0
$\lambda \in k^*$	$0 < \lvert 1 - \lambda \rvert \leq q^{-h}$	$\dfrac{1}{\lvert \lambda - \lambda^{-1} \rvert} - q^{h-1}$
$\begin{pmatrix} \alpha & \beta \\ \tau\beta & \alpha \end{pmatrix}$, $\begin{pmatrix} \alpha & -\beta\epsilon \\ -\tau\beta\epsilon^{-1} & \alpha \end{pmatrix}$	$0 < \lvert 1 - \lambda \rvert^2 \leq q^{-2h+1}$	$-q^{h-1}$
$\lambda = \alpha + \sqrt{\tau}\beta$	$q^{-2h+1} < \lvert 1 - \lambda \rvert^2 < 1$	0
$\begin{pmatrix} \alpha & \beta \\ \epsilon\tau\beta & \alpha \end{pmatrix}$, $\begin{pmatrix} \alpha & -\epsilon\beta \\ -\tau\beta & \alpha \end{pmatrix}$	$0 < \lvert 1 - \lambda \rvert^2 \leq q^{-2h+1}$	$-q^{h-1}$
$\lambda = \alpha + \sqrt{\epsilon\tau}\beta$	$q^{-2h+1} < \lvert 1 - \lambda \rvert^2 < 1$	0
$\begin{pmatrix} \alpha & \beta \\ \epsilon\beta & \alpha \end{pmatrix}$	$0 < \lvert 1 - \lambda \rvert \leq q^{-h}$	$-q^{h-1} + H(\Phi,V)\, \dfrac{\mathrm{sgn}_\epsilon\left(\dfrac{\lambda - \lambda^{-1}}{2\sqrt{\epsilon}}\right)}{\lvert \lambda - \lambda^{-1} \rvert}$
$\lambda = \alpha + \sqrt{\epsilon}\beta$	$q^{-h} < \lvert 1 - \lambda \rvert \leq 1$ $\lvert 1 + \lambda \rvert = 1$	$\dfrac{1}{2}\, \mathrm{sgn}_\epsilon\left(\dfrac{\lambda - \lambda^{-1}}{2\sqrt{\epsilon}}\right) \dfrac{\psi(\lambda) + \psi(\lambda^{-1})}{\lvert \lambda - \lambda^{-1} \rvert} \times [(-1)^h + H(\Phi,V)]$
$\begin{pmatrix} \alpha & \beta\tau \\ \epsilon\beta\tau^{-1} & \alpha \end{pmatrix}$ $\lambda = \alpha + \sqrt{\epsilon}\beta$		REPLACE $H(\Phi,V)$ BY $-H(\Phi,V)$ IN THE ABOVE EXPRESSIONS FOR $\begin{pmatrix} \alpha & \beta \\ \epsilon\beta & \alpha \end{pmatrix}$

Predating Dat's (2000) result (and Deligne's (1970's) result for cuspidal representations too), the Sally-Shalika 1968 table shows $\Omega_\pi = \deg_\pi \Theta_\pi$ has support in $\mathcal{C}$.

Tediously combining the M-Tadić results for the PS components with the Sally-Shalika results for the cuspidal components yields e_0 vanishes on $\mathcal{C} \backslash \mathcal{U}_{\rm top} = -\mathrm{I}_{2\times 2}\mathcal{U}_{\rm top} \coprod$ REST, so

$$\mathrm{supp}(\, e_0 \,) \subset \mathcal{U}_{\rm top} \,.$$

9 Actual values of e_0.

On $\mathcal{U}_{\mathrm{top}}$

$$e_0(y) = (q^2 - 1) \begin{cases} \left(\dfrac{2}{|\alpha - \alpha^{-1}|_F} - 1\right) & \\ \qquad \text{when } y \text{ is split with eigenvalues } \alpha,\ \alpha^{-1} & \\ & \\ -1 & \\ \qquad \text{when } y \text{ is elliptic} & \end{cases}$$

QUESTION: For SL(2), the possible depths are the integers and half integers $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. What can be said for the other idempotents e_d?

For $d > 0$, matters are less delicate. The sum of just the PS component projectors (of depth d) already has support in $\mathcal{U}_{\text{top}}$. In particular,

$$\text{supp}(\, e_d \,) \ \subset \ \mathcal{U}_{\text{top}} \ .$$

But for $d > 0$, the tedious computations of the exact values of e_d have not yet been computed.

Saying something for the lowest depth 0 was always considered a key obstruction.

Important take away is:

$$\delta_{1_G} = \sum_d e_d$$

is an expansion of the delta distribution into a sum of G-invariant essentially compact distributions, each representable by a locally L^1 function, and supported in $\mathcal{U}_{\mathrm{top}}$.

I finished the support computations in December 2013, and looking at my e-mails, I wrote to Roger and Ju-Lee Kim about the answer on December 22. I believe Paul would have appreciated the use of the Sally-Shalika characters tables in the computation, and especially the harmonic analysis that I think occurs when we move the distributions e_d to the Lie algebra.

Pull the distribution e_d to the topological nilpotent set $\mathcal{N}_{\text{top}}$:

$$\mathcal{N}_{\text{top}} \xrightarrow{\exp} \mathcal{U}_{\text{top}} \xrightarrow{e_d} \mathbb{C} \;.$$

QUESTION: What is the Fourier transform $FT(\,e_d \circ \exp\,)$?

Since the G-invariant distribution $e_d \circ \exp$ is presumably an (essentially compact) idempotent distribution on $\mathfrak{g}$, its Fourier Transform should be the characteristic function of a G-invariant set Ξ_d.

- The orthogonality of the idempotents e_d, means, up to measure zero, the sets Ξ_d are disjoint.
- The expansion $\delta_{1_G} = \sum_d e_d$ means, up to measure zero, the union of the sets Ξ_d is all of $\mathfrak{g}$.

Is there a nice description of the sets Ξ_d?

10 Fourier Transform.

For $k \geq 0$ a integer or half integer, consider the G-invariant set:

$$\mathfrak{g}_{-k} \;=\; \{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid \det = (-a^2 - bc) \in \mathcal{P}_F^{-2k} \} \,.$$

We have

$$\cdots \supset 1_{\mathfrak{g}_{-1}} \supset 1_{\mathfrak{g}_{-\frac{1}{2}}} \supset 1_{\mathfrak{g}_0}$$

$$1_{\mathfrak{g}_{-\ell}} \;=\; \varpi^{-1}\, 1_{\mathfrak{g}_{-\ell+1}} \,.$$

In the general setting, there is a definition of the sets $\mathfrak{g}_s$ for $s \in \mathbb{R}$ as:

$$\mathfrak{g}_s \;:=\; \bigcup_{x \in B} \mathfrak{g}_{x,s} \,.$$

Let $1_{\mathfrak{g}_s}$ be the characteristic function. Then,

- (Harish-Chandra) The Fourier Transform $FT(\,1_{\mathfrak{g}_s}\,)$ can be represented by a locally L^1 function supported on the regular set.
- The distributions $FT(\,1_{\mathfrak{g}_s}\,)$ on $\mathfrak{g}$ are essentially compact:

$$\forall f \in C_c^\infty(\mathfrak{g}),\ \text{ the function } FT(\,1_{\mathfrak{g}_s}\,) \star f \ \text{ is in } C_c^\infty(\mathfrak{g}).$$

So, $FT(\,1_{\mathfrak{g}_s}\,)$ is in the Lie algebra Bernstein center.

There is a expansive period relationship $\mathfrak{g}_{s-1} = \varpi^{-1}\mathfrak{g}_s$, which gives an opposite contractive relationship between $FT(\,1_{\mathfrak{g}_{s-1}}\,)$ and $FT(\,1_{\mathfrak{g}_s}\,)$.

In the situation of SL(2), we have the following fact about the supports of $FT(\,1_{\mathfrak{g}_0}\,)$, and $FT(\,1_{\mathfrak{g}_{-\frac{1}{2}}}\,)$:

Proposition 10.1. *The Fourier Transforms $FT(\,1_{\mathfrak{g}_0}\,)$, and $FT(\,1_{\mathfrak{g}_{-\frac{1}{2}}}\,)$ have support in the topologically nilpotent set $\mathcal{N}_{\mathrm{top}}$.*

Corollary 10.2. *For k a non-positive integer or half integer, $FT(\,1_{\mathfrak{g}_k}\,)$ has support in the topologically nilpotent set $\mathcal{N}_{\mathrm{top}}$.*

Here is a sketch of the proof for $FT(1_{\mathfrak{g}_0})$. We are allowed to evaluate $FT(1_{\mathfrak{g}_0})$ at any convenient element in a regular conjugacy class.

Split elements: For $X = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$, and $y = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, we have:

$$\operatorname{trace} (y X) \ = \ 2 \, a \, A \tag{10.3}$$

so,

$$\begin{aligned} FT(1_{\mathfrak{g}_0}) (X) \ &= \ \int_{\mathfrak{g}} 1_{\mathfrak{g}_0}(y) \ \psi(\operatorname{trace} (yX)) \ dy \ = \ \int_{\mathfrak{g}} 1_{\mathfrak{g}_0}(y) \ \psi(2 \, a \, A) \ da \, db \, dc \\ &= \mathrm{PV} \left(\int_{\mathfrak{g}_0} \psi(2 \, a \, A) \ da \, db \, dc \ \right) \end{aligned} \tag{10.4}$$

We need to show for $A \notin \mathcal{P}_F$, the PV integral is zero. For r a (large) positive integer, define B_r to be the 'box':

$$B_r \ := \ \{ \ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \ | \ a, \ b, \ c \ \in \mathcal{P}_F^{-r} \ \} \tag{10.5}$$

We show the integral vanishes over $\mathcal{L}_{(0,r)} := B_r \cap \mathfrak{g}_0$. We focus on the (1,1)-entry a.

Partition $\mathcal{L}_{(0,r)}$ into

$$\begin{aligned}
\mathcal{L}_{(0,r),+} \;&:=\; \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathcal{L}_{(0,r)} \middle|\; a \in \mathcal{R}_F \right\} \\
\mathcal{L}_{(0,r),-} \;&:=\; \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathcal{L}_{(0,r)} \middle|\; a \in F \setminus \mathcal{R}_F \right\}
\end{aligned}$$

Case $\mathcal{L}_{(0,r),+}$:

If $Y = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathcal{L}_{(0,r),+}$, then for all $x \in \mathcal{R}_F$, the matrix

$$Y_x \;=\; \begin{bmatrix} a+x & b \\ c & -(a-x) \end{bmatrix} \tag{10.6}$$

is also in $\mathcal{L}_{(0,r),+}$ and $\psi(\mathrm{trace}(\,Y_x\,X\,)\,) \;=\; \psi(\,2\,(a+x)\,A)$. We can fix the variables b and c and integrate the variable a over the set $\mathcal{R}_F$ to see the integral over a vanishes. Therefore, the integral over $\mathcal{L}_{(0,r),+}$ vanishes.

Case $\mathcal{L}_{(0,r),-}$:
Write an element $y \in \mathcal{L}_{(0,r),-}$ as $y = \begin{bmatrix} u\,\varpi^{-k} & B \\ C & -\,u\,\varpi^{-k} \end{bmatrix}$, with u a unit and $k > 0$. The condition $y \in \mathcal{L}_{(0,r),-}$ is $k \leq r$, and

$$u^2\,\varpi^{-2k} \,+\, BC \,\in \mathcal{R}_F\,.$$

If $x \in \mathcal{R}_F$, then the element

$$\begin{bmatrix} u\,\varpi^{-k} \,+\, x & B' \\ C' & -\,(\,u\,\varpi^{-k} \,+\, x\,) \end{bmatrix} \tag{10.7}$$

lies in $\mathcal{L}_{(0,r),-}$ precisely when

$$-\,B'C' \in v\,\varpi^{-2k} \,+\, \mathcal{R}_F\ , \tag{10.8}$$

where v is the unit $v = (u^2 \,+\, 2\,x\,\varpi^k)$.

For each $x \in \mathcal{R}_F$, the measure of the set of elements $(b,c) \in \mathcal{P}_F^{-r} \times \mathcal{P}_F^{-r}$ satisfying $-bc \in (u^2 + 2x\varpi^k)\varpi^{-2k} + \mathcal{R}_F$ does not depend on x, and thus the integral

$$\int_{\mathcal{L}_{(0,r),-}} \psi(\,2aA\,)da\,db\,dc \tag{10.9}$$

vanishes.

Elliptic elements:

The proofs for unramified and ramified elliptic elements are similar. We only show case of an unramfied elliptic element. Such an elliptic element is GL(2)-conjugate to:

$$X = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \quad \text{with } B, C \text{ units and } BC \text{ a non-square} \tag{10.10}$$

For $y = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, we have $\operatorname{trace}(\, y\, X\,) = (\, b\, C + c\, B\,)$ and so,

$$\begin{aligned} FT(\, 1_{\mathfrak{g}_0}\,)\,(\, X\,) &= \int\limits_{\mathfrak{g}} 1_{\mathfrak{g}_0}(y)\; \psi(\, b\, C + c\, B\,)\; da\, db\, dc \\ &= \mathrm{PV}\left(\int_{\mathfrak{g}_0} \psi(\, b\, C + c\, B\,)\; da\, db\, dc \right) \end{aligned} \tag{10.11}$$

We show the integral

$$\int_{\mathfrak{g}_0 \cap B_r} \psi(\, b\, C \;+\; c\, B\,)\; da\, db\, dc$$

vanishes for $r >> 0$, where the box B_r is (10.5).

We fiber the set $\mathfrak{g}_0 \cap B_r$ by values of a, and show the integral over a fiber is zero.

Case $a \in \mathcal{R}_F$:

Then the condition $-a^2 - bc \;\in\; \mathcal{R}_F$ becomes $bc \;\in\; \mathcal{R}_F$. We further decompose this into the subcases where $k = -\mathrm{val}(b)$, satisfies either $k \geq 1$, or $k \leq 0$, i.e., $b \in \mathcal{R}_F$.

- Case $k \geq 1$. The condition $b\, c \;\in\; \mathcal{R}_F$ is equivalent to $c \in\; \mathcal{P}_F^k$, and so $\psi\,(\, b\, C \;\;+\;\; c\, B\,) \;=\; \psi\,(\, b\, C\,)$, so integration over the c variable gives

$$\psi(\, b\, C\,)\mathrm{meas}(\, \mathcal{P}_F^k\,)\;.$$

Then, replacing b by $b + x$ with $x \in \mathcal{R}_F$ gives

$$\psi(\,x\,C\,)\,\psi(\,b\,C\,)\mathrm{meas}(\,\mathcal{P}_F^k\,)\ . \tag{10.12}$$

This means over a $\mathcal{P}_F$-coset $b+x+\mathcal{P}_F$ with $x \in \mathcal{R}_F$, integration over the c variable gives

$$\psi(\,x\,C\,)\,\psi(\,b\,C\,)\ \mathrm{meas}(\,\mathcal{P}_F\,)\ \mathrm{meas}(\,\mathcal{P}_F^k\,)$$

We then deduce that the integral over (b, c) with $-\mathrm{val}(b) \geq 1$ is zero.

- When $-\mathrm{val}(b) \not\geq 1$, then $b \in \mathcal{R}_F$. The variable c is allowed to run over a (fractional) ideal containing $\mathcal{R}_F$. With a and b fixed the integration over this ideal is zero.

Case $a \notin \mathcal{R}_F$:
Here, $a^2 \notin \mathcal{P}_F^{-1}$. The condition $a^2 + bc \in \mathcal{R}_F$ combined with p odd means the product $-bc$ must be a square, and $2\,\mathrm{val}(a) = \mathrm{val}(b) + \mathrm{val}(c)$. Write $-bc = a_0^2$; so, $a^2 \in a_0^2 + \mathcal{R}_F$. We deduce

$$a \;=\; \pm\, a_0\, u\;, \quad \text{where } u \in 1 + \mathcal{P}_F^{\mathrm{val}(b)+\mathrm{val}(c)} \tag{10.13}$$

Suppose $-\mathrm{val}(b) \geq -\mathrm{val}(c)$, i.e., $|b| \geq |c|$. Then $-\mathrm{val}(b) \geq 1$ and we can think to perturb b to $b + x$ where $x \in \mathcal{R}_F$. Since $-bc$ is a square, so too is $-(b+x)c$ (say $-(b+x)c = a_\dagger^2$). The set of a' so that $(a')^2 + (b+x)c \in \mathcal{R}_F$ is then as in (10.13) with a_0 replaced by $a_\dagger$. Crucially the measure of the two sets over which a and a' are allowed is the same. Therefore the integral over those $y \in B_r$ with $a \notin \mathcal{R}_F$ can be decomposed into integrals fibered over the product $-bc$ a square. For a given $\mathrm{val}(b) + \mathrm{val}(c)$, the measure of the set of qualifying a depends only on $\mathrm{val}(b) + \mathrm{val}(c)$. We can then perturb the variable b or c which has maximum size by elements $x \in \mathcal{R}_F$, while fixing the other. The resulting decomposition of the integral results in vanishing integrals over subsets which partition the subset of B_r with $a \notin \mathcal{R}_F$. We conclude the Fourier transform integral (10.11) vanishes.

This completes the sketch the Fourier transform $FT(\, 1_{\mathfrak{g}_0}\,)$ has support in the topologically nilpotent set $\mathcal{N}_{\mathrm{top}}$. $\square$

Expectation/Conjecture:

- $e_0 \circ \exp = FT(1_{\mathfrak{g}_0})$.
- *More generally, if $k \geq 0$ is an integer or half-integer:*

$$\Big(\sum_{0 \leq j \leq k} e_j\Big) \circ \exp = FT(1_{\mathfrak{g}_{-k}})$$

A tedious, but elementary calculation will determine the status of the above in the situation of SL(2). If true, one then can speculate what happens in higher rank.

11 Excerpt from an e-mail Paul sent after the death of Shalika in Fall 2010.

"I was at the Institute in Autumn, 1967, lecturing on p-adic SL(2) following the works of Bruhat, Gel'fand-Graev, and a few others, including Shalika. Joe was at Princeton. We finally got together in early 1968 and started working. It was an incredibly exciting adventure for two non-tenured, rambunctious rookies. We soon discovered the road map for our project (Harish-Chandra, Plancherel formula for the 2×2 real unimodular group, Proc.N.A.S., 1952). We thought we could do it all: Characters, Plancherel Theorem, and the Fourier Transform of Elliptic Orbital Integrals. We also had the Big Guy down the hall, for regular advice and direction.

We worked mainly in the Seminar Room in Building C, computing, shouting, and wrangling for eight to ten hours at a time. It was spring, and the days were getting longer. So after we finished work, we would walk across the golf course to Andy's Bar on Alexander Street. There, we would drink four or five beers, eat two or three cheeseburgers, revel in the day's successes, and look forward to the same effort the next day. For those who have been in the chase there is no need to talk further about the exhilaration that accompanies this. I went home to Chicago for the 1968-1969 academic year and I made many trips to Princeton. I returned to the Institute for the summer of 1969 to finish the Orbital Integral project, which ultimately used Shalika Germs to derive the Plancherel Formula again.

During the summers of 1968 and 1969, Joe would come occasionally with me to the Nassau Swim Club. Besides being a swimmer, much to my amazement, Joe could do a really nice full summersault off the diving board.

Joe was a good buddy. He gone. Too bad."

Sept 2014: Ditto for Paul.