Matrices almost of order two

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and in memory of Paul Sally, Jr.

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Outline

Adeles

Arithmetic problems $\leftrightsquigarrow$ matrices over $\mathbb{Q}$.

Example: count $\left\{ v \in \mathbb{Z}^2 \;\middle|\; {}^t v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v \leq N \right\}$.

Hard: no analysis, geometry, topology to help.

Possible solution: use $\mathbb{Q} \hookrightarrow \mathbb{R}$.

Example: find area of $\left\{ v \in \mathbb{R}^2 \;\middle|\; {}^t v \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v \leq N \right\}$.

Same idea with $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ leads to

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p = \prod_{v \in \{p, \infty\}}{}' \mathbb{Q}_v,$$

locally compact ring $\supset \mathbb{Q}$ discrete subring.

Arithmetic $\leftrightsquigarrow$ analysis on $GL(n, \mathbb{A})/GL(n, \mathbb{Q})$.

Background about $GL(n,\mathbb{A})/GL(n,\mathbb{Q})$

Gelfand: analysis re $G \leftrightsquigarrow$ irr (unitary) reps of G.

$$\text{analysis on } GL(n,\mathbb{A})/GL(n,\mathbb{Q})$$
$$\leftrightsquigarrow \text{irr reps } \pi \text{ of } \prod_{v\in\{p,\infty\}}{}' GL(n,\mathbb{Q}_v)$$
$$\leftrightsquigarrow \pi = \bigotimes_{v\in\{p,\infty\}}{}' \pi(v), \qquad \pi(v) \in \widehat{GL(n,\mathbb{Q}_v)}$$

Building block for harmonic analysis is one irr rep $\pi(v)$ of $GL(n,\mathbb{Q}_v)$ for each v.

Contributes to $GL(n,\mathbb{A})/GL(n,\mathbb{Q}) \leftrightsquigarrow$ tensor prod has $GL(n,\mathbb{Q})$-fixed vec.

Local Langlands

Big idea from Langlands unpublished[1] 1973 paper:

$$\widehat{GL(n,\mathbb{Q}_v)} \overset{?}{\leftrightsquigarrow} n\text{-diml reps of } \mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v). \qquad \text{(LLC)}$$

Big idea actually goes back at least to 1967; 1973 paper proves it for $v = \infty$.

Caveat: need to replace Gal by Weil-Deligne group.

Caveat: "Galois" reps in (LLC) not irr.

Caveat: Proof of (LLC) for finite v took another 25 years (finished[2] by Harris[3] and Taylor 2001).

Conclusion: irr rep π of $GL(n,\mathbb{A}) \leftrightsquigarrow$ one n-diml rep $\sigma(v)$ of $\mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v)$ for each v.

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[1] Paul Sally did not believe that "big idea" and "unpublished" belonged together. In 1988 he arranged publication of this paper.

[2] History: "finished by HT" is too short. But Phil's not here, so. . .

[3] Not that one, the other one.

Background about arithmetic

$\{\mathbb{Q}_2, \mathbb{Q}_3, \ldots, \mathbb{Q}_\infty\}$ loc cpt fields where $\mathbb{Q}$ dense.

If $E/\mathbb{Q}$ algebraic extension field, then

$$E_v =_{\text{def}} E \otimes_{\mathbb{Q}} \mathbb{Q}_v$$

is a commutative algebra over $\mathbb{Q}_v$.

E_v is direct sum of algebraic extensions of $\mathbb{Q}_v$.

If $E/\mathbb{Q}$ Galois, summands are Galois exts of $\mathbb{Q}_v$.

$\Gamma = \mathrm{Gal}(E/\mathbb{Q})$ transitive on summands.

Choose one summand $E_\nu \subset E \otimes_{\mathbb{Q}} \mathbb{Q}_v$, define

$$\Gamma_v = \mathrm{Stab}_\Gamma(E_\nu) = \mathrm{Gal}(E_\nu/\mathbb{Q}_v) \subset \Gamma.$$

$\Gamma_v \subset \Gamma$ closed, unique up to conjugacy.

Conclusion: n-diml σ of $\Gamma \rightsquigarrow$ n-diml $\sigma(v)$ of Γ_v.

Čebotarëv: knowing almost all $\sigma(v) \rightsquigarrow \sigma$.

Global Langlands conjecture

Write $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \supset \mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v) = \Gamma_v$.

analysis on $GL(n,\mathbb{A})/GL(n,\mathbb{Q})$

$\leftrightsquigarrow$ irr reps π of $\prod'_{v\in\{p,\infty\}} GL(n,\mathbb{Q}_v)$ $\quad \pi^{GL(n,\mathbb{Q})} \neq 0$

$\leftrightsquigarrow \pi = \bigotimes'_{v\in\{p,\infty\}} \pi(v), \qquad \pi^{GL(n,\mathbb{Q})} \neq 0$

$\overset{\text{LLC}}{\leftrightsquigarrow}$ n-diml rep $\sigma(v)$ of Γ_v, each v $\quad$ which $\sigma(v)$??

GLC: $\pi^{GL(n,\mathbb{Q})} \neq 0$ if reps $\sigma(v)$ of $\Gamma_v \rightsquigarrow$ one n-diml representation σ of Γ.

If Γ finite, most $\Gamma_v = \langle g_v \rangle$ cyclic, all g_v occur.

Arithmetic prob: how does conj class g_v vary with v?

Starting local Langlands for $GL(n,\mathbb{R})$

All that was why it's interesting to understand

$$\widehat{GL(n,\mathbb{R})} \overset{\text{LLC}}{\leftrightsquigarrow} n\text{-diml reps of } \mathrm{Gal}(\mathbb{C}/\mathbb{R})$$

$$\leftrightsquigarrow n\text{-diml reps of } \mathbb{Z}/2\mathbb{Z}$$

$$\leftrightsquigarrow \left\{ n \times n \text{ cplx } y,\ y^2 = \mathrm{Id} \right\} / GL(n,\mathbb{C}) \text{ conj}$$

Langlands: more reps of $GL(n,\mathbb{R})$ (Galois $\rightsquigarrow$ Weil).

But what have we got so far?

$y \rightsquigarrow m, \quad 0 \le m \le n \quad (\dim(-1 \text{ eigenspace}))$

$$\rightsquigarrow \text{unitary char } \xi_m \colon B \to \{\pm 1\}, \quad \xi_m(b) = \prod_{j=1}^{m} \mathrm{sgn}(b_{jj})$$

$$\rightsquigarrow \text{unitary rep } \pi(y) = \mathrm{Ind}_B^{GL(n,\mathbb{R})} \xi_m.$$

This is all irr reps of infl char zero.

Integral infinitesimal characters

Infinitesimal char for $GL(n,\mathbb{R})$ is unordered tuple

$$(\gamma_1,\ldots,\gamma_n), \qquad (\gamma_i \in \mathbb{C}).$$

Assume first γ integral: all $\gamma_i \in \mathbb{Z}$. Rewrite

$$\gamma = (\underbrace{\gamma_1,\ldots,\gamma_1}_{m_1 \text{ terms}},\ldots,\underbrace{\gamma_r,\ldots,\gamma_r}_{m_r \text{ terms}}) \qquad (\gamma_1 > \cdots > \gamma_r).$$

A flat of type γ consists of

1. flag $\mathcal{V} = \{V_0 \subset V_1 \subset \cdots \subset V_r = \mathbb{C}^n\}, \quad \dim V_i/V_{i-1} = m_i$;
2. and the set of linear maps

$$\mathcal{F} = \{T \in \mathrm{End}(V) \mid TV_i \subset V_i, T|_{V_i/V_{i-1}} = \gamma_i \mathrm{Id}\}.$$

Such T are diagonalizable, eigenvalues γ.

Each of $\mathcal{V}$ and $\mathcal{F}$ determines the other (given γ).

Langlands param of infl char $\gamma =$ pair $(y,\mathcal{F})$ with $\mathcal{F}$ a flat of type γ, y $n \times n$ matrix with $y^2 = Id$.

Integral local Langlands for $GL(n,\mathbb{R})$

$$\gamma = (\underbrace{\gamma_1,\ldots,\gamma_1}_{m_1\text{ terms}},\ldots,\underbrace{\gamma_r,\ldots,\gamma_r}_{m_r\text{ terms}}) \quad (\gamma_1 > \cdots > \gamma_r) \text{ ints.}$$

Langlands parameter of infl char γ = pair $(y,\mathcal{V})$, $y^2 = \mathrm{Id}$, $\mathcal{V}$ flag, dim $V_i/V_{i-1} = m_i$.

$$\pi \in \widehat{GL(n,\mathbb{R})}, \text{ infl char } \gamma \overset{\text{LLC}}{\leftrightsquigarrow} \{(y,\mathcal{V}\}/\text{conj by } GL(n,\mathbb{C}).$$

So what are these $GL(n,\mathbb{C})$ orbits?

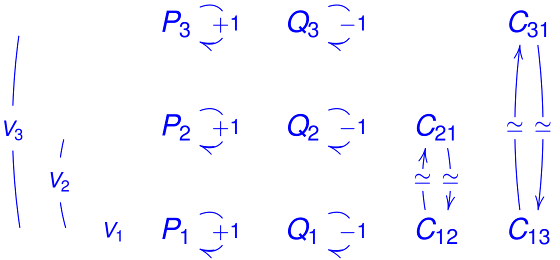
Proposition Suppose $y^2 = \mathrm{Id}_n$ and $\mathcal{V}$ is a flag in $\mathbb{C}^n$. There are subspaces P_i, Q_i, and C_{ij} ($i \neq j$) s.t.

1. $y|_{P_i} = +\mathrm{Id}$, $y|_{Q_i} = -\mathrm{Id}$.
2. $y\colon C_{ij} \xrightarrow{\sim} C_{ji}$.
3. $V_i = \sum_{i' \leq i}(P_{i'} + Q_{i'}) + \sum_{i' \leq i,j} C_{i',j}$.
4. $p_i = \dim P_i$, $q_i = \dim Q_i$, $c_{ij} = \dim C_{ij} = \dim C_{ji}$ depend only on $GL(n,\mathbb{C}) \cdot (y,\mathcal{V})$.

Action of involution y on a flag

Last i rows represent subspace V_i in flag.

Arrows show action of y.



Represent diagram symbolically (Barbasch)

$$\Big(\underbrace{\gamma_1^+,\ldots,\gamma_1^+}_{\text{dim } P_1 \text{ terms}},\underbrace{\gamma_1^-,\ldots,\gamma_1^-}_{\text{dim } Q_1 \text{ terms}},\ldots,\ \underbrace{\gamma_r^+,\ldots}_{\text{dim } P_r \text{ terms}}\ ,\ \underbrace{\gamma_r^-,\ldots}_{\text{dim } Q_r \text{ terms}}\ ,$$

$$\underbrace{(\gamma_1\gamma_2),\ldots,(\gamma_1\gamma_2)}_{\text{dim } C_{12} \text{ terms}},\ldots,\underbrace{(\gamma_{r-1}\gamma_r),\ldots}_{\text{dim } C_{r-1,r} \text{ terms}}\Big)$$

This is involution in S_n plus some signs.

General infinitesimal characters

Recall infl char for $GL(n,\mathbb{R})$ is unordered tuple

$$(\gamma_1,\ldots,\gamma_n), \qquad (\gamma_i \in \mathbb{C}).$$

Organize into congruence classes mod $\mathbb{Z}$:

$$\gamma = \Big(\underbrace{\gamma_1,\ldots,\gamma_{n_1}}_{\text{cong mod }\mathbb{Z}}, \underbrace{\gamma_{n_1+1},\ldots,\gamma_{n_1+n_2}}_{\text{cong mod }\mathbb{Z}}, \ldots, \underbrace{\gamma_{n_1+\cdots+n_{s-1}+1},\ldots,\gamma_n}_{\text{cong mod }\mathbb{Z}}\Big),$$

then in decreasing order in each congruence class:

$$\gamma = \Big(\underbrace{\underbrace{\gamma_1^1,\ldots,\gamma_1^1}_{m_1^1\text{ terms}},\ldots,\underbrace{\gamma_{r_1}^1,\ldots,\gamma_{r_1}^1}_{m_{r_1}^1\text{ terms}}}_{n_1\text{ terms}},\ldots,\underbrace{\underbrace{\gamma_1^s,\ldots,\gamma_1^s}_{m_1^s\text{ terms}},\ldots,\underbrace{\gamma_{r_s}^s,\ldots,\gamma_{r_s}^s}_{m_{r_s}^1\text{ terms}}}_{n_s\text{ terms}}\Big)$$

$$\gamma_1^1 > \gamma_2^1 > \cdots > \gamma_{r_1}^1, \quad \cdots \quad \gamma_1^s > \gamma_2^s > \cdots > \gamma_{r_s}^s.$$

Nonintegral flats

Start with general infinitesimal character

$$\gamma = (\underbrace{\underbrace{\gamma_1^1, \ldots, \gamma_1^1}_{m_1^1 \text{ terms}}, \ldots, \underbrace{\gamma_{r_1}^1, \ldots, \gamma_{r_1}^1}_{m_{r_1}^1 \text{ terms}}}_{n_1 \text{ terms}}, \ldots, \underbrace{\underbrace{\gamma_1^s, \ldots, \gamma_1^s}_{m_1^s \text{ terms}}, \ldots, \underbrace{\gamma_{r_s}^s, \ldots, \gamma_{r_s}^s}_{m_{r_s}^1 \text{ terms}}}_{n_s \text{ terms}})$$

A flat of type γ consists of

1a. direct sum decomp $\mathbb{C}^n = V^1 \oplus \cdots \oplus V^s$, $\dim V^k = n_k$;
1b. flags $\mathcal{V}^k = \{V_0^k \subset \cdots \subset V_{r_k}^k = V^k\}$, $\dim V_i^k / V_{i-1}^k = m_i^k$;
2. and the set of linear maps
$$\mathcal{F}(\{\mathcal{V}^k\}, \gamma) = \{T \in \mathrm{End}(\mathbb{C}^n) \mid TV_i^k \subset V_i^k, T|_{V_i^k/V_{i-1}^k} = \gamma_i^k \mathrm{Id}\}.$$

Such T are diagonalizable, eigenvalues γ.

Each of (1) and (2) determines the other (given γ).

invertible operator $e(T) =_{\text{def}} \exp(2\pi i T)$ depends only on flat: eigenvalues are $e(\gamma_i^k)$ (ind of i), eigenspaces $\{V^k\}$.

Langlands param of infl char γ = pair $(y, \mathcal{F})$ with $\mathcal{F}$ a flat of type γ, y $n \times n$ matrix with $y^2 = e(T)$.

Langlands parameters for $GL(n,\mathbb{R})$

Infl char $\gamma = (\gamma_1, \dots, \gamma_n)$ ($\gamma_i \in \mathbb{C}$ unordered).

Recall Langlands parameter $(y, \mathcal{F})$ is

1. direct sum decomp of $\mathbb{C}^n$, indexed by $\{\gamma_i + \mathbb{Z}\}$;
2. flag in each summand
3. $y \in GL(n,\mathbb{C})$, $y^2 = e(\gamma_i)$ on summand for $\gamma_i + \mathbb{Z}$.

Proposition $GL(n,\mathbb{C})$ orbits of Langlands parameters of infl char γ are indexed by

1. pairing some (γ_i, γ_j) with $\gamma_i - \gamma_j \in \mathbb{Z} - 0$; and
2. labeling each unpaired γ_k with $+$ or $-$.

Example infl char $(3/2, 1/2, -1/2)$:

$[(3/2, 1/2), (-1/2)^{\pm}]$, two params

$[(3/2, -1/2), (1/2)^{\pm}]$, two params

$[(1/2, -1/2), (3/2)^{\pm}$, two params

$[(3/2)^{\pm}, (1/2)^{\pm}, (-1/2)^{\pm}]$ eight params

$[(\gamma_1, \gamma_2)] \leftrightsquigarrow$ disc ser, HC param $\gamma_1 - \gamma_2$ of $GL(2,\mathbb{R})$

$[\gamma^{+ \text{ or } -}] \leftrightsquigarrow$ character $t \mapsto |t|^{\gamma}(\text{sgn } t)^{0 \text{ or } 1}$ of $GL(1,\mathbb{R})$.

Other reductive groups

$G(\mathbb{R})$ real red alg group, ${}^\vee G$ dual (cplx conn red alg).

Semisimple conj class $\mathcal{H} \subset {}^\vee\mathfrak{g} \leftrightsquigarrow$ infl char. for G.

For semisimple $T \in {}^\vee\mathfrak{g}$ and integer k, define

$$\mathfrak{g}(k, T) = \{X \in {}^\vee\mathfrak{g} \mid [T, X] = kX\}.$$

Say $T \sim T'$ if $T' \in T + \sum_{k>0} \mathfrak{g}(k, T)$.

Flats in ${}^\vee\mathfrak{g}$ are the equivalence classes (partition each semisimple conjugacy class in ${}^\vee\mathfrak{g}$).

Exponential $e(T) = \exp(2\pi i T) \in {}^\vee G$ const on flats.

If $G(\mathbb{R})$ split, Langlands parameter for $G(\mathbb{R})$ is $(y, \mathcal{F})$ with $\mathcal{F} \subset {}^\vee\mathfrak{g}$ flat, $y \in {}^\vee G$, $y^2 = e(\mathcal{F})$.

Theorem (LLC—Langlands, 1973) Partition $\widehat{G(\mathbb{R})}$ into finite L-packets $\leftrightsquigarrow$ ${}^\vee G$ orbits of $(y, \mathcal{F})$.

Infl char of L-packet is ${}^\vee G \cdot \mathcal{F}$.

Future ref: $(y, \mathcal{F}) \rightsquigarrow$ inv $w(y, \mathcal{F}) \in W$.

For infl char 0, Theorem says irr reps partitioned by conj classes of homs $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to {}^\vee G$.

and now for something completely different. . .

G cplx conn red alg group.

Problem: real forms of $G/(\text{equiv})$?

Soln (Cartan): $\leftrightsquigarrow \{x \in \mathrm{Aut}(G \mid x^2 = 1\}/\text{conj}$.

Details: given aut x, choose cpt form σ_0 of G s.t. $x\sigma_0 = \sigma_0 x =_{\text{def}} \sigma$.

Example.

$G = GL(n, \mathbb{C})$, $x_{p,q}(g) = \text{conj by} \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.

Choose $\sigma_0(g) = {}^t\overline{g}^{-1}$ (real form $U(n)$).

$$\sigma_{p,q} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}^t\overline{A} & -{}^t\overline{C} \\ -{}^t\overline{B} & {}^t\overline{D} \end{pmatrix}^{-1},$$

real form $U(p, q)$.

Another case of matrices almost of order two.

Cartan involutions

G cplx conn red alg group.

Galois parameter is $x \in G$ s.t. $x^2 \in Z(G)$.

$\theta_x = \mathrm{Ad}(x) \in \mathrm{Aut}(G)$ Cartan involution.

Say x has central cochar $z = x^2$.

$G = SL(n,\mathbb{C})$, $x_{p,q} = \begin{pmatrix} e(-q/2n)I_p & 0 \\ 0 & e(p/2n)I_q \end{pmatrix}$.

$x_{p,q} \leftrightsquigarrow$ real form $SU(p,q)$, central cocharacter $e(p/n)I_n$.

Theorem (Cartan) Surjection $\{$Galois params$\} \rightsquigarrow$ $\{$equal rk real forms of $G(\mathbb{C})\}$.

$G = SO(n,\mathbb{C})$, $x_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ allowed iff $q = 2m$ even.

$x_{n-2m,m} \leftrightsquigarrow$ real form $SO(n-2m,2m)$, central cochar I_n.

$G = SO(2n,\mathbb{C})$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $x_J = n$ copies of J on diagonal.

$x_J \leftrightsquigarrow$ real form $SO^*(2n)$, central cochar $-I_{2n}$.

Imitating Langlands

Since Galois param $\leftrightsquigarrow$ part of Langlands param, why not complete to a whole "Langlands param"?

Start with $z \in Z(G)$

Choose reg ss class $\mathcal{G} \subset \mathfrak{g}$ so $e(g) = z$ $(g \in \mathcal{G})$.

Define Cartan parameter of infl cochar $\mathcal{G}$ as pair $(x, \mathcal{E})$, with $\mathcal{E} \subset \mathcal{G}$ flat, $x \in G(\mathbb{C})$, $x^2 = e(\mathcal{E})$.

Equivalently: pair $(x, \mathfrak{b})$ with $\mathfrak{b} \subset \mathfrak{g}$ Borel.

As we saw for Langlands parameters for $GL(n)$,

$$\text{Cartan param } (x, \mathcal{E}) \rightsquigarrow \text{involution } w(x, \mathcal{E}) \in W;$$

const on $G \cdot (x, \mathcal{E})$; $w(x, \mathfrak{b}) =$ rel pos of $\mathfrak{b}, x \cdot \mathfrak{b}$.

Langlands params $\leftrightsquigarrow$ repns.

Cartan params $\leftrightsquigarrow$???

What Cartan parameters count

Fix reg ss class $\mathcal{G} \subset \mathfrak{g}$ so $e(g) \in Z(G)$ $(g \in \mathcal{G})$.

Define Cartan parameter of infl cochar $\mathcal{G} = (x, \mathcal{E})$, with $\mathcal{E} \subset \mathcal{G}$ flat, $x \in G$, $x^2 = e(\mathcal{E})$.

Theorem Cartan parameter $(x, \mathcal{E}) \leftrightsquigarrow$

1. real form $G(\mathbb{R})$ (with Cartan inv $\theta_x = \mathrm{Ad}(x)$;
2. θ_x-stable Cartan $T(\mathbb{R}) \subset G(\mathbb{R})$;
3. Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$.

That is: $\{(x, \mathcal{E})\}/(G \text{ conj})$ in 1-1 corr with $\{(G(\mathbb{R}), T(\mathbb{R}), \mathfrak{b})\}/(G \text{ conj})$.

Involution $w = w(x, \mathcal{E}) \in W \leftrightsquigarrow$ action of θ_x on $T(\mathbb{R})$.

Conj class of $w \in W \leftrightsquigarrow$ conj class of $T(\mathbb{R}) \subset G(\mathbb{R})$.

How many Cartan params over involution $w \in W$?

Answer uses structure theory for reductive gps. . .

Counting Cartan params

Max torus $T \subset G \rightsquigarrow$

cowgt lattice $X_*(T) =_{\text{def}} \mathrm{Hom}(\mathbb{C}^\times, T)$.

Weyl group $W \simeq N_G(T)/T \subset \mathrm{Aut}(X_*)$.

Each $w \in W$ has Tits representative $\sigma_w \in N(T)$.

Lie algebra $\mathfrak{t} \simeq X_* \otimes_{\mathbb{Z}} \mathbb{C}$, so W acts on $\mathfrak{t}$.

$\mathfrak{g}_{ss}/G \simeq \mathfrak{t}/W$; $\mathcal{G}$ has unique dom rep $g \in \mathfrak{t}$.

Theorem Fix dom rep g for $\mathfrak{G}$, involution $w \in W$.

1. Each G orbit of Cartan params over w has rep $e((g-\ell)/2)\sigma_w$, $\ell \in X_*$ s.t. $(w-1)(g-\rho^\vee-\ell)=0$.
2. Two such reps are G-conj iff $\ell' - \ell \in (w+1)X_*$.
3. set of orbits over w is

$$\begin{cases} \text{princ homog/ } X_*^w/(w+1)X_* & (w-1)(g-\rho^\vee) \in (w-1)X_* \\ \text{empty} & (w-1)(g-\rho^\vee) \notin (w-1)X_* \end{cases}$$

If $g \in X_* + \rho^\vee$, get canonically

Cartan params of infl cochar $\mathcal{G} \simeq X_*^w/(w+1)X_*$.

Integer matrices of order 2

X_* lattice ($\mathbb{Z}^n$), $w \in \mathrm{Aut}(X_*)$, $w^2 = 1$.

$$X_* = \mathbb{Z}, \quad w_+ = (1),$$
$$X_* = \mathbb{Z}, \quad w_- = (-1),$$
$$X_* = \mathbb{Z}^2, \quad w_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note: $-w_s$ differs from w_s by chg of basis $e_1 \mapsto -e_1$.

Theorem Any $w \simeq$ sum of copies of w_+, w_-, w_s.

$$X_*^w/(1+w)X_* = \begin{cases} \mathbb{Z}/2\mathbb{Z} & w = w_+ \\ 0 & w = w_- \\ 0 & w = w_s \end{cases}.$$

Corollary If $w = (w_+)^p \oplus (w_-)^q \oplus (w_s)^r$, then

$$\operatorname{rk} X_*^w = p + r$$
$$\operatorname{rk} X_*^{-w} = q + r$$
$$\dim_{\mathbb{F}_2} X_*^w/(1+w)X_* = p.$$

So p, q, and r determined by w; decomp of X_* is not.

Calculations for classical groups

In classical grp ($GL(n)$, $SO(2n+\epsilon)$, $Sp(2n)$) $X_* = \mathbb{Z}^n$.

$W \subset S_n \rtimes \{\pm 1\}^n$ (perm, sgn chgs of coords).

Involution $w \leftrightsquigarrow$ partition coords $\{1, \ldots, n\}$ into

1. pos coords $a_{i_1}, \ldots, a_{i_p}$, $we_{a_i} = e_{a_i}$;
2. neg coords $b_{j_1}, \ldots, a_{j_q}$, $we_{b_j} = -e_{b_j}$;
3. w_s pairs $(c_{k_1}, c'_{k_1}), \ldots, (c_{k_{r_+}}, c'_{k_{r_+}})$, $we_{c_k} = e_{c'_k}$; and
4. $-w_s$ pairs $(d_{l_1}, d'_{l_1}), \ldots, (d_{l_{r_-}}, d'_{l_{r_-}})$, $we_{d_k} = -e_{d'_k}$.

Consequently $n = p + q + 2(r_+ + r_-)$.

So $\dim_{\mathbb{F}_2} X_*^w/(1+w)X_* = p$.

Example Suppose $G = GL(n, \mathbb{C})$,
$w = (-\mathrm{Id}) \cdot$ (prod of r transp):
$n = 0 + (n-2r) + 2(r+0)$, $X_*^w/(1+w)X_* = \{0\}$.

Conclude: if infl cochar $g \in \rho^\vee + \mathbb{Z}^n$, one Cartan param for each involution w;

Putting it all together

So suppose G cplx reductive alg, ${}^\vee G$ dual.

Fix infl char (semisimple ${}^\vee G$ orbit) $\mathcal{H} \subset {}^\vee\mathfrak{g}$, infl cochar (reg integral ss G orbit) $\mathcal{G} \subset \mathfrak{g}$.

Definition. Cartan param $(x, \mathcal{E})$ and Langlands param $(y, \mathcal{F})$ said to match if $w(x, \mathcal{E}) = -w(y, \mathcal{F})$

Example of matching:
$w(y, \mathcal{F}) = 1 \iff$ rep is principal series for split G;
$w(x, \mathcal{E}) = -1 \iff T(\mathbb{R})$ is split Cartan subgroup.

Theorem. Irr reps (of infl char $\mathcal{H}$) for real forms (of infl cochar $\mathcal{G}$) are in 1-1 corr with matching pairs $[(x, \mathcal{E}), (y, \mathcal{F})]$ of Cartan and Langlands params.

Corollary. L-packet for Langlands param $(y, \mathcal{F})$ is (empty or) princ homog space for $X_*^{-w}/(1-w)X_*$, $w = w(y, \mathcal{F})$.

What did I leave out?

Two cool slides called **Background about arithmetic** and **Global Langlands conjecture** discussed assembling local reps to make global rep, and when the global rep should be automorphic.

Omitted two cool slides called **Background about rational forms** and **Theorem of Kneser *et al.***, about ratl forms of each $G/\mathbb{Q}_v \rightsquigarrow$ ratl form $G/\mathbb{Q}$.

Omitted interesting extensions of local results over $\mathbb{R}$ to study of unitary reps.

Fortunately trusty sidekick Jeff Adams addresses this in 23 hours 15 minutes.

Trusty sidekicks are a very Paul Sally way to get things done.

HAPPY BIRTHDAY BECKY!