Pieces of Cake

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October 15, 2009

Some things mathematicians do

1. **Ask “natural questions”**
2. Explore and experiment
3. **Represent (multiply)**
4. Look for structure (patterns, symmetry, etc.)
5. **Connect** (exactly, analogously, metaphorically, etc.)
6. Conjecture
7. **Consult** (experts, literature, Google, etc.)
8. Seek proofs
9. **Be opportunistic**
10. Prove
11. **Attend to rigor, scientific integrity, aesthetics, taste**

Our ("Natural") **Mathematical Question**

(From the study of fractions)

- Some cakes (c cakes) are to be equally shared by some students (s students)

(Jeff Lagarias prefers that I say something other than "students," like "secretaries," or "surgeons.")

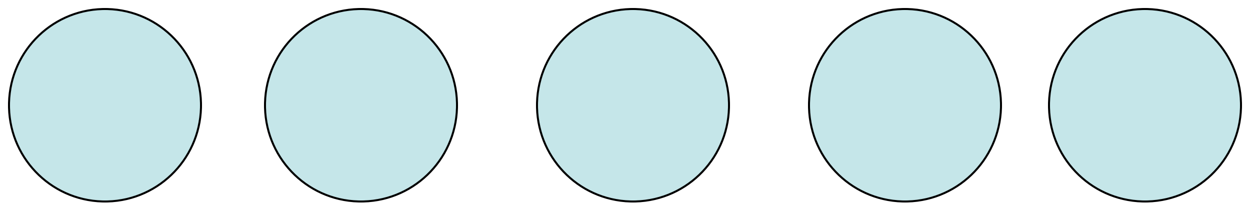
- What is the <u>least number of cake pieces</u> needed to make this equal distribution?

- [Don't confuse with other "fair share" problems.]

(Explore) Examples:
5 cakes shared by 7 students: $c = 5 \quad s = 7$

7 Students: Ä Ä Ä Ä Ä Ä Ä

5 Cakes:

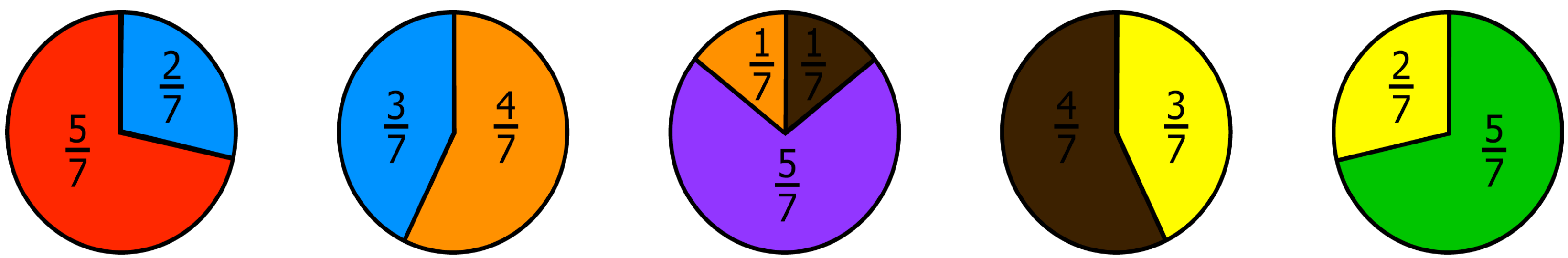


How many cake pieces are needed to share?

- Represent the problem, visually.
- What are your ideas about how to do this?

The “Linear Distribution”

Ä Ä (2 pieces each) Ä Ä

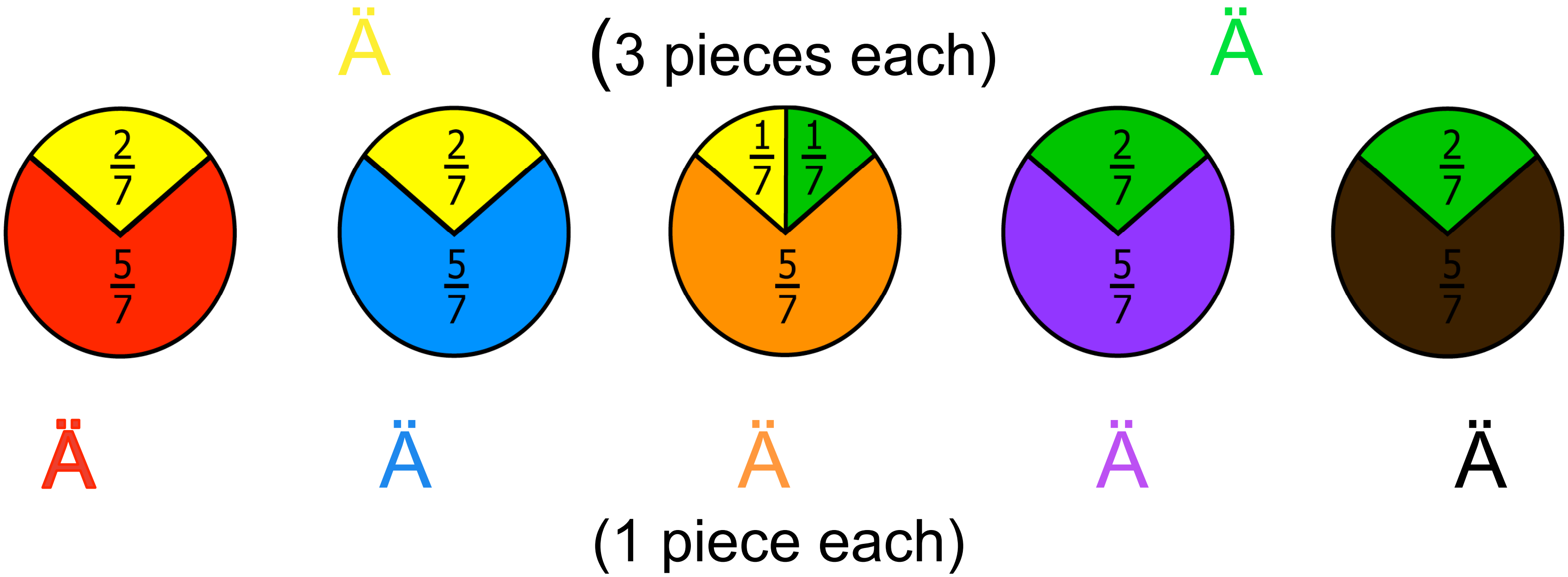


Ä Ä Ä

(1 piece each)

<u>Altogether: 11 pieces</u>

The “Euclidean Distribution”



<u>Altogether: 11 pieces</u>

Remarkable; coincidence?

More natural questions.

Is 11 the minimum possible?

Natural questions; continued exploration

What do you expect the answer to be for general *c* and *s*?

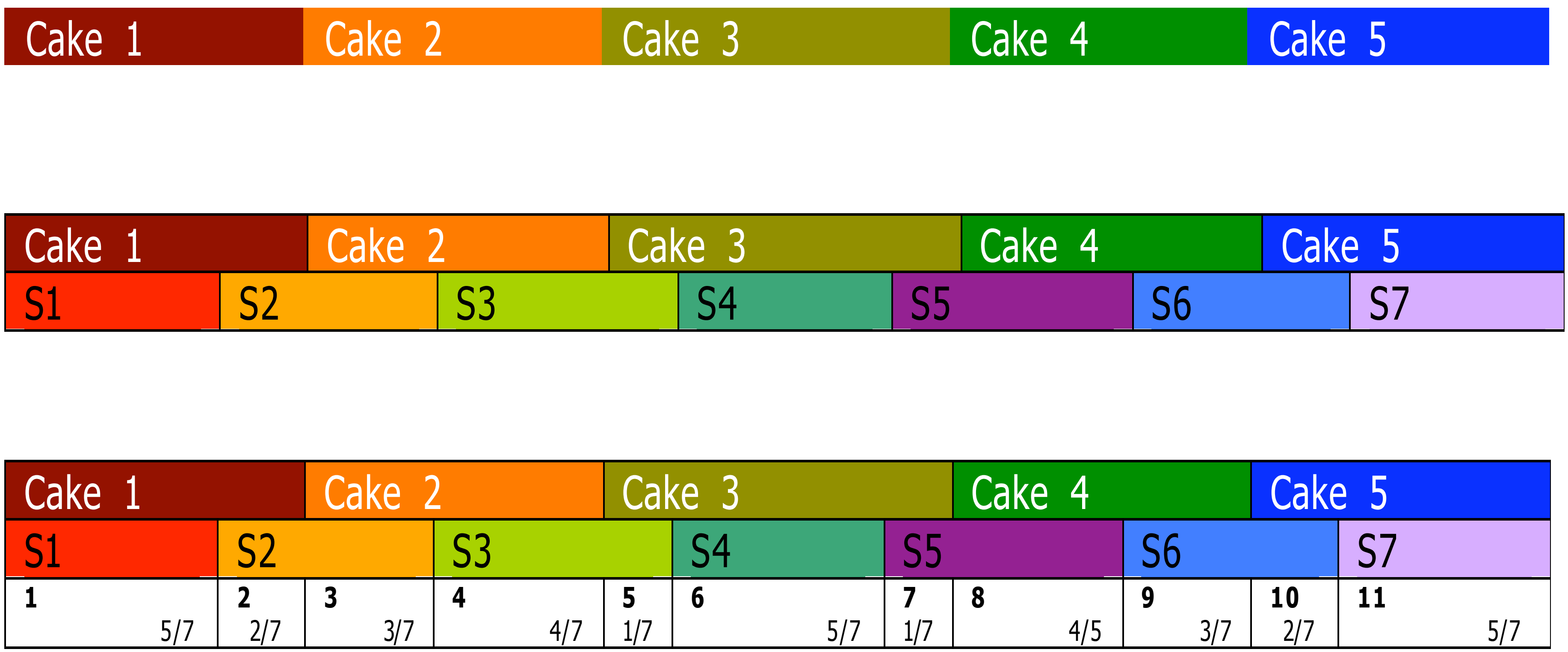
(How is 11 related to 5 and 7?)

For example, what happens for 33 students sharing 17 cakes?

Or 14 students sharing 10 cakes?

From Round to Rectangular Food: I. The linear distribution

A more illuminating representation (“1-dimensional”)



From Round to Rectangular Food: II. The Euclidean distribution

A more illuminating representation (2-dimensional)

<u>5 cakes:</u> Cake 1, Cake 2, … , Cake 5

<u>for 7 students:</u> S1, S2, …, S7

<u>Horizontal</u>: Cake separations; Cakes are the 5 rows

<u>Vertical</u>: Cake cuts

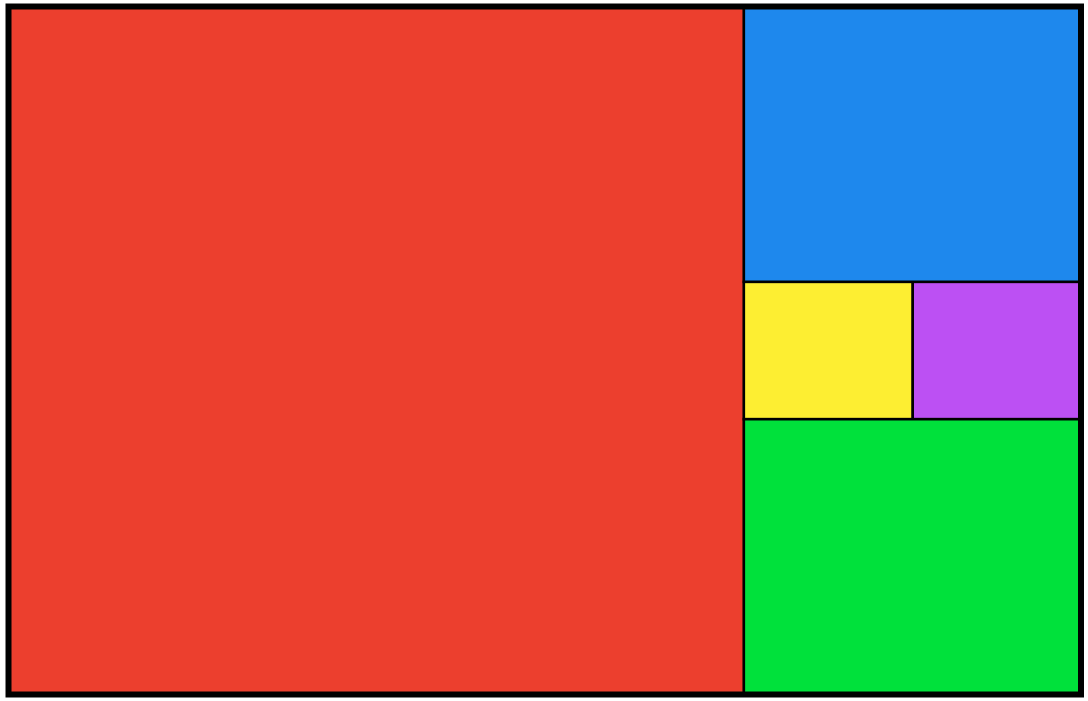
<u>Colors</u>: Student shares

<u>With 11 pieces</u>:
P1, P2, … , P11

A Bi-product of the Euclidean Distribution

Connections; Opportunism

Square Tiling of the rectangle



Euclidean algorithm:

7 = 1•5 + 2

(1 5x5 square)

5 = 2•2 + 1

(2 2x2 squares)

2 = 2•1 + 0

(2 1x1 squares)

Sum of the side lengths of the tiles: 5 + 2 + 2 + 1 + 1 = **11**

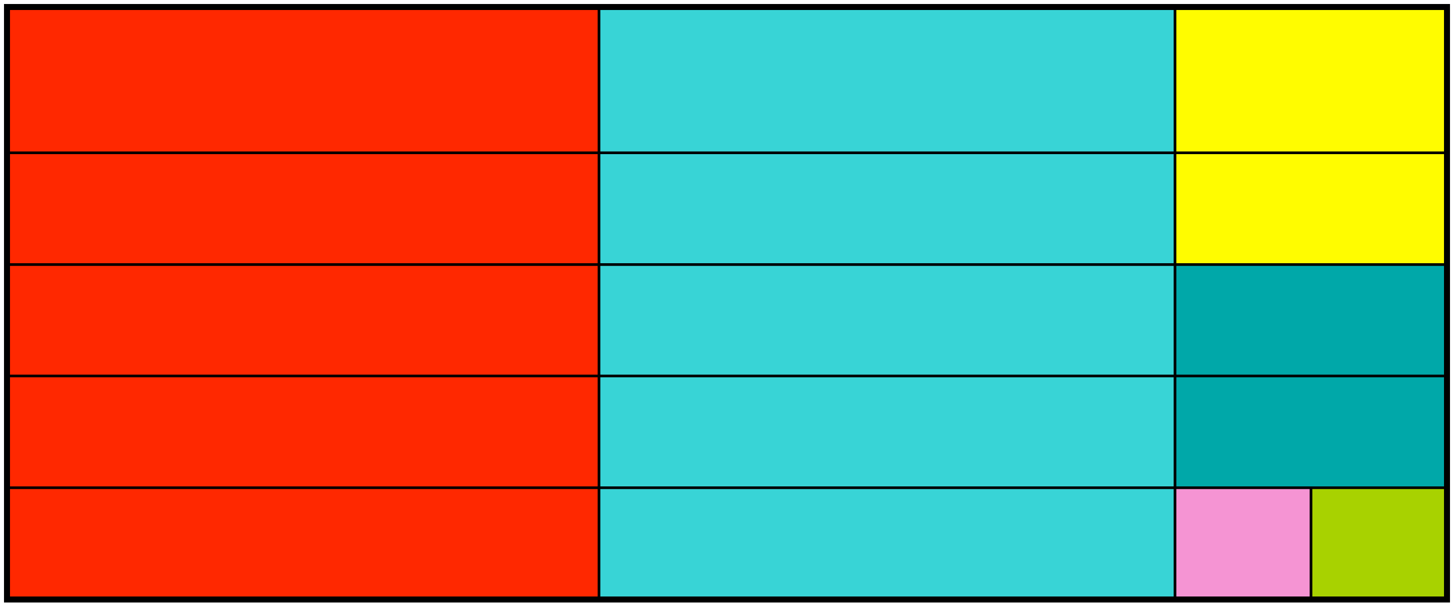
(Connection)

The “complete perimeter” (more later) = 2•(5 + 7) + 5 + 1 + 2 + 2 = 34

5 cakes for 12 students: Euclidean Distribution

More exploration, connection

12 = 2•5 + 2   5 = 2•2 + 1   2 = 2•1 + 0



tiles = 2 + 2 + 2 = 6

pieces = 2•5 + 2•2 + 2•1 = **16**

Sum of side lengths of tiles = 5 + 5 + 2 + 2 + 1 + 1 = **16** (Connection)

The Minimum Number of Pieces

At first a Conjecture; a long time to prove

Theorem. (a) The minimum number, p(c, s), of cake pieces required to equally distribute c cakes among s students is

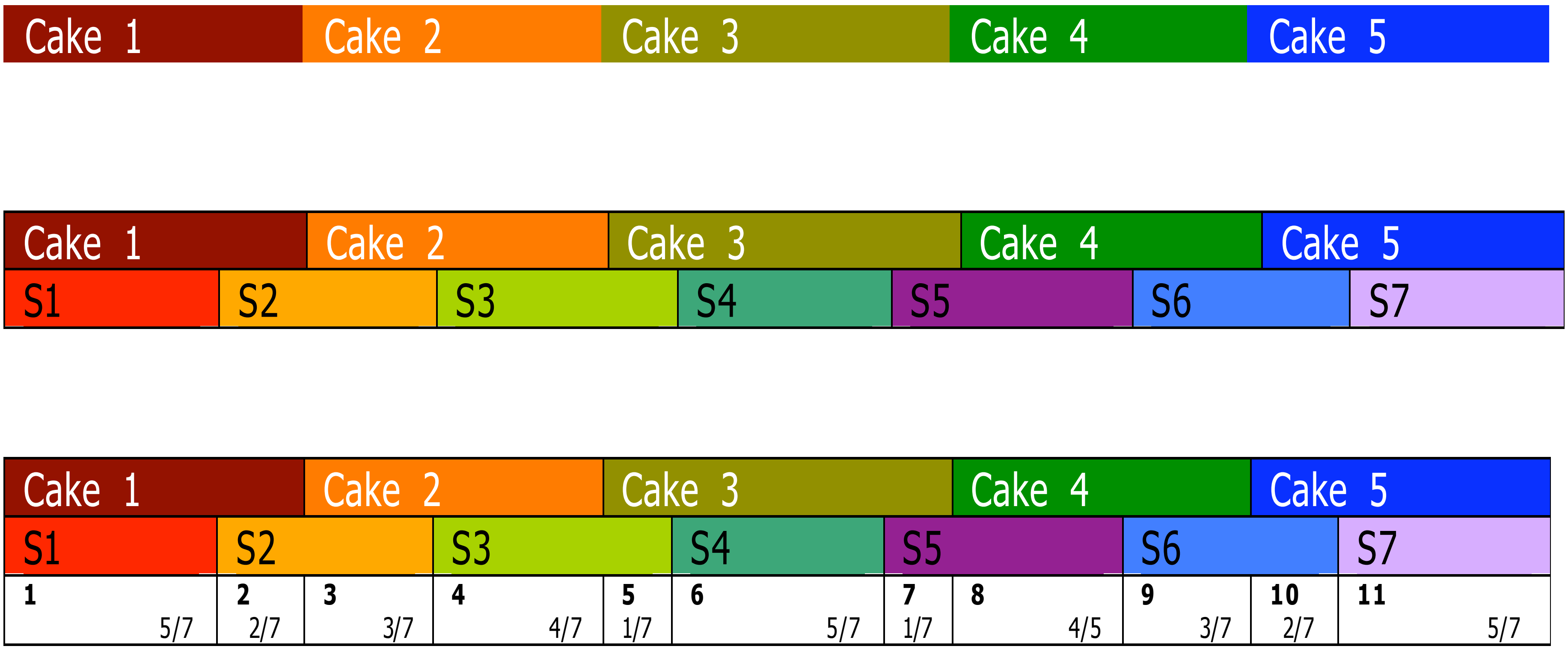
$$\mathbf{p(c, s) = c + s - d},$$

where

$$d = \mathrm{gcd}(c, s).$$

(b) This minimum number of pieces is achieved by both the linear and the Euclidean distributions.

Proof of (b) for the linear distribution



Proof that the linear distribution produces c + s - d pieces.

Note how well suited the linear representation is to the argument.

Place the c (rectangular) cakes, each of length s, end to end, so that total length is c•s units. Treat this as one long cake of length c•s. Cake separations are at multiples of s:

c - 1 of these.

Student share separations are at multiples of c:

s - 1 of these.

These cuts coincide at multiples of m = lcm(c,s): (c•s = d•m)

d - 1 of these.

So the total number of cuts is

$$(c - 1) + (s - 1) - (d - 1) = c + s - d - 1,$$

So, the total number of pieces is

$$c + s - d.$$

Proof that the euclidean distribution produces c + s - d pieces.

- This relies on an inductive argument based on the Euclidean algorithm.
- No time for the details here.

- We have shown that

$$p(c,s) \leq c + s - d$$

This was proved early, using the linear and euclidean distributions

- It remains to show that,

$$p(c,s) \geq c + s - d$$

This took much longer

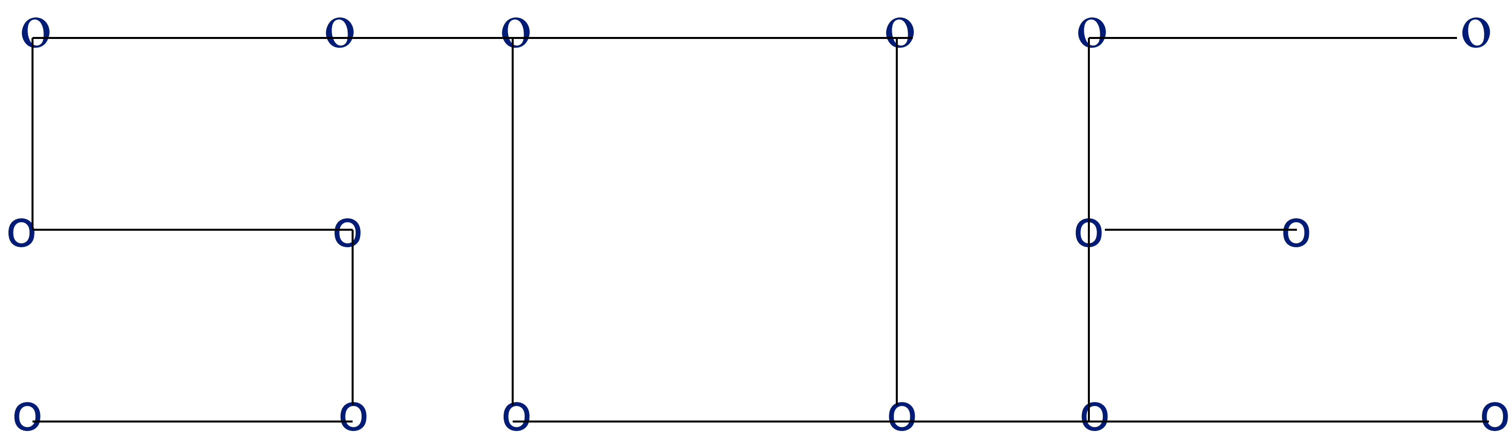
- Needed a new idea

A new, combinatorial (rather than geometric) representation of a distribution

Graphs Γ

Vertices: V   Edges: E

End points: E --- > VxV



The Basic Inequality for a connected graph:

$$\#E \geq \#V - 1$$

Equality < = > Γ is a tree

- Start with a single vertex: #V = 1, #E = 0
- Adjoin edges, one at a time
- Attach only one endpoint:
    - Both #V & #E increase by 1.
    - Still a tree.
- Attach both endpoints:
    - #V unchanged, #E increases by 1.
    - No longer a tree.

The (bipartite) graph Γ(D) of a cake distribution D

V = {cakes} |_| {students}

#V = c + s

E = {cake pieces}

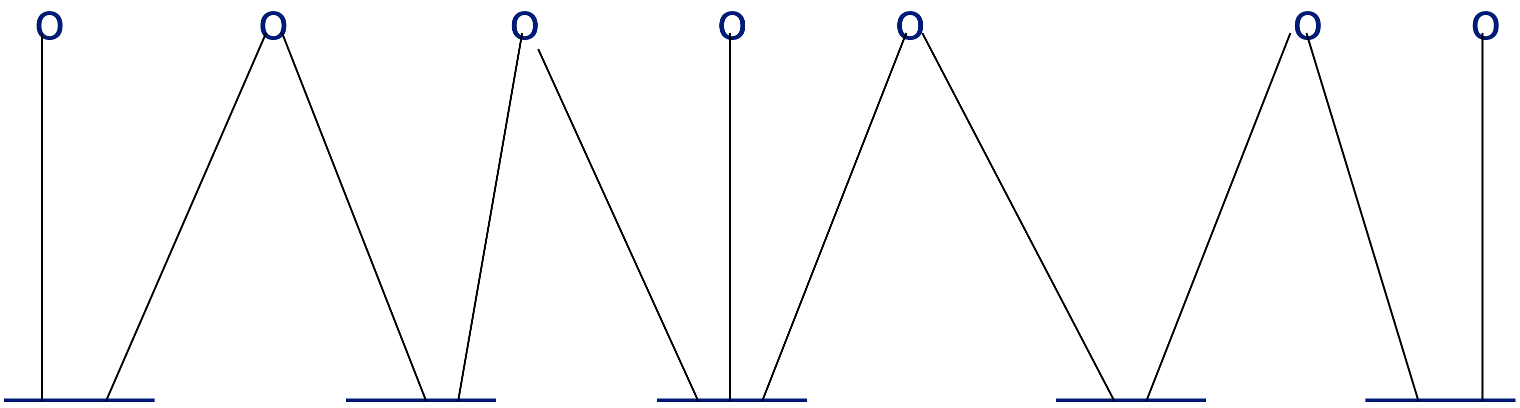
#E = the number of cake pieces in D

End points of p in E:

p joins the cake from which it came

to the student to which it is given

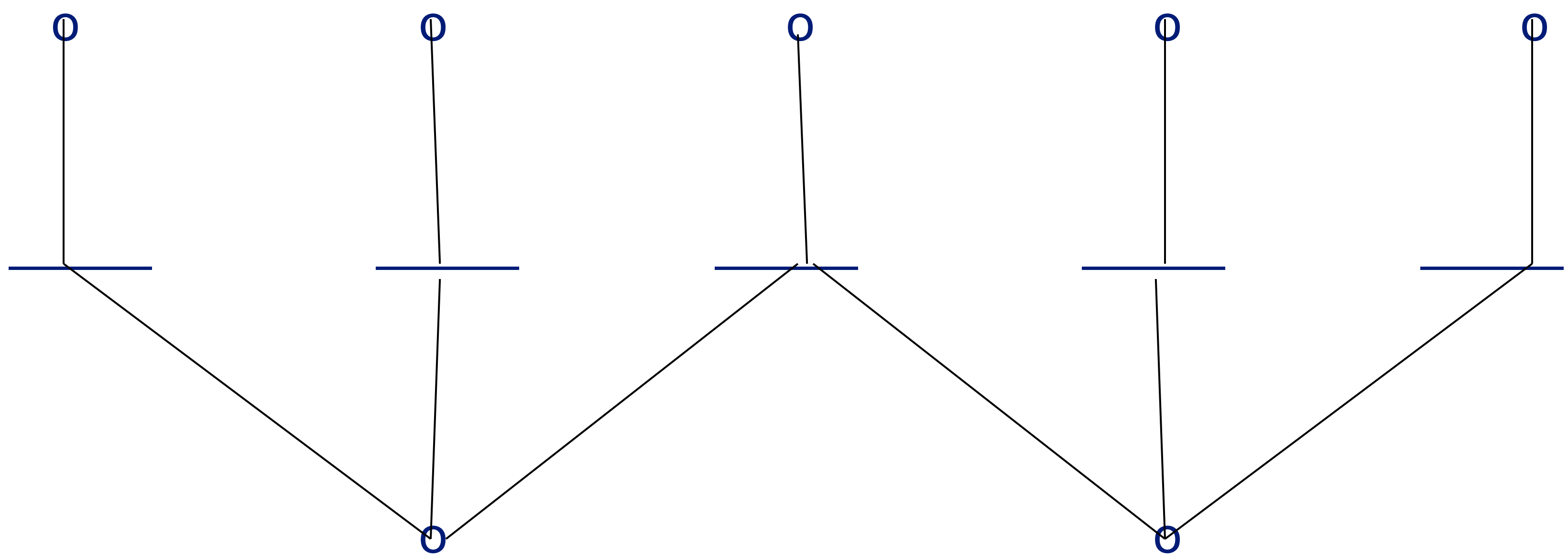
The graph $\Gamma(D_L)$ of the linear distribution D_L of 5 cakes for 7 students



Each student vertex has degree ≤ 2

One cake vertex has degree 3

The graph Γ(D_E) of the Euclidean distribution D_E of 5 cakes for 7 students



Degree 1 vertices: 5 students

Degree 2 vertices: 4 cakes

Degree 3 vertices: 2 students & 1 cake

Proof that: $p(c,s) \geq c + s - d$

1) Let D be a distribution of c cakes among s students using the minimal number,
   p = p(c,s) of cake pieces, and let
   Γ = Γ(D) be its graph. We have seen already that:

2) $p \leq c + s - d$

3) We have
   $\#V = c + s$, & $\#E = p$

Proof that $p(c,s) \geq c + s - d$ (cont)

4) Let Γ' be a connected component of Γ, with vertices V' and edges E'. Say

$$\#V' = c' + s' \qquad \text{and} \qquad \#E = p',$$

where Γ' has c' cake vertices, s' student vertices, and p' edges.

5) Then Γ' is the graph of a distribution D' of c' cakes among s' students, and D', like D, is clearly still minimal (uses the smallest possible number of pieces); otherwise the number of pieces in D could be reduced. Since all students, not just those in Γ', receive the same share, we must have,

6) $$c'/s' = c/s.$$

Let c_0/s_0 be the reduced form of this fraction. Then

$$c = dc_0, \quad s = ds_0, \quad c' = d'c_0, \quad \text{and} \quad s' = d's_0, \qquad \text{where}$$

$$d = \gcd(c, s) \qquad \text{and} \qquad d' = \gcd(c', s').$$

Proof that $p(c,s) \geq c + s - d$ (**conclusion**)

7) Since the distribution D' is minimal we have

$$p' \leq c' + s' - d'$$

8) On the other hand, since Γ' is connected, it follows from the Basic Inequality for graphs that $p' \geq c' + s' - 1$,

with equality iff Γ' is a tree.

$$c' + s' - 1 \leq p' \leq c' + s' - d'$$

9) It follows that: $d' = 1$, and so: $(c', s') = (c_0, s_0)$, $p' = c_0 + s_0 - 1$, and Γ' is a tree. All this is independent of the connected component Γ'.

10) Thus, Γ is a disjoint union of d trees, each with c_0 cake vertices, s_0 student vertices, and $c_0 + s_0 - 1$ edges.

It follows that $p = d \cdot (c_0 + s_0 - 1) = c + s - d$

QED

Square Tilings of Rectangles

Returning to opportunism

We have seen that the Euclidean Algorithm “is” a kind of “greedy algorithm” for square tiling a rectangle.

Is it optimal? For example, does it produce a tiling using the least number of square tiles?

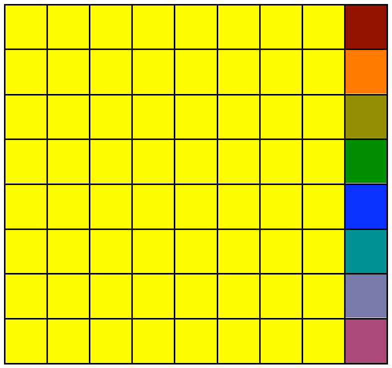
No (not always.)

But it does for ‘Fibonacci rectangles.’

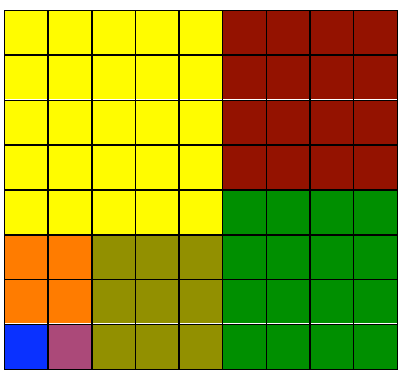
And we shall see that it is optimal for minimizing the “complete perimeter” of the tiling. (“Isoperimetric tiling”)

How many squares to tile a rectangle? (The 8 x 9 case)

The Greedy (Euclidean) tiling: 9 tiles



Fewer (7) tiles



Perimeter measures of square tilings

R = a (c x s)-rectangle, tiled by a set T of square tiles

For each (square) tile σ in T let

$$s(\sigma) = \text{side length of } \sigma,$$

and put

$$p(T) = \Sigma_{\sigma \text{ in } T}\, s(\sigma)$$

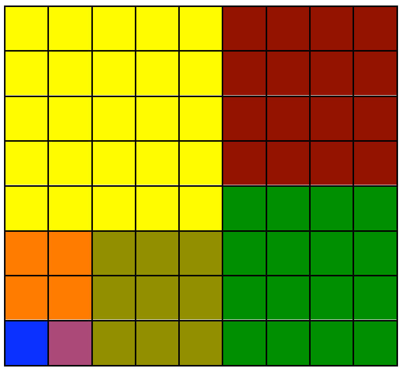
Thus,

$$\text{Area}(R) = \Sigma_{\sigma \text{ in } T}\, s(\sigma)^2 = c \bullet s$$

The "complete perimeter," CP(T)

CP(T) = the sum of the lengths of all line segments in the diagram of the square tiling T of R.

Example: **c = 8, s = 9**, $T = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$

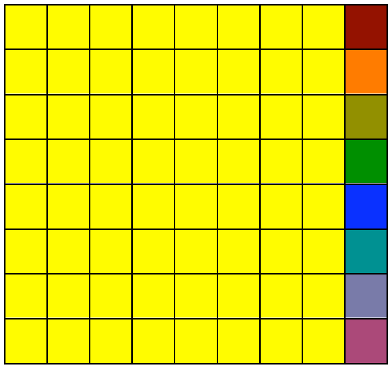


$s(\sigma_1) = s(\sigma_2) = 1$, $s(\sigma_3) = 2$, $s(\sigma_4) = 3$, $s(\sigma_5) = s(\sigma_6) = 4$, $s(\sigma_7) = 5$

So p(T) = 1+1+2+3+4+4+5 = **20**, and

CP(T) = (8 + 8 + 8 + 4 + 1) + (9 + 9 + 5 + 4 + 2) = **58**

The Euclidean tiling makes CP smaller



Example: **c = 8, s = 9** again.

T_E = the Euclidean tiling = $\{\sigma_0, \sigma_i \,|\, 1 \leq i \leq 8\}$

$s(\sigma_0) = 8$, $s(\sigma_i) = 1 \;\; (1 \leq i \leq 8)$,

$p(T_E) = 8 + 8 = \mathbf{16} = 8 + 9 - 1$

$CP(T_E) = (8 + 8 + 8) + (9 + 9 + 8) = \mathbf{50}$

The Iso-Complete-Perimetric Theorem

Let T be a square tiling of a (c x s)-rectangle R

1. CP(T) = 2•p(T) + (c + s),

so

CP(T) and p(T) are simultaneously minimized by T.

2. p(T) ≥ p(c, s) = c + s - d,

with equality for the Euclidean tiling T_E, for which

$CP(T_E)$ = 3(c + s) - 2d.

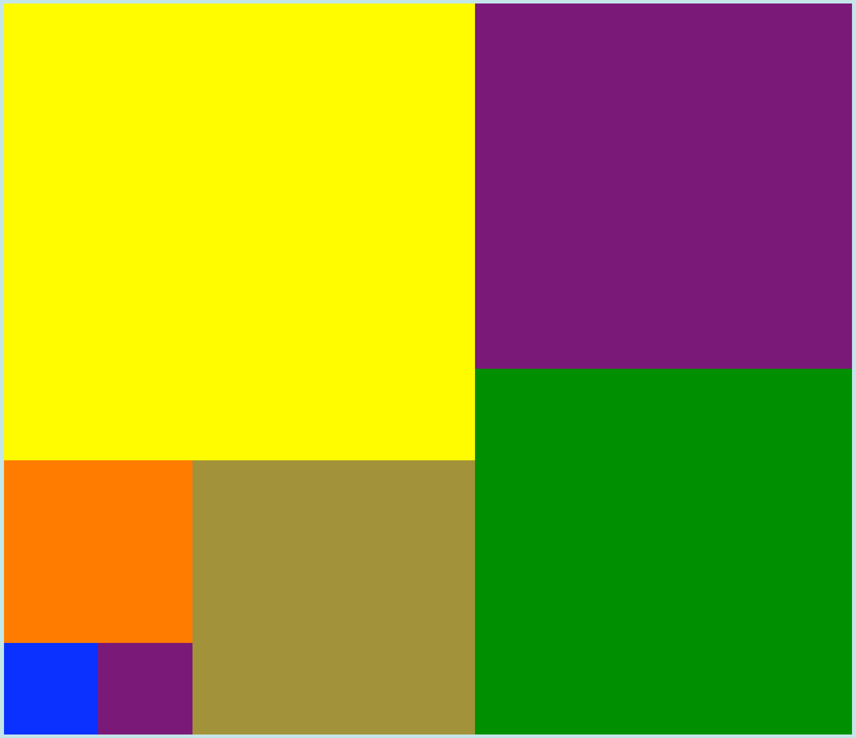
Proof: Make a cake distribution from a square tiling

- Assume that we have a square tiling of a c x s rectangle such that c, s, and all of the tile side lengths are integers.

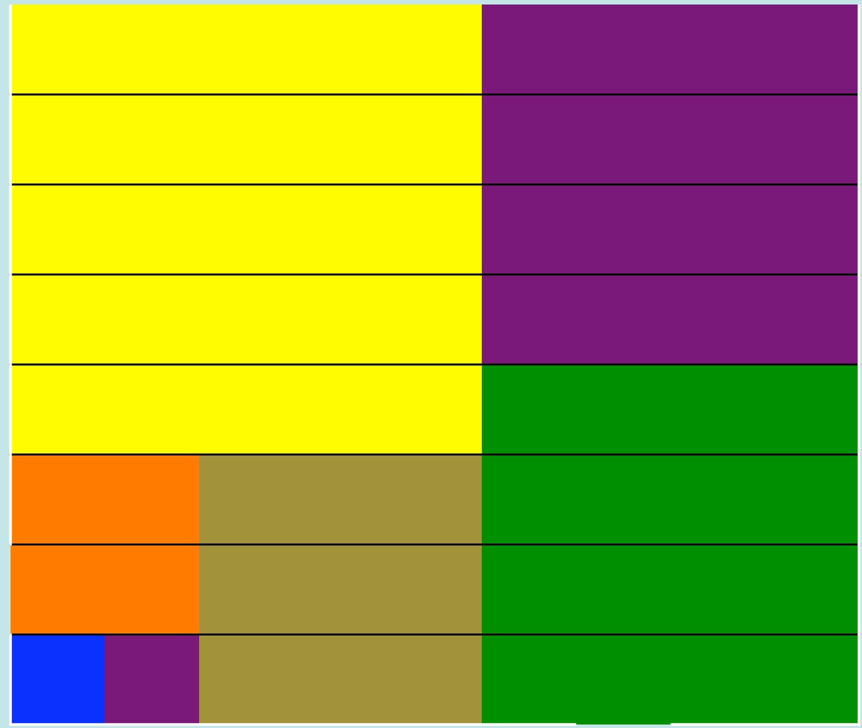
- Using a theorem of Max Dehn, the general case can be reduced to this one

- Cut the rectangle into c horizontal (1 x s)-rectangles, that we consider to be the “cakes”

c = 8   s = 9

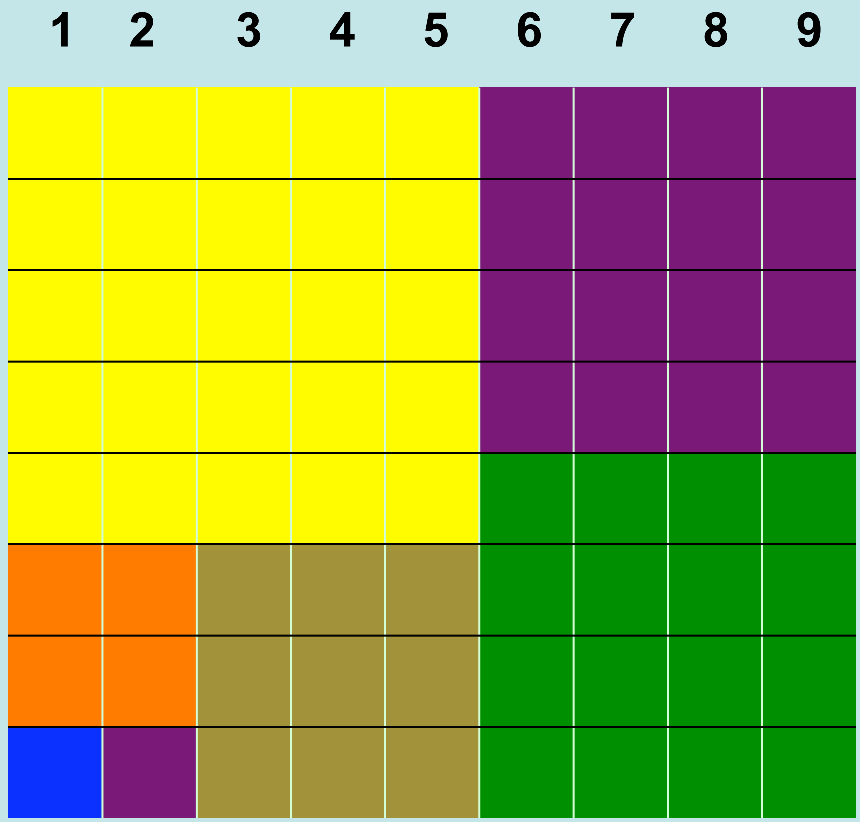


Rows = Cakes



Vertical cuts make the Cake Pieces

 # Pieces = p(T) = $\Sigma_{\sigma \text{ in } T}\, s(\sigma)$



Column labels are the students

The Distribution

Corollary. $p(T) \geq c + s - d$,
with equality for the Euclidean tiling.

What about non integer rectangles??

Square tiling of any rectangle?

- Why doesn't this theorem apply to any square tiling of any (c x s)-rectangle R? (c, s real numbers)
- What then would be the meaning of d = gcd(c, s) if c and s are not integral?
- It is easy to extend the meaning of d whenever c and s are commensurable (c/s is rational). Then d is the positive generator of the (cyclic) group **Z**c + **Z**s.
- An old theorem of Max Dehn says that:

  "The rectangle R admits a (finite) square tiling iff "R is commensurable," (c/s is rational)."

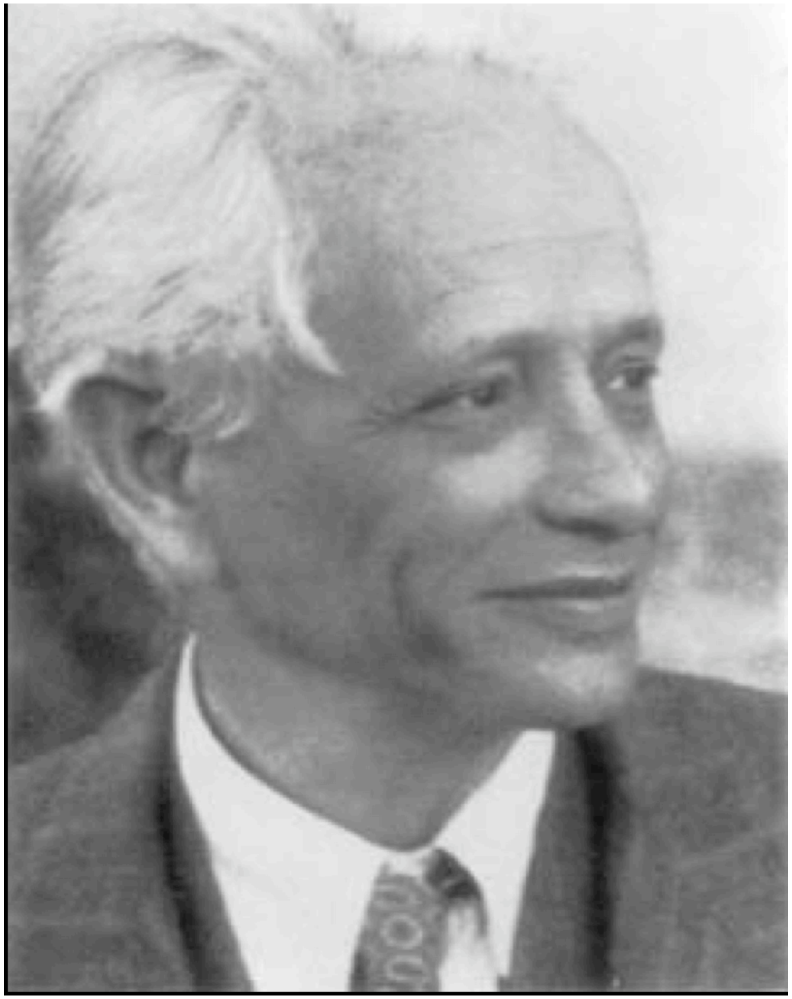
- In this case the Iso-CP Theorem remains meaningful, and is true.

Max Dehn (1878-1952)

A German mathematician who studied under David Hilbert at Gottingen. Dehn did deep and fundamental work in geometry, topology, and group theory. He was the first to solve one of Hilbert's famous list of 23 problems. Giving a negative solution to Problem #3, Dehn showed that a cube and a regular tetrahedron of the same volume could not be cut into polyhedra that are pairwise congruent. This contrasts with what happens in the plane, where two polygons of the same area can be decomposed into triangles that are pairwise congruent.

In 1938 Dehn, a Jew, was forced by the Nazis to leave his professorship in Frankfurt. In 1945 he became the unique math professor at Black Mountain College in North Carolina, where he remained till his death. There was no opportunity there to teach advanced mathematics, but he also taught Latin, Greek, and Philosophy. The Black Mountain faculty included such figures as John Cage, Merce Cunningham, Willem de Kooning, Buckminster Fuller (of whom Dehn became a close friend), Walter Gropius, and many other artists.

Max Dehn (1878-1952)



Thanks

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Noticing the symmetry of c and s

Patterns; structure

Symmetric Reformulation

- Given <u>one</u> cake, a group of s_1 (= c) students, and a group of s_2 (= s) students. What is the smallest number of cake pieces into which we can cut the cake so that we can make both an equal share distribution D_1 to the s_1 students, and also an equal share distribution D_2 to the s_2 students? (Using the same pieces in each case.)

  [Noticed also by Man-Keung Siu]

- Why just <u>two</u> groups? What about n?

n simultaneous equal distributions

Opportunism; natural questions; generalization

- Let M be a number > 0.

- By a partition P of M, with p = #P pieces, we mean a sequence

  $(m_1, m_2, \ldots, m_p)$ of numbers $m_i > 0$ such that

  $$m_1 + m_2 + \ldots + m_p \quad = \quad M$$

- For an integer s > 0, call P an s-equi-partition if there is a map D: [p] → [s] such that, for each h, $1 \leq h \leq s$, we have

  $$\Sigma \{m_k \mid D(k) = h\} \quad = \quad M/s$$ (“equal shares,” independent of h)

- For a sequence $(s_1, s_2, \ldots, s_n)$ of integers > 0, we call P an

  $(s_1, s_2, \ldots, s_n)$-equi-partition of M

  if it is an s_j-equi-partition of M for each j = 1, 2, … , n.

Generalized cake-pieces problem

- Let $(s_1, s_2, \dots, s_n)$ be a sequence of integers > 0, and let M be a number > 0. What is the minimum number,

$$p_n = p_n(s_1, s_2, \dots, s_n),$$

  of pieces of an $(s_1, s_2, \dots, s_n)$-equi-partition of M?

- The problem does not depend on M; we can rescale M and P.

- For n = 1, it is clear that: $p_1(s) = s$.

  For n = 2, we have proved that:

$$p_2(s_1, s_2) = s_1 + s_2 - \gcd(s_1, s_2).$$

Remarks

- Consider a p-uniform partition P of M, that decomposes M into a sum of p equal pieces, each equal to M/p. Then this is an $(s_1, s_2, \ldots, s_n)$-equi-partition of M if and only if each s_j divides p, i.e. if and only if p is a multiple of

$$[s_1, s_2, \ldots, s_n] = \mathrm{lcm}(s_1, s_2, \ldots, s_n)$$

- Since we can rescale M and P, we shall, for the present discussion, take M to be

$$S = \Pi_{1 \le j \le n} s_j = s_1 \bullet s_2 \bullet \ldots \bullet s_n$$

- For n = 1, the s_1-uniform partition is clearly a minimal equi-partition, and so

$$p_1(s_1) = s_1$$

- For n = 2, we have $p_2(s_1, s_2) = s_1 + s_2 - (s_1, s_2)$

  where we here use the notation $(s_1, s_2, \ldots, s_n) = \gcd(s_1, s_2, \ldots, s_n)$
- The linear equi-partition LP: For each j, cut the interval [0, S] into s_j subintervals of equal length $s_j' = S/s_j$. Then the union of all of these cuts creates the linear equi-partition, LP, consisting of the sequence of lengths of the resulting subintervals of [0, S].

PROPOSITION 1.

$$\# LP \quad = \quad \Sigma_J \; (-1)^{\#J - 1} \, (s_J)$$

where the sum is over all non-empty subsets J of [n], and

$$(s_J) \; = \; \gcd \{s_j \mid j \text{ in } J\}$$

New Questions

1. **Conjecture**:

$$p_n(s_1, s_2, \ldots, s_n) = \Sigma_J (-1)^{\#J - 1} (s_J)$$

2. What n-dimensional equi-partition corresponds to the Euclidean distribution for n = 2? Is there some connection with n-dimensional cubical tilings?