TOPOI AND COMPUTATION

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ABSTRACT
This is the written version of a talk given at the C.I.R.M. Workshop on Logic in Computer Science at Marseille, France in June, 1988. Its purpose is to argue that there is a close connection between the theory of computation and the geometric side of topos theory. We begin with a brief outline of the history and basic concepts of topos theory.

Introduction to Topos Theory

Topos theory had two beginnings. Its first incarnation was due to Grothendieck [GV], who made two fundamental observations in an attempt to extend concepts like cohomology from algebraic topology to algebraic geometry over arbitrary fields, even finite fields, so as to obtain number-theoretic consequences (specifically the Weil conjectures).

Grothendieck's first observation was that all the information needed to define the cohomology of a space X is conveniently contained in the category $Sh(X)$ of sheaves over X. A sheaf on X is best visualized as a space Y equipped with a local homeomorphism $p : Y \to X$, but for Grothendieck's purposes the following equivalent definition was more useful. A sheaf on X consists of (1) for each open $U \subseteq X$ a set $F(U)$ (of "sections" over U) and (2) for all open sets $U \subseteq V \subseteq X$ a "restriction" operation $F(V) \to F(U)$ satisfying some coherence requirements (making F a contravariant functor from the poset of open subsets of X to the category $\mathcal{S}$ of sets) plus the following patching condition: Let an open set U be covered by open subsets U_i; then any given sections over the U_i's, such that the sections over U_i and U_j have the same restriction to $U_i \cap U_j$, are restrictions of a single section over U. (A fairly typical example is obtained by letting $F(U)$ be the set of all continuous real-valued functions on U.) The crucial thing about this

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definition is that the space X enters the picture only through its poset of open subsets and the notion of a cover.

Grothendieck's second basic observation was that sheaf theory, including cohomology, makes good sense even if the poset of open subsets of a space is replaced with an arbitrary (small) category, provided a suitable notion of cover is available. He gave axioms to specify what is suitable here, and such notions of cover are now called *Grothendieck topologies*. A category equipped with a particular Grothendieck topology is called a *site*, and a category of all sheaves over some site is what Grothendieck called a *topos* and what is now called a *Grothendieck topos*.

The primary examples of Grothendieck topoi are thus the categories $Sh(X)$ associated to topological spaces X, and it is appropriate to think of all Grothendieck topoi as generalized topological spaces. An example of a Grothendieck topos not arising from a space is the topos $\mathcal{S}^G$ of sets equipped with an action of a particular group (or monoid) G. It should perhaps be emphasized that, when a topos is viewed as a generalized space, the objects of the topos must be viewed as the sheaves over (not the points in) the generalized space. Notice in this connection that the category $\mathcal{S}$ of all sets is to be viewed as a one-point space, since it is the category of sheaves over such a space.

The second beginning of topos theory was work of Lawvere and Tierney [L]. They noticed that the objects in an arbitrary Grothendieck topos are enough like ordinary sets (= objects in $\mathcal{S}$) to allow interpretation of higher-order intuitionistic logic. (Strictly speaking, the phrase "higher-order intuitionistic" makes little sense, since it is not clear that constructions like the power set are intuitionistically acceptable. What is meant by the phrase is a type theory built on a formally intuitionistic underlying logic.) This means, for example, that any two subsheaves of a sheaf have a union and an intersection (to interpret disjunction and conjunction). It also means that, for any sheaves A and B, there are a sheaf B^A of maps from A to B and a sheaf $\mathcal{P}(A)$ of subsheaves of A, having the right formal properties to deserve the names we have given them. $\mathcal{P}(A) = \Omega^A$, where $\Omega = \mathcal{P}(1)$ is the sheaf of truth values (which need not consist of just **true** and **false** since the logic is intuitionistic).

Lawvere and Tierney axiomatized this structure, defining an *elementary topos* (now usually called simply a *topos*) to be a category with finite limits, finite colimits, exponentials B^A, and Ω (and therefore also power objects $\mathcal{P}(A)$). Later, Mikkelsen and Kock showed that all these axioms follow from the existence of finite limits and power objects. The word "elementary" in connection with the Lawvere-Tierney definition refers to the fact that their axioms are expressed in the first-order language of categories; in fact the axioms are essentially algebraic.

Of course, every Grothendieck topos is an elementary topos, but there are other examples as well. Any (possibly) non-standard model of set theory (or type theory) yields a topos, namely the category of sets and functions of the model. One can also form non-standard versions of sheaf categories. There is also a topos, called the *effective topos* [H], whose internal logic amounts to Kleene's recursive realizability. Were it not for time restrictions, this talk would have contained an extensive discussion of the effective topos and its recent applications to the semantics of polymorphism.

An interpretation $\mathcal{M}$ of a higher-order language in a topos $\mathcal{E}$ assigns to each sort S of the language an object $S_{\mathcal{M}}$ of $\mathcal{E}$, to each predicate symbol P of sorts $S_1, \dots, S_n$ a subobject $P_{\mathcal{M}}$ of the product $S_{1\mathcal{M}} \times \dots \times S_{n\mathcal{M}}$, and to each function symbol F of argument sorts $S_1, \dots, S_n$ and value sort S a morphism $F_{\mathcal{M}} : S_{1\mathcal{M}} \times \dots \times S_{n\mathcal{M}} \to S_{\mathcal{M}}$. Then any formula ϕ of higher-order logic, with free variables among $x_1, \dots, x_k$ (henceforth abbreviated as $\mathbf{x}$) of sorts $S_1, \dots, S_k$, is interpreted by means of the topos structure as a subobject $\|\phi\|_{\mathcal{M},\mathbf{x}}$ of $S_{1\mathcal{M}} \times \dots \times S_{k\mathcal{M}}$. Furthermore, these interpretations respect intuitionistic logic and type theory (extensionality and comprehension axioms). This means that, if $\forall\mathbf{x}(\phi \implies \psi)$ is provable in intuitionistic higher-order logic, then $\|\phi\|_{\mathcal{M},\mathbf{x}}$ is a subobject of $\|\psi\|_{\mathcal{M},\mathbf{x}}$.

The fact that the internal logic of topoi is intuitionistic is sometimes taken as an indication of a connection with computation. In a sense, this is correct. For example, there are topoi (like the effective topos) in which all functions from natural numbers to natural numbers are recursive; this would be impossible if the internal logic were classical. But there is another way of viewing the situation. The intuitionists' primary criticism of classical mathematics is that logical principles valid for finite domains are unjustifiably extrapolated to infinite domains. Thus, if a language talks about computational matters, which are essentially finite, then one might reasonably expect that for this language classical and intuitionistic logic would agree.

For our purposes, the crucial point in the preceding discussion of topos theory is that topos theory connects a geometric aspect (at the forefront of the Grothendieck theory) with a logical aspect (at the forefront of the Lawvere-Tierney theory). Since it is well known that logic is also connected with computation theory, we have the picture of logic reaching out in one direction toward the geometry of topoi and in another direction toward the theory of computation; if it is not torn apart by reaching in these two directions, logic therefore connects topoi to computation. The main thesis of this talk is that logic will not be torn apart because it is not really reaching in two different directions.

THESIS: From the point of view of logic, the geometric side of topos theory lies in the same direction as the theory of computation.

In order to provide evidence for this thesis, it will be necessary to discuss in more detail the connection between the logic and the geometry of topoi, and this requires a discussion of morphisms between topoi, to which we now turn.

Morphisms

In view of the essentially algebraic nature of the definition of (elementary) topoi, it is fairly clear what a homomorphism of topoi should be: a functor that preserves finite limits, finite colimits, exponentiation, and Ω (and therefore also $\mathcal{P}$ — in view of the results of Mikkelsen and Kock cited earlier, it suffices to preserve finite limits and $\mathcal{P}$). Such functors are called *logical morphisms*, because they preserve the internal logic of topoi in the sense that, for any such functor $f : \mathcal{E} \to \mathcal{F}$, any formula ϕ of higher-order logic, and any interpretation $\mathcal{M}$ in $\mathcal{E}$, we have

$$f(\|\phi\|_{\mathcal{M},\mathbf{x}}) = \|\phi\|_{f(\mathcal{M}),\mathbf{x}}.$$

The notion of logical morphism is clearly the right notion of morphism between elementary topoi. But it is wrong from the geometrical viewpoint of Grothendieck topoi. If topoi are generalized spaces, then their morphisms should be generalized continuous functions. A continuous map of topological spaces $f : X \to Y$ induces two functors between the corresponding sheaf topoi, the direct image $f_* : Sh(X) \to Sh(Y)$ and the inverse image $f^* : Sh(Y) \to Sh(X)$, but neither of these is, in general, a logical morphism. And this failure to obtain logical morphisms from continuous maps has nothing to do with any pathology in the topological situation; it occurs even when f is the (unique) map from the real line to a one-point space. In fact, for nice spaces, f_* is logical if and only if f is a homeomorphism, and f^* is logical if and only if f is a local homeomorphism.

Abstracting the properties of the two functors induced by a continuous map, Grothendieck defined a *morphism*, nowadays called a *geometric morphism*, from $\mathcal{F}$ to $\mathcal{E}$ to be a pair of functors

$$f^* : \mathcal{E} \to \mathcal{F} \quad \text{and} \quad f_* : \mathcal{F} \to \mathcal{E}$$

such that f^* is left adjoint to f_* and f^* preserves finite limits. (f^* automatically preserves colimits since it has a right adjoint.) The geometric morphisms between topoi of sheaves over reasonable spaces correspond exactly to the continuous functions between those spaces. Furthermore, the geometric morphisms from $\mathcal{S}^G$ to $\mathcal{S}^H$, where G and H are groups, correspond to the homomorphisms $G \to H$. And geometric morphisms from $Sh(X)$ to $\mathcal{S}^G$ correspond to one-dimensional cohomology classes of X with coefficients in G. Examples like these provide convincing evidence that this is the correct definition of morphism between Grothendieck topoi. If topoi are viewed as generalized spaces, then geometric morphisms should be viewed as generalized continuous functions. In particular, since a point in a topological space is equivalent to a continuous map into that space from a one-point space, and since the topos $\mathcal{S}$ of sets corresponds to a one-point space, one regards geometric morphisms from $\mathcal{S}$ to an arbitrary topos $\mathcal{E}$ as the *points* of the generalized space $\mathcal{E}$.

I suspect that the success of this concept of morphism somewhat delayed the development of elementary topos theory by suggesting that the important concepts of topos theory must be preserved by geometric morphisms. Since exponentiation, power objects, and Ω are not preserved by geometric morphisms, they did not get much attention until Lawvere and Tierney used them as the basis for the theory of elementary topoi.

The definition of geometric morphisms, although originally formulated in the context of Grothendieck topoi, makes good sense also for elementary topoi.

The Geometric Logic Problem

For a geometric morphism $f : \mathcal{F} \to \mathcal{E}$, the inverse-image part f^* preserves the truth values of finite conjunctions (because it preserves finite limits), disjunctions (even infinite ones, because it has a right adjoint), and existential quantification (again because of the adjoint), but not negation, implication, universal quantification, or the higher-order aspects of the logic.

These observations lead to the problem of describing exactly how much of logic is preserved by geometric morphisms. Specifically, we say that a formula ϕ is *strongly preserved* if

$$f^*(\|\phi\|_{\mathcal{M},\mathbf{x}}) = \|\phi\|_{f^*(\mathcal{M}),\mathbf{x}}$$

for all geometric morphisms $f : \mathcal{F} \to \mathcal{E}$, and all interpretations $\mathcal{M}$ in $\mathcal{E}$. Our primary interest will be in strong preservation, but, in view of its name, we should also mention weak preservation. A sentence (i.e. a formula without free variables) ϕ is *weakly preserved* if, whenever f and $\mathcal{M}$ are as above and $\|\phi\|_{\mathcal{M}} = 1$, then $\|\phi\|_{f^*(\mathcal{M})} = 1$. An equivalent characterization of weak preservation is that, for f and $\mathcal{M}$ as above,

$$\|\phi\|_{f^*(\mathcal{M})} \geq f^*(\|\phi\|_{\mathcal{M}});$$

the equivalence would be clear if the only truth values were 0 and 1, but, because of the intuitionistic nature of the logic, the proof is not entirely trivial.

If $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are strongly preserved, then $\forall\mathbf{x}(\phi(\mathbf{x}) \implies \psi(\mathbf{x}))$ is weakly preserved. Such a universally quantified implication between strongly preserved formulas will be called a geometric sequent. (Warning: Other authors use this term in a more restricted sense; see below.) There are weakly preserved sentences that cannot be expressed as geometric sequents, for example the sentences in higher-order logic that express that the universe of discourse is finite or that a particular relation is well-founded. (Each of these examples has classically equivalent but intuitionistically inequivalent formulations; the versions intended here are: A set is finite if every family of subsets that contains $\emptyset$ and is closed under adjoining single elements contains the whole set. A relation $<$ on a set A is well-founded if the only subset X of A such that $(\forall b < a) b \in X \implies a \in X$ for every $a \in A$ is A itself.)

As mentioned earlier, our primary interest is in strong preservation, and, in particular, in finding reasonably large, syntactically described classes of formulas that are guaranteed to be strongly preserved. We consider this problem, the geometric logic problem, as a precise form of (at least part of) the vague project of determining how the logical and geometric sides of topos theory interact.

We have already described such a class implicitly, namely the fragment of first-order logic obtained by closing the class of atomic formulas under conjunction, disjunction, and existential quantification, i.e., existential positive first-order logic. (We identify the propositional constants **true** and **false** with the empty conjunction and disjunction respectively, so these are included.) This fragment of first-order logic is therefore sometimes called geometric logic (but other authors include infinite disjunctions); its formulas are then called geometric formulas and universally quantified implications between these formulas are called (in disagreement with our definition above) geometric sequents. This terminology is, in my opinion, not justified. It is true that the other connectives and quantifiers of first-order logic lead out of the class of strongly preserved formulas, but first-order logic is not all of logic. By looking beyond first-order logic, one can find a natural larger class of strongly invariant formulas, namely the existential fixed-point formulas.

Let δ and ϕ be two formulas that are in a language L except for containing positive occurrences of an n-ary predicate symbol P not in L. Then we introduce a new formula

$$LET\ P(\mathbf{x}) \leftarrow \delta(\mathbf{x}, \mathbf{y}, P)\ THEN\ \phi(\mathbf{y}, P),$$

whose meaning is that ϕ holds when P is interpreted as the least fixed-point of the (monotone) operator defined by δ. Since P also occurs only positively in ϕ, this meaning is easily expressed by the second-order formula

$$\forall P\big(\forall \mathbf{x}(\delta \Longrightarrow P(\mathbf{x})) \Longrightarrow \phi\big).$$

It is shown in [Bl] that, if ϕ and δ are strongly preserved, then so is $LET\ P(\mathbf{x}) \leftarrow \delta\ THEN\ \phi$. Thus, the class of strongly preserved formulas includes all of existential fixed point logic, the logic obtained from the existential positive fragment of first-order logic by adding the "$LET\ \ldots\ THEN$" construction.

This logic goes beyond first-order logic. For example, the natural numbers (with 0 and the successor operation) cannot be characterized up to isomorphism by any first-order sentences, but they are characterized by the existential fixed-point sequents

$$\forall x(Sx = 0 \Longrightarrow \mathbf{false})$$

$$\forall x, y(Sx = Sy \Longrightarrow x = y)$$

$$\forall y\big(\mathbf{true} \Longrightarrow LET\ P(x) \leftarrow (x = y \vee P(Sx))\ THEN\ P(0)\big).$$

Thus we recover the known fact that for any geometric morphism f, the inverse image functor f^* preserves the object of natural numbers.

The least fixed-point construction can be expressed in infinitary logic by means of an infinite disjunction, so the preservation theorem for it follows from the known preservation of infinite disjunctions as long as the topoi in question are complete, so that infinite disjunctions can be interpreted in them. But in fact the preservation theorem for the least fixed-point construction holds for all elementary topoi. In view of this result, it would be reasonable to extend the meaning of "geometric logic" to include this construction and perhaps other, as yet undiscovered operations on formulas that do not take us out of the class of strongly preserved formulas.

At the moment, the fragment of higher-order logic that is known to be strongly preserved by geometric morphisms is existential fixed-point logic. Negations of some predicates could be included, to match the conventions of [BG], by introducing them as new predicates subject to the geometric sequents

$$\forall \mathbf{x}\big(P(\mathbf{x}) \wedge P'(\mathbf{x}) \Longrightarrow \mathbf{false}\big) \qquad \text{and} \qquad \forall \mathbf{x}\big(\mathbf{true} \Longrightarrow P(\mathbf{x}) \vee P'(\mathbf{x})\big).$$

This same fragment of higher-order logic has arisen naturally in several places in computation theory. It expresses the queries computable by pure PROLOG programs [CH] and it is the natural language for the pre- and post-conditions in Hoare logic [BG]. In a sense, it expresses those properties of a database that will not change if new basic facts are added as long as the basic facts already in the database remain true.

These remarks provide some evidence for the thesis that, from the point of view of logic, the geometric side of topos theory and the theory of computation lie in the same direction. I conjecture that, if additional logical operations on formulas are discovered that don't lead out of the class of strongly preserved formulas, then those operations will also have computational significance. In the next section, I shall briefly describe some of the known facts about preserved formulas that lead me to this conjecture.

Consequences of Geometric Preservation

A theorem of Barr [Ba] about geometric morphisms of topoi implies that classical and intuitionistic logic agree on formulas with enough preservation properties. Specifically, suppose that Σ is a set of weakly preserved sentences (e.g. geometric sequents) and that σ is a geometric sequent. If σ is a consequence of Σ in classical logic, then it is also a consequence in intuitionistic logic. (See [J] for a proof in the case that σ and the members of Σ are geometric sequents in the traditional sense.) As mentioned earlier, this sort of agreement between classical and intuitionistic logic is to be expected on philosophical grounds when one has a language whose sentences are about finite phenomena, as in computations.

Here is some additional evidence for the finitary nature of existential fixed-point logic and, in fact, of any logic with the same geometric preservation properties. There is a forcing relation $\Vdash$ between finite conjunctions of atomic formulas (often called conditions in this context) and formulas of higher-order logic with the following properties. First, forcing is defined by induction on the formula being forced, the definition being a special case of Kripke-Joyal semantics (see [J]) and containing the clause for forcing negations that is characteristic of forcing. Second, for all strongly preserved formulas ϕ, forcing equals truth in the sense that, for all models $\mathcal{M}$ and all $\mathbf{a} \in \mathcal{M}$, we have $\mathcal{M} \models \phi(\mathbf{a})$ if and only if there exist a condition α and elements $\mathbf{b} \in \mathcal{M}$ such that $\mathcal{M} \models \alpha(\mathbf{a}, \mathbf{b})$ and $\alpha(\mathbf{x}, \mathbf{y}) \Vdash \phi(\mathbf{x})$. Furthermore, when ϕ is strongly preserved, then $\alpha \Vdash \phi$ if and only if ϕ is a logical consequence of α (with respect to ordinary models or with respect to models in arbitrary topoi).

One consequence of the existence of such a forcing relation is that, if a strongly preserved formula is true of some elements in a structure $\mathcal{M}$, then it is true on the basis of a finite part of $\mathcal{M}$, namely the part involved in the condition that forces the formula to be true. For existential fixed-point logic (and in fact for a somewhat broader fragment of higher-order logic), such finiteness results were given in [M]; what is important from our present point of view is that such results follow from the topos-theoretic property of strong preservation.

A further consequence is that the inductive constructions of the least fixed-points involved in "*LET* ... *THEN*" take at most an ordinary infinite sequence of steps; no transfinite iterations are needed. Again, this holds not only for existential fixed-point logic (as in [BG]) but whenever the δ and the ϕ in the "*LET* ... *THEN*" are strongly preserved.

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