

AN INDUCTION PRINCIPLE AND PIGEONHOLE PRINCIPLES FOR K-FINITE SETS

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ABSTRACT. We establish a course-of-values induction principle for K-finite sets in intuitionistic type theory. Using this principle, we prove a pigeonhole principle conjectured by Bénabou and Loiseau. We also comment on some variants of this pigeonhole principle.

1. INTRODUCTION

The pigeonhole principle says that a finite set cannot be mapped one-to-one into a proper subset. There is a dual principle saying that a finite set cannot be mapped onto a proper superset. We consider these principles in the context of constructive logic. The motivation for these considerations came from a weak version of the dual pigeonhole principle proved constructively by Bénabou and Loiseau, who noted that their argument does not establish a natural stronger version of the principle.

Throughout this paper we work in an intuitionistic type theory of the sort that arises as the internal logic of an elementary topos [3, 5, 6, 7, 8, 10, 12]. Although the questions we consider originated in the course of topos-theoretic work of Bénabou and Loiseau [4], many of our results are theorems of intuitionistic type theory (in fact of intuitionistic third-order logic) and involve no reference to topoi.

Of the several concepts of finiteness that are equivalent in classical logic but not in intuitionistic logic, we shall use the one commonly called K-finiteness or Kuratowski-finiteness [1, 10, 11]. The definition and some comments on it are given in Section 2. Henceforth, we omit the prefix K and refer simply to finiteness.

Bénabou and Loiseau showed [4, Prop. 5.3] that, if X is finite and inhabited, then no function $f : X \rightarrow X \times 2$ can be surjective. In other words, for every such f it is not the case that every element of $X \times 2$ is in its image. They pointed out that this version of the dual pigeonhole principle is weaker (in constructive logic) than the statement that for every such f there is an element of $X \times 2$ not in its image. They remarked that the latter, stronger statement “seems to be true, but we do not have a general proof of it.” One purpose of the present paper is to give a general proof of it and in fact of the stronger statement obtained by replacing $X \times 2$ with $X + 1$.

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The proof uses an induction principle whereby, when one proves a property for an arbitrary finite set X , one can assume the property for all complemented, proper subsets of X . This principle seems to be of interest independently of the application that motivated it.

After explaining our terminology and presenting some preliminary facts in Section 2, we devote Section 3 to proving the induction principle. The application to the stronger version of the dual pigeonhole principle conjectured by Bénabou and Loiseau is in Section 4. In Section 5, we consider the (undualized) pigeonhole principle, showing that a weak version is intuitionistically provable but a strong version is not. The final Section 6 is about some variants of the dual pigeonhole principle.

2. PRELIMINARIES

The set theory and logic used in this paper are an intuitionistic type theory of the sort described in [3, 5, 6, 7, 8, 10, 12]. These references differ in some details, but the differences will not matter in our work. We shall work with elements, subsets, and families of subsets of some fixed but arbitrary type U , as well as (partial) functions from U to U .

A set is called *inhabited* if there exists an element in it. This is stronger (in intuitionistic logic) than not being empty. A subset A of a set B is a *proper* subset if $B - A$ is inhabited, i.e., if B has an element that is not in A . We say that A is *complemented* in its superset B if $B = A \cup (B - A)$, i.e., if every element of B is either in A or not in A .

A function f is *one-to-one* if $f(x) = f(y)$ implies $x = y$. This definition is intuitionistically stronger and more natural than the classically equivalent but negation-filled definition that $x \neq y$ implies $f(x) \neq f(y)$.

We call a set A *finite* if it belongs to every family \mathcal{X} that contains the empty set \emptyset and is closed under adjoining single elements in the sense that if $Z \in \mathcal{X}$ and $a \in A$ then $Z \cup \{a\} \in \mathcal{X}$.

The definition trivially implies that \emptyset is finite and that, if A is finite, then so is $A \cup \{p\}$ for every p .

This definition also immediately implies an induction principle. To prove that all finite sets have some property, it suffices to prove that the empty set has the property (induction basis) and that, whenever Z has the property and a is any element (in U), then $Z \cup \{a\}$ also has the property (induction step). We shall refer to this sort of induction as *ordinary* induction on finite sets, to distinguish it from the new induction principle to be established in Section 3.

The definition of finiteness is (intuitionistically) equivalent (cf. [4, Lemma 5.2]) to the definition of K-finiteness given in [10], namely that A belongs to every family \mathcal{Y} that contains \emptyset and $\{a\}$ for all $a \in A$ and is closed under binary union. To see the equivalence, note first that any \mathcal{Y} as in the second definition is also an \mathcal{X} as in the first definition, so all finite sets are K-finite. For the other direction, one shows that the family of finite sets is closed under binary union, so the finite subsets of A form a \mathcal{Y} as in the second definition. To prove that, if x and y are finite then so is

$x \cup y$, one proceeds by ordinary induction on x ; both the basis and the induction step are trivial.

An equally trivial induction establishes that every finite set is either empty or inhabited.

A complemented subset A of a finite set B is finite. To see this, proceed by induction on B , the basis ($B = \emptyset$) being trivial. So suppose the result is true for B and that A is a complemented subset of $B \cup \{p\}$. Then $A \cap B$ is a complemented subset of B , so by induction hypothesis it is finite. If $p \notin A$, then $A = A \cap B$ and thus A is finite. If $p \in A$ then $A = (A \cap B) \cup \{p\}$ and again A is finite. Since A is complemented in $B \cup \{p\}$, the cases considered in the preceding two sentences exhaust the possibilities, so the proof is complete.

It follows from the preceding two paragraphs that, if A is a complemented subset of B then either it is a proper subset or it equals B . Indeed, $B - A$ is also complemented, hence finite, and hence either inhabited or empty. If $B - A$ is inhabited, then A is a proper subset of B . If $B - A$ is empty then, as A is complemented in B , we have $B = A \cup (B - A) = A$.

We emphasize that the finite sets we work with need not have a decidable equality relation. That is, we do not assume that $x = y$ or $x \neq y$. In fact, by [1], such an assumption would allow us to work in a sub-universe (sub-topos) in which classical logic holds and would thus remove the whole point of working in intuitionistic logic.

3. AN INDUCTION PRINCIPLE

This section is devoted to establishing an induction principle, different from the one given by the definition of finiteness, for proving properties of finite sets.

Theorem 1. *Let \mathcal{X} be a family of finite sets such that*

- (1) *for all finite A , if all complemented proper subsets of A are in \mathcal{X} , then also $A \in \mathcal{X}$.*

Then \mathcal{X} contains all finite sets.

The theorem says that, in order to prove a statement for all finite sets, it suffices to prove it for an arbitrary finite set A assuming that it holds for all complemented proper subsets of A . It is related to ordinary induction for finite sets much as course-of-values induction is related to ordinary induction for natural numbers.

Proof. We show, by ordinary induction on finite sets B (in the sense explained in Section 2) that

$$(2) \quad \forall \mathcal{X} [(1) \implies B \in \mathcal{X}].$$

The basis is easy, for if $B = \emptyset$ then B has no proper subset, so (1) applied with $A = B$ immediately gives $B \in \mathcal{X}$.

For the induction step, we assume (2) for a particular finite B ; we wish to prove (2) for $B \cup \{p\}$. So fix an \mathcal{X} satisfying (1); we must prove $B \cup \{p\} \in \mathcal{X}$. Define

$$\mathcal{Y} = \{A \mid A \in \mathcal{X} \text{ and } A \cup \{p\} \in \mathcal{X}\}.$$

We claim that (1) holds with \mathcal{Y} in place of \mathcal{X} . Assuming the claim for a moment, we can apply the induction hypothesis (2) for B with \mathcal{X} instantiated as \mathcal{Y} . So we get $B \in \mathcal{Y}$, from which the desired $B \cup \{p\} \in \mathcal{X}$ immediately follows by definition of \mathcal{Y} .

So all that remains is to prove the claim that (1) holds with \mathcal{Y} in place of \mathcal{X} . So let A be finite and assume that all its complemented proper subsets are in \mathcal{Y} . In particular, all its complemented proper subsets are in \mathcal{X} and so $A \in \mathcal{X}$ since (1) holds for \mathcal{X} . It remains to prove that $A \cup \{p\} \in \mathcal{X}$, and we shall do this by applying the assumption (1) for \mathcal{X} .

So let C be any complemented proper subset of $A \cup \{p\}$; we must show $C \in \mathcal{X}$. As C is complemented, we have $p \in C$ or $p \notin C$. Also, as $C \cap A$ is a complemented subset of A , it is either equal to A or a proper subset of A . We consider the various cases.

If $C \cap A$ is a proper subset of A , we use the assumption that \mathcal{Y} contains all the complemented proper subsets of A to conclude that $C \cap A \in \mathcal{Y}$. Then C , being equal to $(C \cap A) \cup \{p\}$ or to $C \cap A$ (according to whether $p \in C$), is in \mathcal{X} by definition of \mathcal{Y} .

If $C \cap A = A$ and $p \notin C$ then $C = A$, and we already saw that $A \in \mathcal{X}$.

The remaining case, $C \cap A = A$ and $p \in C$, is impossible as C is a proper subset of $A \cup \{p\}$.

Thus, we have $C \in \mathcal{X}$ in all cases, which completes the proof. \square

Although Theorem 1 suffices for the proofs in the following sections, it seems natural to ask whether it could be strengthened by replacing “complemented” with “finite” in (1). It is not difficult to prove this strengthened induction principle if the axiom of infinity is available, that is, if the type \mathbb{N} of natural numbers is available in the intuitionistic type theory. The proof begins by showing, by ordinary induction on finite sets A , that there is a natural number n such that $\bar{n} = \{0, 1, \dots, n-1\}$ has no one-to-one map into A . Then one shows by induction on n that any set A admitting no one-to-one map from \bar{n} must be in every class \mathcal{X} that satisfies the weakened version of (1).

It is not clear to me whether one can obtain the same result without an axiom of infinity, but it seems that any proof would have to be substantially different from the one just given. To see this, consider the statement “If A is finite then there is a finite B such that the equality relation on B is decidable and B has no one-to-one map into A .” This statement is a reformulation, in the absence of \mathbb{N} , of the result of the first half of the proof given above. (See [1] for the connection between the sets \bar{n} and finite sets B with decidable equality.) But this statement is not provable in intuitionistic type theory without the axiom of infinity. More precisely, there exist a topos \mathcal{E} and a finite object A in it such that, for any object C of \mathcal{E} , the statement “ C has a finite subset with decidable equality admitting no one-to-one map into A ” fails to be internally valid.

To construct such a topos, let P be a three-element partially ordered set with a top element 1 and two incomparable elements a and b below it. Let \mathcal{E}' be the topos of presheaves on P in some non-standard model of set theory, and let \mathcal{E}

be the subtopos consisting of those presheaves whose values at 1 and a are finite in the sense of that non-standard model and whose values at b are really finite. Fix a set S that is finite in the sense of the non-standard model but is not really finite, and let A be the presheaf whose values at 1 and a are S with the identity as transition map between them and whose value at b is a singleton. This A is finite in \mathcal{E} . If C is any other object of E , then one can calculate, using Kripke-Joyal semantics, that a finite subset of C with decidable equality at 1 would have all three of its components really finite (the b component by definition of \mathcal{E} , then the 1 component because decidability makes the transition maps one-to-one, and then the a component because finiteness makes the transition maps surjective). So, after restriction to a , it could be mapped one-to-one into A since $A(a)$ is really infinite.

4. THE STRONG DUAL PIGEONHOLE PRINCIPLE

The purpose of this section is to establish, in intuitionistic type theory, the stronger version of the dual pigeonhole principle conjectured by Bénabou and Loiseau, namely that if X is finite and inhabited and $f : X \rightarrow X \times 2$ then there is an element of $X \times 2$ that is not in the image of f . In fact, our proof gives a stronger statement with $X + 1$ instead of $X \times 2$. (To see that the $X + 1$ result is indeed stronger than the $X \times 2$ result, it suffices to observe that, since X is inhabited, there is a surjection $X \times 2 \rightarrow X + 1$ sending one copy of X in $X \times 2$ onto X and the other copy onto 1.)

Theorem 2. *If A is a finite, complemented, proper subset of B and if $f : A \rightarrow B$, then $B - \text{Image}(f)$ is inhabited.*

Proof. Let \mathcal{X} be the family of those finite sets A such that, for every B in which A is a complemented, proper subset and for every $f : A \rightarrow B$, there is an element of B not in the image of f . We prove that \mathcal{X} contains all finite sets by applying Theorem 1. So it suffices to prove $A \in \mathcal{X}$ under the assumptions that A is finite and that every complemented, proper subset of A is in \mathcal{X} .

To do this, suppose $f : A \rightarrow B$ where A is a complemented, proper subset of B . Since A is complemented in B , $f^{-1}(A)$ is complemented in $f^{-1}(B) = A$. As was pointed out in Section 2, it follows that $f^{-1}(A)$ either equals A or is a proper subset of A . If $f^{-1}(A) = A$, then $\text{Image}(f) \subseteq A$, so the complement in B of this image includes $B - A$, which is inhabited because A is a proper subset of B . So the desired conclusion holds in this case.

There remains the case that $f^{-1}(A)$ is a complemented, proper subset of A and is therefore in \mathcal{X} . Apply the definition of \mathcal{X} with $f^{-1}(A)$, A and $f \upharpoonright f^{-1}(A)$ in the roles of A , B , and f . It shows that $A - \text{Image}(f \upharpoonright f^{-1}(A))$ is inhabited. But this set equals $A - \text{Image}(f) \subseteq B - \text{Image}(f)$. \square

5. THE UNDUALIZED PIGEONHOLE PRINCIPLE

In this section we consider the principle that a finite set cannot be mapped one-to-one into a proper subset. More precisely, we consider two intuitionistically inequivalent versions of this principle. The weaker version says that if X is finite

then a map $f : X + 1 \rightarrow X$ cannot be one-to-one. The stronger version says that if X is finite and $f : X + 1 \rightarrow X$ then there exist $x, y \in X + 1$ such that $x \neq y$ but $f(x) = f(y)$.

Of course in classical logic these are equivalent and easy to prove. We shall show that the weaker version is intuitionistically provable but the stronger is not. In fact, the stronger version implies the law of the excluded middle.

Theorem 3. *If X is finite then there is no one-to-one function from $X + 1$ into X .*

Proof. Suppose X were a finite set with a one-to-one function $f : X + 1 \rightarrow X$. By (the proof of) Theorem 5.44 of [9], there is a natural numbers object N that is a subobject of X (namely the intersection of all $Y \subseteq X$ such that $f(Y + 1) \subseteq Y$). This is clearly absurd, but for the sake of completeness we sketch an intuitionistic proof that it is absurd. Since X is finite, it is a surjective image of $[n] = \{a \in N \mid a < n\}$ for some $n \in N$, by [10, Theorem 9.20]. So its subset $[n + 1]$ is the surjective image of some subset of $[n]$. But the usual classical proof, by induction on n , that there is no surjection from a subset of $[n]$ to $[n + 1]$ is intuitionistically correct because equality of natural numbers is decidable. \square

I thank the referee for pointing out the relevance of [9, Theorem 5.44]; the first version of this paper used the less direct method of the following paragraph, which seems still worth mentioning because of its applicability to other, similar situations.

The intuitionistic provability of Theorem 3 and some related results can be established without actually giving intuitionistic proofs. Consider some situation, which we want to show is impossible. For example, in Theorem 3, the situation would be that X is finite and $f : X + 1 \rightarrow X$ is one-to-one. If the situation is impossible in classical type theory and if it is preserved by inverse images of geometric morphisms of topoi, then it is impossible in intuitionistic type theory also. Indeed, if it were intuitionistically consistent and therefore had a non-zero truth value in some elementary topos, then, by Barr's theorem [2, 10], it would have a non-zero truth value in some Boolean topos and would therefore be consistent in classical type theory.

This argument, inferring the existence of an intuitionistic proof from a classical one, applies to Theorem 3 because “ X is finite,” “ $f : X + 1 \rightarrow X$,” and “ f is one-to-one” are preserved by inverse images of geometric morphisms. It also applies to the weak form of the dual pigeonhole principle because “ $f : X \rightarrow X + 1$,” and “ f is surjective” are preserved by inverse images of geometric morphisms.

In fact, we can do a bit better and replace “ X is finite” by the intuitionistically weaker “ X is a subset of a finite set” and still conclude that there is no surjection $X \rightarrow X + 1$ and no one-to-one map $X + 1 \rightarrow X$. This is because the weaker hypothesis suffices for the classical proof and its internal validity is also preserved by geometric inverse images.

Also provable by the same method are the facts, used in the proof of Theorem 3, that the set of natural numbers (if it exists) cannot be mapped one-to-one into a finite set and that, for any natural number n , $[n]$ cannot be mapped onto $[n + 1]$.

In contrast to the situation with the dual pigeonhole principle, where the stronger form turned out to be provable (Theorem 2), the undualized pigeonhole principle cannot be similarly strengthened without going to classical logic.

Theorem 4. *Assume that, for all finite X and all $f : X + 1 \rightarrow X$, there exist x and y in $X + 1$ with $f(x) = f(y)$ but $x \neq y$. Then the law of the excluded middle holds.*

Proof. Let u be an arbitrary truth value, and let X be a set whose elements are exactly a and b where $a = b$ if and only if u . Such a set X can be obtained as the quotient of $1 + 1$ by an equivalence relation containing all pairs if u and also containing the diagonal pairs; the equivalence classes of the two distinct elements of $1 + 1$ serve as a and b . Clearly, X is finite.

Writing c for the unique element of 1 , we define a map f from $X + 1 = \{a, b, c\}$ to $X = \{a, b\}$ by sending a to itself, b to a , and c to b . This is well-defined even though a might equal b , since they are sent to the same element a .

By assumption, there are $x, y \in \{a, b, c\}$ with $f(x) = f(y)$ but $x \neq y$. We have $x = a$ or $x = b$ or $x = c$ and similarly for y , so we can consider the nine resulting (exhaustive though not necessarily exclusive) cases. Three “diagonal” cases have $x = y$ contrary to the choice of x and y . Two other cases have $x = c$ while $y = a$ or $y = b$; in these cases $f(x) = f(y)$ means that $b = a$ and therefore u holds. The two similar cases with $y = c$ also give that u holds. There remain two cases, one with $x = a$ and $y = b$ and the symmetric one with $x = b$ and $y = a$. In either of these two cases, $x \neq y$ means that $a \neq b$ and therefore not u . Thus, in all cases, we have u or not u . As u was an arbitrary truth value, the proof is complete. \square

6. VARIANTS OF THE DUAL PIGEONHOLE PRINCIPLE

This section is devoted to refuting two possible strengthenings of Theorem 2. The first is to weaken the hypothesis from “finite” to “subset of a finite set,” as we did with the weak pigeonhole principles in the remarks following Theorem 3. For the strong dual pigeonhole principle, this further strengthening is not only unprovable intuitionistically but equivalent to classical logic.

Theorem 5. *Assume that, whenever A is a subset of a finite set and $f : A \rightarrow A + 1$ then there is an element of $A + 1$ not in the image of f . Then the law of the excluded middle holds.*

Proof. Let u be any truth value, and let U be the corresponding subobject of 1 (inhabited if and only if u). Since 1 is finite, we can apply the hypothesis of the theorem with $A = U$. Let $f : U \rightarrow U + 1$ be the inclusion of U in the second summand 1 of $U + 1$. By hypothesis, $U + 1$ has an element $x \notin \text{Image}(f)$. If x is in the first summand U , then (as U is thereby inhabited) u holds. If x is in the second summand 1 , then, by definition of f , it would belong to the image of f with truth value u . Since it does not belong to this image, we conclude that not u . As x must be in one of the two summands, we have proved that u or not u . \square

Finally, we consider an attempt to extend Theorem 2 from the internal logic of topoi (intuitionistic type theory) to the external logic. Specifically, if A is a finite

object in a topos \mathcal{E} and if $f : A \rightarrow A+1$ is a morphism in \mathcal{E} , must $A+1$ have a global section disjoint from the image of f ? The answer is negative. For a counterexample, consider a topos with an inhabited finite object A having no global section, and let $f : A \rightarrow A+1$ map A onto the summand 1. A global section of $A+1$ disjoint from the image of f would be a global section of A and thus does not exist. For a simple example of a topos containing such an A , use the topos of presheaves on a four-element poset $\{p, q, r, s\}$ where p is incomparable with q , r is incomparable with s , and both p and q are below both r and s . Let A be the presheaf whose value at each point is $1+1$ and whose transition maps are the identity map of $1+1$ except for one, say from r to p , that interchanges the elements of $1+1$. Then A clearly has no global section, but it is inhabited and finite and in fact internally isomorphic to $1+1$.

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