ORDERINGS OF ULTRAFILTERS

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INTRODUCTION

The research leading to this thesis was originally motivated by the following considerations. Intuitively, all non-principal ultrafilters on the set ω of natural numbers look pretty much alike. If one attempts to formalize this intuitive feeling, one might conjecture that any two such ultrafilters are isomorphic (i. e., correspond to each other under a suitable permutation of ω), but such a conjecture is quickly destroyed by a simple cardinality argument: There are too many ultrafilters and not enough permutations. Knowing that there are non-isomorphic (i. e., "essentially different") non-principal ultrafilters on ω, one naturally asks what is the difference between them. What properties, invarinat under isomorphism, are possessed by some, but not all, non-principal ultrafilters on ω? Or are there perhaps no such properties (expressible in the usual language of set theory)? The questions can be generalized to refer to uniform ultrafilters on sets of arbitrary cardinality. A partial answer was known, for Rudin had shown [14] that some, but not all, non-principal ultrafilters on ω are P-points (see Definition 7.2) provided the continuum hypothesis is true, and Keisler had shown [7] that some, but not all, uniform ultrafilters on a set of cardinality $\kappa > \omega$ are κ^+-good (see Section 1) provided $2^\kappa = \kappa^+$. If one does not assume any instances of the generalized continuum hypothesis, the problem appears to be much more difficult. We shall show (Theorem 18.1) that a certain property applies to some but not all uniform ultrafilters on sets of certain cardinalities, but I know of no set theoretically definable properties which can be shown, without using the continuum hypothesis or some other special assumption, to apply to some, but not all, non-principal ultrafilters on ω.

In considering this problem, I was led to consider the weak partial ordering of ultrafilters which places one ultrafilter below another if and only if the former is the image of the latter under some function (Definition 2.1). The first results I obtained about this ordering (existence of minimal elements, directedness, and Corollary 9.10) convinced me that it deserved further study. That study is the principal subject of this thesis. It turned out that this ordering and its simpler properties (Sections 5 and 11), as well as Corollary 8.8 (with GCH in place of FRH), had been known to Keisler and others, though nothing had been published on the subject. (As mentioned in Section 9, Corollary 9.10 also follows from work of Booth [2].) However, Keisler suggested three open questions about this ordering (all are answered negatively in Sections 17 and 18) and other questions suggested themselves (e.g., is the ordering an upper semi-lattice).

The results obtained indicate that the structure of the ordering is quite irregular. For example, if we assume the generalized continuum hypothesis and restrict our attention to non-principal ultrafilters on ω, the partially ordered set $RK(\omega)$, obtained by identifying isomorphic ultrafilters, has the following properties. It has cardinality $\aleph_2$. Every element has $\aleph_2$ immediate successors (in a strong sense; see Definition 16.2) but at most $\aleph_1$ predecessors. The long line and the Boolean algebra of all subsets of ω can both be order-isomorphically embedded in $RK(\omega)$. There are $\aleph_2$ distinct minimal elements. There are two elements which have no least upper bound but have exactly n minimal upper bounds, for any given natural number $n \geq 2$.

In addition to the ordering described above, certain other (stronger) orderings of ultrafilters (see Definitions 18.1 and 15.8) and related properties of ultrafilters are considered.

The thesis is divided into four chapters as follows. Chapter I consists of basic definitions and fundamental theorems, almost all of which were known in some form or another but many of which are not in the literature. In particular, Theorem 2.5, which is perhaps the most basic result in the field, has been discovered independently by nearly everyone who has worked in the subject, but no complete and general proof seems to have been published. Chapter II consists of results obtained with essentially one tool -- construction of ultrafilters by transfinite induction. This tool, in conjunction with the (generalized) continuum hypothesis, had been used by Keisler and Rudin to obtain the theorems mentioned above. We show that, in some cases, it is possible to replace the continuum hypothesis by a weaker hypothesis, and we prove some results about the arrangement of the P-points in our ordering. Chapter III concerns ultrapowers and the connection between their model-theoretic properties and the ordering-theoretic properties of the ultrafilters used to create them. Finally, Chapter IV consists of results depending on the ideas of limit, sum, and product of ultrafilters.

The thesis is also divided into sections, which are numbered consecutively without reference to the chapters containing them. Definitions, lemmas, propositions, theorems, corollaries, and remarks are numbered in a single sequence within each section, starting over at the beginning of each section. The seventh numbered item of Section 15, being a lemma, is called Lemma 7 within that section and Lemma 15.7 elsewhere; the third of its eight parts is Lemma 7(3) or Lemma 15.7(3).

CHAPTER I.

THE CATEGORY OF ULTRAFILTERS.

§1. Notation and preliminaries. For any notation which we use and which is not standard, see Shoenfield [15], especially Chapter 9, Problems 28 and 29 of Chapter 5, and Section 2.5. The common notation $f(a)$, where f is a function, is ambiguous, denoting either $f'a$ [15, p. 245] or $\{f'x \mid x \in a \cap Do(f)\} = f''a$. We shall usually write $f(a)$, as it will be clear which meaning is intended, but if confusion seems likely we will use the precise notations $f'a$ and $f''a$. The letter κ will always denote an infinite cardinal, and κ^+ is the least cardinal $> \kappa$. (G)CH is the (generalized) continuum hypothesis. We use the usual symbol $\models$ for satisfaction; thus, if L is a (first-order) language, $\mathfrak{A}$ a structure for L, and φ a sentence of $L(\mathfrak{A})$, then $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A}(\varphi) = T$. If D is an ultrafilter on I and $f \in \prod_{i \in I} A_i$, we use the notation $[f]_D$ or sometimes just $[f]$ (rather than Shoenfield's $\phi(f)$) for

$$\left\{g \in \prod_{i \in I} A_i \mid \{i \in I \mid f(i) = g(i)\} \in D\right\} \quad ;$$

we call $[f]_D$ the germ of f on D. If $g \in [f]_D$, we shall say that f and g are equal modulo D ($f = g \bmod D$). The set of germs is called $D\text{-}\mathrm{prod}_i A_i$, with a similar notation for ultraproducts of structures. For any set X , $P(X)$ is the set of all subsets of X , and $P_\kappa(X)$ is the set of those subsets of X whose cardinal is $< \kappa$. In particular, $P_\omega(X)$ is the set of finite subsets of X .

We assume the set theory ZFC , Zermelo-Fränkel set theory including the axiom of choice. For convenience, we shall occasionally speak of specific proper classes.

A filter in a Boolean algebra $\mathcal{B}$ is a nonempty set $F \subseteq \mathcal{B}$ such that, for all $A, B \in \mathcal{B}$, $A \cap B \in F \Longleftrightarrow A, B \in F$, and $0 \notin F$. An ultrafilter in $\mathcal{B}$ is a maximal filter in $\mathcal{B}$. A basis for a filter F is a set $G \subseteq F$ such that $F = \{A \mid (\exists B \in G) B \subseteq A\}$. A subset G of $\mathcal{B}$ has the finite intersection property if and only if no finite meet of elements of G is 0 . By Zorn's lemma, every such G is contained in an ultrafilter. G is said to generate, or to be a sub-basis for, the smallest filter containing it. Every filter is the intersection of the ultrafilters that contain it. A filter (or ultrafilter) on a set I is a filter (or ultrafilter) in the Boolean algebra $P(I)$.

Let F be a filter on a set I . We say that F-most elements $i \in I$ (or most i with respect to F) have a property φ , and we write $(\forall iF)\varphi(i)$, if and only if $\{i \in I \mid \varphi(i)\} \in F$. We say that

F-<u>many</u> i have φ, and we write $(\exists iF)\varphi(i)$, if and only if, for all $A \in F$, $A \cap \{i \mid \varphi(i)\} \neq \emptyset$. We then have

$$(\forall iF)(\varphi(i) \text{ and } \psi(i)) \Longleftrightarrow (\forall iF)\varphi(i) \text{ and } (\forall iF)\psi(i) \quad ,$$

$$(\exists iF)(\varphi(i) \vee \psi(i)) \Longleftrightarrow (\exists iF)\varphi(i) \vee (\exists iF)\psi(i) \quad ,$$

$$(\forall iF)\varphi(i) \Longleftrightarrow \sim (\exists i F) \sim \varphi(i) \quad .$$

If F is the principal filter generated by $\{J\}$ with $J \subseteq I$, then

$$(\forall iF)\varphi(i) \Longleftrightarrow (\forall i \in J)\varphi(i)$$

and

$$(\exists iF)\varphi(i) \Longleftrightarrow (\exists i \in J)\varphi(i) \quad .$$

In particular, if D is the principal ultrafilter containing $\{j\}$ then

$$(\forall iD)\varphi(i) \Longleftrightarrow (\exists iD)\varphi(i) \Longleftrightarrow \varphi(j) \quad .$$

For any filter F,

$$F \text{ is an ultrafilter} \Longleftrightarrow \left[\text{For arbitrary } \varphi,\ (\forall iF)\varphi(i) \Longleftrightarrow (\exists iF)\varphi(i)\right]$$

$$\Longleftrightarrow \left[\text{For abritrary } \varphi, \sim (\forall iF)\varphi(i) \Longleftrightarrow (\forall iF)\sim\varphi(i)\right] \quad .$$

If $(\forall i)(\varphi(i) \rightarrow \psi(i))$ and $(\forall iF)\varphi(i)$, then $(\forall iF)\psi(i)$.

Warning: The "quantifiers" $(\forall iF)$ do not commute with each other. If F consists of the cofinite subsets of ω, then $(\forall xF)(\forall yF)x < y$, but not $(\forall yF)(\forall xF)x < y$.

The fundamental theorem on ultraproducts is, in this notation,

$$\left[\text{D-prod}\ \mathfrak{A}_i \models \varphi([f_1], \cdots, [f_n])\right] \Longleftrightarrow (\forall iD)\mathfrak{A}_i \models \varphi\big(f_1(i), \cdots, f_n(i)\big) \quad .$$

The <u>size</u> of a filter F on a set I is the least of the cardinalities of the sets in F. F is <u>uniform</u> if and only if $\text{size}(F) = \text{Card Un}(F)$, i.e., all the sets in F have the same cardinal. F is κ-complete if and only if it is closed under formation of intersections of fewer than κ elements at a time. Thus, all filters are ω-complete. Those that are $\aleph_1$-complete are also called <u>countably complete.</u> An ultrafilter D on a set I is κ-<u>regular</u> if and only if there is a function $f : \text{Un}(D) \to P_\omega(\kappa)$ such that $(\forall\alpha \in \kappa)(\forall iD)\alpha \in f(i)$. D is <u>regular</u> if and only if it is $\text{size}(D)$-regular. D is κ^+-good if and only if, given any map $\Phi : \text{Un}(D) \to P(P_\omega(\kappa))$ satisfying $(\forall x \in P_\omega(\kappa))(\forall iD)x \in \Phi(i)$, there exists an $f : \text{Un}(D) \to P_\omega(\kappa)$ such that $(\forall\alpha \in \kappa)(\forall iD)\alpha \in f(i)$ and $(\forall iD)f(i) \in \Phi(i)$. D is <u>good</u> if and only if it is $\text{size}(D)^+$-good.

We now list a number of facts which we shall need. Since most of these are standard, we give references or brief hints rather than proofs for them.

1. Any set on which there is a non-principal countably complete ultrafilter must be very large. It is (relatively) consistent with ZFC to suppose that there is no such set. (See Shoenfield [15, Section 9.10 and Problem 9.14] and Keisler-Tarski [11].)

2. If D is κ-regular, then $\mathrm{size}(D) \geq \kappa$. (For any $A \in D$, $\bigcup_{i \in A} f(i) = \kappa$.)

3. If D is κ-regular, then it is countably incomplete. (The sets

$$A_n = \{i \mid \mathrm{Card}(f(i)) \geq n\}$$

are in D and have empty intersection.)

4. If D is κ^+-good, then it is κ-regular. (In the definition of κ^+-good, set $\Phi(i) = P_\omega(\kappa)$ for all i.)

5. There is a κ-regular ultrafilter on $P_\omega(\kappa)$, hence on any set of cardinality κ. (The collection $\{A_\alpha \mid \alpha \in \kappa\}$, where $A_\alpha = \{x \in P_\omega(\kappa) \mid \alpha \in x\}$, has the finite intersection property. Any ultrafilter containing it is κ-regular, for we may take $f = \mathrm{id}$ in the definition of regular.)

6. Any uniform filter on an infinite set is contained in a uniform ultrafilter. (Adjoin to the filter all sets whose complement has smaller cardinality than the sets of the filter. Extend the resulting family to an ultrafilter.)

7. An ultrafilter is κ^+-good if and only if if is countably incomplete and satisfies the following condition. Given any $g : P_\omega(\kappa) \to D$ such that

$$F \subseteq F' \in P_\omega(\kappa) \Longrightarrow g(F) \supseteq g(F') \quad ,$$

there is an $h : P_\omega(\kappa) \to D$ such that, for all F , $F' \in P_\omega(\kappa)$, $h(F \cup F') = h(F) \cap h(F')$ and $h(F) \subseteq g(F)$. (Proof postponed.)

8. If $2^\kappa = \kappa^+$, then there is a κ^+-good ultrafilter on any set of cardinality κ . (Keisler [7].)

9. Every countably incomplete ultrafilter is $\aleph_0$-regular and $\aleph_1$-good. (Keisler [7].)

Let L be a language and $\mathfrak{A}$ a structure for L . $\mathfrak{A}$ is κ-saturated if and only if, given any set Γ of formulas of $L(\mathfrak{A})$, with a single free variable, such that $\mathrm{Card}(\Gamma) < \kappa$ and every finite subset of Γ is simultaneously satisfiable in $\mathfrak{A}$, the whole set Γ is simultaneously satisfiable in $\mathfrak{A}$.

10. D is κ^+-good if and only if for every language L and every family of structures $\mathfrak{A}_i (i \in Un(D))$ for L , D-prod$_i\mathfrak{A}_i$ is κ^+-saturated. (Proof postponed.)

11. Any two elementarily equivalent κ-saturated structures of power κ , for a language with fewer than κ symbols, are isomorphic. (The proof is like the proof that all countable dense linear orderings without endpoints are isomorphic. See also [15, Problem 5.26].)

We now prove (7) and (10). In Keisler [8], goodness was defined by (essentially) the condition in (7) and proved equivalent to the condition in (10), so we need only prove (10). First suppose D is κ^+-good, let $\mathfrak{A}$ = D-prod $\mathfrak{A}_i$, and let Γ be as in the definition of κ^+-saturated. In particular, $Card(\Gamma) \leq \kappa$. For each $[a] \in |\mathfrak{A}|$, choose a representing function $a' \in \prod_i |\mathfrak{A}_i|$. Interpret $L(\mathfrak{A})$ in $\mathfrak{A}_i$ by letting $\underline{[a]}$ denote $a'(i)$. For $i \in Un(D)$, let

$$\Phi(i) = \{x \in P_\omega(\Gamma) \mid x \text{ is simultaneously satisfiable in } \mathfrak{A}_i\} ,$$

and, for $x \in \Phi(i)$, let $b(i,x)$ be an element of $|\mathfrak{A}_i|$ satisfying all $\varphi \in x$. Thus

$$\varphi \in x \in \Phi(i) \Longrightarrow b(i,x) \text{ satisfies } \varphi \text{ in } \mathfrak{A}_i .$$

Any finite subset of Γ is, by hypothesis on Γ and the fundamental theorem on ultraproducts, satisfiable in D-most of the $\mathfrak{A}_i$. (For the satisfiability of a finite set can be expressed by a single sentence, the existential quantification of the conjunction.) Hence,

$$(\forall x \in P_\omega(\Gamma))(\forall i D) x \in \Phi(i) \quad .$$

Using the κ^+-goodness of D, we can obtain $f : \mathrm{Un}(D) \to P_\omega(\Gamma)$ such that $(\forall i D) f(i) \in \Phi(i)$ and $(\forall \varphi \in \Gamma)(\forall i D)\varphi \in f(i)$. If we let $b(i) = b(i, f(i))$ (when $f(i) \in \Phi(i)$; $b(i)$ arbitrary otherwise), then the properties of f and the implication displayed above show that $(\forall \varphi \in \Gamma)(\forall i D) b(i)$ satisfies φ in $\mathfrak{A}_i$. Hence [b] satisfies every $\varphi \in \Gamma$ in $\mathfrak{A}$. Thus, $\mathfrak{A}$ is κ^+-saturated.

For the converse, let L have two binary predicate symbols, ϵ and $\subseteq$. Let $\mathfrak{A}$ have universe $P_\omega(\kappa) \cup P(P_\omega(\kappa))$, and interpret ϵ and $\subseteq$ in the obvious way. Let $i : \mathfrak{A} \to$ D-prod $\mathfrak{A}$ be the canonical embedding (taking a to the germ of the function constantly a). Let Φ be as in the definition of κ^+-good. For $\alpha \in \kappa$, let φ_α be the formula: $\underline{i(\{\alpha\})} \subseteq x$ and $x \in \underline{[\Phi]}$. The set $\Gamma = \{\varphi_\alpha | \alpha \in \kappa\}$ has cardinality $< \kappa^+$, and any finite subset $\{\varphi_\alpha | \alpha \in m\} (m \in P_\omega(\kappa))$ is satisfied by $i(m)$. So, as we are assuming D-prod $\mathfrak{A}$ κ^+-saturated, choose an [f] satisfying Γ. It is trivial to verify that f (or a function equal to it modulo D) has the properties required in the definition of κ^{+}-good .

§2. The category of ultrafilters. DEFINITION 1. Let D be an ultrafilter, and let $f : \mathrm{Un}(D) \to Y$ be a function. The image of D under f is defined to be the ultrafilter

$$f(D) = \{B \subseteq Y \mid f^{-1}(B) \in D\} .$$

The following lemma is obvious.

LEMMA 2. (1) If $\mathcal{B}$ is a basis for D, then $\{f(A) \mid A \in \mathcal{B}\}$ is a basis for $f(D)$.

(2) If $g : Y \to Z$, then $(g \circ f)(D) = g(f(D))$.

(3) If id is the identity map of $\mathrm{Un}(D)$, then $\mathrm{id}(D) = D$.

(4) $f(D)$ is principal (with basis $\{\{y\}\}$) if and only if f is constant (with value y) on some set in D. In particular, if D is principal (with basis $\{\{x\}\}$), then $f(D)$ is principal (with basis $\{\{f(x)\}\}$).

For the remainder of the lemma, let $f, f' : \mathrm{Un}(D) \to Y$, and let $g, g' : Y \to Z$.

(5) If $f = f' \bmod D$ then $g \circ f = g \circ f' \bmod D$.

(6) $g = g' \bmod f(D)$ if and only if $g \circ f = g' \circ f \bmod D$.

(7) If $f = f' \bmod D$, then $f(D) = f'(D)$. □

In view of part (7) of this lemma, it makes sense to speak of the image of D under a germ $[f]_D$; we shall also say that $[f]_D$ maps D to $f(D)$.

Example 3. We note that the converse of (5) is false; take f and f' to be different constant maps and take g to be any constant map. The converse of (7) is also false, as shown by the following example. Let E be a non-principal ultrafilter on ω. The sets $(A \times A) - \Delta$, where $A \in E$ and $\Delta = \{(x, x) \mid x \in \omega\}$, form a filterbase on $\omega \times \omega$. If D is any ultrafilter containing this filterbase, and if $\pi_1, \pi_2 : \omega \times \omega \to \omega$ are the projections, then $\pi_1(D) = \pi_2(D) = E$, but $\pi_1 \neq \pi_2 \bmod D$.

We define a category $\mathfrak{U}$ of ultrafilters as follows. The objects of $\mathfrak{U}$ are all ultrafilters (on arbitrary sets). A morphism from D to E is a germ $[f]_D$ which maps D to E. If $[f]_D : D \to E$ and $[g]_E : E \to F$ are morphisms (so $f(D) = E$ and $g(E) = F$) then, according to the lemma, $[g \circ f]_D$ is a morphism from D to F, depending only on $[f]_D$ and $[g]_E$ (not on the choice of representatives f and g), and we define the composite $[g]_E \circ [f]_D$ to be $[g \circ f]_D$. It is clear that composition is associative and that $[id_{Un(D)}]_D$ is an identity morphism for D, so $\mathfrak{U}$ is a category. To simplify the notation, we shall sometimes refer to a map $f : \mathrm{Un}(D) \to \mathrm{Un}(E)$ as a morphism, when we really mean that $[f]_D$ is a morphism. This practice should cause no confusion.

PROPOSITION 4. <u>In $\mathfrak{U}$, every morphism is an epimorphism (in the category-theoretic sense)</u>.

<u>Proof.</u> This proposition just restates the "if" part of statement (6) of the lemma. □

THEOREM 5. <u>The only morphism from an ultrafilter to itself is the identity</u>.

<u>Proof.</u> Let $[f]_D : D \to D$ where D is an ultrafilter on $X = \mathrm{Un}(D)$ and $f : X \to X$. We have $f(D) = D$, and we must show that $f = \mathrm{id}_X \bmod D$.

Let f^n be the nth iterate of f $(n \geq 0)$; $f^0 = \mathrm{id}_X$, $f^{n+1} = f \circ f^n$. For $x, y \in X$, define $x \simeq y$ if and only if for some n and $m (\geq 0)$ $f^n(x) = f^m(y)$. Clearly this is an equivalence relation, and $f(x) \simeq x$. Say that x is periodic if and only if, for some $k \geq 1$, $f^k(x) = x$. Let $A \subseteq X$ be a choice set for the partition of X into equivalence classes (i.e., for each equivalence class E,

$\mathrm{Card}(A \cap E) = 1$), and arrange that, if an equivalence class E contains a periodic element, then the element of $A \cap E$ is periodic. (Clearly, such an A exists, by the axiom of choice.)

Let us temporarily confine our attention to one (arbitrary) equivalence class E, and let a be the element of $A \cap E$. For each $x \in E$, let $m(x)$ be the least m such that for some n $f^n(x) = f^m(a)$, and let $n(x)$ be the least n such that $f^n(x) = f^{m(x)}(a)$. (These exist because $x \simeq a$.) Let $d(x) = m(x) - n(x)$. I claim that $d(f(x)) = d(x) + 1$ or $x = a$ (or both).

Let $y = f(x)$. Then

(1) $$f^{n(y)+1}(x) = f^{n(y)}(y) = f^{m(y)}(a) \quad .$$

By definition of $m(x)$, we conclude $m(x) \leq m(y)$.

<u>Case</u> 1: $m(x) < m(y)$. If $n(x) \geq 1$, then

(2) $$f^{n(x)-1}(y) = f^{n(x)}(x) = f^{m(x)}(a) \quad ,$$

contrary to the definition of $m(y)$. So in fact $n(x) = 0$, and $x = f^{m(x)}(a)$. Then $y = f(x) = f^{m(x)+1}(a)$, so $m(y) \leq m(x) + 1$. As $m(y) > m(x)$, we conclude first that $m(y) = m(x) + 1$, and second (by definition of $n(y)$) that $n(y) = 0$. So $n(x) = n(y) = 0$ and

$m(x) + 1 = m(y)$. Therefore $d(y) = d(x) + 1$, as claimed.

Case 2: $m(x) = m(y) = m$. Let $b = f^m(a)$. Equation (1) now shows that $n(x) \leq n(y) + 1$. If equality holds, then $d(y) = d(x) + 1$ as claimed. So suppose now that $n(x) \leq n(y)$. If $n(x) \geq 1$, then we have (2) which now contradicts the fact that $n(x) - 1 < n(y)$, so in fact $n(x) = 0$, $x = f^m(a) = b$. Since $f^{n(y)+1}(b) = f^{n(y)}(y) = b$, b is periodic; by definition of A , a is also periodic, say $f^k(a) = a$ $(k \geq 1)$. Choose p so that $pk \geq m$, and observe

$$f^{pk-m}(x) = f^{pk-m}f^m(a) = f^{pk}(a) = a .$$

By definition of $m(x)$, $m = 0$, and $x = b = a$, as claimed.

Since the equivalence class E was arbitrary, we have in fact defined d on all of X and proved that $d(f(x)) = d(x) + 1$ unless $x \in A$. Let

$$X_i = \{x \in X \mid d(x) \equiv i \pmod 2\} \qquad i = 0,1 .$$

Thus, $X_i \cap f^{-1}(X_i) \subseteq A$. As D is an ultrafilter on $X = X_0 \cup X_1$, $X_i \in D$ for $i = 0$ or for $i = 1$. As $f(D) = D$, we also have $f^{-1}(X_i) \in D$, and therefore $A \in D$. Again, $f^{-1}(A) \in D$, so $A \cap f^{-1}(A) \in D$.

But if $x \in A \cap f^{-1}(A)$, then x and $f(x)$ are both in A and both in the same equivalence class. By definition of A, this implies $x = f(x)$. Therefore, $\{x \mid x = f(x)\} \in D$, and $f = id_X \bmod D$. □

COROLLARY 6. <u>If there are morphisms</u> $f : D \to E$ <u>and</u> $g : E \to D$, <u>then</u> D <u>and</u> E <u>are isomorphic; indeed,</u> f <u>and</u> g <u>are inverse isomorphisms. Furthermore, under these circumstances,</u> f <u>is the only morphism from</u> D <u>to</u> E (<u>and</u> g <u>is the only morphism from</u> E <u>to</u> D).

<u>Proof</u>: For the first statement, apply the theorem to the morphisms $gf : D \to D$ and $fg : E \to E$. For the second statement, observe that any $f' : D \to E$ would, like f, be an inverse for g, but g can have only one inverse. □

The second statement of the corollary provides a partial converse for part (7) of Lemma 2.

PROPOSITION 7. $[f]_D : D \to E$ <u>is an isomorphism if and only if, for some</u> $A \in D$, $f \restriction A$ <u>is one-to-one</u>.

<u>Proof</u>: Suppose $A \in D$ and $f \restriction A$ is one-to-one. Extend its inverse $f(A) \to A$ arbitrarily to a map $g : Un(E) \to Un(D)$. Then $g \circ f = id \bmod D$, so $g(E) = g(f(D)) = D$ (by Lemma 2), and $[g]_E : E \to D$. Therefore

$[f]_D$ is an isomorphism by Corollary 6. Conversely, suppose $[f]_D$ is an isomorphism with inverse $[g]_E$. Then $g \circ f = \text{id} \bmod D$, i.e.,

$$A = \{x \mid gf(x) = x\} \in D \quad ,$$

and clearly $f \restriction A$ is one-to-one. □

The following lemma often permits simplification of notation. In effect, it says that any morphism might as well be the projection of a product of two sets to one of the factors.

LEMMA 8. *Let* $[f]_D : D \to E$ *be any morphism, and let* κ *be the cardinal of* $\mathrm{Un}(D)$ *or* $\mathrm{Un}(E)$, *whichever is larger. Then there are ultrafilters* D' *on* $\kappa \times \kappa$ *and* E' *on* κ, *isomorphic to* D *and* E *respectively, such that the diagram*

$$\begin{array}{ccc} D & \xrightarrow{\cong} & D' \\ \Big\downarrow [f]_D & & \Big\downarrow [\pi]_{D'} \\ E & \xrightarrow{\cong} & E' \end{array}$$

commutes, where $\pi : \kappa \times \kappa \to \kappa$ *is projection to the first factor.*

Proof: Let $\beta : \mathrm{Un}(E) \to \kappa$ be an injection, and set $E' = \beta(E)$. By Proposition 7, $[\beta]_E : E \to E'$ is an isomorphism. Now map

$\mathrm{Un}(D) \xrightarrow{\alpha} \kappa \times \kappa$ by $\alpha(x) = (\beta f(x), \lambda)$ where x is the λth element of $f^{-1}(f(x))$ in some (fixed) well-ordering of $\mathrm{Un}(D)$ of order type $\leq \kappa$. Then $\pi\alpha = \beta f$ so the diagram commutes, and α is one-to-one so $[\alpha]_D : D \to D' = \alpha(D)$ is an isomorphism. □

Essentially the same proof gives the following corollary.

COROLLARY 9. In the situation of the lemma, let $\kappa_1 \geq \mathrm{size}(E)$, and suppose that, on some set of D, f is at-most-κ_2-to-one. Then there exist D' on $\kappa_1 \times \kappa_2$ and E' on κ_1 such that all conclusions of the lemma hold. □

PROPOSITION 10. In $\mathcal{U}$, every monomorphism is an isomorphism.

Proof: In view of Lemma 8, we may begin by supposing that D is an ultrafilter on $\kappa \times \kappa$, E is an ultrafilter on κ, π is the projection to the first factor $\kappa \times \kappa \to \kappa$, $E = \pi(D)$, and $[\pi]_D$ is not an isomorphism. We must show that $[\pi]_D$ is not a monomorphism.

Let $p : \kappa \times \kappa \times \kappa \to \kappa \times \kappa$ be the projection to the first two factors and $q : \kappa \times \kappa \times \kappa \to \kappa \times \kappa$ be projection to the first and third factors. Let

$$\Delta = \{(x,y,y) \mid x, y \in \kappa\} = \{t \in \kappa \times \kappa \times \kappa \mid p(t) = q(t)\} .$$

For any $A \in D$, let

$$A' = p^{-1}(A) \cap q^{-1}(A) - \Delta .$$

I claim the sets $A'(A \in D)$ form a filterbase. Clearly $(A \cap B)' = A' \cap B'$, so we need only prove $A' \neq \emptyset$. Suppose the contrary, namely $A \in D$ and $A' = \emptyset$. By definition of A', we find

$$(x, y) \in A \quad \text{and} \quad (x, z) \in A \quad \Rightarrow y = z \quad .$$

Then π is one-to-one on A, which, by Proposition 7, contradicts the assumption that $[\pi]_D$ is no isomorphism. Therefore, there is an ultrafilter F containing A' for every $A \in D$. It immediately follows that $p(F) = q(F) = D$. Thus $[p]_F$ and $[q]_F$ are morphisms $F \to D$; they are distinct because $\Delta \notin F$ (since Δ is disjoint from A'). But $[\pi]_D[p]_F = [\pi]_D[q]_F$ because $\pi p = \pi q$. Therefore, $[\pi]_D$ is not a monomorphism. □

§3. Cartesian products of filters. There are two ways of defining the product of two filters. One definition will be considered in Chapter IV. It has the property that the product of two ultrafilters is always an ultrafilter. In this section, we consider the other definition, which we call the cartesian product. In most cases the cartesian product $D \times E$ of two ultrafilters D and E will not be an ultrafilter. It turns out that $D \times E$ is an ultrafilter if and only if D and E have a product in $\mathcal{U}$ (in the category-theoretic sense of product), and then $D \times E$ is this product.

DEFINITION 1. Let $\{F_i | i \in I\}$ be an indexed family of filters on sets $X_i = \mathrm{Un}(F_i)$. Let $X = \prod_{i\in I} X_i$ with projection maps $\pi_i : X \to X_i$. The sets $\pi_i^{-1}(A)$ $(i \in I\ ;\ A \in F_i)$ form a sub-basis of a filter which we call the cartesian product $\prod_{i\in I} F_i$ of the filters F_i. We use the notations $F_1 \times F_2$, $F_1 \times \cdots \times F_n$ with the obvious meaning.

LEMMA 2. An ultrafilter D on X contains $\prod_{i\in I} F_i$ if and only if, for each $i \in I$, $F_i \subseteq \pi_i(D)$.

Proof: Both conditions say that, for each $i \in I$ and each $A \in F_i$, $\pi_i^{-1}(A) \in D$. □

PROPOSITION 3. Let $\{D_i | i < n\}$ be a finite family of ultrafilters on sets $X_i = \mathrm{Un}(D_i)$. Let $F = \prod_{i<n} D_i$ on $X = \prod_{i<n} X_i$. For any

ultrafilter E and any family of morphisms $[f_i]_E : E \to D_i$ $(i < n)$, there are a unique ultrafilter E' on X and a unique morphism $[f]_E : E \to E'$ such that

$$[f_i]_E = [\pi_i]_{E'} \circ [f]_E \qquad (i < n) \quad .$$

Furthermore, $F \subseteq E'$.

Proof: Existence: Let $Y = \mathrm{Un}(E)$. Let $f : Y \to X$ be the (unique) map whose coordinates are the f_i (i.e., $f_i = \pi_i f$), and let $E' = f(E)$. $F \subseteq E'$ by Lemma 2, and the other conclusions are clear.

Uniqueness: Suppose f and $\tilde{f}$ were two maps satisfying all the conditions, with $E' = f(E)$, $\tilde{E}' = \tilde{f}(E)$. Then $\pi_i f = f_i \bmod E$ and $\pi_i \tilde{f} = f_i \bmod E$. By Section 1, $(\forall x E)$

$$\pi_0 f(x) = f_0(x) \text{ and} \cdots \text{and } \pi_{n-1} f(x) = f_{n-1}(x)$$

$$\text{and} \quad \pi_0 \tilde{f}(x) = f_0(x) \text{ and} \cdots \text{and } \pi_{n-1} \tilde{f}(x) = f_{n-1}(x) \quad .$$

Therefore $(\forall x E) f(x) = \tilde{f}(x)$, so $f = \tilde{f} \bmod E$ and $E' = \tilde{E}'$. □

In the language of category theory, the last proposition says

$$\prod_{i<n} \operatorname{Hom}(E, D_i) \cong \coprod_{E' \supseteq F} \operatorname{Hom}(E, E') \quad ,$$

where $\coprod$ means disjoint union, and the bijection is natural with respect to E. As an immediate consequence, we have

COROLLARY 4. _If, in the situation of Proposition_ 3, F _is an ultrafilter, then_ F _(together with the morphisms_ $[\pi_i]_F : F \to D_i$ _) is a product of the_ D_i's _in the category-theoretic sense._ □

Conversely, we have

PROPOSITION 5. _With the notation of Proposition_ 3, _suppose that the_ D_i _have a product in the category-theoretic sense. Then_ F _is an ultrafilter, and_ F _is isomorphic to the category-product of the_ D_i _(with the_ $[\pi_i]_F$ _corresponding to the projections of the category-product)._

Proof: Let the category-product be E with projections $[f_i]_E : E \to D_i$, and let f, E' be as given by Proposition 3. Let E'' be any ultrafilter containing F. By Lemma 2, $[\pi_i]_{E''} : E'' \to D_i$, so, by definition of category-products, there is a morphism $[g]_{E''} : E'' \to E$ such that $\pi_i = f_i g \bmod E''$. Since $\pi_i f = f_i \bmod E$ and $E = g(E'')$, we conclude (using Lemma 2.2)

$$\pi_i fg = f_i g = \pi_i \qquad \text{mod } E'' \quad .$$

As in the proof of Proposition 3, we obtain $fg = \text{id} \bmod E''$, and, in particular, $E'' = fg(E'') = f(E) = E'$. Thus E' is the only ultrafilter containing F. By Section 1, F is an ultrafilter. The remainder of the proposition may now be proved either by direct verification or by appealing to Corollary 4 and the uniqueness of category-products. □

THEOREM 6. For any two ultrafilters D and E, the following are equivalent.

(1) D and E have a category-product (in $\mathcal{U}$).

(2) $D \times E$ is an ultrafilter.

(3) For every function $\Gamma : \text{Un}(D) \to E$, there is a set $A \in D$, with $\bigcap_{x \in A} \Gamma(x) \in E$.

Proof: (1) $\Longleftrightarrow$ (2) has just been proved.

(2) $\Longrightarrow$ (3) : Given Γ, let

$$Z = \{(x, y) \in \text{Un}(D) \times \text{Un}(E) \mid y \in \Gamma(x)\} \quad .$$

If $A \in D$ and $B \in E$, choose any $x \in A$ and any $y \in \Gamma(x) \cap B$. ($A \neq \emptyset$ and $\Gamma(x) \cap B \neq \emptyset$ because $A \in D$ and $\Gamma(x) \cap B \in E$.) Then $(x, y) \in (A \times B) \cap Z$. Thus, every set of $D \times E$ meets Z. As $D \times E$ is an ultrafilter, $Z \in D \times E$. Thus, there exist $A \in D$, $B \in E$ such that $A \times B \subseteq Z$. Then, for any $x \in A$ and $y \in B$, $y \in \Gamma(x)$, so $B \subseteq \bigcap_{x \in A} \Gamma(x)$. As $B \in E$, $\bigcap_{x \in A} \Gamma(x) \in E$.

(3) $\Rightarrow$ (2): Let $Z \subseteq \mathrm{Un}(D) \times \mathrm{Un}(E)$ be given. We must show that Z or its complement is in $D \times E$. Since the quantifiers $(\forall xD)$ and $(\forall yE)$ commute with negation (see Section 1), either

$$(\forall xD)(\forall yE)\ (x, y) \in Z \tag{4}$$

or the same statement holds when Z is replaced by its complement. Considering the complement rather than Z if necessary, we may assume that (4) holds. Let $\Gamma(x) = \{y \mid (x, y) \in Z\}$ if this set is in E (which happens for D-most x, by (4)), and $\Gamma(x) = \mathrm{Un}(E)$ otherwise. Thus $\Gamma : \mathrm{Un}(D) \to E$. By (3) there is a set $A_1 \in D$ with $\bigcap_{x \in A_1} \Gamma(x) \in E$. Let $A_2 = \{x \mid (\forall yE)(x, y) \in Z\} \in D$, so that, for $x \in A_2$, $\Gamma(x) = \{y \mid (x, y) \in Z\}$, and let $A = A_1 \cap A_2 \in D$. Then let

$$B = \{y \mid (\forall x \in A)(x, y) \in Z\} = \bigcap_{x \in A} \{y \mid (x, y) \in Z\}$$

$$= \bigcap_{x \in A} \Gamma(x) \supseteq \bigcap_{x \in A_1} \Gamma(x) \in E .$$

Then $A \times B \in D \times E$, and $A \times B \subseteq Z$, so $Z \in D \times E$. □

COROLLARY 7. If D is principal and E is arbitrary, then E is a category-product of D and E. □

COROLLARY 8. If E is size $(D)^+$-complete, then $D \times E$ is an ultrafilter. □

COROLLARY 9. Condition (3) of the theorem is, despite its appearance, symmetrical in D and E. □

COROLLARY 10. If D and E are countably incomplete, then they have no category-product in $\mathcal{U}$.

Proof: Let $A_1, A_2, \cdots \in D$; $\bigcap_{i<\omega} A_i \notin D$; $B_1, B_2, \cdots \in E$, $\bigcap_{i<\omega} B_i \notin E$. Replacing A_i by $A_i - \bigcap_{i<\omega} A_i$, we may suppose $\bigcap_{i<\omega} A_i = \emptyset$. For each $x \in \mathrm{Un}(D)$, let $n(x)$ be the least i such that $x \notin A_i$. Observe that $n(x)$ is not bounded on any set of D, for if $n(x) < N$ for all $x \in A$ then A is disjoint from $\bigcap_{i=1}^{N-1} A_i$ which is in D. Let $\Gamma(x) = \bigcap_{i<n(x)} B_i \in E$. Then, if $A \in D$,

$$\bigcap_{x \in A} \Gamma(x) = \bigcap_{i<\omega} B_i \notin E \quad ,$$

so condition (3) of the proposition fails. □

§4. Size, regularity, and completeness of ultrafilters in $\mathcal{U}$.

In this section we investigate the correlation between the existence or non-existence of a morphism in $\mathcal{U}$ from D to E and various properties of D and E.

PROPOSITION 1. *If D is an ultrafilter and f is any function on Un(D), then size(f(D)) $\leq$ size(D). If D $\cong$ E, then size(D) = size(E). Every ultrafilter D is isomorphic to a uniform ultrafilter on the cardinal size(D).*

Proof: The first assertion is immediate from the definition of size, and the second follows from the first. For the third assertion, let κ = size(D) = Card(A) with $A \in D$. Take a bijection $A \to \kappa$ and extend it arbitrarily to a map $f : \text{Un}(D) \to \kappa$. By Proposition 3.7, D is isomorphic (via f) to f(D), and, by the second assertion, f(D) is uniform on κ. □

This proposition shows that we may, without loss of generality, restrict our attention to uniform ultrafilters on cardinals. To be precise, the inclusion, into $\mathcal{U}$, of the full subcategory whose objects are uniform ultrafilters on cardinals, is an equivalence of categories. Observe that, although all the ultrafilters isomorphic to a given D form a proper class, those that are on size(D) form a set (of cardinality at most $2^{\text{size}(D)}$).

DEFINITION 2. $\mathcal{U}(\kappa)$, $\mathcal{U}(<\kappa)$, $\mathcal{U}(\leq\kappa)$ are the full subcategories of $\mathcal{U}$ whose objects are the ultrafilters of size κ, size $<\kappa$, and size $\leq\kappa$, respectively.

PROPOSITION 3. If $[g]_D : D \to E$ is a morphism and E is κ-regular, then D is κ-regular.

Proof: If $f : \text{Un}(E) \to P_\omega(\kappa)$ is as in the definition of κ-regular, then $f \circ g : \text{Un}(D) \to P_\omega(\kappa)$ shows that D is κ-regular. □

PROPOSITION 4. D is κ-complete if and only if, whenever $[g]_D : D \to E$ is a morphism and $\text{size}(E) < \kappa$, E is principal.

Proof: Suppose D is κ-complete, $[g]_D : D \to E$, and $\text{size}(E) < \kappa$. Let $A \in E$ be such that $\text{Card}(A) = \text{size}(E) < \kappa$. Then $\bigcap_{a\in A}(A - \{a\}) = \emptyset$, so $\bigcap_{a\in A} g^{-1}(A - \{a\}) = \emptyset \notin D$. As D is κ-complete, there is an $a \in A$ such that $g^{-1}(A - \{a\}) \notin D$, so $A - \{a\} \notin g(D) = E$. But $(A - \{a\}) \cup \{a\} \in E$, so $\{a\} \in E$, and E is principal.

For the converse, suppose D is not κ-complete. Then, for some $\lambda < \kappa$, we have a family $\{A_\alpha | \alpha < \lambda\} \subseteq D$ with $\bigcap_{\alpha<\lambda} A_\alpha \notin D$.

As usual, we may replace A_α by $A_\alpha - \bigcap_{\alpha<\lambda} A_\alpha$ and thus assume $\bigcap_{\alpha<\lambda} A_\alpha = \emptyset$. Let

$$g : \mathrm{Un}(D) \longrightarrow \lambda : x \longmapsto (\mu\alpha)(x \notin A_\alpha) \quad ,$$

and let $E = g(D)$. Then $[g]_D : D \to E$ and $\mathrm{size}(E) \leq \lambda < \kappa$. To complete the proof, we shall show that E is non-principal. Otherwise, we would have $\{\alpha\} \in E$ for some $\alpha < \lambda$. By Lemma 2.2(4), there is an $A \in D$ such that

$$x \in A \Longrightarrow g(x) = \alpha \quad .$$

By definition of g, it follows that $A \cap A_\alpha = \emptyset$, contrary to the fact that both A and A_α are in D. $\square$

§5. The Rudin-Keisler ordering. DEFINITION 1. *Let* D *and* E *be ultrafilters.* $D \leq E$ *if and only if there is a morphism from* E *to* D *in* $\mathcal{U}$. *The relation* $\leq$ *is called the Rudin-Keisler ordering.*

PROPOSITION 2. (1) $\leq$ *is reflexive and transitive.*

(2) $D \cong E \Longleftrightarrow D \leq E$ *and* $E \leq D$.

Proof: (1) follows from the fact that $\mathcal{U}$ is a category, as does half of (2). The remaining implication (right to left) follows from Corollary 2.6. □

Intuitively speaking, the relation $\leq$ induces a partial ordering of isomorphism classes of ultrafilters. Unfortunately, too many things here are proper classes, so we define instead

DEFINITION 3. $\overline{D}$ *is the set of (uniform) ultrafilters on* size(D) *which are isomorphic to* D . $\overline{D} \leq \overline{E}$ *if and only if* $D \leq E$. RK *is the class of all sets of the form* $\overline{D}$, *partially ordered by* $\leq$.

Remark 4. By Proposition 2, the relation $\leq$ on RK has all the properties of a partial order (except that it isn't a set). Obviously, $\overline{D} = \overline{E}$ if and only if $D \cong E$, so $\overline{D}$ is "as good as the isomorphism class of D ." We shall sometimes act as though the ultrafilters themselves, rather than the sets $\overline{D}$, were elements of RK .

Translating the results of the preceding section, we get

PROPOSITION 5. Size is a well-defined order preserving map of RK to the class of cardinals. If $\overline{D} \leq \overline{E}$ and D is κ-regular, then so is E ; in particular, it makes sense to say that an element of RK is κ-regular. D is κ-complete if and only if the only $\overline{E} \leq \overline{D}$ with $\text{size}(\overline{E}) < \kappa$ is $\overline{E} = \{\{\{0\}\}\}$. Hence, if $\overline{D'} \leq \overline{D}$ and D is κ-complete, then so is D' , and it makes sense to say that an element of RK is κ-complete. □

Remark 6. E is principal if and only if $\overline{E} = \{\{\{0\}\}\}$. We sometimes write $\overline{0}$ for $\{\{\{0\}\}\}$. $\overline{0}$ is the least element of RK .

DEFINITION 7. $RK(\kappa)$, $RK(<\kappa)$, $RK(\leq \kappa)$ are the sets of all $\overline{D}$ where D is an ultrafilter of size κ , size $< \kappa$, size $\leq \kappa$, respectively. (Note that these are really sets.)

We now begin an investigation of the structure of the partially ordered class RK .

PROPOSITION 8. (1) For any $\alpha \in RK(\leq \kappa)$, $\text{Card}\{\beta \in RK \mid \beta \leq \alpha\} \leq 2^{\kappa}$.

(2) $\text{Card } RK(\leq \kappa) = 2^{2^{\kappa}}$.

(3) For any $\alpha \in RK(\leq \kappa)$, $\text{Card } \{\beta \in RK(\leq \kappa) \mid \beta \geq \alpha\} = 2^{2^{\kappa}}$.

(4) Card $RK(\kappa) = 2^{2^\kappa}$.

Proof: (1) Any $\alpha \in RK(\leq \kappa)$ is $\overline{D}$ for some D on κ, and any $\beta \leq \alpha$ is $\overline{f(D)}$ for some $f : \kappa \to \kappa$. Since there are only 2^κ functions from κ to κ, (1) follows.

(2) It is well-known that there are 2^{2^κ} ultrafilters on κ (see, e. g., Čech [3]). The argument given for part (1) shows that each isomorphism class contains at most 2^κ ultrafilters. Therefore, there must be 2^{2^κ} isomorphism classes.

(3) Let $\alpha = \overline{D}$ where $Un(D) = \kappa$. For each of the 2^{2^κ} ultrafilters E on κ, let E^* be an ultrafilter on $\kappa \times \kappa$ such that $E^* \supseteq D \times E$. Then $E = \pi_2(E^*)$ (by Lemma 3.2; π_1 and π_2 are the projections $\kappa \times \kappa \to \kappa$), so distinct E's give distinct E^*'s, and there are 2^{2^κ} E^*'s. $\overline{E^*} \geq \alpha$ because $\pi_1(E^*) = D$. As there are at most 2^κ E^*'s in any isomorphism class, (3) follows.

(4) This is immediate from (3) and the fact that $\beta \geq \alpha \Rightarrow \text{size}(\beta) \geq \text{size}(\alpha)$ (and the fact that there is a uniform ultrafilter on κ). □

COROLLARY 9. $RK(\kappa)$ has no maximal elements.

Proof: Clear from (3) of the proposition. □

PROPOSITION 10. Every subset of RK has an upper bound. In $RK(\leq \kappa)$, any subset of cardinality $\leq \kappa$ has an upper bound.

Proof: Let $\{D_i \mid i \in I\}$ be a family of ultrafilters. By Lemma 3.2, any ultrafilter containing $F = \prod_{i \in I} D_i$ is $\geq D_i$ for every $i \in I$. This proves the first assertion. For the second, we may suppose $Un(D_i) = \kappa$ and $Card(I) \leq \kappa$, so F is a filter on κ^I. A basis for F is given by finite intersections of sets of the form $\pi_i^{-1}(A)$ with $A \in D_i$. It follows that the set

$$B = \left\{f \in \kappa^I \mid f_i = 0 \quad \text{for all but finitely many} \quad i \in I\right\}$$

meets every set in F, so there is an ultrafilter $E \supseteq F \cup \{B\}$. As before, $E \geq D_i$ for all $i \in I$, and $size(E) \leq Card(B) = \kappa$. □

COROLLARY 11. $RK(\leq \kappa)$ contains a chain of order type κ^+. In fact, any element of $RK(\leq \kappa)$ is the first element of such a chain.

Proof: Let $\alpha \in RK(\leq \kappa)$. Define a strictly increasing function $f : \kappa^+ \to RK(\kappa)$ as follows. $f(0) = \alpha$. If f is already defined for all $\xi < \eta (< \kappa^+)$, use the proposition to get an upper bound β for $f''\eta$. In view of Corollary 9, there is an element of $RK(\leq \kappa)$ which is $> \beta$.

Let $f'\eta$ be such an element. Then clearly $f''\kappa^+$ is a chain of the required type whose first element is α . □

Having shown the existence of upper bounds in RK , we might naturally ask whether least upper bounds exist in RK . This question also arises from the following consideration. As we shall see, the structure of RK , or even RK(ω) , is rather wild. When confronted with a wild partially ordered set one naturally tries to compare it with others of its kind, and the first one that comes to mind is the ordering of the degrees of recursive unsolvability (Turing degrees). This ordering has the one pleasant property of being an upper semi-lattice, and one might hope that RK shares this property.

It is clear that, if two ultrafilters D and E have a category-product $D \times E$, then $\overline{D \times E}$ is a least upper bound for $\overline{D}$ and $\overline{E}$; unfortunately, by Corollary 3.10, this only happens if D or E is countably complete. It is also obvious that if D and E are comparable, then the larger of the two serves as a least upper bound; unfortunately, Kunen has shown [12] that RK(ω) contains 2^{ω} pairwise incomparable elements, and we shall show in Chapter II that, assuming GCH (or certain weaker hypotheses) there are $2^{2^{\kappa}}$ pairwise incomparable (in fact minimal) elements of RK(κ) , for all κ .

Thus, the trivial ways of obtaining least upper bounds do not suffice to make RK an upper semi-lattice. We shall show in Chapter IV that, assuming CH , RK is in fact not an upper semi-lattice, and it is not a lower semi-lattice either. We shall obtain two elements of RK(ω) which have neither a least upper bound nor a greatest lower bound in RK .

§6. Ultrafilters omitting cardinals. DEFINITION 1. *An ultrafilter* D *omits an infinite cardinal* κ *if and only if, for every* $E \le D$, $\mathrm{size}(E) \neq \kappa$.

PROPOSITION 2. (1) *If* $D \le D'$ *and* D' *omits* κ *then* D *omits* κ.

(2) D *does not omit* size(D).

(3) D *omits all cardinals* $>$ size(D).

(4) D *is* κ-*complete if and only if* D *omits all infinite cardinals* $< \kappa$.

(5) *If* D *is a uniform ultrafilter on* κ *and* E *is a* λ-*regular ultrafilter with* $\lambda \ge 2^{\kappa}$, *then* $D \le E$. *In particular, a* 2^{κ}-*regular ultrafilter does not omit* κ.

Proof: (1) and (2) are obvious, (3) is contained in Proposition 4.1, and (4) is Proposition 4.4. For (5), let $f : \mathrm{Un}(E) \to P_{\omega}(\lambda)$ be such that, for all $\alpha \in \lambda$, $\{x \mid \alpha \in f(x)\} \in E$ (as in the definition of λ-regular), and let $h : P(\kappa) \to \lambda$ be an injection. For each $x \in \mathrm{Un}(E)$, let $g(x)$ be an arbitrary element of

$$\bigcap_{\substack{h(A) \in f(x) \\ \text{and } A \in D}} A \in D.$$

Then, if $A \in D$,

$$g^{-1}(A) = \{x \mid g(x) \in A\} \supseteq \{x \mid h(A) \in f(x)\} \in E ,$$

so $g(E) = D$. □

The following theorem is a slight generalization of a theorem of Chang [4].

THEOREM 3. Let κ *be a regular cardinal. There is a cardinal* λ *such that* $\kappa^+ \leq \lambda \leq 2^\kappa$ *and no ultrafilter of size* λ *omits* κ. (*The proof will yield an explicit definition of* λ.)

Proof: Let X be the set of all maps $\kappa \to \kappa$, so $\mathrm{Card}(X) = 2^\kappa$. Let λ be the least cardinal such that, for some set $F \subseteq X$, $\mathrm{Card}(F) = \lambda$ and

(1) $$(\forall g \in X)(\exists f \in F)(\forall \xi < \kappa)(\exists x)(\xi < x < \kappa \text{ and } g(x) < f(x)) .$$

If we let $\mathcal{F}$ be the filter generated by the set of sets of the form $\{x \mid \xi < x < \kappa\}$ for $\xi < \kappa$ (i.e., $\mathcal{F} = \{A \subseteq \kappa \mid \mathrm{Card}(\kappa - A) < \kappa\}$ because κ is regular), then (1) may be rewritten

(1′) $$(\forall g \in X)(\exists f \in F)(\exists x \mathcal{F})\ g(x) < f(x) .$$

It is clear from the definition of λ that $\lambda \leq 2^\kappa$.

<u>Claim:</u> $\lambda \geq \kappa^+$, and λ is regular.

<u>Proof of claim</u>: Suppose $\lambda < \kappa^+$, so $F = \{f_\beta \mid \beta < \kappa\}$ for an appropriate indexing (possibly with repetitions). Define $g : \kappa \to \kappa$ by letting $g(\alpha)$ be any element of κ larger than $f_\beta(\alpha)$ for all $\beta < \alpha$; as κ is regular, such an element exists. Then

$$g(x) < f_\gamma(x) \Rightarrow x \leq \gamma \quad ,$$

and from (1′) we get

$$(\exists \gamma < \kappa)(\exists x \mathcal{F}) g(x) < f_\gamma(x) \quad .$$

Hence,

$$(\exists \gamma < \kappa)(\exists x \mathcal{F})\ x \leq \gamma \quad ,$$

contrary to the definition of $\mathcal{F}$. Therefore $\lambda \geq \kappa^+$.

Now suppose λ were singular, so $F = \bigcup_{i \in I} F_i$ where $\mathrm{Card}(I) < \lambda$ and $\mathrm{Card}(F_i) < \lambda$. By the minimality of λ, we can, for each $i \in I$, choose $g_i \in X$ so that

$$(\forall f \in F_i)(\forall x \mathcal{F}) g_i(x) \geq f(x) \quad .$$

Again by minimality of λ, we can choose $g \in X$ so that

$$(\forall i \in I)(\forall x\mathcal{F})g(x) \geq g_i(x) \quad .$$

Thus,

$$(\forall i \in I)(\forall f \in F_i)(\forall x\mathcal{F}) \quad g(x) \geq g_i(x) \geq f(x) \quad ,$$

so

$$(\forall f \in F)(\forall x\mathcal{F}) \quad g(x) \geq f(x) \quad ,$$

contrary to $(1')$. This proves the claim.

Let $\prec$ be a well-ordering of F, of order type λ. For each $f \in F$, the set of its predecessors has cardinality $< \lambda$, so, by minimality of λ, choose $y_f \in X$ such that

(2) $$(\forall g \prec f)(\forall x\mathcal{F}) \quad y_f(x) \geq g(x) \quad .$$

Note that, if one function y were y_f for arbitrarily large f's (in the ordering $\prec$), then (2) would imply

$$(\forall g \in F)(\forall x\mathcal{F}) \quad y(x) \geq g(x) \quad ,$$

contrary to $(1')$. It follows that each y is of the form y_f for only a bounded set of f's. By regularity of λ, the set

$$Y = \{y_f \mid f \in F\}$$

has cardinality λ .

Any ultrafilter of size λ is isomorphic to a uniform ultrafilter on Y , so to prove the theorem we must show that no uniform ultrafilter on Y omits κ .

Claim: Any uniform ultrafilter D on Y contains a decreasing chain of sets, of length κ , with intersection $\notin D$.

Proof of claim: For each β , $\eta < \kappa$, let

$$A^{\beta}_{\eta} = \{y \in Y \mid y(\beta) \geq \eta\} \quad .$$

If we fix β , then $\{A^{\beta}_{\eta} \mid \eta < \kappa\}$ is a decreasing chain, of length κ , with empty intersection. So, if $\exists \beta \, \forall \eta \, A^{\beta}_{\eta} \in D$, then the claim is true. Suppose, however, that this is not the case. Then, for each $\beta < \kappa$, let $h(\beta) < \kappa$ be such that $A^{\beta}_{h(\beta)} \notin D$. By $(1')$, we can pick $g \in F$ such that $(\exists x \mathfrak{J}) h(x) < g(x)$. But

$$h(x) < g(x) \Rightarrow A^{x}_{g(x)} \subseteq A^{x}_{h(x)} \notin D$$

$$\Rightarrow \{y \in Y \mid y(x) < g(x)\} = Y - A^{x}_{g(x)} \in D$$

$$\Rightarrow B_x = \{y \in Y \mid (\exists \xi \geq x) y(\xi) < g(\xi)\} \in D \quad .$$

Thus, $(\exists x \mathfrak{J}) B_x \in D$. Since the B_x form a decreasing chain, it follows that $(\forall x \in \kappa) B_x \in D$. Thus, we have a chain $\{B_x \mid x \in \kappa\}$ in D, of order type κ, and we need only show $\bigcap_{x\in\kappa} B_x \notin D$. But

$$\bigcap_{x\in\kappa} B_x = \{y \in Y \mid (\exists \xi \in \mathcal{F}) y(\xi) < g(\xi)\} \quad \text{(by definition of } \mathcal{F}\text{)}$$

$$\subseteq \{y_f \mid f \in F \text{ and } f \preceq g\} \quad \text{(by (2))},$$

and this set has cardinality $< \lambda$. As D is uniform, $\bigcap_{x\in\kappa} B_x \notin D$, and the claim is proved.

Let $\{A_\alpha \mid \alpha < \kappa\}$ be a decreasing chain in D with $\bigcap_{\alpha<\kappa} A_\alpha \notin D$. As usual, we replace A_α by $A_\alpha - \bigcap_{\alpha<\kappa} A_\alpha$, and henceforth assume $\bigcap_{\alpha<\kappa} A_\alpha = \emptyset$. Now define, for each $y \in Y$,

$$f(y) = \mu\alpha (y \notin A_\alpha)$$

so $f : Y \to \kappa$.

<u>Claim:</u> If $A \in D$, then $f(A)$ is an unbounded subset of κ.

<u>Proof of claim</u>: Suppose not. Say, for all $y \in A$, $f(y) < \alpha < \kappa$. Then, for all $y \in A$, $A_\alpha \subseteq A_{f(y)}$, and, as $y \notin A_{f(y)}$, $y \notin A_\alpha$. Thus A and A_α are disjoint, contradicting the fact that they are in D.

Therefore, $f(D)$ is a uniform ultrafilter on κ, and D does not omit κ. □

CHAPTER II

INDUCTIVE CONSTRUCTIONS

§7. The filter reduction hypothesis. When one tries to prove the existence of ultrafilters having certain special properties, one often finds that the necessary constructions can be carried out if one assumes GCH , but apparently not if one only uses ZFC . Hence, many existence theorems in the theory of ultrafilters have GCH , or some special case of GCH , as a hypothesis. As typical examples we cite the following two well-known theorems.

THEOREM 1 (Keisler [7]). *If $2^{\kappa} = \kappa^{+}$ then there is a κ^{+}-good ultrafilter on κ .*

DEFINITION 2. *An ultrafilter D of size ω is a P-point if and only if, for every morphism $[f]_D$ of D into a non-principal ultrafilter, there is a set $A \in D$ such that $f \restriction A$ is finite-to-one.*

THEOREM 3 (Rudin [14]). *Assuming CH , there is a P-point.*

Unfortunately, there seems to be no convincing reason for believing GCH , so it is desirable to find weaker hypotheses which suffice to prove these and other theorems. In his thesis [2], Booth finds it possible to replace CH in many theorems by a proposition called Martin's axiom, which we will not state here, because it is complicated and we shall not need it. A theorem of Solovay (cited in [2]) asserts that Martin's axiom is strictly weaker than CH. (Since he considers only ultrafilters of size $\leq \omega$, Booth never needs GCH for larger cardinals.) We shall find it convenient to use the following substitute for GCH.

DEFINITION 4. FRH(κ) (= "filter reduction hypothesis for κ") is the following statement. If a uniform filter F on κ has a basis of cardinality $< 2^{\kappa}$, then there is a uniform filter $F' \supseteq F$ having a basis of cardinality $\leq \kappa$.

Remark 5. It is obvious that $2^{\kappa} = \kappa^{+} \Rightarrow$ FRH(κ). One also sees easily that FRH(ω) is equivalent to the following statement P_0: If a uniform filter F on ω has a basis of cardinality $< 2^{\omega}$, then there is an infinite $B \subseteq \omega$ such that, for all $A \in F$, $B - A$ is finite. It is known (see [2, Theorem 3.5]) that P_0 follows from Martin's axiom. Thus, at least for $\kappa = \omega$, FRH(κ) is strictly weaker than $2^{\kappa} = \kappa^{+}$. On the other hand, Kunen has obtained a model of ZFC in which CH is false but there is a uniform ultrafilter on ω with a basis of cardinality

$\aleph_1$. Since no uniform ultrafilter on ω can have a countable basis, FRH(ω) must be false in this model. Thus, FRH(ω) is not a theorem of ZFC (if ZFC is consistent).

Most proofs using FRH (or GCH) construct the desired ultrafilters by transfinite induction (see for example Keisler's and Rudin's proofs of the theorems quoted above). To avoid repeating the same ideas in many proofs, we will prove one very general theorem which isolates these ideas, and then, whenever a proof would require the same ideas, we can appeal instead to the general theorem. This theorem is perhaps best stated in topological language. It then closely resembles the Baire category theorem. We therefore turn now to the definition of the relevant topologies.

DEFINITION 6. Let X be any infinite set. We define βX to be the set of all ultrafilters on X , and we consider the following two topologies on βX . The standard topology has as its basic open sets all sets of the form

$$\hat{A} = \{D \in \beta X \mid A \in D\}$$

where $A \subseteq X$. The fine topology has as its basic open sets all sets of the form $\bigcap_{A \in \mathcal{C}} \hat{A}$ where $\mathcal{C} \subseteq P(X)$ and $\mathrm{Card}(\mathcal{C}) \leq \mathrm{Card}(X)$. When we speak of βX as a topological space without specifying the topology, we mean the standard topology. Let unif(X) be the set of all uniform ultrafilters on X . As a subset of βX , it also has a standard and a fine topology, but, when we refer to it as a space without specifying the topology, we mean the fine topology.

Remark 7. βX is the Stone-Cech compactification of X with the discrete topology. It is also the Stone space of the Boolean algebra P(X) . In particular, it is a totally disconnected compact Hausdorff space. The fine topology is strictly finer than the standard topology, because the set of principal ultrafilters is closed in the fine topology but not closed (dense, in fact) in the standard topology. When discussing unif(X) , we shall use $\hat{A}$ $(A \subseteq X)$ to mean $\hat{A} \cap \text{unif}(X)$; this should not cause any confusion. Observe that the basic open set $\bigcap_{A \in \mathcal{A}} \hat{A}$ in unif(X) is nonempty if and only if every finite subfamily of $\mathcal{A}$ has intersection of cardinality Card(X) .

THEOREM 8 ("Baire category"). Assume FRH(κ) . Then, in unif(κ) , any intersection of 2^κ or fewer dense open sets is dense.

Proof: Let U_α $(\alpha < 2^\kappa)$ be dense open subsets of unif(κ) , say

$$U_\alpha = \bigcup_{i \in I_\alpha} \bigcap_{B \in \mathcal{B}_{\alpha,i}} \hat{B} \quad ,$$

and let V be any nonempty basic open set in unif(κ) , say

$$V = \bigcap_{A \in \mathcal{A}} \hat{A} \quad .$$

(Here $\mathcal{B}_{\alpha, i}$ and $\mathcal{A}$ are subsets of $P(\kappa)$ of cardinality $\leq \kappa$.) We must show $V \cap \bigcap_{\alpha<2^\kappa} U_\alpha \neq \emptyset$. Since $V \neq \emptyset$, the filter F_0 generated by $\mathcal{A}$ is uniform on κ and has a basis (consisting of finite intersections of sets in $\mathcal{A}$) of cardinality $\leq \kappa$.

By induction of $\alpha < 2^\kappa$ we define an increasing sequence of uniform filters F_α on κ with bases of cardinality $\leq \kappa$. F_0 is already defined. If α is a limit ordinal and F_β is defined for $\beta < \alpha$, then $F = \bigcup_{\beta<\alpha} F_\beta$ is uniform and has a basis (namely the union of the bases of F_β of cardinality $\leq \kappa$) of cardinality $\leq \mathrm{Card}(\kappa \times \alpha) < 2^\kappa$. By FRH(κ) , F is contained in a uniform filter F' with a basis of cardinality $\leq \kappa$. Let $F_\alpha = F'$. Now suppose $\alpha = \beta + 1$ and F_β is already defined. Let $\mathcal{C}_\beta$ be a basis for F_β of cardinality $\leq \kappa$. Then

$$W = \bigcap_{C \in F_\beta} \hat{C} = \bigcap_{C \in \mathcal{C}_\beta} \hat{C} = \{D \in \mathrm{unif}(\kappa) \mid F_\beta \subseteq D\}$$

is a nonempty basic open set in $\mathrm{unif}(\kappa)$. As U_β is dense, there is a $D \in W \cap U_\beta$. As

$$U_\beta = \bigcup_{i \in I_\beta} \bigcap_{B \in \mathcal{B}_{\beta, i}} \hat{B} \quad .$$

there is an $i \in I_\beta$ such that

$$D \in W \cap \bigcap_{B \in \mathcal{B}_{\beta, i}} \hat{B} \quad .$$

Then the filter F generated by $F_\beta \cup \mathcal{B}_{\beta, i}$ is contained in D, so it is uniform, and it has a basis (consisting of sets of the form $C \cap B$ with $C \in \mathcal{C}_\beta$, $B \in \mathcal{B}_{\beta, i}$) of cardinality $\leq \kappa$. Let

$$F_\alpha = F_{\beta+1} = F \quad .$$

Now the filter $\bigcup_{\alpha < 2^\kappa} F_\alpha$ is uniform, so let D be a uniform ultrafilter containing it. Then $F_0 \subseteq D$, so $\mathcal{A} \subseteq D$, so

$$D \in \bigcap_{A \in \mathcal{A}} \hat{A} = V \quad .$$

Also, for each $\alpha < 2^\kappa$, $F_{\alpha+1} \subseteq D$, so, for some $i \in I_\alpha$, $\mathcal{B}_{\alpha, i} \subseteq D$ (by definition of $F_{\alpha+1}$), so

$$D \in \bigcap_{B \in \mathcal{B}_{\alpha, i}} \hat{B} \subseteq U_\alpha \quad .$$

Therefore,

$$V \cap \bigcap_{\alpha < 2^\kappa} U_\alpha \neq \emptyset \quad . \square$$

DEFINITION 9. A subset of unif(κ) is meager if and only if it is contained in the union of a family of 2^κ or fewer nowhere dense closed sets. A subset is comeager if and only if its complement (in unif(κ))

is meager.

Remark 10. This terminology will not cause any confusion, because we shall never use the words meager and comeager in their ordinary sense (with ω in place of 2^κ in the definition). Clearly, a set is comeager if and only if it contains the intersection of a family of 2^κ or fewer open dense sets. Assuming FRH(κ), the "Baire category" theorem shows that comeager sets are dense; in particular they are nonempty. The comeager sets thus form a $(2^\kappa)^+$-complete filter on unif(κ). One should think of comeager sets as being large and meager sets as being small. The next proposition, a refinement of the category theorem, shows that comeager sets are also large in the sense of cardinality.

PROPOSITION 11. Assume FRH(κ). Every comeager set in unif(κ) has cardinality 2^{2^κ}. In fact, the intersection of any comeager set and any nonempty open set has cardinality 2^{2^κ}.

Proof: We first remark that a uniform filter F on κ which has a basis of cardinality $\leq \kappa$ cannot be an ultrafilter. Indeed, let $\mathcal{B} = \{B_i \mid i < \kappa\}$ be such a basis for F, and choose inductively, for each $i < \kappa$, two distinct elements $x_i, y_i \in B_i$ such that

$$x_i, y_i \notin \{x_j \mid j < i\} \cup \{y_j \mid j < i\} \quad .$$

This is possible because

$$\mathrm{Card}\{x_j, y_j \mid j < i\} < \kappa = \mathrm{Card}(B_i) .$$

Then $X = \{x_i \mid i < \kappa\}$ and $Y = \{y_i \mid i < \kappa\}$ are disjoint sets, each meeting every B_i, hence also every set in F. In fact, if we choose the enumeration $\{B_i \mid i < \kappa\}$ so that each $B \in \mathfrak{B}$ is B_i for κ distinct values of i, we can arrange that the filters $F^{(1)}$ and $F^{(2)}$, generated by $F \cup \{X\}$ and $F \cup \{Y\}$ respectively, are uniform.

Thus, for all uniform filters F on κ with a basis of cardinality $\leq \kappa$, we have two other such filters $F^{(1)}$ and $F^{(2)}$, containing F, and not both contained in any ultrafilter D. Suppose for each F definite $F^{(1)}$ and $F^{(2)}$ have been selected.

Now let $f : 2^{\kappa} \to \{1, 2\}$ be any function. In the proof of the Baire category theorem, change the inductive conditions defining F_α as follows. For each α, let G_α be defined from the F_β, $\beta < \alpha$ exactly as F_α was defined before, but then let $F_\alpha = G_\alpha^{(f(\alpha))}$. Let D^f be the ultrafilter finally obtained in this way. (Whenever any choices had to be made, e.g., the choice of F' in the induction at limit ordinals, we assume that an appropriate choice function is selected once and for all, independently of f.) Then $D^f \in V \cap \bigcap_{\alpha < 2^\kappa} U_\alpha$, and

I claim that $f \neq g \Rightarrow D^f \neq D^g$. Suppose $f \neq g$ and suppose $\alpha < 2^\kappa$ is the first place where they differ; $f(\alpha) \neq g(\alpha)$ but $\beta < \alpha \Rightarrow f(\beta) = g(\beta)$. Then, for $\beta < \alpha$, $F_\beta(f) = F_\beta(g)$, and hence $G_\alpha(f) = G_\alpha(g)$. But then no ultrafilter contains both $F_\alpha(f) = G_\alpha(f)^{(f(\alpha))}$ and $F_\alpha(g) = G_\alpha(g)^{(g(\alpha))}$. Since $D^f \supseteq F_\alpha(f)$ and $D^g \supseteq F_\alpha(g)$, we conclude $D^f \neq D^g$. Hence,

$$\mathrm{Card}\Big(V \cap \bigcap_{\alpha < 2^\kappa} U_\alpha\Big) = 2^{2^\kappa} \quad . \ \square$$

<u>Remark</u> 12. If we did not assume $\mathrm{FRH}(\kappa)$, we could still prove that the intersection of κ^+ or fewer dense open sets in $\mathrm{unif}(\kappa)$ is dense and, in fact, meets every nonempty open set at least $2^{(\kappa^+)}$ times. The proofs are practically identical to the ones we have given.

§8. <u>Some comeager sets.</u> THEOREM 1. <u>The set of</u> κ^+<u>-good ultrafilters on</u> κ <u>is comeager</u>.

<u>Proof</u>: Say that a map $g : P_\omega(\kappa) \to P(\kappa)$ is order-reversing if and only if, for all $F \subseteq F' \in P_\omega(\kappa)$, $g(F) \supseteq g(F')$; say that g is multiplicative if and only if, for all F, $F' \in P_\omega(\kappa)$, $g(F \cup F') = g(F) \cap g(F')$; and say that $h : P_\omega(\kappa) \to P(\kappa)$ is under g if and only if, for all $F \in P_\omega(\kappa)$, $h(F) \subseteq g(F)$. Let

$$U_h = \bigcap_{F \in P_\omega(\kappa)} \widehat{h(F)}$$

in $\mathrm{unif}(\kappa)$, and let

$$V_g = \mathrm{Un}\left\{\widehat{\kappa - g(F)} \,\middle|\, F \in P_\omega(\kappa)\right\}$$

$$\cup\ \mathrm{Un}\left\{U_h \,\middle|\, h \text{ is multiplicative and under } g\right\} .$$

Then U_h is a basic open set, and V_g is open in $\mathrm{unif}(\kappa)$. Furthermore, the main lemma in Keisler's proof of Theorem 7.1 [7, Lemma 4C] easily implies that V_g is dense for all order-reversing g. Hence,

$$G = \bigcap_{g \text{ order-reversing}} V_g$$

is comeager, because there are only 2^κ functions $P_\omega(\kappa) \to P(\kappa)$. Now let $D \in G$ and suppose g is an order-reversing map $P_\omega(\kappa) \to D$.

Then, for $F \in P_\omega(\kappa)$, $D \notin \widehat{\kappa - g(F)}$, so, for some multiplicative h under g, $D \in U_h$ (by definition of V_g). This means $h : P_\omega(\kappa) \to D$ (by definition of U_h). Therefore, by Section 1, D is κ^+-good. The set of κ^+-good ultrafilters on κ contains the comeager set G. □

COROLLARY 2. If FRH(κ), then there are 2^{2^κ} κ^+-good ultrafilters on κ. □

THEOREM 3. The set of uniform ultrafilters D on κ, such that, for every $f : \kappa \to \kappa$, there is an $A \in D$ with either $\mathrm{Card}(f(A)) < \kappa$ or $f \restriction A$ one-to-one, is comeager.

Proof: For every $f : \kappa \to \kappa$, let

$$V_f = \mathrm{Un}\left\{\hat{A} \subseteq \mathrm{unif}(\kappa) \mid \mathrm{Card}\ f(A) < \kappa \text{ or } f \restriction A \text{ is one-to-one}\right\} .$$

As each $\hat{A}$ is open, V_f is open. I claim V_f is dense. Let $U = \bigcap_{B \in \mathcal{B}} \hat{B}$ be a nonempty basic open set, where $\mathrm{Card}(\mathcal{B}) \leq \kappa$. Let F be the filter generated by $\mathcal{B}$. It has a basis (consisting of finite intersections of sets in $\mathcal{B}$) of cardinality $\leq \kappa$, say $\{C_i \mid i < \kappa\}$, and it is uniform on κ because $U \neq \emptyset$. We must find a $D \in V_f$ such that $D \in U$, i.e., such that $F \subseteq D$. If, for some $i < \kappa$, $\mathrm{Card}\ f(C_i) < \kappa$, then any D containing F satisfies $D \in \hat{C}_i \subseteq V_f$, and we are done. So suppose, for all i, $\mathrm{Card}\ f(C_i) = \kappa$. Choose, by induction on i, elements $x_i \in C_i$ such that

$$f(x_i) \notin \{f(x_j) \mid j < i\} \quad ;$$

this is possible because

$$\mathrm{Card}\{f(x_j) \mid j < i\} \leq \mathrm{Card}(i) < \kappa = \mathrm{Card}\, f(C_i) \quad .$$

Then, if $X = \{x_i \mid i < \kappa\}$, $f \restriction X$ is obviously one-to-one. Furthermore, by choosing the enumeration $\{C_i \mid i < \kappa\}$ so that each C_i is also C_j for κ different values of j, we can ensure that $\mathrm{Card}(X \cap C_i) = \kappa$ for all $i < \kappa$. Hence, there is a uniform ultrafilter $D \supseteq F \cup \{X\}$. So $D \in U$, and $D \in \hat{X} \subseteq V_f$.

Therefore V_f is dense and $\bigcap_{f:\kappa\to\kappa} V_f$ is comeager. But this set is precisely the set asserted to be comeager in the theorem. □

To show why the ultrafilters considered in this theorem are of interest, we prove the following. (See also Section 10.)

PROPOSITION 4. Let D be a uniform ultrafilter on κ. The following are equivalent.

(1) For every $f : \kappa \to \kappa$, there is an $A \in D$ such that $\mathrm{Card}(f(A)) < \kappa$ or $f \restriction A$ is one-to-one.

(2) $\overline{D}$ <u>is a minimal element of</u> $RK(\kappa)$.

<u>Proof</u>: Statement (2) says that, for any map f of κ into any set, either $\overline{f(D)} \notin RK(\kappa)$ or $\overline{f(D)} = \overline{D}$, i.e., either size $f(D) < \kappa$ or $f(D) \cong D$. The former possibility means that, for some $A \in D$, Card $f(A) < \kappa$ (see Lemma 2.2(1)); the latter possibility means that, for some $A \in D$, $f \restriction A$ is one-to-one (see Corollary 2.6 and Proposition 2.7). Since it is clearly no loss of generality to assume $X = \kappa$ (if necessary, compose f with an injection $f(\kappa) \to \kappa$) , (2) is equivalent to (1) . □

<u>Remark</u> 5. We shall sometimes refer to D , rather than $\overline{D}$, as minimal in $RK(\kappa)$. We shall also speak of minimal elements of RK ; we mean minimal elements of $RK - \{\overline{0}\}$. (Recall that $\overline{0}$ is the least element of RK .) A non-principal ultrafilter D is minimal if and only if every function on $Un(D)$ is constant or one-to-one on a set in D . (The proof is like that of the last proposition.) Notice that, by Proposition 4.4, if D is minimal, it is size(D)-complete, so size(D) is either ω or a measurable cardinal. Any ultrafilter minimal in $RK(\omega)$ is minimal (clearly), but for measurable κ there may exist ultrafilters minimal in $RK(\kappa)$ but not $\aleph_1$-complete (see Corollary 8 below), hence surely not minimal.

COROLLARY 6. _The set of ultrafilters on_ κ _minimal in_ $RK(\kappa)$ _is comeager._ □

COROLLARY 7. _The set of_ κ^+_-good ultrafilters on_ κ _which are minimal in_ $RK(\kappa)$ _is comeager._ □

COROLLARY 8. _Assume_ $FRH(\kappa)$. _There are_ 2^{2^κ} κ^+_-good (hence_ κ_-regular and countably incomplete) ultrafilters on_ κ _which are minimal in_ $RK(\kappa)$. $RK(\kappa)$ _has_ 2^{2^κ} _distinct (as equivalence classes) minimal elements consisting of_ κ^+_-good ultrafilters._

Proof: For the last assertion, recall that each equivalence class has at most 2^κ elements. □

COROLLARY 9. _Assume_ $FRH(\omega)$. $RK(\omega)$ _has exactly_ 2^{2^ω} _minimal elements. There are_ 2^{2^ω} _P-points on_ ω. □

Proof: For the last assertion, observe that any ultrafilter minimal in $RK(\omega)$ is clearly a P-point. □

Note the contrast between the present results, which say that "most" uniform ultrafilters on κ are minimal in $RK(\kappa)$, and 5.8(3), 5.10, 5.11, which say that there are a great many non-minimal (in fact very far from minimal) uniform ultrafilters on κ.

§9. P-points. In the preceding section, the existence of P-points was obtained as an immediate corollary of the existence of minimal ultrafilters in $RK(\omega)$. For all we have shown so far, it might be that all P-points are minimal, or perhaps that all uniform ultrafilters on countable sets are P-points. The latter possibility is easily disposed of by means of the following counter-example. On $\omega \times \omega$, the sets

$$A(f, n) = \{(x, y) \mid x \geq n \text{ and } y > f(x)\},$$

for $n < \omega$ and $f : \omega \to \omega$, form a filterbase $\mathcal{B}$. If $\pi : \omega \times \omega \to \omega$ is projection to the first factor, then any set B on which π is constant, say $B \subseteq \{a\} \times \omega$, is disjoint from $A(f, a+1)$ for arbitrary f, and any set B on which π is finite-to-one is disjoint from $A(f, 0)$ where $f(x) = \max\{y \mid (x, y) \in B\}$. Hence, no ultrafilter containing $\mathcal{B}$ can be a P-point. (Another proof, using topological methods, is in Rudin [14].) The possibility that all P-points are minimal has also been disproved, assuming CH, by Booth [2, Theorem 1.11]. The existence of non-minimal P-points will also follow from the main results of this section.

We begin with a proposition whose main purpose is to justify the name P-point; see Gillman-Jerison [6, Exercise 4L].

PROPOSITION 1. Let $D \in \beta\omega - \omega$. D is a P-point if and only if, in $\beta\omega - \omega$, every G_δ-set containing D is a neighborhood of D.

Proof: We remind the reader that the use of the notation $\beta\omega - \omega$, rather than $\text{unif}(\omega)$, means that we are using the standard (Stone-Čech) topology.

Suppose D is a P-point and A is a G_δ-set, say $A = \bigcap_{i<\omega} A_i$, containing D. For each $i < \omega$, choose a basic open set $\hat{G}_i$ such that $D \in \hat{G}_i \subseteq A_i$; $G_i \subseteq \omega$. Let

$$f : \omega \longrightarrow \omega + 1 : n \longrightarrow \begin{cases} \mu i(n \notin G_i) & \text{if } n \notin \bigcap_{i<\omega} G_i \\ \omega & \text{if } n \in \bigcap_{i<\omega} G_i \end{cases} .$$

As D is a P-point, f is finite-to-one or constant on some $B \in D$. If $B \subseteq f^{-1}(i)$ for some $i < \omega$, then B and G_i are disjoint sets in D, a contradiction. If

$$B \subseteq f^{-1}(\omega) = \bigcap_{i<\omega} G_i ,$$

then

$$(1) \qquad D \in \hat{B} \subseteq \bigcap_{i<\omega} \hat{G}_i \subseteq A ,$$

and A is a neighborhood of D. If f is finite-to-one on B, then, for each i ,

$$B - G_i = \{n \in B \mid n \notin G_i\} \subseteq B \cap f^{-1}\{0,1,\cdots,i\}$$

is finite, so any uniform ultrafilter which contains B also contains G_i . Therefore (1) holds again, and A is a neighborhood of D in $\beta\omega - \omega$.

Conversely, suppose any G_δ-set containing D is a neighborhood of D , and suppose $f : \omega \to \omega$ is not constant on any set of D . Then, for each $n \in \omega$,

$$A_n = \{k \mid f(k) \geq n\} \in D \quad .$$

So $D \in \bigcap_{n\in\omega} \hat{A}_n$, and, by assumption, there is a set $B \subseteq \omega$ such that

$$D \in \hat{B} \subseteq \bigcap_{n\in\omega} \hat{A}_n \quad .$$

Thus, $B \in D$, and, for each n , every uniform ultrafilter containing B also contains A_n . Hence $B - A_n$ is finite, and f is finite-to-one on B . Therefore, D is a P-point. $\square$

We now turn to the construction of P-points with further special properties (including non-minimality). These constructions are by transfinite induction, but they are a bit more subtle than the construction summarized by the Baire category theorem. It is, of course, clear that we cannot obtain non-minimal ultrafilters by a direct application of the Baire category theorem, for the set of non-minimal ultrafilters on ω is meager. We shall need the following lemma for the construction of special P-points.

LEMMA 2. Assume CH. Suppose C is a nonempty closed subset of $\beta\omega - \omega$ with the property that, whenever a G_δ-set meets C, its interior also meets C. Then C contains a P-point. (It suffices to consider G_δ-sets of the form $\bigcap_{B\in\mathcal{B}}\hat{B}$ with $\mathcal{B}$ countable.)

Proof: The number of G_δ-sets of the form $\bigcap_{B\in\mathcal{B}}\hat{B}$ with $\mathcal{B}$ countable is $2^\omega = \omega^+$; let $\{X_i \mid i < \omega^+\}$ be the set of all such G_δ-sets. We define inductively nonempty closed sets $C_i = C \cap \hat{B}_i$ for certain $B_i \subseteq \omega$, such that $i < j \Rightarrow C_i \supseteq C_j$. We begin by taking $C_0 = C = C \cap \hat{\omega}$. If α is a limit ordinal $< \omega^+$ and $C_\beta = C \cap \hat{B}_\beta$ is defined for all $\beta < \alpha$, then

$$C \cap \bigcap_{\beta<\alpha} \hat{B}_\beta = \bigcap_{\beta<\alpha} C_\beta \neq \emptyset$$

because it is a nested intersection of nonempty compact sets. By hypothesis, there is a basic open set $\hat{B}_\alpha$ $(B_\alpha \subseteq \omega)$ such that

$$\hat{B}_\alpha \subseteq \bigcap_{\beta<\alpha} \hat{B}_\beta \qquad \text{and} \qquad \hat{B}_\alpha \cap C \neq \emptyset \quad .$$

We define $C_\alpha = C \cap \hat{B}_\alpha$. This is closed (because $\hat{B}_\alpha$ is), nonempty, and $\subseteq C_\beta$ for all $\beta < \alpha$. If α is a successor, $\beta + 1$, $C_\beta = C \cap \hat{B}_\beta$ is already defined, and $X_\beta \cap C_\beta = \emptyset$, let $C_\alpha = C_\beta$. But if

$$X_\beta \cap C_\beta = C \cap (\hat{B}_\beta \cap X_\beta) \neq \emptyset ,$$

then, by hypothesis, we can find $B_\alpha \subset \omega$ such that $C \cap \hat{B}_\alpha \neq \emptyset$ and $\hat{B}_\alpha \subseteq \hat{B}_\beta \cap X_\beta$; let $C_\alpha = C \cap \hat{B}_\alpha$. This completes the definition of the decreasing sequence C_α. By compactness, there is a $D \in \bigcap_{\alpha<\omega^+} C_\alpha$; obviously $D \in C_0 = C$, and I claim that D is a P-point. If X is any G_δ-set containing D, then, for some $i < \omega^+$, $D \in X_i \subseteq X$. (Replace the open sets whose intersection is X by basic open subsets containing D.) Thus $D \in C_i \cap X_i$, and C_{i+1} was defined as $C \cap \hat{B}_{i+1}$, where $\hat{B}_{i+1} \subseteq X_i$. Therefore,

$$D \in C_{i+1} \subseteq \hat{B}_{i+1} \subseteq X_i \subseteq X ,$$

and X is a neighborhood of D, as claimed. □

Restating the lemma in non-topological language, we obtain

COROLLARY 3. <u>Assume</u> CH. <u>Let</u> F <u>be a filter on</u> ω <u>containing all cofinite sets. Assume that, for every decreasing sequence</u> $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ <u>of sets</u> Y_i <u>each of which meets every set in</u> F, *there is a set* S , *meeting every set in* F , *and such that* $S - Y_i$ *is finite for all* $i < \omega$. *Then there is a* P-*point containing* F . □

Obviously the lemma and its corollary apply to all countable sets, not just to ω .

THEOREM 4. *Assume* CH . *For every* P-*point* D , *there is a* P-*point* $E > D$.

Proof: Without loss of generality, assume $Un(D) = \omega$. Let $\pi : \omega \times \omega \to \omega$ be the first projection. For any set $A \subseteq \omega \times \omega$, define $f_A : \omega \to \omega + 1$ by

$$f_A(n) = \mathrm{Card}(A \cap \pi^{-1}(n)) = \mathrm{Card}\{y \mid (n, y) \in A\} .$$

Let F be the family of all sets $A \subseteq \omega \times \omega$ such that $f_{\omega \times \omega - A}$ is bounded by some $n < \omega$ on some set in D . It is trivial that F is a filter on $\omega \times \omega$ containing all cofinite sets.

We shall verify that F satisfies the assumptions of Corollary 3. Let $Y_0 \supseteq Y_1 \supseteq \cdots$ be a sequence of sets such that each Y_i meets every set in F . If we let $f_i = f_{Y_i}$, this assumption means that each of the f_i is not bounded by any $n < \omega$ on any set of D . Let

$$h(k) = (\mu n < k) f_n(k) < n .$$

(For the μ-notation, see Shoenfield [15, p. 112]; $h(k) = k$ if $(\forall n < k) f_n(k) \geq n$.) Thus,

$$(1) \qquad k \geq 1 \Rightarrow h(k) \geq 1 \quad \text{and} \quad f_{h(k)-1}(k) \geq h(k) - 1 \quad .$$

Suppose h were constant on a set of D, say $h^{-1}(a) \in D$. Since D is non-principal,

$$A = \{x \mid x > a \text{ and } h(x) = a\} \in D \quad .$$

The definition of h shows that, for $x \in a$, $f_a(x) < a$, contradicting the fact that f_a is not bounded by any finite number on any set of D. So h is not constant on any set of D, and, because D is a P-point, h is finite-to-one on some set $A \in D$. Without loss of generality, say $0 \notin A$. For each $x \in A$, (1) and the definition of f_n show that there is a set S_x of cardinality $h(x) - 1$ such that

$$\{x\} \times S_x \subseteq \overline{Y_{h(x)-1}} \quad .$$

For $x \notin A$, let $S_x = \emptyset$. Let

$$S = \{(x, y) \mid y \in S_x\} = \bigcup_{x \in A} (\{x\} \times S_x) \quad .$$

I claim that S has the properties required by the hypothesis of Corollary 3.

First, for each n,

$$f_S(n) = \text{Card } S_n = h(n) - 1 \qquad \text{if } n \in A$$

$$= 0 \qquad \text{if } n \notin A \quad .$$

Hence, the set

$$f_S^{-1}\{0,1,\cdots,k-1\} \cap A = \{x \in A \mid h(n) \leq k\}$$

is finite (because h is finite-to-one) and thus not in D. But $A \in D$, so

$$f_S^{-1}\{0,\cdots,k-1\} \notin D \quad .$$

We have shown that f_S is not bounded by any $k < \omega$ on any set of D. Therefore, S meets every set of F.

Second, if $h(x) > n$, then

$$\{x\} \times S_x \subseteq Y_{h(x)-1} \subseteq Y_n \quad ,$$

so

$$S - Y_n \subseteq \left(\bigcup_{x \in A} \{x\} \times S_x\right) - \left(\bigcup_{h(x)>n} \{x\} \times S_x\right)$$

$$= \bigcup_{\substack{x \in A \\ \text{and } h(x) \leq n}} \left(\{x\} \times S_x\right) .$$

Since h is finite-to-one on A and each S_x is finite, $S - Y_n$ is contained in a finite union of finite sets, hence is finite.

Thus, Corollary 3 applies, and there is a P-point $E \supset F$. If $B \in D$, then $f_{\omega\times\omega - \pi^{-1}(B)}$ is identically zero on B, so $\pi^{-1}(B) \in F \subseteq E$. Thus $\pi(E) = D$ and $E \geq D$. Furthermore, if $A \subseteq \omega \times \omega$ is such that $\pi \restriction A$ is one-to-one, then f_A takes only the values 0 and 1, so

$$\omega \times \omega - A \in F \subseteq E \quad ,$$

and $A \notin E$. Since π is not one-to-one on any set of E, $[\pi]_E : E \to D$ is not an isomorphism, by Proposition 2.7. By Corollary 2.6, $D \ngeq E$, so $E > D$. □

COROLLARY 5 (Booth[2]). <u>Assume</u> CH. <u>There are non-minimal P-points</u>. □

COROLLARY 6. <u>Assume</u> CH. <u>There are increasing ω-sequences of P-points. In fact, every P-point is the first term of such a sequence.</u> □

We shall see, in Chapter IV, that the set of P-points is not directed; in fact there are two minimal ultrafilters no common upper bound of which is a P-point (assuming FRH(ω)).

PROPOSITION 7. *If* D *is non-principal,* E *is a* P-*point, and* $D \leq E$, *then* D *is a* P-*point.*

Proof: Size(D) = ω by Proposition 4.1. Suppose f(D) is non-principal, and let D = g(E). Then fg(E) = f(D) is non-principal, so, as E is a P-point, fg is finite-to-one on some set $A \in E$. But then f is finite-to-one on $g(A) \in D$. □

THEOREM 8. *Assume* CH. *There is a set of* P-*points which, with the Rudin-Keisler ordering, is isomorphic to the real line with its usual ordering.*

Proof: We use the usual notations $\mathbb{R}$ and $\mathbb{Q}$ for the sets of real and rational numbers respectively. We must find, for each $\xi \in \mathbb{R}$, a P-point D_ξ such that

$$\xi < \eta \Longrightarrow D_\xi < D_\eta \quad .$$

Let X be the set of functions $x : \mathbb{Q} \to \omega$ such that $x(r) = 0$ for all but finitely many $r \in \mathbb{Q}$. Note that Card(X) = ω. For each $\xi \in \mathbb{R}$, define $f_\xi : X \to X$ by

$$f_\xi(x)(r) = \begin{cases} x(r) & \text{if } r \leq \xi \\ 0 & \text{if } r > \xi \end{cases} \quad .$$

Clearly,

$$f_\xi \circ f_\eta = f_\eta \circ f_\xi = f_{\min(\xi,\eta)} \quad .$$

Eventually, the required ultrafilters D_ξ will be defined to be $f_\xi(D)$ for some particular ultrafilter D on X. Observe that, if $\xi < \eta$, then

$$f_\xi(D_\eta) = f_\xi f_\eta(D) = f_\xi(D) = D_\xi \quad ,$$

so

$$[f_\xi]_{D_\eta} : D_\eta \longrightarrow D_\xi \quad .$$

Hence $D_\xi \leq D_\eta$. We must choose D so that in fact $D_\xi < D_\eta$ and so that each D_ξ is a P-point. By Corollary 2.6, the first objective will be accomplished if $[f_\xi]_{D_\eta}$ is not an isomorphism, and, by Proposition 7, the second objective will be accomplished if D itself is a P-point.

We consider first the problem of making sure that $f_\xi : D_\eta \to D_\xi$ is not an isomorphism. What we want is that, for each $g : X \to X$, $g \circ f_\xi \neq \text{id} \bmod D_\eta$. (See Corollary 2.6.) In other words, when $\xi < \eta$,

$$\left\{x \in X \mid g f_\xi(x) \neq x\right\} \in D_\eta = f_\eta(D) \quad ,$$

or

$$\{x \in X \mid gf_\xi(x) \neq f_\eta(x)\} = \{x \in X \mid gf_\xi(f_\eta(x)) \neq f_\eta(x)\}$$

$$= f_\eta^{-1}\{x \in X \mid gf_\xi(x) \neq x\}$$

$$\in D \quad .$$

Let

$$B(g, \xi, \eta) = \{x \in X \mid gf_\xi(x) \neq f_\eta(x)\}$$

for any $g : X \to X$ and any $\xi < \eta \in \mathbb{R}$. We have just seen that, in order that $D_\xi < D_\eta$ whenever $\xi < \eta$, we must have

$$B(g, \xi, \eta) \in D \qquad \text{for all} \quad g, \xi, \eta \quad .$$

Hence we will surely want to know

LEMMA 9. The family

$$\{B(g, \xi, \eta) \mid g : X \longrightarrow X \;\; ; \;\; \xi < \eta \in \mathbb{R}\}$$

has the finite intersection property.

Proof: We first observe that, if $\xi \leq \xi' < \eta' \leq \eta$, then

$$B(f_{\eta'} gf_\xi, \xi', \eta') \subseteq B(g, \xi, \eta) \quad .$$

For

$$x \notin B(g,\xi,\eta) \implies f_\eta(x) = gf_\xi(x)$$

$$\implies f_{\eta'}(x) = f_{\eta'}f_\eta(x) = f_{\eta'}gf_\xi(x) = f_{\eta'}gf_\xi f_{\xi'}(x)$$

$$\implies x \notin B(f_{\eta'}gf_\xi, \xi'; \eta') \quad .$$

Now consider a finite intersection $\bigcap_{i=1}^{n} B(g_i, \xi_i, \eta_i)$. By the observation just made, this set contains another of the same form but with the intervals $[\xi_i, \eta_i]$ disjoint. By renumbering, we may suppose

$$\xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdot \cdot \cdot < \xi_n < \eta_n \quad .$$

For each i , let r_i be a rational number such that $\xi_i < r_i < \eta_i$. We define a function $x : \mathbb{Q} \to \omega$ as follows. First, $x(r) = 0$ for all values of r except $r_1, \cdots, r_n$. $x(r_i)$ is defined by induction on i , so suppose $x(r_j)$ is already defined for $j < i$. Then $f_{\xi_i}(x)$ is already determined. Choose $x(r_i)$ to be any number different from $g_i f_{\xi_i}(x)(r_i)$. Then $f_{\eta_i}(x)$ and $g_i f_{\xi_i}(x)$ have different values at r_i , so

$$x \in \bigcap_{i=1}^{n} B(g_i, \xi_i, \eta_i) \quad .$$

This completes the proof of the lemma. □

Before continuing with the proof of the theorem, we remark that what we have already done suffices to prove (without CH)

COROLLARY 10. There is a subset of $RK(\omega)$ order-isomorphic to the real line. □

COROLLARY 11. There is a subset of $RK(\omega)$, order-isomorphic to the real line, above any prescribed element of $RK(\omega)$.

Proof: Let E be any prescribed ultrafilter on ω. Adjoin $-\infty$ to $\mathbb{Q}$ with $-\infty < r$ for all rational r; call the result $\mathbb{Q}^*$, and let $\mathbb{R}^*$ be similarly defined. Define X^* and $B^*(g, \xi, \eta)$ as before (ξ may now be $-\infty$). For each $A \in E$, let $A' \subseteq X^*$ be $\{x \in X \mid x(-\infty) \in A\}$. A trivial modification of Lemma 9 shows that

$$\left\{B^*(g, \xi, \eta) \mid g : X^* \longrightarrow X^* \text{ and } \xi < \eta \in \mathbb{R}^*\right\} \cup \left\{A' \mid A \in E\right\}$$

has the finite intersection property. If D is an ultrafilter containing this family, $D_\xi = f_\xi(D)$ gives the required chain above E, for

$$E = f_{-\infty}(D) = f_{-\infty}(D_\xi) \quad . \ \square$$

COROLLARY 12. There is a subset of $RK(\omega)$, order-isomorphic to the long line, above any prescribed element of $RK(\omega)$.

Proof: Use Corollary 11 and Proposition 5.10. □

Returning to the theorem, let F be the filter generated by the sets $B(g, \xi, \eta)$. It has a basis consisting of finite intersections $\bigcap_{i=1}^{n} B(g_i, \xi_i, \eta_i)$ where, as in the proof of the lemma, we may assume

$$\xi_1 < \eta_1 < \cdots < \xi_n < \eta_n$$

and, if we wish, that the ξ_i and η_i are rational. To complete the proof, we must find a P-point $D \supseteq F$. For this we use Corollary 3, whose hypotheses we now intend to verify. F contains all cofinite sets, for otherwise we could find a principal $D \supseteq F$, but then all the D_ξ are principal, contradicting the fact that no two of them are isomorphic. (A more direct proof is clearly also possible.)

Now let $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ be subsets of X each of which meets every set in F. We must find a set $S \subseteq X$ such that S meets every set in F and, for all i, $S - Y_i$ is finite. Let $\sigma_0, \sigma_1, \sigma_2, \cdots$ be an enumeration of all the (countably many) sequences of rationals of the form

$$p_1 < q_1 < p_2 < q_2 < \cdots < p_m < q_m$$

for arbitrary $m < \omega$. Let $\lambda(i)$ be half the number of terms of σ_i

(i.e., m if σ_i is the sequence just exhibited).

Let n be a (temporarily) fixed natural number. Let σ_n be $p_1 < q_1 < \cdots < p_\lambda < q_\lambda$ where $\lambda = \lambda(n)$. For each i such that $1 \leq i \leq \lambda + 1$, we will call certain elements of X i-<u>acceptable</u>. The definition of i-acceptability is by downward induction on i. An element $x \in X$ is $\lambda + 1$-acceptable if and only if $x \in Y_n$. For $1 \leq i \leq \lambda$, $x \in X$ is i-acceptable if and only if there are two $i+1$-acceptable elements, x_0 and x_1, such that $f_{q_i}(x_0) \neq f_{q_i}(x_1)$ but $f_{p_i}(x_0) = f_{p_i}(x_1) = f_{p_i}(x)$. I claim that, for each i such that $1 \leq i \leq \lambda + 1$, the set $\mathrm{Acc}(i)$ of i-acceptable elements of X meets every set in F. This claim is true for $i = \lambda + 1$ because we are assuming that Y_n meets every set in F. We proceed by downward induction on i. Suppose $\mathrm{Acc}(i+1)$ meets every set in F but $\mathrm{Acc}(i)$ does not. Say $\mathrm{Acc}(i)$ is disjoint from $C \in F$. By definition of i-acceptability,

$x \in C \Longrightarrow$ All those $i+1$-acceptable y's which have the same image as x under f_{p_i} have the same image under f_{q_i}.

For each $x \in C$, let $g(x)$ be the image under f_{q_i} of one (hence of every) $i+1$-acceptable $y \in C$ such that $f_{p_i}(y) = f_{p_i}(x)$. Clearly, $g(x)$ depends only on $f_{p_i}(x)$, so let $g(x) = h(f_{p_i}(x))$. Then, for all

$x, y \in X$,

$$x \in C \text{ and } y \in Acc(i+1) \text{ and } f_{p_i}(y) = f_{p_i}(x) \Rightarrow$$

$$f_{q_i}(y) = g(x) = h(f_{p_i}(x)) = h(f_{p_i}(y)) \Rightarrow$$

$$y \notin B(h, p_i, q_i) \quad .$$

In particular, letting $y = x$, we find that

$$Acc(i+1) \cap C \cap B(h, p_i, q_i) = \emptyset \quad ,$$

contrary to the induction hypothesis that $Acc(i+1)$ meets every set in F . This proves the claim.

Thus, there is a 1-acceptable $x \in X$. By definition of acceptability, there are 2-acceptable x_0 and x_1 , 3-acceptable x_{00}, x_{01}, x_{10}, and $x_{11}, \cdots$, $\lambda + 1$-acceptable x_J , where J is a λ-tuple of zeroes and ones, such that $f_{p_k}(x_{\dots})$ depends only on the first $k - 1$ components of $\cdots$, but $f_{q_k}(x_{\dots})$ depends also on the kth component. Let S_n be the set of $2^{\lambda(n)}$ elements x_J of Y_n thus obtained (from a specific $x \in Acc(1)$) .

Now let n no longer be fixed, and define $S = \bigcup_{n<\omega} S_n$. As $S_n \subseteq Y_n$ and the Y_n form a decreasing sequence, $S - Y_i \subseteq \bigcup_{n<i} S_n$, which is finite. All we still have to prove is that S meets every set

in F. By previous observations, it suffices to show that S meets every set of the form $\bigcap_{i=1}^{\lambda} B(g_i, p_i, q_i)$ where $p_i, q_i \in \mathbb{Q}$ and $p_i < q_i < p_{i+1}$. Choose n so that σ_n is $p_1 < q_i < \cdots < p_\lambda < q_\lambda$, so $\lambda(n) = \lambda$. With this particular value of n, we may use the notation of the preceding two paragraphs where a fixed n was considered. In particular, $x_{\cdots}$ is defined, where $\cdots$ is any sequence of λ or fewer zeroes and ones. Choose $j_1 = 0$ or 1 so that $f_{q_1}(x_{j_1}) \neq g_1 f_{p_1}(x)$; this can be done because $f_{q_1}(x_0) \neq f_{q_1}(x_1)$. After $j_1, \cdots, j_{i-1}$ have been chosen (for $2 \leq i \leq \lambda$), choose $j_i = 0$ or 1 so that $f_{q_i}(x_{j_i \ldots j_i}) \neq g_i f_{p_i}(x_{j_i \ldots j_{i-1}})$; this can be done because $f_{q_i}(x_{j_1 \ldots j_{i-1}0}) \neq f_{q_i}(x_{j_1 \ldots j_{i-1}1})$. Then $y = x_{j_1 \ldots j_\lambda}$ satisfies, for all $i (1 \leq i \leq \lambda)$,

$$f_{q_i}(y) = f_{q_i}(x_{j_1 \cdots j_i}) \neq g f_{p_i}(x_{j_1 \cdots j_{i-1}}) = g_i f_{p_i}(y) \quad ;$$

that is,

$$y \in \bigcap_{i=1}^{\lambda} B(g_i, p_i, q_i) \quad .$$

Also, $y \in S_n \subseteq S$. This completes the proof that the hypotheses of Corollary 3 hold and hence also the proof of Theorem 8. □

§10. Minimal ultrafilters. We have remarked (in 8.5) that ultrafilters minimal in RK are characterized by the fact that every function on Un(D) is constant or one-to-one on some set of D, and that, for minimal D, size(D) is either ω or a measurable cardinal. In this section, we collect various (mostly known) facts giving equivalent characterizations of minimality.

DEFINITION 1. If A is a set and $n \in \omega$, $[A]^n$ is the set of all subsets of A of cardinality n. If A is linearly ordered, we identify $[A]^n$ with the subset of A^n consisting of those n-tuples whose components are in strictly increasing order. If $\{P_1, P_2\}$ is a partition of $[A]^n$ (i.e., $P_2 = [A]^n - P_1$), a subset $X \subseteq A$ is homogeneous for $\{P_1, P_2\}$ if and only if $[X]^n \subseteq P_1$ or $[X]^n \subseteq P_2$. A filter F is a Ramsey filter if and only if it is uniform and every partition of $[\text{Un } F]^n$ (for any $n < \omega$) admits a homogeneous set in F.

DEFINITION 2. A uniform ultrafilter D on κ is normal if and only if, for any $f : \kappa \to \kappa$ such that $(\forall x D) f(x) < x$, there is a $\lambda < \kappa$ such that $(\forall x D) f(x) = \lambda$. A uniform ultrafilter D on κ is quasi-normal if and only if, for every map $\Gamma : \kappa \to D$, there is an $A \in D$ such that

$$x, y \in A \text{ and } x < y \implies y \in \Gamma(x) \quad .$$

In the definition of Ramsey filter, the case $n = 0$ is vacuous, and the case $n = 1$ yields

LEMMA 3. Every Ramsey filter is an ultrafilter. □

PROPOSITION 4. Every Ramsey ultrafilter is minimal in RK.

Proof: Let F be a Ramsey ultrafilter, and let f be any function on $Un(F)$. Partition $[Un(F)]^2$ by

$$\{x, y\} \in P_1 \iff f(x) = f(y)$$

$$\{x, y\} \in P_2 \iff f(x) \neq f(y) \quad .$$

Let $X \in F$ be homogeneous for $\{P_1, P_2\}$. Then f is either constant on X (if $[X]^2 \subseteq P_1$) or one-to-one on X (if $[X]^2 \subseteq P_2$). □

PROPOSITION 5. Every quasi-normal ultrafilter D on κ is Ramsey.

Proof: We must show that D contains a homogeneous set for any partition of $[\kappa]^n$. This is clear if $n = 0$ or 1; we proceed by induction on n. Suppose the assertion is true for n (≥ 1), and let $\{P_1, P_2\}$ be a partition of $[\kappa]^{n+1}$. As discussed above, we view $[\kappa]^{n+1}$ as the set of properly ordered $n+1$-tuples from κ. For each $x \in \kappa$ define a partition of $[\kappa]^n$ by setting, for each $y_1 < \cdots < y_n \in \kappa$,

$$(y_1, \cdots, y_n) \in P_1(x) \iff x < y_1 \quad \text{and} \quad (x, y_1, \cdots, y_n) \in P_1$$

$$(y_1, \cdots, y_n) \in P_2(x) \iff x \geq y_1 \quad \text{or} \quad (x, y_1, \cdots, y_n) \in P_2 \quad .$$

By induction hypothesis, there is a $\Gamma(x) \in D$ such that

$$[\Gamma(x)]^n \subseteq P_{i(x)}(x) \qquad \text{where} \quad i : \kappa \longrightarrow 2 \quad .$$

As D is uniform, we may suppose that $y \in \Gamma(x) \Rightarrow y > x$. Then

$$y_1 < \cdots < y_n \in \Gamma(x) \Rightarrow (x, y_1, \cdots, y_n) \in P_{i(x)} \quad .$$

Now let A be as in the definition of quasi-normality, and let $B \in D$ be a set on which i is constant. Then $A \cap B \in D$, and

$$\begin{aligned} x < y_1 < \cdots < y_n \in A \cap B &\Rightarrow y_1 < \cdots < y_n \in \Gamma(x) \quad \text{and} \quad x \in B \\ &\Rightarrow (x, y_1, \cdots, y_n) \in P_i \end{aligned}$$

where i is the constant value of $i(x)$ for $x \in B$. Therefore, $A \cap B$ is the required homogeneous set. □

PROPOSITION 6 (Kunen, see [2]). Every minimal uniform ultrafilter D on κ is quasi-normal.

Proof: Let $\Gamma : \kappa \to D$; we must find an $A \in D$ such that

$$x < y \quad \text{and} \quad x, y \in A \quad \Rightarrow y \in \Gamma(x) \quad .$$

If $\bigcap_{x\in\kappa}\Gamma(x) \in D$, then this intersection can serve as A. So assume from now on that $\bigcap_{x\in\kappa}\Gamma(x) \notin D$. By subtracting the intersection from each $\Gamma(x)$, we may assume without loss of generality that $\bigcap_{x\in\kappa}\Gamma(x) = \emptyset$. Then we can define $f : \kappa \to \kappa$ by

$$f(y) = \mu x (y \notin \Gamma(x)) \quad .$$

As each $\Gamma(x) \in D$, f cannot be constant on any set of D; by minimality, f is one-to-one on a set $B \in D$. For $x < \kappa$, let

$$g(x) = \sup\Big(\{y \in B \mid f(y) \leq x\} \cup \{x+1\}\Big) \quad ;$$

as f is one-to-one on B, the set whose supremum we are taking has cardinality $\leq x + 2$, and, as κ is regular (being ω or measurable), $g(x) < \kappa$. Thus g is a well-defined map $\kappa \to \kappa$. Clearly

(1) $$g(x) > x \quad .$$

(2) $$y \in B \text{ and } y > g(x) \implies f(y) > x$$
$$\implies y \in \Gamma(x) \quad .$$

Define a sequence α_k $(k < \kappa)$ by $\alpha_0 = 0$, $\alpha_{k+1} = g(\alpha_k)$, and $\alpha_k = \bigcup_{j<k}\alpha_j$ for limit k. Then $\alpha_k < \kappa$ by regularity of κ, and $\bigcup_{k<\kappa}\alpha_k = \kappa$. For any $y \in \kappa$, let $h(y)$ be the least k for which

$y \leq \alpha_k$. Any set on which h is constant is bounded (by a suitable α_k), hence is not in D . Therefore, h is one-to-one on some $C \in D$. Since D is an ultrafilter, it contains a set $A \subseteq B \cap C$ such that no two consecutive ordinals are in $h(A)$. Now suppose $x, y \in A$ and $x < y$. As h is one-to-one on A and is obviously monotone, $h(x) < h(y)$. As no two consecutive ordinals are in $h(A)$, $h(x) + 1 < h(y)$. By definition of $h(x)$,

$$x \leq \alpha_{h(x)} \quad ,$$

and, as g is monotone,

$$g(x) \leq g(\alpha_{h(x)}) = \alpha_{h(x)+1} \quad .$$

By definition of $h(y)$, $h(x) + 1 < h(y)$ implies

$$\alpha_{h(x)+1} < y \quad ,$$

so

$$g(x) < y \quad .$$

By (2) , $y \in \Gamma(x)$. Thus A has the properties required in the definition of quasi-normality. □

Summarizing, we have

THEOREM 7. Let D be a uniform ultrafilter on κ. The following are equivalent.

(1) D is minimal in RK.

(2) D is Ramsey.

(3) D is quasi-normal.

As a corollary, we observe that quasi-normality is invariant under isomorphism, which is not clear from the definition, as the ordering of κ was used there.

To relate normal ultrafilters to minimal ones, we cite

PROPOSITION 8. (1) (Scott; see [11]). If D is a uniform κ-complete ultrafilter on $\kappa > \omega$, then there is a normal ultrafilter $\leq$ D on κ.

(2) (see [16]) Normal ultrafilters are Ramsey.

COROLLARY 9. If $\kappa > \omega$, then the list of equivalent conditions in Theorem 7 can be extended to include

(4) D is isomorphic to a normal ultrafilter.

We remark that, unlike quasi-normality, normality is not invariant under isomorphism. In fact at most one ultrafilter in any isomorphism class is normal. We remark also that, in contrast to the case $\kappa = \omega$, when κ is a measurable cardinal the existence of minimal ultrafilters on κ has been proved (Proposition 8) without any special assumptions like CH or FRH(ω).

It is easy to see that, if D is a uniform ultrafilter on κ, then $D \times D$ is not an ultrafilter. In fact, each of the three disjoint sets

$$A = \{(\alpha, \beta) \mid \alpha < \beta\}$$

$$B = \{(\alpha, \beta) \mid \alpha > \beta\}$$

$$\Delta = \{(\alpha, \alpha) \mid \alpha \in \kappa\}$$

in $\kappa \times \kappa$ meets every set of $D \times D$. Therefore, $D \times D$ is contained in at least three distinct ultrafilters, namely any ultrafilters containing $D \times D \cup \{A\}$, $D \times D \cup \{B\}$, $D \times D \cup \{\Delta\}$; furthermore, every ultrafilter containing $D \times D$ must contain one of these sets. Now $D \times D \cup \{\Delta\}$ generates an ultrafilter, namely $\delta(D)$, where $\delta : \kappa \to \kappa \times \kappa$ is the diagonal map $\alpha \to (\alpha, \alpha)$. If $D \times D \cup \{A\}$ (and, symmetrically, $D \times D \cup \{B\}$) generates an ultrafilter too, then there will be exactly three ultrafilters containing $D \times D$; that is, $D \times D$ will be contained in as few ultrafilters as possible. The next proposition tells us when this happens.

PROPOSITION 10. Let D be a uniform ultrafilter on κ . $D \times D$ is contained in at least three ultrafilters on $\kappa \times \kappa$. The number is exactly three if and only if D is minimal.

Proof: In view of the preceding remarks and Theorem 7, it suffices to show that $D \times D \cup \{A\}$ generates an ultrafilter if and only if D is minimal. For $D \times D \cup \{A\}$ to generate an ultrafilter means that, given any partition $\{P_1, P_2\}$ of $A = [\kappa]^2$, there is a set $X \in D$ such that $[X]^2 = X^2 \cap A \subseteq P_1$ or P_2 . This is just the case $n = 2$ of the definition of Ramsey. Hence (Theorem 7), it follows from D being minimal. Conversely, it implies minimality, for only this case $(n = 2)$ was used in the proof of Proposition 4. □

Remark 11. It is known that an uncountable cardinal κ is inaccessible and weakly compact if and only if every partition of $[\kappa]^2$ into two pieces admits a homogeneous set of cardinality κ . Although this condition on κ requires κ to be quite large, it is much weaker than measurability. For example, if κ is measurable and D is a normal ultrafilter on κ , then

$$\{\lambda < \kappa \mid \lambda \text{ inaccessible and weakly compact}\}$$

is in D , hence has cardinality κ . The next proposition shows that an apparently mild additional condition on κ is, in reality, very strong.

PROPOSITION 12. Let κ be an uncountable cardinal, and suppose it is possible to assign to each partition of $[\kappa]^2$ a homogeneous set of cardinality κ in such a way that the collection of these assigned homogeneous sets has the finite intersection property. Then κ is measurable. In fact, the filter F generated by the assigned homogeneous sets is a κ-complete ultrafilter isomorphic to a normal ultrafilter on κ.

Proof: First note that, if $A \subseteq \kappa$, then A or $\kappa - A$ is in F. For we have a partition of $[\kappa]^2$ given by

$$\{\alpha, \beta\} \in P_1 \Longleftrightarrow \min\{\alpha, \beta\} \in A \quad ,$$

and clearly any homogeneous set for this partition is a subset of A or of $\kappa - A$ (except for its last element, but the assigned homogeneous sets have no last element). Thus, F is an ultrafilter. Further, if $\mathrm{Card}(A) < \kappa$, then the homogeneous set assigned to this partition, having cardinality κ, cannot be a subset of A, so $\kappa - A \in F$. Thus, F is uniform. Clearly, F satisfies the case $n = 2$ of the definition of Ramsey filters, and, as in the proof of Proposition 10, this suffices to show that F is minimal. Therefore, F is κ-complete, and, since $\kappa > \omega$, κ is measurable. By Corollary 9, F is isomorphic to a normal ultrafilter on κ. □

CHAPTER III.

ULTRAPOWERS

§ 11. Ultrapowers and morphisms.

DEFINITION 1 : Let $\mathfrak{a}$ be any structure for any language L, and let $[f]_E : E \to D$ be a morphism in $\mathcal{U}$. We define the induced map, $[f]_E^*$ or f^*, from $D\text{-prod}\,|\mathfrak{a}|$ to $E\text{-prod}\,|\mathfrak{a}|$ by $f^*([g]_D) = [g \circ f]_E$, for any $g : Un(D) \to |\mathfrak{a}|$.

LEMMA 2 : (1) $[g \circ f]_E$ depends only on $[g]_D$ and $[f]_E$, so * is well-defined.

(2) f^* is one-to-one.

(3) $id_{Un(D)}{}^* = id_{D\text{-prod}\,|\mathfrak{a}|}$.

(4) If also $[f']_F : F \to E$, then $(f \circ f')^* = f'^* \circ f^*$.

Proof : (3) and (4) are obvious. (1) and (2) follow from parts (5) and (6) of Lemma 2.2. □

PROPOSITION 3 : f^* is an elementary embedding of $D\text{-prod}\,\mathfrak{a}$ into $E\text{-prod}\,\mathfrak{a}$

Proof : Let $\varphi(x_1, \ldots, x_n)$ be a formula of L , all of whose free variables are among $x_1, \ldots, x_n$, and let $[g_1]_D, \ldots, [g_n]_D$ be arbitrary elements of D-prod $|\mathfrak{A}| = |$D-prod $\mathfrak{A}|$.

$$\text{D-prod } \mathfrak{A} \models \varphi([g_1]_D, \ldots, [g_n]_D) \Longleftrightarrow$$

$$\{i \mid \mathfrak{A} \models \varphi(g_1(i), \ldots, g_n(i))\} \in D = f(E) \Longleftrightarrow$$

$$\{j \mid \mathfrak{A} \models \varphi(g_1 f(j), \ldots, g_n f(j))\} = f^{-1}\{i \mid \mathfrak{A} \models \varphi(g_1(i), \ldots, g_n(i))\} \in E \Longleftrightarrow$$

$$\text{E-prod } \mathfrak{A} \models \varphi([g_1 f]_E, \ldots, [g_n f]_E) \Longleftrightarrow$$

$$\text{E-prod } \mathfrak{A} \models \varphi(f^*([g_1]_D), \ldots, f^*([g_n]_D)) . \quad \square$$

It is not in general true that every elementary embedding of D-prod $\mathfrak{A}$ into E-prod $\mathfrak{A}$ is of the form f^*. Trivial counterexamples are obtained by taking $\mathfrak{A}$ finite and $D \not\geq E$. For a less trivial example, assume GCH , and let D and E be non-isomorphic κ^+-good ultrafilters minimal in RK(κ) (see Corollary 8.8), where κ exceeds the cardinalities of $|\mathfrak{A}|$ and L . Then there are no morphisms at all from E to D , yet D-prod $\mathfrak{A}$ and E-prod $\mathfrak{A}$ are isomorphic (see Section 1.) Roughly, elementary embeddings of the form f^* are natural with respect to $\mathfrak{A}$, while the isomorphisms between saturated structures tend to be unnatural, as one sees from the inductive "picking and choosing" argument by which they are obtained. (This heuristic idea can be made precise by defining an appropriate category of models, on which "D-prod" and "E-prod" are functors. Then the natural transformations from D-prod into E-prod are exactly the f^*'s where $f : E \to D$.)

If, however, the structure $\mathfrak{A}$ is "sufficiently rich" (in comparison with D and E) then all elementary embeddings D-prod $\mathfrak{A} \to$ E-prod $\mathfrak{A}$ are of the form f^*. We proceed now to define certain "rich" structures.

DEFINITION 4 : Let A be any set. Let L be the language which has a predicate or function symbol, $\underline{R}$ or $\underline{f}$, for every predicate R or function f on A. The complete structure on A is the structure $\mathfrak{A}$ for L which has universe A and in which $\underline{R}$ denotes R and $\underline{f}$ denotes f for all predicates and functions on A. When we speak of a set as though it were a structure, we mean the complete structure on that set.

Note that every element $a \in A$ has a name $\underline{a}$ (a 0-place function symbol) in the language of the complete structure on A. Therefore, every structure elementarily equivalent to A has an elementary submodel isomorphic to A.

PROPOSITION 5 : Let D and E be ultrafilters, and let $A = Un(D)$. Any elementary embedding

$$e : D\text{-prod } A \to E\text{-prod } A$$

is f^* for some $f : E \to D$. $[f]_E$ is unique.

Proof : The identity map, $id : A \to A$, of $Un(D)$ determines an element $[id]_D$ of D-prod A and thus an element

$$e([id]_D) \in E\text{-prod } A \quad .$$

Let that element be $[f]_E$, where $f : Un(E) \to A = Un(D)$.

For any $B \subseteq A$,

$$\begin{aligned} B \in D &\Longleftrightarrow \{i \mid A \models \underline{B}(i)\} \in D \\ &\Longleftrightarrow D\text{-prod } A \models \underline{B}([id]_D) \\ &\Longleftrightarrow E\text{-prod } A \models \underline{B}([f]_E) \\ &\Longleftrightarrow \{j \mid A \models \underline{B}(f(j))\} \in E \\ &\Longleftrightarrow f^{-1}(B) \in E \quad , \end{aligned}$$

so $D = f(E)$, and $[f]_E$ is a morphism from E to D. We now show that f^* coincides with e. If $[g]_D \in D\text{-prod } A$, then $g : A \to A$, and

$$\{i \mid A \models g(i) = \underline{g}\,(id(i))\} = A \in D \quad ,$$

so

$$D\text{-prod } A \models [g]_D = \underline{g}\,([id]_D) \quad .$$

As e is an elementary embedding,

$$E\text{-prod } A \models e([g]_D) = \underline{g}([f]_E) \quad .$$

If we let $e([g]_D) = [h]_E$, we obtain

$$\{i \mid A \models h(i) = \underline{g}\,(f(i))\} \in E \quad ,$$

so $h = g \circ f \bmod E$, and

$$e([g]_D) = [h]_E = [g \circ f]_E = f^*([g]_D) \quad .$$

Finally, suppose $f' : E \to D$ were another morphism such that $f'^* = e$. Then

$$[f]_E = e([id]_D) = f'^*([id]_D) = [f']_E \quad .$$

Therefore, $[f]_E$ is unique. □

It is easy to modify the proof of this proposition to obtain the same result when A is any set of cardinality $\geq$ size (D) . Observe that, by functoriality of * , an isomorphism of ultrafilters induces isomorphisms of ultraproducts of arbitrary structures. As a partial converse, we observe

COROLLARY 6 : *With* D, E, A *as in the proposition, let* $g : D \to E$ *be such that* g^* *is an isomorphism from* E-prod A *to* D-prod A . *Then* $[g]_D$ *is an isomorphism* .

Proof : By the proposition, $(g^*)^{-1}$ is f^* for some $f : E \to D$. Now apply Corollary 2.6. □

Collecting the preceding results, we obtain the following characterization of the Rudin-Keisler ordering.

PROPOSITION 7 : *Let* D *and* E *be ultrafilters, and let* $\kappa \geq$ size (D) (*resp.*, $\kappa \geq$ size (D) *and* $\kappa \geq$ size (E)). *The following are equivalent.*

(1) $D \leq E$ (*resp.*, $D \cong E$) .

(2) *For all structures* $\mathfrak{A}$, D-prod $\mathfrak{A}$ *can be elementarily embedded in* (*resp.*, *is isomorphic to*) E-prod $\mathfrak{A}$.

(3) D-prod κ *can be elementarily embedded in* (*resp.*, *is isomorphic to*) E-prod κ . □

§ 12. Ultrapowers of ω. In this section, we shall be concerned with ultrapowers of (the complete model on) ω with respect to ultrafilters on ω. In defining the complete model on a set, we used $\underline{R}$ and $\underline{f}$ as the symbols of the language L which denote R and f. This notation is often inconvenient and sometimes (as when R is the binary relation $<$) confusing, so we will often just use R and f as symbols of L. It is also convenient to identify an element a of A with the corresponding element of D-prod A, namely the denotation of $\underline{a}$, which is represented by the function $\mathrm{Un}(D) \rightarrow A$ which is constantly a.

PROPOSITION 1 : Let D be a non-principal ultrafilter on ω. D is minimal if and only if the only proper elementary submodel of D-prod ω is ω.

Proof : If D is not minimal, say $E < D$, E non-principal, then, by the results of the preceding section, E-prod ω is isomorphic to a proper elementary submodel of D-prod ω. Since E has size ω, it cannot be $\aleph_1$-complete, so E-prod ω is not isomorphic to ω.

Conversely, suppose D-prod ω had a proper elementary submodel M different from (hence properly containing) ω. Let $[f]_D \in M - \omega$, $[g]_D \in (\text{D-prod } \omega) - M$, where f and g are maps $\omega \rightarrow \omega$. f cannot be constant on any set of D, for if it were, $[f]_D$ would be in ω.

Suppose f were one-to-one on some set $A \in D$. Then there would be an h ($= g \circ f^{-1}$ on $f(A)$) such that $g = h \circ f$ on A. But then, in D-prod ω,

$$[g]_D = [h \circ f]_D = \underline{h}([f]_D) \quad .$$

But $[f]_D \in M$, and M is closed under the function denoted by $\underline{h}$ (since M is an elementary submodel), so $[g]_D \in M$, a contradiction. Therefore, f is neither constant nor one-to-one on any set of D, so D is not minimal. □

COROLLARY 2 : Assume CH (or only FRH(ω). Then the complete model on ω has a proper elementary extension ω' such that no proper elementary extension of ω is a proper elementary submodel of ω'. (ω' is a minimal proper elementary extension of ω.) In fact, there are $2^{2^{\omega}}$ pairwise non-isomorphic such extensions ω'.

Proof : Use the preceding proposition, Corollary 8. 9, and Proposition 11. 7. □

It is true that every minimal proper elementary extension of ω is isomorphic to D-prod ω for some minimal ultrafilter on ω. This fact follows immediately from the following

PROPOSITION 3 : Every proper elementary extension of (the complete model on) a set A contains an elementary submodel isomorphic to D-prod A for some non-principal D on A. In fact, the extension is the union of all such submodels .

Proof : Let A' be a proper elementary extension of A , and let $a \in A'$; we must show that a is in an elementary submodel of A' isomorphic to D-prod A for some D. (If $a \notin A$, then D will clearly have to be non-principal.) We let D be defined by

$$B \in D \Longleftrightarrow A' \models \underline{B}\,(a)$$

for any $B \subseteq A$. First, we must check that D is an ultrafilter. For any $B_1, B_2 \subseteq A$,

$$A \models \forall x\,(\underline{B_1 \cap B_2}(x) \Longleftrightarrow \underline{B_1}(x) \text{ and } \underline{B_2}(x)),$$

so A' satisfies the same sentence, and

$$\begin{aligned} B_1 \cap B_2 \in D &\Longleftrightarrow A' \models \underline{B_1 \cap B_2}(a) \\ &\Longleftrightarrow A' \models \underline{B_1}\,(a) \text{ and } A' \models \underline{B_2}(a) \\ &\Longleftrightarrow B_1 \in D \text{ and } B_2 \in D\ . \end{aligned}$$

Similarly,

$$B \in D \Longleftrightarrow A - B \notin D \quad .$$

Next, we must define an elementary embedding

$$e : \text{D-prod } A \to A' \quad .$$

If $[f]_D \in$ D-prod A, $e([f]_D)$ is defined to be the unique $b \in A'$ for which $A' \models b = \underline{f}(a)$. (Intuitively, $e(f)$ is $f(a)$.) This is well-defined, for if $f = f' \bmod D$, then

$$C = \{x \mid f(x) = f'(x)\} \in D \ ,$$

so $A' \models \underline{C}(a)$. But

$$A' \models (\forall x)\ (\underline{C}\ (x) \Longleftrightarrow \underline{f}(x) = \underline{f}'(x))$$

because this sentence is true in A. Therefore, $A' \models \underline{f}(a) = \underline{f}'(a)$, and $e([f]_D) = e([f']_D)$.

To verify that e is an elementary embedding, let $\varphi(x_1, \ldots, x_n)$ be a formula, and let $[f_1]_D, \ldots, [f_n]_D \in$ D-prod A (f_i : Un(D) $\to A$). Since

$$A' \models e([f_i]_D) = \underline{f_i}\ (a) \quad ,$$

we compute (with an obvious "vector notation")

$$\begin{aligned} \text{D-prod } A \models \varphi([f]_D) &\iff \{i \mid A \models \varphi(f(i))\} \in D \\ &\iff A' \models \underline{\{i \mid A \models \varphi(f(i))\}}\,(a) \\ &\iff A' \models \varphi(\underline{f}(a)) \\ &\iff A' \models \varphi(e([f]_D)) \quad , \end{aligned}$$

where the third equivalence is because the sentence

$$(\forall x)(\underline{\{i \mid A \models \varphi(f(i))\}}\,(x) \iff \varphi(\underline{f}(x)))$$

is true in A , hence in A' .

Finally, a is in the image of e , for $a = e([id]_D)$. □

PROPOSITION 4 : <u>Let</u> D <u>and</u> E <u>be ultrafilters on</u> ω , $f : E \to D$ <u>a morphism</u> . $f^*(\text{D-prod } \omega)$ <u>is cofinal in</u> E-prod ω <u>(with respect to the natural order) if and only if</u> f <u>is finite-to-one on some set of</u> E .

<u>Proof</u> : $f^*(\text{D-prod } \omega)$ is cofinal in E-prod ω if and only if, for every $g : \omega \to \omega$, there is an $h : \omega \to \omega$ such that

$$[g]_E \leq f^*([h]_D) = [h \circ f]_E$$

in E-prod ω. If this is the case for $g = id_\omega$, we have an h such that

$$A = \{x \mid hf(x) \geq x\} \in E \quad .$$

Then, for $x \in A$ and $y \in \omega$,

$$f(x) = y \Rightarrow x \leq h(y) \quad ,$$

so f takes the value y at most $h(y)+1$ times on A. Therefore, f is finite-to-one on A. Conversely, suppose f is finite-to-one on some $A \in E$, and let any $g : \omega \to \omega$ be given. Define

$$h(x) = \max \{g(y) \mid y \in A \text{ and } f(y) = x\} \quad ;$$

this is the maximum of a finite set, so h is well-defined. Clearly, for $y \in A$, $g(y) \leq hf(y)$, so $[g]_E \leq [h \circ f]_E$ as required. □

From the preceding two propositions, we obtain

COROLLARY 5 : A non-principal ultrafilter D on ω is a P-point if and only if every elementary submodel of D-prod ω, except ω itself, is cofinal in D-prod ω. □

§13. The initial segment ordering. Starting with the characterization of the Rudin-Keisler ordering in Proposition 11.7, we define a stronger ordering by requiring one ultraproduct to be not only an elementary submodel but also an initial segment of the other.

DEFINITION 1 : A morphism $[f]_D : D \to E$ in $\mathcal{U}$ is an IS(κ)-morphism if and only if $f^*(E\text{-prod } \kappa)$ is an initial segment of D-prod κ (with respect to the natural order). If there is such an f, then we write $E \leq_\kappa D$.

Clearly, identity morphisms and composites of IS(κ)-morphisms are IS(κ)-morphisms. Hence ultrafilters and IS(κ)-morphisms form a subcategory of $\mathcal{U}$, and $\leq_\kappa$ is (or rather, induces) a partial ordering of RK, stronger than the Rudin-Keisler ordering $\leq$.

PROPOSITION 2 : Suppose $\lambda < \kappa$ and $f : D \to E$ is an IS(κ)-morphism. Then $f^* : E\text{-prod } \lambda \to D\text{-prod } \lambda$ is an isomorphism.

Proof : Since $f^* : E\text{-prod } \lambda \to D\text{-prod } \lambda$ is an elementary embedding, we need only check that it is surjective. Let $[g]_D$ be any element of D-prod λ, so $g : \text{Un}(D) \to \lambda$. Let $\ell : \text{Un}(E) \to \kappa$ be the constant function with value λ. Then, for all $i \in \text{Un}(D)$, $g(i) < \lambda = \ell f(i)$, so $[g]_D < [\ell f]_D = f^*([\ell]_E)$ in D-prod κ. Since $f^*(E\text{-prod } \kappa)$ is an initial segment, there must be an $h : Un(E) \to \kappa$ such that $[g]_D = f^*([h]_E) = [hf]_D$. Because g maps into λ,

$$f^{-1}\{i \mid h(i) < \lambda\} = \{i \mid hf(i) < \lambda\} \supseteq \{i \mid hf(i) = g(i)\} \in D ,$$

so $\{i \mid h(i) < \lambda\} \in f(D) = E$. Redefining h on the complement of this set in E (which does not affect $[h]_E$), we may suppose $h(i) < \lambda$ for all i. Then $[h]_E \in E\text{-prod}\,\lambda$, and $f^*([h]_E) = [g]_D$. □

COROLLARY 3 : If size $(D) < \kappa$, then any $IS(\kappa)$-morphism with domain D is an isomorphism.

Proof : Apply the proposition, with $\lambda = \text{size}(D)$, and then use Corollary 11.6. □

COROLLARY 4 : If $\lambda \leq \kappa$, any $IS(\kappa)$-morphism is an $IS(\lambda)$-morphism. □

PROPOSITION 5 : Let $f : D \to E$ and $f' : D \to E'$ be $IS(\kappa)$-morphisms. If there is a morphism $g : E \to E'$ such that $f' = g \circ f$, then g is also an $IS(\kappa)$-morphism. If both E and E' have size $\leq \kappa$, then either there is a unique such g or there is a unique $g' : E' \to E$ such that $f = g' \circ f'$. (If both g and g' exist, they are inverse isomorphisms by Corollary 2.6.)

Proof : Assume g is given and $f' = gf$. Then the order-preserving embedding f^*, of E-prod κ into D-prod κ, sends g^*(E$'$-prod κ) to f'^*(E$'$-prod κ) which is an initial segment of D-prod κ and a subset of f^*(E-prod κ). Therefore f^*g^*(E$'$-prod κ) is an initial segment of f^*(E-prod κ), so g^*(E$'$-prod κ) is an initial segment of E-prod κ. This proves the first assertion.

Now assume both E and E$'$ have size $\leq \kappa$. Since f^*(E-prod κ) and f'^*(E$'$-prod κ) are initial segments of D-prod κ, one is contained in the other; say f'^*(E$'$-prod κ) $\subseteq f^*$(E-prod κ). Then $f^{*-1} \circ f'^*$: E$'$-prod $\kappa \to$ E-prod κ is an elementary embedding (because f^* and f'^* are elementary embeddings). By Proposition 11.5, there is a unique $g : E \to E'$ such that $f^{*-1} \circ f'^* = g^*$, i.e. $(gf)^* = f'^*$, i.e. (by Proposition 11.5 again) $gf = f'$. □

COROLLARY 6 : In the subcategory of $\mathcal{U}(\kappa)$ whose morphisms are the IS(κ)-morphisms, there is at most one morphism from any object to any other.

Proof : Suppose f and f' were morphisms $D \to E$ in this subcategory. By the proposition, we have $f' = gf$ or $f = gf'$ for some $g : E \to E$. But the only such g is the identity, by Theorem 2.5, so $f = f'$. □

The corollary shows that the category of ultrafilters of size κ and $IS(\kappa)$-morphisms, which we denote by $IS(\kappa)$, is essentially nothing more than a partially ordered set (after identification of isomorphic ultrafilters), namely $RK(\kappa)$ with the $IS(\kappa)$ ordering $\leq_{\kappa}$. Thus, no confusion will arise if we also let $IS(\kappa)$ denote this partially ordered set. From the last proposition, we obtain immediately

COROLLARY 7 : $IS(\kappa)$ *is a (not necessarily well-founded) tree; that is, the predecessors of any element are linearly ordered.* □

PROPOSITION 8 : *Let* κ *be a measurable cardinal, and let* P *be the subset of* $IS(\kappa)$ *consisting of equivalence classes of* κ-*complete ultrafilters. Then* P *(with ordering* $\leq_{\kappa}$*) is well-founded.*

Proof : If D is a κ-complete ultrafilter, D-prod κ is well-ordered (by its natural ordering; see [15, p. 311].). Let $\ell(D)$ be its order type. Clearly, if $D \leq_{\kappa} E$, then $\ell(D) \leq \ell(E)$ with equality if and only if $D \cong E$ (by Corollary 11. 6). Thus ℓ maps P to ordinals in a strictly monotone manner. Hence, given a nonempty subset of P, we obtain a minimal element simply by taking one with minimum possible ℓ. □

REMARKS 9 : $IS(\omega)$ is not well-founded; see Corollary 15. 18 and [2, Theorem 2. 12].

It is not obvious that $IS(\kappa)$ is non-trivial, i.e. that there exist D and E (of size κ, say) such that $D <_\kappa E$. Indeed, we shall later give a heuristic argument showing that $D <_\kappa E$ is a rather strange situation unless $\kappa = \omega$ or κ is measurable. Nevertheless, if κ is regular and $2^\kappa = \kappa^+$, such D and E do exist.

PROPOSITION 10 : Let D and E be ultrafilters on X and $X \times Y$ respectively, with $D = \pi(E)$ where $\pi : X \times Y \to X$ is the projection. $[\pi]_E$ is an $IS(\kappa)$-morphism if and only if, given any function f on $X \times Y$ for which $\mathrm{Card}(f''\pi^{-1}(x)) < \kappa$ for all $x \in X$ (or even for all $x \in B$ where $B \in D$), there is a set $A \in E$ on which $f(x,y)$ depends only on x, i.e. $\mathrm{Card}\, f''(A \cap \pi^{-1}(x)) \leq 1$ for all x.

Proof : First suppose π is an $IS(\kappa)$-morphism, and let f be given. Clearly, we may replace f by any f' such that

$$(\forall x \in X)\,(\forall y, z \in Y)\quad f(x,y) = f(x,z) \iff f'(x,y) = f'(x,z) \quad ,$$

since such a replacement affects neither the hypothesis on f nor the property required of A. Thus, we may suppose $f''\pi^{-1}(x)$ is an initial segment of κ for each $x \in X$, and let $g(x) \in \kappa$ be an upper bound for $f''\pi^{-1}(x)$. For all $x \in X$, $y \in Y$, $f(x,y) \leq g(x) = g\pi(x,y)$, so, in E-prod κ, $[f]_E \leq [g\pi]_E = \pi^*[g]_D$. As π is an $IS(\kappa)$-morphism,

$[f]_E$ must be $\pi^*[h]_D = [h\pi]_E$ for some $h : X \to \kappa$. Then the required set A is $\{(x,y) \mid f(x,y) = h(x) = h\pi(x,y)\} \in D$.

Conversely, suppose every f with Card $f''\pi^{-1}x < \kappa$ for all x depends only on the first coordinate on some set of E. We must show π is an IS(κ)-morphism, so let $[f]_E \le \pi^*[g]_D = [g\pi]_E$, where $f : X \times Y \to \kappa$ and $g : X \to \kappa$. Let $f' : X \times Y \to \kappa$ agree with f on $\{(x,y) \mid f(x,y) \le g(x) = g\pi(x,y)\} \in E$, and let f' be 0 elsewhere. Then $f' = f \bmod E$, and, for each x, $f'''\pi^{-1}(x)$ has cardinality $<\kappa$ because it is bounded by $g(x)$. By hypothesis, there is an $A \in E$ such that f' assumes at most one value on $\pi^{-1}(x) \cap A$; let $h(x)$ be that value. ($h(x)$ is arbitrary if $\pi^{-1}(x) \cap A = \emptyset$.) Then

$$\{(x,y) \mid f'(x,y) = h(x) = h\pi(x,y)\} \supseteq A \in E ,$$

so

$$[f]_E = [f']_E = [h\pi]_E = \pi^*([h]_D) \in \pi^*(D\text{-prod } \kappa). \quad \square$$

Observe that the restrictions that D and E be on X and $X \times Y$ and that the morphism $D \to E$ be π are inessential by Lemma 2.8.

THEOREM 11 : Let κ be a regular cardinal such that $2^\kappa = \kappa^+$, and let D be a κ^+-good ultrafilter on κ. There is an ultrafilter E

on $\kappa \times \kappa$ such that $\pi(E) = D$, $[\pi]_E$ is an IS(κ)-morphism, and $[\pi]_E$ is not an isomorphism. Thus $D <_\kappa E$, so the partially ordered set IS(κ) is not trivial.

Proof : Since $2^\kappa = \kappa^+$, the family $\mathcal{F}$, of functions $f : \kappa \times \kappa \rightarrow \kappa$ such that for all $x \in \kappa$ Card $f''\pi^{-1}x < \kappa$, can be well-ordered so that each f has at most κ predecessors ; let $\prec$ be such a well-ordering, and let f^+ be the immediate successor of f in $\prec$. We define, by transfinite induction with respect to $\prec$, filters F_f on $\kappa \times \kappa$ such that

(1) F_f has a basis of cardinality $\leq \kappa$.

(2) Each set $A \in F_f$ has the property that $(\forall x\ D)\ \{y \mid (x, y) \in A\}$ has cardinality κ .

(3) If $f \prec g$ then $F_f \subseteq F_g$.

(4) F_{f^+} contains a set A such that, for all x , f is constant on $A \cap \pi^{-1}(x)$.

If f is the first element of $\mathcal{F}$, let B_f consist of all the sets $\{(x, y) \mid x > \alpha\}$ for all $\alpha < \kappa$, and let F_f be the filter generated by B_f . This satisfies (2) because D , being κ^+-good, must be uniform, and (1), (3), (4) are trivial. If f is a limit element of $\mathcal{F}$, let $F_f = \bigcup_{g \prec f} F_g$

and $B_f = \bigcup_{g \prec f} B_g$ This satisfies (1) since f has at most κ predecessors, and the other three conditions are trivial. Now suppose F_f and B_f are defined; we must define F_{f^+}. F_{f^+} will be generated by $B_{f^+} = B_f \cup \{A\}$ where A is as in (4); thus (1), (3), (4) will hold. For (2), we must make sure that, for all $X \in B_f$, $\text{Card}\{y \mid (x,y) \in A \cap X\} = \kappa$ for most x with respect to D. Let $B_f = \{X_\alpha \mid \alpha < \kappa\}$, by (1). For each $x < \kappa$, let

$$\Phi(x) = \{G \in P_\omega(\kappa) \mid \text{Card}\{y \mid (x,y) \in \bigcap_{\alpha \in G} X_\alpha\} = \kappa\}.$$

Given any $G \in P_\omega(\kappa)$, $\bigcap_{\alpha \in G} X_\alpha \in F_f$, so, by (2), $\{x \mid G \in \Phi(x)\} \in D$. Since D is κ^+-good, there is a function $g : \kappa \to P_\omega(\kappa)$ such that $\{x \mid g(x) \in \Phi(x)\} \in D$ and, for all $\alpha \in \kappa$, $\{x \mid \alpha \in g(x)\} \in D$. If we let g' agree with g on $\{x \mid g(x) \in \Phi(x)\}$ and be ϕ elsewhere, then $g'(x) \in \Phi(x)$ for all $x \in \kappa$, and, for all $\alpha \in \kappa$,

$$\{x \mid \alpha \in g'(x)\} = \{x \mid \alpha \in g(x)\} \cap \{x \mid g(x) \in \Phi(x)\} \in D \quad .$$

For each $x \in \kappa$, let $Y_x = \{y \mid (x,y) \in \bigcap_{\alpha \in g'(x)} X_\alpha\}$. Thus $\text{Card}\, Y_x = \kappa$, but f takes fewer then κ values on $\{x\} \times Y_x$. Since κ is regular, $\{x\} \times Y_x$ has a subset Z_x of cardinal κ, on which f is constant. Let $A = \bigcup_{x \in \kappa} Z_x$. Clearly A is as required in (4). We must still check that

$$(\forall x D)\text{Card}\, \{y \mid (x,y) \in A \cap X_\alpha\} = \kappa \quad , \quad \text{for every } \alpha < \kappa .$$

Let α be given. $\text{Card}\{y \mid (x,y) \in A \cap X_\alpha\} = \text{Card } Z_x \cap X_\alpha$ for all x. But, for most x (with respect to D), $\alpha \in g'(x)$, so $Z_x \subseteq \{x\} \times Y_x \subseteq X_\alpha$ so, for most x, $\text{Card } \{y \mid (x,y) \in A \cap X_\alpha\} = \text{Card } Z_x = \kappa$, as required.

Let $F = \bigcup_{f \in \mathcal{F}} F_f$. If we adjoin to F all the sets $\pi^{-1}(C)$ for $C \in D$ and all the sets $A \subseteq \kappa \times \kappa$ such that π is one-to-one on $\kappa \times \kappa - A$(or even fewer-than-κ-to-one), the resulting set F' has the finite intersection property, by (2), so let E be an ultrafilter containing F'. $\pi(E) = D$ because, for all $C \in D$, $\pi^{-1}(C) \in F' \subseteq E$. $[\pi]_E$ is not an isomorphism, because if π is one-to-one on A, then $\kappa \times \kappa - A \in F' \subseteq E$, and $A \notin E$. Finally, $[\pi]_E$ is an $IS(\kappa)$-morphism because of (4) and Proposition 10. □

REMARK 12 : Since, in this proof, we could include in F' the complements of all sets on which π is fewer-than-κ-to-one, we could require in the theorem that π not be fewer-than-κ-to-one on any set of E.

§ 14. Non-standard ultrafilters In this section we shall develop another way of viewing morphisms and IS(κ)-morphisms. Apart from being interesting in its own right, this viewpoint will provide the promised "implausibility argument" for Theorem 13.11. It will also help to motivate the definition of sums of ultrafilters and the Rudin-Frolik ordering, and it will be useful in the proof that the ordering IS(ω) differs from the Rudin-Frolik ordering.

Throughout this section, D will be an ultrafilter on a set I, and V will be a very large set. Intuitively, we think of V as "the universe", but to avoid technical problems we want V to be a set, say Stg (λ) (see [15, p. 303]) for some λ so large that V contains all the sets in which we shall be interested below. We remind the reader of our convention that, when a set is treated as a structure, we mean the complete structure on the set, so the language has symbols for all predicates and functions on that set. We shall use the notation Hom(X, Y) for the set of functions from X into Y.

We consider the "non-standard universe" D-prod V. It has V as an elementary submodel via the embedding $x \to {}^*x$, where ${}^*x$ is the denotation in D-prod V of the name $\underline{x}$ of x, namely the germ on D of the constant function with value x. An element of

D-prod V is standard if and only if it is ${}^*x$ for some $x \in V$. A subset S of D-prod V is internal if and only if for some $s \in$ D-prod V

$$(\forall a \in \text{D-prod } V) \quad a \in S \Longleftrightarrow \text{D-prod } V \models a \underline{\in} s \quad ;$$

then we say that s represents S. (Clearly s is unique.) Subsets of D-prod V that are not internal are external. By abuse of language, we often use the same symbol to denote corresponding relations or functions on V and D-prod V; thus, for $a, b \in$ D-prod V, we may write $a \in b$ instead of D-prod V $\models a \underline{\in} b$. Similarly, we may use the same symbol for an internal set and its representative. We shall also write $[f]$ for $[f]_D$, since D is fixed.

If X is a set (tacitly understood to be $\in V$) and $A \subseteq I \times X$, we obtain $\tilde{A} : I \to P(X)$ by $\tilde{A}(i) = \{x \mid (i,x) \in A\}$. Then, in D-prod V, $[\tilde{A}] \in {}^*P(X)$, and any element of ${}^*P(X)$ (i.e. any internal subset of ${}^*X$) is $[\tilde{A}]$ for some A. Similarly, if $f : I \times X \to Y$, we define $\tilde{f} : I \to \text{Hom}(X,Y)$ by $\tilde{f}(i)(x) = f(i,x)$. Then $[\tilde{f}] \in {}^*\text{Hom}(X,Y)$, and all internal functions ${}^*X \to {}^*Y$ are of this form.

Now suppose E is an ultrafilter on $I \times X$, and $\pi(E) = D$, where $\pi : I \times X \to I$ is the projection. We define $E/D \subseteq {}^*P(X)$ by

DEFINITION 1 : $[\tilde{A}] \in E/D \Longleftrightarrow A \in E$.

Observe that, if $\tilde{A} = \tilde{A}' \bmod D$, then the complement of the symmetric difference of A and A' is in E, so $A \in E \Leftrightarrow A' \in E$; therefore the definition is legitimate. From trivial identities like $[\tilde{A}] \cap [\tilde{B}] = [\widetilde{A \cap B}]$ and $^*X - [\tilde{A}] = [\widetilde{(I \times X) - A}]$ it follows that E/D is an ultrafilter in the Boolean algebra $^*P(X)$. Note that E/D need not be internal. In fact, any ultrafilter in $^*P(X)$ is E/D for a unique ultrafilter E on $I \times X$ such that $\pi(E) = D$; the required E is defined by Definition 1, read from right to left.

If $f : {}^*X \to {}^*Y$ is internal and $A \subseteq {}^*Y$ is internal, then $f^{-1}(A) \subseteq {}^*X$ is internal. Thus, if F is an ultrafilter in $^*P(X)$ we can define an ultrafilter

$$f(F) = \{A \in {}^*P(Y) \mid f^{-1}(A) \in F\}$$

in $^*P(Y)$. One thus obtains an analog $\mathcal{U}_D$ of the category $\mathcal{U}$ by taking as objects all ultrafilters in $^*P(X)$ for arbitrary X and as morphisms germs of internal maps. Note that $\mathcal{U}_D$ is not just $^*\mathcal{U}$ since the objects of $\mathcal{U}_D$ may be external; $^*\mathcal{U}$ is equivalent to the full subcategory of $\mathcal{U}_D$ whose objects are internal ultrafilters. We have seen that the objects of $\mathcal{U}_D$ correspond to ultrafilters E on $I \times X$ (for arbitrary X) with $\pi(E) = D$. If E is such an ultrafilter and $g : I \times X \to I \times Y$ is a function commuting with π, then one easily computes $[\widetilde{\pi' g}](E/D) = g(E)/D$, where $\pi' : I \times Y \to Y$ is the projection. Using Lemma 2. 8, one then finds that $\mathcal{U}_D$ is equivalent to the category of $\mathcal{U}$-objects over D , whose objects are morphisms in $\mathcal{U}$ with codomain D and whose morphisms are commutative triangles

$$\begin{array}{ccc} E & \to & E' \\ & \searrow \swarrow & \\ & D & \end{array}$$

Translating Proposition 13. 10 into the present terminology, we obtain

COROLLARY 2 : Let E be an ultrafilter on $I \times X$ with $\pi(E) = D$. The following condition is necessary and sufficient for $[\pi]_E$ to be an IS(κ)-morphism. Given any internal function f on ${}^*X$ such that Card $f''{}^*X < {}^*\kappa$ in D-prod V, there is a (necessarily internal) $A \in E/D$ such that $f \upharpoonright A$ is constant in D-prod V. □

Observe that, when E/D is internal, the condition in the corollary says that E/D is ${}^*\kappa$-complete . One easily checks that E/D is principal if and only if $\pi : E \to D$ is an isomorphism. Hence, $D <_\kappa E$ via π and E/D is internal, if and only if E/D is a non-principal ${}^*\kappa$-complete ultrafilter on ${}^*X$. Since V is an elementary submodel of D-prod V , this condition can hold for some E/D if and only if $\kappa = \omega$ or there is a measurable cardinal λ such that

$\kappa \leq \lambda \leq$ Card (X) . Hence,

COROLLARY 3 : If $\kappa = \omega$ or κ is measurable, then the conclusion of Theorem 13.11 holds without the assumptions that $2^{\kappa} = \kappa^{+}$ and D is κ^{+}-good . □

On the other hand, E/D is uniform if and only if π is not fewer-than-κ-to-one on any set of E , so we find

COROLLARY 4 : If $\kappa \neq \omega$ and κ is not measurable, and if D and E satisfy the conclusion of Theorem 13.11 and the remark following it, then E/D is external. □

Heuristic remark : Suppose κ is regular but neither measurable nor countable, and suppose $2^{\kappa} = \kappa^{+}$ and D is κ^{+}-good. According to a person living in D-prod V , there are no uniform ${}^{*}\kappa$-complete ultrafilters on ${}^{*}\kappa$ (i. e. in ${}^{*}P(\kappa)$), because ${}^{*}\kappa$ is neither measurable nor countable. But, looking at his universe from the outside, we can see that there is such an ultrafilter; it just does not happen to be in his world (i. e. to be internal). If the resident of D-prod V is willing to believe us when we tell him about this ultrafilter, he will say that ${}^{*}\kappa$ although not measurable, is pseudo-measurable, in the sense that a ${}^{*}\kappa$ -complete uniform ultrafilter "exists in another world."

CHAPTER IV

LIMIT CONSTRUCTIONS

§ 15. Limits, sums, and products of ultrafilters Recall from elementary topology that an ultrafilter D on a topological space X is said to converge to a point $x \in X$, and x is called a limit of D, if and only if every neighborhood of x is in D. If D has a unique limit, we call it $\lim D$; on a compact Hausdorff space, every ultrafilter has a unique limit. If D is an ultrafilter on a set I and f is a function from I to a topological space X, then we write $D\text{-}\lim f$ or $D\text{-}\lim_i f(i)$ for $\lim f(D)$. We shall be concerned mainly with the case that X is the Stone-Čech compactification of some (discretely topologized) set J. (See 7.7)

LEMMA 1 : Let I and J be sets, D an ultrafilter on I, and E a function assigning to each $i \in I$ an ultrafilter E_i on J, i.e. $E : I \to \beta J$. For any $A \subseteq J$,

$$A \in D\text{-}\lim_i E_i \Longleftrightarrow (\forall i\, D) \quad A \in E_i \quad .$$

Proof : $\hat{A}$ is both open and closed in βJ. Hence

$$(1) \qquad \hat{A} \in E(D) \Longleftrightarrow \lim E(D) \in \hat{A} \qquad .$$

The right side of (1) is equivalent to

$$A \in \lim E(D) = D\text{-}\lim_i E_i \quad .$$

The left side is equivalent to

$$\{i \mid E_i \in \hat{A}\} = E^{-1}(\hat{A}) \in D \quad ,$$

which means $(\forall i\, D)\ A \in E_i$. □

PROPOSITION 2 : Let I, J, D, and E be as in the lemma, and let $E' : I \to \beta J$ be another function. If $E = E' \bmod D$, then $D\text{-}\lim_i E_i = D\text{-}\lim_i E'_i$.

Proof : Obvious from the lemma or from the fact that $E(D) = E'(D)$. □

PROPOSITION 3 : Let I, J, D, and E be as in the lemma and let $f : J \to J'$. Then $f(D\text{-}\lim_i E_i) = D\text{-}\lim_i f(E_i)$.

Proof : Applying the lemma, we compute for any $A \subseteq J'$,

$$\begin{aligned} A \in f(D\text{-}\lim_i E_i) &\iff f^{-1}(A) \in D\text{-}\lim_i E_i \\ &\iff (\forall i\, D)\ f^{-1}(A) \in E_i \\ &\iff (\forall i\, D)\ A \in f(E_i) \\ &\iff A \in D\text{-}\lim_i f(E_i) \quad . \quad \square \end{aligned}$$

PROPOSITION 4 : Let $f : I \to I'$, let D be an ultrafilter on I , let $D' = f(D)$, and let $E : I' \to X$ for any space X . Then $D\text{-}\lim_i E_{f(i)} = D'\text{-}\lim_{i'} E_{i'}$, in the sense that, if either limit exists and is unique, so does the other, and they agree .

Proof : Both are $\lim (E \circ f)(D)$. □

DEFINITION 5 : Let I be a set and D an ultrafilter on I . For each $i \in I$, let J_i be a set and E_i an ultrafilter on J_i . The disjoint union of the J_i is

$$\coprod_{i \in I} J_i = \{(i,j) \mid i \in I \ , \ j \in J_i\} \ ;$$

there are canonical injections

$$\varphi_i : J_i \to \coprod_{i \in I} J_i : j \mapsto (i,j)$$

and a canonical projection

$$\pi : \coprod_{i \in I} J_i \to I : (i,j) \mapsto i \ .$$

The sum of the E_i with respect to D is defined to be the ultrafilter

$$D\text{-}\Sigma_i E_i = D\text{-}\lim_i \varphi_i(E_i)$$

on $\coprod_i J_i$. If all the J_i are the same set J and all the E_i are the same ultrafilter E , then $\coprod_i J_i = I \times J$, and $D\text{-}\Sigma_i E$ will be called the product of D and E (in that order) and denoted by $D \cdot E$.

REMARKS 6 : (1) $\tilde{A}(i)$ and $\tilde{f}(i)$, as defined in Section 14, are, in the present notation , $\varphi_i^{-1}(A)$ and $f \circ \varphi_i$ respectively .

(2) Do not confuse the product $D \cdot E$ defined here with the cartesian product $D \times E$ defined in Section 3 . Note that $D \cdot E$, unlike $D \times E$, is always an ultrafilter.

(3) In much of the literature, $D \cdot E$ is called $E \times D$.

LEMMA 7 : (1) For all $A \subseteq \coprod_{i \in I} J_i$,

$$A \in D - \Sigma_i E_i \iff (\forall i\, D)(\forall j\, E_i)\ (i, j) \in A .$$

Thus the quantifier $(\forall (i,j)\ D - \Sigma_i E_i)$ is equivalent to $(\forall i\, D)(\forall j\, E_i)$.

(2) For all $A \subseteq I \times J$,

$$A \in D \cdot E \iff (\forall i\, D)(\forall j\, E)\ (i, j) \in A \ ;$$

$(\forall (i,j)\ D \cdot E)$ is equivalent to $(\forall i\, D)(\forall j\, E)$.

(3) For each $(i, j) \in \coprod_i J_i$, let F_{ij} be an ultrafilter on a set K_{ij} . The natural bijection between

$$\coprod_{i \in I} \Big(\coprod_{j \in J_i} K_{ij} \Big) \quad \text{and} \quad \coprod_{(i,j) \in \coprod_{i \in I} J_i} K_{ij}$$

(namely $(i, (j, k)) \longleftrightarrow ((i, j), k))$ maps

$$D - \Sigma_i (E_i - \Sigma_j F_{ij}) \quad \text{to} \quad (D - \Sigma_i E_i) - \Sigma_{i,j} F_{ij} \ ;$$

we usually identify these two via this bijection. In particular, multiplication of ultrafilters is associative.

(4) The projection $\pi : \coprod_i J_i \to I$ maps $D - \Sigma_i E_i$ to D. If all the J_i are the same set J and π' is the projection $I \times J \to J$, then $\pi'(D - \Sigma_i E_i) = D\text{-}\lim_i E_i$. If all the E_i are the same E, then $\pi'(D \cdot E) = E$.

(5) $\pi : D - \Sigma_i E_i \to D$ is an isomorphism if and only if $(\forall i\, D)\ E_i$ is principal. $\pi' : D \cdot E \to E$ is an isomorphism if and only if D is principal.

(6) $(\forall i\, D)\ E_i = E'_i$ (i. e. $E = E' \bmod D$) $\Longleftrightarrow D - \Sigma_i E_i = D - \Sigma_i E_i'$.

(7) Suppose, for each $i \in I$, $f_i : J_i \to J_i'$. The induced map

$$f : \coprod_{i \in I} J_i \to \coprod_{i \in I} J'_i : (i, j) \mapsto (i, f_i(j))$$

takes $D - \Sigma_i E_i$ to $D - \Sigma_i f_i(E_i)$. If $(\forall i\, D) f_i$ is an isomorphism, then f is an isomorphism.

(8) Suppose $g : I' \to I$ and suppose D' is an ultrafilter on I' with $g(D') = D$. Then

$$\overline{g} : \coprod_{i' \in I'} J_{g(i')} \to \coprod_{i \in I} J_i : (i', j) \mapsto (g(i'), j)$$

maps $D' - \Sigma_{i'} E_{g(i')}$ to $D - \Sigma_i E_i$. If g is an isomorphism, then so is $\bar{g}$.

Proof : Straightforward verification, omitted. □

According to (7) of the lemma, we may unambiguously define sums of isomorphism classes by

$$D - \Sigma_i \bar{E}_i = \overline{D - \Sigma_i E_i} .$$

Note however, that (8) does not suffice to permit an analogous definition of $\bar{D} - \Sigma_i E_i$, since when D is replaced by an isomorphic ultrafilter the E_i's must be re-indexed. Of course, if all the E_i are equal, then there is no such difficulty and we define $\bar{D} \cdot \bar{E} = \overline{D \cdot E}$.

DEFINITION 8 : $D \leq_{RF} E$ if and only if, for some ultrafilters F_i $(i \in \mathrm{Un}(D))$, $D - \Sigma_i F_i \cong E$. The relation $\leq_{RF}$ is called the Rudin-Frolik ordering.

Part (8) of the last lemma shows that the relation $D \leq_{RF} E$ depends only on the isomorphism classes of D and E, so we get an induced relation $\bar{D} \leq_{RF} \bar{E}$ on the class RK. This relation is reflexive by (5) and transitive by (3) of the lemma. By (4), $\leq_{RF}$ implies $\leq$, so it is anti-symmetric. RK with the partial ordering

$\leq_{RF}$ will be called RF ; similarly for RF(κ), etc. RF(ω) has been studied in detail by Booth [2] . To connect our definition with his , we need the following

PROPOSITION 9 : Let D be an ultrafilter on I , $E : I \to \beta J$. If the points $E_i \in \beta J$ have a system of pairwise disjoint neighborhoods (in βJ), then $D\text{-}\lim_i E_i \cong D - \Sigma_i E_i$.

Proof : The pairwise disjoint neighborhoods can be taken to basic open sets $\hat{A}_i$; $A_i \in E_i$, and

$$i \neq i' \Longrightarrow A_i \cap A_{i'} = \emptyset \quad .$$

Define a function $g : J \to I$ to have value i on A_i (and to have arbitrary value on $J - \bigcup_i A_i$), and let

$$f : J \to I \times J : j \mapsto (g(j), j) \quad .$$

By choice of g , f agrees with φ_i on A_i , so $f(E_i) = \varphi_i(E_i)$. Hence, using Proposition 3,

$$\begin{aligned} D - \Sigma_i E_i &= D\text{-}\lim_i \varphi_i(E_i) \\ &= D\text{-}\lim_i f(E_i) \\ &= f(D\text{-}\lim_i E_i) \quad . \end{aligned}$$

Since f is obviously one-to-one, the proof is complete . □

Let us call a family of points in a topological space <u>strongly discrete</u> if and only if the points have a system of pairwise disjoint neighborhoods (as in the last proposition). This property is, in general, stronger than just discreteness. For example, if X is an uncountable set and 2 is the discrete space $\{0,1\}$, then, in the product space 2^X, the points precisely one of whose coordinates is 1 (i.e., the standard "unit vectors") form a discrete but not strongly discrete collection. (Indeed, any family of pairwise disjoint open sets is countable.) Discreteness is often an easier property to deal with than strong discreteness because the former is an intrinsic property while the latter depends on the ambient space. Thus, the following simple result is often useful.

PROPOSITION 10 : <u>In a regular</u> (i.e. T_1 <u>and</u> T_3) <u>space</u> X, <u>any discrete countable set is strongly discrete</u>.

<u>Proof</u> : Let $\{x_i \mid i < \omega\}$ be a countable discrete set ; thus each x_i has an open neighborhood N_i containing no other x_j. Define inductively closed neighborhoods $C_i \subseteq N_i$ of x_i as follows. If C_j has been defined for $j < i$ and $C_j \subseteq N_j$, then

$$\bigcup_{j<i} C_j \subseteq \bigcup_{j<i} N_j$$

is a closed set not containing x_i. By regularity, the neighborhood

$N_i - \bigcup_{j<i} C_j$ of x_i contains a closed neighborhood C_i of x_i. Then the C_i are pairwise disjoint, so $\{x_i \mid i < \omega\}$ is strongly discrete. □

Taken together, the last two propositions show that our definition of $\leq_{RF}$ agrees with Booth's. We continue with two propositions which show that (roughly speaking) when dealing with P-points we need never worry about discreteness.

PROPOSITION 11 : Any countable family of (distinct) P-points is discrete (hence, strongly discrete) in $\beta\omega$.

Proof : Let the P-points in question be E_i $(i < \omega)$. Temporarily consider a fixed i. For each $j \neq i$, let G_j be a neighborhood of E_i in $\beta\omega$ not containing E_j. By Proposition 9.1, $\bigcap_{j\neq i} G_j$ contains a set N which is a neighborhood of E_i in $\beta\omega - \omega$. Clearly N contains no E_j $(j \neq i)$. Thus, $\{E_i \mid i < \omega\}$ is discrete (in $\beta\omega-\omega$, hence in $\beta\omega$, because discreteness is intrinsic). □

PROPOSITION 12 : A convergent P-point on a regular space contains a strongly discrete set.

Proof : Let D be a P-point on the regular space X, and let $p \in X$ be the limit of D. Since D has size ω (by definition of P-point), there is an $A \in D$ with $\text{Card}(A) = \omega$ and $p \notin A$; let $A = \{a_n \mid n < \omega\}$. Since X is regular, let G_n be an open neighborhood of p not containing a_n, and let C_n be a closed neighborhood of p contained in G_n. By choosing G_n and C_n inductively rather than all at once, we can arrange $G_{n+1} \subseteq C_n$. For each n, let $g(a_n)$ be the least k such that $a_n \notin G_k$; $g(a_n)$ exists because $a_n \notin G_n$. Define g arbitrarily on $X - A$. If g is constant on a set $B \subseteq X$, say $g(B) \subseteq \{k\}$, then B is disjoint from $A \cap G_k$ which is in D (as G_k is a neighborhood of lim D), so $B \notin D$. As D is a P-point, g must be finite one-to-one on some $B \in D$; since D is an ultrafilter, we may choose B so that $B \subseteq A$ and g takes only even or only odd values on B, say even values. The finitely many points of B where g takes the value $2k$ are, by definition of g on A, in G_{2k-1} but not in G_{2k}, so they lie in $G_{2k-1} - C_{2k}$ (as $C_{2k} \subseteq G_{2k}$). Since $G_{2k+1} \subseteq C_{2k}$, the various sets $G_{2k-1} - C_{2k}$ $(k < \omega)$ are pairwise disjoint open sets which cover B, and only finitely many points of B lie in each of those sets. Using the fact that X is Hausdorff, we easily conclude that B is strongly discrete. □

REMARK 13: The hypothesis of convergence is not needed in the proposition. The proof of this proceeds by first observing that it suffices to prove discreteness (by Proposition 10) which is intrinsic, so, without loss of generality, the space X may be replaced by a countable subspace, and we may as well assume that X itself is countable. But any countable regular space is completely regular, so X has a Stone-Čech compactification βX, and in βX the proposition can be applied because any ultrafilter converges. We omit the details of this proof, because in practice we shall only need the case where the given ultrafilter converges; in fact, the space X will be compact in applications.

By Propositions 9 and 12, we get

COROLLARY 14 : _If_ D _is a P-point on_ I , _and if_ $E : I \to \beta J$ _is one-to-one on a set of_ D , _then_ $D\text{-}\lim_i E_i \cong D\text{-}\Sigma_i E_i$. □

If E is not one-to-one on any set of D , then let $f : I \to I'$ be a surjection such that

$$f(i) = f(j) \Leftrightarrow E_i = E_j$$

(e. g. let I' be obtained from I by dividing by an appropriate equivalence), let $D' = f(D)$, and let $F_{f(i)} = E_i$, so $F : I' \to \beta J$. By Proposition 9. 7, D' is a P-point or principal, and clearly the $F_{i'}$ are all distinct. Hence, using the corollary and Proposition 4,

$$\begin{aligned} D\text{-}\lim_i E_i &= D\text{-}\lim_i F_{f(i)} = D'\text{-}\lim_{i'} F_{i'} \\ &\cong D'\text{-}\Sigma_{i'} F_{i'} . \end{aligned}$$

Thus, in any case,

COROLLARY 15 : *If* D *is a P-point, then any ultrafilter of the form* $D\text{-}\lim_i E_i$ *is isomorphic to one of the form* $D'\text{-}\Sigma_{i'} F_{i'}$, *where* $D' \leq D$ *and the* $F_{i'}$ *'s are among the* E_i *'s* .

We conclude this section by examining the connection between sums of ultrafilters and the non-standard ultrafilters considered in Section 14 .

PROPOSITION 16 : *Let* D *be an ultrafilter on a set* I , *and let* $E : I \to \beta J$. *Then* $(D\text{-}\Sigma_i E_i)/D$ *is the internal ultrafilter (represented by)* $[E]_D$.

Proof : If $A \subseteq I \times J$, we compute (using $\tilde{A}(i) = \varphi_i^{-1}(A)$)

$$\begin{aligned}
[\tilde{A}]_D \in [E]_D &\iff (\forall i\, D)\ \ \tilde{A}(i) \in E_i \\
&\iff (\forall i\, D)\ \ A \in \varphi_i(E_i) \\
&\iff A \in D\text{-}\lim_i \varphi_i(E_i) = D\text{-}\Sigma_i E_i \\
&\iff [\tilde{A}] \in (D\text{-}\Sigma_i E_i)/D \quad . \ \square
\end{aligned}$$

Since an ultrafilter F on $I \times J$ with $\pi(F) = D$ is completely determined by F/D , COROLLARY 17 : If F is an ultrafilter on $I \times J$ such that $\pi(F) = D$ and F/D is internal, then $F = D\text{-}\Sigma_i E_i$ for certain maps $E: I \to \beta J$, namely, just those E for which $[E]_D = F/D$. □

As another corollary, we obtain again the "if" part of Lemma 7(6). By Corollary 14. 2,

COROLLARY 18 : The projection

$$\pi : D\text{-}\Sigma_i E_i \to D$$

is an $IS(\omega)$-morphism . Hence $\leq_{RF}$ implies $\leq_\omega$. □

§ 16. Successors in $RK(\omega)$ In this section we make a first application of the ideas of sum and product of ultrafilters to the study of the Rudin-Keisler ordering.

THEOREM 1 : Assume $FRH(\omega)$. For each uniform ultrafilter D on ω, there are an ultrafilter E on ω and a morphism $f : E \to D$ such that any morphism with domain E either is an isomorphism or factors through f, but f itself is not an isomorphism. In fact, there are 2^{2^ω} pairwise non-isomorphic such E's.

Proof : Let $E_i \ (i < \omega)$ be pairwise non-isomorphic minimal ultrafilters on ω; such E_i exist by Corollary 8. 9. Let $E = D\text{-}\Sigma_i E_i$ on $\omega \times \omega$. We shall show that the projection $\pi : E \to D$ is not an isomorphism, and any morphism with domain E either is an isomorphism or factors through π. (An E as required in the theorem can then be obtained by taking any ultrafilter on ω isomorphic to the E we have defined.) First, by Lemma 15. 7(5), π is not an isomorphism, for none of the E_i are principal. Now let g be any function $\omega \times \omega \to \omega$. Recall that $\tilde{g} : \omega \to \mathrm{Hom}(\omega, \omega)$ is defined by $\tilde{g}(i)(n) = g(i, n)$. Since each E_i is minimal, $\tilde{g}(i)$ is constant or one-to-one on some set $A_i \in E_i$.

Case 1 : $\{i \mid \tilde{g}(i) \text{ is constant on } A_i\} = B \in D$.

Then

$$A = \{(i,n) \mid i \in B \ , \ n \in A_i\} \in E \quad ,$$

and, if we let $h(i)$ be the value of $\tilde{g}(i)$ on A_i when $i \in B$ ($h(i)$ arbitrary when $i \notin B$), then

$$(i,n) \in A \implies g(i,n) = \tilde{g}(i)(n) = h(i) = h\pi(i,n) \quad .$$

Therefore, $g = h\pi \bmod E$.

<u>Case 2</u> : $\{i \mid \tilde{g}(i) \text{ is one-to-one on } A_i\} = C \in D$.

Now, using Proposition 15. 3 ,

$$\begin{aligned} g(E) &= g(D\text{-lim}_i \varphi_i(E_i)) \\ &= D\text{-lim}_i g\varphi_i(E_i) \\ &= D\text{-lim}_i \tilde{g}(i)(E_i) \quad . \end{aligned}$$

But, as $\tilde{g}(i)$ is one-to-one on A_i for D-most i , we have

$$(\forall i\, D) \quad \tilde{g}(i)(E_i) \cong E_i \quad .$$

Furthermore, the ultrafilters $\tilde{g}(i)(E_i)$ for $i \in C$ are distinct (for the various E_i were chosen to be non-isomorphic) ; hence they form a strongly discrete set, by Proposition 15. 11. Therefore, we have

$$
\begin{aligned}
g(E) &= D\text{-}\lim_i \tilde{g}(i)(E_i) \\
&\cong D\text{-}\Sigma_i \tilde{g}(i)(E_i) \\
&\cong D\text{-}\Sigma_i E_i \\
&\cong E \quad .
\end{aligned}
$$

By Corollary 2.6, g is an isomorphism.

Since D is an ultrafilter, one of the two cases considered must occur, so the main assertion of the theorem is proved.

By Corollary 8.9, we can choose 2^{2^ω} sequences E_i as above, in such a way that no ultrafilter appears in two distinct sequences. From each sequence, we obtain an E as above, and these 2^{2^ω} E's are distinct by Lemma 15.7(6). Since only 2^ω ultrafilters on $\omega \times \omega$ can be in any one isomorphism class, we obtain 2^{2^ω} pairwise non-isomorphic E's as required. □

DEFINITION 2 : An element a of a partially ordered set P is an immediate successor of $b \in P$ if and only if $b < a$ and

$$(\forall x \in P) \;\; x < a \Longrightarrow x \leq b \quad .$$

Note that this definition requires not only that a is above b with nothing in between, but also that no element incomparable with b

lies below a . The theorem immediately implies

COROLLARY 3 : Every element of RK(ω) has $2^{2^{\omega}}$ immediate successors, assuming FRH(ω) . □

Of course $\bar{0}$, also has $2^{2^{\omega}}$ immediate successors in RK($\leq\omega$), namely the minimal ultrafilters. $\bar{0}$, its immediate successors, their immediate successors, etc. form a tree, of height ω , with $2^{2^{\omega}}$-fold ramification at each node. Using Proposition 5. 10, the tree can be extended until it has height $\aleph_1$. Thus,

COROLLARY 4 : Assume FRH(ω) . Let P be the partially ordered set of maps p from arbitrary countable ordinals into $2^{2^{\omega}}$, ordered by inclusion (i. e. , $p \leq q$ if and only if p is the restriction of q to Do(p)). Thus, P is the "standard" tree of height $\aleph_1$ with $2^{2^{\omega}}$-fold branching from every from every node. Then P can be isomorphically imbedded into RK($\leq\omega$) . □

Observe that the image of P is by no means all of RK($\leq\omega$), because the former is a tree while the latter is directed upward (in a very strong way ; see Proposition 5. 10) and is not a chain (by 8. 9 if FRH(ω) ; by [12] in general). Observe also that the isomorphic embedding of P into RK($\leq\omega$) can be taken to map the least element , $\emptyset$, of P to any prescribed element of RK(ω) .

§ 17 Goodness, sums, and minimality Assuming FRH(κ), there are a great many ultrafilters on κ which are both κ^+-good and minimal in RK(κ). (See Corollary 8.8) The question naturally arises whether there is any necessary connection between κ^+-goodness and minimality in RK(κ). Does one imply the other? Or does the negation of one imply the other? Keisler has proved

THEOREM I : *Assuming* D *is countably incomplete*, $D\text{-}\Sigma_i E_i$ *is* κ^+-*good if and only if* D *is* κ^+-*good*.

Proof : See [9]. □

In particular, for countably incomplete D , $D \cdot E$ is κ^+-good if and only if D is κ^+-good. Given any ultrafilter E on $\kappa > \omega$, we can obtain both κ^+-good ultrafilters and non-κ^+-good ultrafilters $> E$ on κ by choosing D to be κ^+-good in the first case (using Corollary 8.8) and non-κ^+-good in the second case (e.g., let size(D) = ω). If $\kappa = \omega$ then all ultrafilters of size κ are κ^+-good (see [7]).

The only possible implication not ruled out by these considerations is "minimal $\Rightarrow$ good". Keisler has asked whether this implication holds (for $\kappa > \omega$, assuming $2^\kappa = \kappa^+$ if necessary), and also whether every κ-regular ultrafilter is $\geq$ some κ^+-good one. (An affirmative answer to the latter question would imply that, for any two elementarily equivalent structures $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality $\leq \kappa$ for a language with $\leq \kappa$ symbols, and for any κ-regular ultrafilter D on κ, D-prod $\mathfrak{A}$ and D-prod $\mathfrak{B}$ are isomorphic. See [1].) We shall answer both questions negatively, assuming $2^\kappa = \kappa^+$, or even just FRH(κ), by constructing an ultrafilter on κ which is minimal in RK(κ), κ-regular, but not κ^+-good, provided κ is not cofinal with ω. Note that, if κ is measurable, a normal κ-complete ultrafilter on κ is a counterexample for the first question, but not for the second.

LEMMA 2 : If all the E_i are κ-regular, then so is $D\text{-}\Sigma_i E_i$.

Proof : Let Un(D) = I, Un(E_i) = J_i ($i \in I$). Suppose that, for each i, $f_i : J_i \to P_\omega(\kappa)$ is as in the definition of κ-regular. Then one trivially checks that

$$f : \coprod_{i \in I} J_i \to P_\omega(\kappa) : (i,j) \to f_i(j)$$

also satisfies that definition, so $D\text{-}\Sigma_i E_i$ is κ-regular. □

THEOREM 3 : Let κ be a cardinal of cofinality $> \omega$, and assume FRH(κ). Then there is an ultrafilter E on $\omega \times \kappa$ which is κ-regular (hence uniform), minimal in RK(κ), but not κ^+-good.

Proof : Begin by letting D be any uniform ultrafilter on ω. We shall define certain κ-regular ultrafilters E_i on κ $(i < \omega)$, and then we shall let E be $D\text{-}\Sigma_i E_i$. By the lemma, E will be κ-regular; by Keisler's theorem (and the facts in Section 1), E will not be κ^+-good. Most of the following proof is therefore concerned with ensuring the minimality of E in $RK(\kappa)$.

LEMMA 4 : In unif(κ), let C be any comeager set. Assume FRH(κ). Then C contains a countable sequence E_i $(i < \omega)$ of ultrafilters with the following property. If $f_i : \kappa \to \kappa$ $(i < \omega)$ are maps such that f_i is one-to-one on a set of E_i, then the set of ultrafilters $\{f_i(E_i) \mid i < \omega\}$ is strongly discrete in $\beta\kappa$.

Proof of lemma : Suppose $C \supseteq \bigcap_{\alpha < 2^\kappa} C^\alpha$ where each C^α is open and dense in unif (κ) (with the fine topology). There are 2^κ systems $\{f_i : \kappa \to \kappa \mid i < \omega\}$ of countably many self-maps of κ; well-order them with order type 2^κ, and let the α^{th} system be $\{f_i^\alpha \mid i < \omega\}$. We define uniform filters $\mathcal{F}_i^\alpha$ $(i < \omega)$ on κ, simultaneously for all i, by induction on α, so that

(1) $\mathcal{F}_i^\alpha \subseteq \mathcal{F}_i^\beta$ for $\alpha \leq \beta$.

(2) $\mathcal{F}_i^\alpha$ has a basis of cardinality $\leq \kappa$.

(3) Any ultrafilter containing $\mathcal{F}_i^{\alpha+1}$ is in C^α.

(4) If, for each i, f_i^α takes κ distinct values on each set of $\mathcal{F}_i^{\alpha+1}$, then there are pairwise disjoint sets $A_i \subseteq \kappa$ such that

$$(f_i^\alpha)^{-1}(A_i) \in \mathcal{F}_i^{\alpha+1} .$$

Begin by setting $\mathcal{F}_i^0 = \{\kappa\}$. If γ is a limit ordinal $< 2^\kappa$ and $\mathcal{F}_i^\alpha$ is defined for all $\alpha < \gamma$, then $\bigcup_{\alpha<\gamma} \mathcal{F}_i^\alpha$ has a basis of cardinality $\leq \text{Card}(\kappa \times \gamma) < 2^\kappa$, so by FRH($\kappa$), it can be extended to a filter with a basis of cardinality $\leq \kappa$. Let that filter be $\mathcal{F}_i^\gamma$.

Now suppose $\gamma = \alpha + 1$ and $\mathcal{F}_i^\alpha$ is already defined. By (2), the set V_i of uniform ultrafilters containing $\mathcal{F}_i^\alpha$ is open and nonempty (as $\mathcal{F}_i^\alpha$ is uniform) in unif(κ). Since C^α is dense, V_i meets C^α, so V_i meets some basic open set

$$\bigcap_{A \in \mathcal{G}_i} \hat{A} \subseteq C^\alpha$$

where $\text{Card}(\mathcal{G}_i) \leq \kappa$. Let $\mathcal{J}_i$ be the filter generated by $\mathcal{F}_i^\alpha \cup \mathcal{G}_i$. $\mathcal{J}_i$ has a basis of cardinality $\leq \kappa$, say $\{G_{i,\mu} \mid \mu < \kappa\}$. If, for some i, f_i^α does not take κ values on each $G_{i,\mu}$, then we may set $\mathcal{F}_i^{\alpha+1} = \mathcal{J}_i$, as (4) will hold vacuously. From now on, suppose, for each i, f_i^α takes κ values on every $G_{i,\mu}$. Well-order the triples $(i, \mu, \nu) \in \omega \times \kappa \times \kappa$ with order type κ. Inductively choose

$a(i,\mu,\nu) \in G_{i,\mu}$ so that

$$f_i^{\alpha}(a(i,\mu,\nu)) \neq f_{i'}^{\alpha}(a(i',\mu',\nu'))$$

for all earlier triples (i',μ',ν'). This can be done because f_i^{α} takes κ values on $G_{i,\mu}$ and there are fewer than κ earlier (i',μ',ν'). Let

$$B_i = \{ a(i,\mu,\nu) \mid \mu, \nu \in \kappa \} \quad .$$

Since B_i meets $G_{i,\mu}$ κ times (at least once for each ν), the filter $\mathcal{F}_i^{\alpha+1}$ generated by $\mathcal{G}_i \cup \{B_i\}$ is uniform. Conditions (1) and (2) are obviously satisfied, and (3) holds because $G_i \subseteq \mathcal{G}_i \subseteq \mathcal{F}_i^{\alpha+1}$. For (4), let $A_i = f_i^{\alpha}(B_i)$. By choice of $a(i,\mu,\nu)$, the A_i are pairwise disjoint, and

$$(f_i^{\alpha})^{-1}(A_i) \supseteq B_i \in \mathcal{F}_i^{\alpha+1} \quad .$$

Now if we let E_i be any uniform ultrafilter containing $\bigcup_{\alpha < 2^{\kappa}} \mathcal{F}_i^{\alpha}$, condition (3) implies

$$E_i \in \bigcap_{\alpha < 2^{\kappa}} C^{\alpha} \subseteq C \quad .$$

If $f_i : \kappa \to \kappa$ are maps, say $f_i = f_i^{\alpha}$, and each f_i is one-to-one on a set of E_i, then f_i must take at least κ values on each set of $\mathcal{F}_i^{\alpha+1}$, for otherwise E_i contains a set on which f_i is one-to-one and takes fewer than κ values, contrary to the fact that E_i is uniform. Then, by (4), we have disjoint neighborhoods $\hat{A}_i$ of $f_i(E_i)$ in $\beta\kappa$. Thus, the lemma is proved. □

REMARK 5 : We could have obtained as many as κ E_i's in C whose images under any one-to-one maps f_i $(i < \kappa)$ are strongly discrete, by the same proof.

Returning to the proof of the theorem, use the lemma with C = the set of κ^+-good ultrafilters on κ minimal in $RK(\kappa)$. C is comeager by Corollary 8. 7. Since the E_i provided by the lemma are κ^+-good, they are κ-regular, so, as remarked above, $E = D\text{-}\Sigma_i E_i$ is κ-regular but not κ^+-good. We now show that E is minimal in $RK(\kappa)$.

Let $g : \omega \times \kappa \to \kappa$ be any function. We must show that g is one-to-one or takes fewer than κ values on some set of E. Since the E_i are minimal in $RK(\kappa)$, we have sets $A_i \in E_i$ such that $\tilde{g}(i) = g\,\varphi_i$ takes fewer than κ values or is one-to-one on A_i.

<u>Case 1</u> : $\{i \mid \tilde{g}(i)$ takes fewer than κ values on $A_i\} = C \in D$. Then $\coprod_{i \in C} A_i \in E$, and

$$g(\coprod_{i \in C} A_i) = \bigcup_{i \in C} \tilde{g}(i)(A_i)$$

has cardinality $< \kappa$ because κ is not cofinal with $\mathrm{Card}(C) = \omega$.

Case 2 : $\{i \mid \tilde{g}(i) \text{ is one-to-one on } A_i\} \in D$. Modifying $\tilde{g}$ on the complement of this set (which doesn't affect $[g]_E$), we may assume that all the $\tilde{g}(i)$ are one-to-one on A_i. Hence, by the choice of E_i according to the lemma, $\{\tilde{g}(i)\,(E_i) \mid i < \omega\}$ is strongly discrete. Using 15.3, 15.9, and 15.7(7),

$$\begin{aligned} g(E) &= g(D\text{-}\lim_i \varphi_i(E)) \\ &= D\text{-}\lim_i \tilde{g}(i)\,(E_i) \\ &\cong D\text{-}\Sigma_i\, \tilde{g}(i)\,(E_i) \\ &\cong D\text{-}\Sigma_i E_i \\ &= E \end{aligned} ,$$

so, by Corollary 2.6, $[g]_E$ is an isomorphism ; g is one-to-one on a set of E.

Since D is an ultrafilter, one of the two cases happens, and the theorem is proved. □

REMARK 6 : If we are willing to assume FRH(ω) in addition to FRH(κ) , then the preceding proof can be greatly simplified. The lemma may be omitted altogether. Choose the E_i to be κ^+-good , minimal in RK(κ), and pairwise non-isomorphic, and choose D to be a P-point. The rest of the proof remains the same, except that in Case 2 the strong discreteness of $\{\tilde{g}(i)(E_i) \mid i \in A\}$ for some $A \in D$ is established by citing Proposition 15. 12, rather than the lemma.

§ 18. Ultrafilters on singular cardinals Keisler has raised the question whether, assuming GCH , every element of RK(κ) lies above a minimal element. In this section, we shall obtain a negative answer to this question in the case $\kappa = \aleph_\omega$. We shall also consider a quite unrelated question whose solution uses the same idea as the solution of Keisler's problem . We now digress for a moment to motivate and present this question.

We know that there are 2^{2^κ} different isomorphism classes of uniform ultrafilters on κ , and we know various properties (e. g. κ-completeness , κ^+-goodness, minimality in RK(κ)) which may distinguish some isomorphism classes from others. However, for each such property considered so far, it apparently cannot be proved in ZFC alone that some uniform ultrafilters on κ have the property and others do not. Thus, unless κ is measurable, we cannot have both κ-complete and κ-incomplete uniform ultrafilters on κ. And we have not been able to prove the existence of κ^+-good ultrafilters (for $\kappa > \omega$) or minimal ultrafilters in RK(κ) without some special hypothesis such as GCH or FRH . Therefore, one might conjecture that no isomorphism-invariant property of ultrafilters, definable by a formula of L(ZF) , can be proved in ZFC to apply to some but not all uniform ultrafilters of size κ. Put another way, one might think that ZFC remains consistent upon addition of the axiom schema

(*) $(\exists D)(\text{size}(D) = \kappa \text{ and } \varphi(D)) \Rightarrow (\forall D)(\text{size}(D) = \kappa \Rightarrow (\exists E)\ E \cong D \text{ and } \varphi(E))$

where $\varphi(D)$ is any formula whose only free variable is D (ranging over ultrafilters.) This restriction on φ is clearly needed, for otherwise, we could take $\varphi(D)$ to be $D \cong F$ and get a trivial contradiction. The preceding remarks show that the schema (*) contradicts the existence of measurable cardinals and every instance of GCH. We shall show that (*) is in fact inconsistent. It may, however, be of some interest to consider weakened forms of (*), e.g. by requiring κ to be regular, or even by taking only the single case $\kappa = \omega$. Intuitively (*) says that all ultrafilters of a given size look alike.

We proceed to the construction of a counterexample to (*).

THEOREM 1 : Let κ be the limit of the ω-sequence of cardinals defined by $\alpha_0 = \omega$, $\alpha_{n+1} = 2^{\alpha_n}$. Then

(1) There are uniform ultrafilters on κ of the form $D\text{-}\lim_i E_i$ where $\text{Un}(D) = \omega$ and, for $i < \omega$, $\text{size}(E_i) = \alpha_i$.

(2) There are uniform ultrafilters on κ not of that form.

Proof: (1) Let D be any uniform ultrafilter on ω and E_i any ultrafilter on κ of size α_i. If $A \in D\text{-}\lim_i E_i$, then for infinitely many i (all i in a set of D) $A \in E_i$, so

$$\text{Card}(A) \geq \text{size } E_i = \alpha_i \quad .$$

Therefore, Card (A) $\geq \kappa$, and $D\text{-lim}_i E_i$ is uniform.

(2) We count how many ultrafilters can have the form in (1). There are $2^{2^\omega} = \alpha_2$ choices for D. For each i, there are (at most) 2^κ choices of a set $A \subseteq \kappa$ of cardinality α_i, and then (at most) $2^{2^{\alpha_i}} = \alpha_{i+2}$ choices of an ultrafilter uniform on A (whose image under the inclusion into κ is to be E_i). Thus, the total number of ultrafilters of the form in (1) is no more than $(2^\kappa)^\omega = 2^\kappa$. But there are 2^{2^κ} uniform ultrafilters on κ altogether. □

Since being of the form in (1) is evidently an isomorphism-invariant property expressible in L(ZF), the theorem disproves (*).

THEOREM 2 : Assume GCH. There are uniform ultrafilters on $\aleph_\omega$ which are not $\geq$ any minimal element of RK($\aleph_\omega$).

Proof : Choose, once and for all, a minimal ultrafilter D on ω (using CH). Let S be the set of all uniform ultrafilters on $\aleph_\omega$ which are of the form $D\text{-lim}_i E_i$ where all the E_i have size $< \aleph_\omega$. As in the proof of part (1) of the preceding theorem, we see that $S \neq \emptyset$. We shall show that any uniform ultrafilter on $\aleph_\omega$ which lies below an element of S is itself in S, and that S contains no minimal element of $RK(\aleph_\omega)$. This clearly will suffice to prove the theorem.

First, suppose F is uniform on $\aleph_\omega$ and $F \leq E$ for some

$$E = D\text{-}\lim_i E_i \in S$$

(size $(E_i) < \aleph_\omega$). Then, for some $f : \aleph_\omega \to \aleph_\omega$,

$$\begin{aligned} F &= f(E) \\ &= f(D\text{-}\lim_i E_i) \\ &= D\text{-}\lim_i f(E_i) \quad , \end{aligned}$$

and

$$\text{size } (f(E_i)) \leq \text{size } (E_i) < \aleph_\omega \quad ,$$

so $F \in S$.

Now suppose $D\text{-}\lim_i E_i = E \in S$ were minimal in $RK(\aleph_\omega)$. If the function

$$\omega \to \aleph_\omega \ : \ i \to \text{size } (E_i)$$

were bounded, say by $\aleph_n$, on a set $A \in D$, then we could choose, for each $i \in A$, a set $X_i \in E_i$ of cardinality $\leq \aleph_n$. Then E would contain the set $\bigcup_{i \in A} X_i$ of cardinality $\leq \aleph_n$, contrary to the fact that S contains only uniform ultrafilters. Hence, size (E_i) must be an unbounded (in $\aleph_\omega$) function of i on each set of D. Since D is minimal, the function $i \to E_i$ must be one-to-one on a set of D, and Corollary 15.14 now shows that

$$E' = D\text{-}\Sigma_i E_i \cong E \quad ,$$

so E' is minimal in $RK(\aleph_\omega)$.

Define $f : \omega \times \aleph_\omega \to \aleph_\omega$ as follows. If size $(E_i) = \omega$, then $\tilde{f}(i)$ is $id_{\aleph_\omega}$. If size $(E_i) = \aleph_{n+1}$, then $\tilde{f}(i)$ is such a map $\aleph_\omega \to \aleph_\omega$ that $\tilde{f}(i)(E_i)$ has size $\aleph_n$; such maps exist by GCH and Chang's Theorem 6.3. Since size (E_i) is unbounded in $\aleph_\omega$ on every set of D, so is size $(\tilde{f}(i)(E_i))$. Hence, as in the proof of Theorem 1(1),

$$\begin{aligned} f(E') &= f(D\text{-}\Sigma_i E_i) \\ &= f(D\text{-}\lim_i \varphi_i(E_i)) \\ &= D\text{-}\lim_i \tilde{f}(i)(E_i) \end{aligned}$$

is uniform. It is obviously $\leq E'$, so by minimality of E', f must be one-to-one on some $A \in E'$. For D-most i, $\tilde{A}(i) \in E_i$ (by definition of E') and $\tilde{f}(i)$ is one-to-one on $\tilde{A}(i)$, so $\tilde{f}(i)(E_i) \cong E_i$. Since isomorphic ultrafilters have the same size, we see from the definition of f that, for D-most i, size $(E_i) = \omega$. This contradicts the fact that size (E_i) is unbounded in $\aleph_\omega$ on every set of D. Hence no element of S is minimal. □

§ 19. Isomorphisms between sums In this section, we shall derive a result, essentially due to Rudin [13] , which says, roughly, that if $D\text{-}\Sigma_i E_i$ and $D'\text{-}\Sigma_i E'_i$ are isomorphic then they are isomorphic for a trivial reason. Apart from the possibility $D = D'$, $E_i = E'_i$ (for most i , up to a permutation) , the only trivial reason is Lemma 15. 7(3).

LEMMA 1 : In $\beta\omega$, any two disjoint countable sets whose union is discrete have disjoint closures .

Proof : Let the sets be $\{D_i | i < \omega\}$ and $\{E_i | i < \omega\}$. By Proposition 15. 10 , we can find $A_i \in D_i$, $B_i \in E_i$ so that all the A_i's and B_i's are pairwise disjoint . Let $A = \bigcup_{i<\omega} A_i$ and $B = \bigcup_{i<\omega} B_i$. Then $A \cap B = \emptyset$. By Lemma 15. 1 , any $D \in \mathrm{Cl}\{D_i\}$ contains A and any $E \in \mathrm{Cl}\{E_i\}$ contains B . Therefore, these two closures are disjoint. □

Now suppose D, D', E_i , and E'_i $(i<\omega)$ are ultrafilters on ω , and suppose $f : \omega\times\omega \to \omega\times\omega$ is an isomorphism from $D\text{-}\Sigma_i E_i$ to $D'\text{-}\Sigma_i E'_i$. Modifying f on the complement of a set in $D\text{-}\Sigma_i E_i$, we may suppose that f is a bijection. Recall that φ_i is the map $\omega \to \omega\times\omega$ mapping j to (i,j) . Define functions $A, B : \omega \to \beta(\omega\times\omega)$ by

$$A_i = f\varphi_i(E_i) = \tilde{f}(i)(E_i)$$

and

$$B_i = \varphi_i(E_i') \quad .$$

Clearly, A and B are one-to-one and have discrete ranges. We have

$$\begin{aligned} D'\text{-}\lim_i B_i &= D'\text{-}\lim_i \varphi_i(E_i') \\ &= D'\text{-}\Sigma_i E_i' \\ &= f(D\text{-}\Sigma_i E_i) \\ &= D\text{-}\lim_i A_i \quad . \end{aligned}$$

By the lemma (with $\omega \times \omega$ in place of ω, which obviously makes no difference) the sets $A(X)$ and $B(Y)$, for any $X \in D$, $Y \in D'$, either are not disjoint or have non-discrete union, for their closures meet.

Consider first the case that, for all $X \in D$ and all $Y \in D'$, $A(X)$ and $B(Y)$ meet. Then the ultrafilters $A(D)$ and $B(D')$ are identical, for each set in one meets each set in the other. Let $[g]_D$ be the composite isomorphism

$$D \xrightarrow{[A]} A(D) = B(D') \xrightarrow{[B]^{-1}} D' \quad .$$

For D-most i,

$$\tilde{f}(i)(E_i) = A_i = B_{g(i)} = \varphi_{g(i)}(E'_{g(i)}) \quad .$$

It follows that $E_i \tilde{=} E'_{g(i)}$ for D-most i . Furthermore, as

$$\{g(i)\} \times \omega \in \varphi_{g(i)}(E'_{g(i)}) \quad ,$$

we have

$$(\forall i D)(\forall j E_i) \quad f(i,j) = \tilde{f}(i)(j) \in \{g(i)\} \times \omega \quad ,$$

so

$$(\forall i D)(\forall j E_i) \quad \pi_1 f(i,j) = g(i) \quad .$$

If we let $h_i(j) = \pi_2 f(i,j)$, then we have shown

$$f(i,j) = (g(i), h_i(j))$$

for most (i,j) with respect to $D\text{-}\Sigma_i E_i$. By construction, g is one-to-one on a set of D , and , because f is one-to-one, h_i is one-to-one on a set of E_i for D-most i . By inessential changes in g and h_i , we may assume that g and all the h_i are bijections. Then [f] is a composition of isomorphisms of the sort given by Lemma 15.7(7) and (8) . Up to permutations, D is D′ and E_i is E'_i for most i ; f consists of the relevant permutations .

Now we turn to the other case, $A(X) \cap B(Y) = \emptyset$ for some $X \in D$, $Y \in D'$. Redefining E_i and $\tilde{f}(i)$ for $i \notin X$ (which does not affect $D\text{-}\Sigma_i E_i$ or the germ of f) , we may as well suppose $A(\omega) \cap B(\omega) = \emptyset$.

Let

$$X = \{i \mid A_i \in \mathrm{Cl}\,B(\omega)\} \quad ; \quad Y = \{i \mid B_i \in \mathrm{Cl}\,A(\omega)\} \ .$$

Clearly, $A(\omega - X)$ and $B(\omega - Y)$ have discrete union and are disjoint; hence we cannot have both $\omega - X \in D$ and $\omega - Y \in D'$. Say $X \in D$. (The case $Y \in D'$ is handled analogously, interchanging primed and unprimed, and replacing f by f^{-1}.) For $i \in X$, $A_i \in \mathrm{Cl}\,B(\omega)$ but $A_i \notin B(\omega)$ (since $A(\omega) \cap B(\omega) = \emptyset$), so there is a non-principal ultrafilter F_i on ω such that

$$\begin{aligned} \tilde{f}(i)(E_i) = A_i &= F_i\text{-}\lim_j B_j \\ &= F_i\text{-}\lim_j \varphi_j(E_j') \\ &= F_i\text{-}\Sigma_j E_j' \end{aligned}$$

Let F_i be arbitrary if $i \notin X$. We have

$$\begin{aligned} D'\text{-}\Sigma_i E_i' &= f(D\text{-}\Sigma_i E_i) \\ &= D\text{-}\lim_i \tilde{f}(i)(E_i) \\ &= D\text{-}\lim_i (F_i\text{-}\Sigma_j E_j') \ . \end{aligned}$$

On the other hand, using Lemma 15.7 (8), (3), and (4),

$$(\pi_2(D\text{-}\Sigma_i F_i))\text{-}\Sigma_j E'_j = \bar{\pi}_2((D\text{-}\Sigma_i F_i)\text{-}\Sigma_{i,j} E'_{\pi_2(i,j)})$$
$$= \bar{\pi}_2(D\text{-}\Sigma_i(F_i\text{-}\Sigma_j E'_j))$$
$$= D\text{-}\lim_i(F_i\text{-}\Sigma_j E'_j) \quad ,$$

where $\pi_2 : (i,j) \to j$, and $\bar{\pi}_2$ is as in 15.7 (8). Therefore,

$$D'\text{-}\Sigma_i E'_i = (\pi_2(D\text{-}\Sigma_i F_i))\text{-}\Sigma_j E'_j \quad .$$

Applying π_1 to both sides,

$$D' = \pi_2(D\text{-}\Sigma_i F_i) = D\text{-}\lim_i F_i \quad .$$

Because $\{F_i \mid i \in X\}$ is discrete (as $\{A_i \mid i \in X\}$ is discrete), we obtain

$$D' \cong D\text{-}\Sigma_i F_i \quad .$$

Thus, up to isomorphism, D' is $D\text{-}\Sigma_i F_i$, and E_i is $F_i\text{-}\Sigma_j E'_j$ for most D-most i . The isomorphism f corresponds to the equality

$$(D\text{-}\Sigma_i F_i)\text{-}\Sigma_j E'_j = D\text{-}\Sigma_i(F_i\text{-}\Sigma_j E'_j) \quad .$$

Summarizing, and omitting the details of the various isomorphisms, we have

THEOREM 2 : For ultrafilters of size $\leq \omega$, if

$$D\text{-}\Sigma_i E_i \cong D' - \Sigma_i E_i' \quad ,$$

then one of the following happens.

(1) $g : D \cong D'$, and, for D-most i , $E_i \cong E'_{g(i)}$.

(2) For some non-principal ultrafilters F_i , $D' \cong D\text{-}\Sigma_i F_i$, and, for D-most i , $E_i \cong F_i\text{-}\Sigma_j E_j'$.

(3) For some non-principal ultrafilters F_i , $D \cong D'\text{-}\Sigma_i F_i$, and, for D′-most i , $E_i' \cong F_i\text{-}\Sigma_j E_j$. □

COROLLARY 3 : If $D \cdot E \cong D' \cdot E$ (where D, D′, E have size ω), then $D \cong D'$.

Proof : Apply the theorem with all $E_i = E_i' = E$. In case (1), the required conclusion is immediate. In case (2), $E \cong F_i \cdot E$. By Lemma 15. 7(5) and Corollary 2. 6, F_i is principal, contrary to the assertion of the theorem in case (2) . Case (3) is the same. □

COROLLARY 4 : If $D \cdot E \cong D' \cdot E'$ (D, D′, E, E′ of size $\leq \omega$) , then one of the following happens.

(1) $D \cong D'$ and $E \cong E'$.

(2) <u>For some non-principal</u> F , $D' \cong D \cdot F$ <u>and</u> $E \cong F \cdot E'$.

(3) <u>For some non-principal</u> F , $D \cong D' \cdot F$ <u>and</u> $E' \cong F \cdot E$.

<u>Proof</u> : Apply the theorem with all $E_i = E$ and all $E_i' = E'$. In case (1) of the theorem, we immediately get conclusion (1) of the corollary. In case (2) we have, for some non-principal F_i , $D' \cong D\text{-}\Sigma_i F_i$ and $E \cong F_i \cdot E'$ for all i in a certain set $X \in D$. If $i, j \in X$, then $F_i \cong F_j$ because of Corollary 3 ; let F be F_i for any $i \in X$. Then

$$D' \cong D\text{-}\Sigma_i F_i \cong D \cdot F \quad ,$$

and

$$E \cong F \cdot E' \quad ,$$

so we have conclusion (2) of the corollary. Case (3) is analogous. □

Corollary 4 says that any isomorphism between products of ultrafilters on ω is either trivial (i. e. , corresponding factors agree) or an instance of the associative law.

COROLLARY 5 : $RF(\leq \omega)$ <u>is a tree</u> ; i. e. , <u>the predecessors of any element are linearly ordered</u>.

<u>Proof</u> : Immediate from the theorem. □

§ 20. Cartesian products of ultrafilters on ω Consider two uniform ultrafilters, D and E, on ω. We know (by Corollary 3.10) that $D \times E$ is not an ultrafilter on $\omega \times \omega$. We can explicitly exhibit two distinct ultrafilters containing it. First, by Lemmas 15.7(4) and 3.2, $D \cdot E \supseteq D \times E$. Secondly, if we let

$$t : \omega \times \omega \to \omega \times \omega : (x, y) \to (y, x) \quad ,$$

then $t(E \cdot D) \supseteq D \times E$ for the same reasons. $D \cdot E$ and $t(E \cdot D)$ are distinct, because the former contains $A = \{(x, y) \mid x < y\}$ while the latter contains $t(A)$ which is disjoint from A. It is natural to ask whether there are any further ultrafilters containing $D \times E$. The case $D = E$ was considered in Section 10, where we saw that $D \times D$ is also contained in $\delta(D)$ where δ is the diagonal map $x \to (x, x)$ (whose range, $(\omega \times \omega) - (A \cup tA)$, we call Δ), and that $D \times D$ is contained in only three ultrafilters if and only if D is minimal. Thus, we have

COROLLARY 1 : $\delta(D)$, $D \cdot D$, and $t(D \cdot D)$ are, for uniform D on ω, distinct ultrafilters containing $D \times D$. There are no others if and only if D is minimal. □

We now turn to the case $D \neq E$.

THEOREM 2 : Assume FRH(ω) . There are uniform ultrafilters D and E , on ω such that $D \cdot E$ and $t(E \cdot D)$ are the only ultrafilters containing $D \times E$.

Proof : The two sets

$$A' = A \cup \Delta = \{(x, y) | x \leq y\}$$

and $t(A')$ cover $\omega \times \omega$, so any ultrafilter on $\omega \times \omega$ contains one of them. Thus, we need only arrange that $(D \times E) \cup \{A'\}$ and $(D \times E) \cup \{t(A')\}$ generate ultrafilters, for then any ultrafilter containing $D \times E$ is one of these two. Consider the set Γ of all pairs (R, ε), where R is a binary relation on $\omega (R \subseteq \omega \times \omega)$ and $\varepsilon = 0$ or 1 . We shall so construct D and E that, for each $(R, \varepsilon) \in \Gamma$, there are sets $X_{R, \varepsilon} \in D$, $Y_{R, \varepsilon} \in E$ with the property that

(1) $\varepsilon = 0$ (resp. , 1) and $x \in X_{R, \varepsilon}$ and $y \in Y_{R, \varepsilon}$ and $x \leq y$ (resp. , $x \geq y$) $\Longrightarrow$ $(x, y) \in R_\varepsilon$

where R_ε is either R or $\omega \times \omega - R$. Then $(X_{R,0} \times Y_{R,0}) \cap A'$ is a set in the filter generated by $(D \times E) \cup \{A'\}$ and is contained in R or in $\omega \times \omega - R$, so this filter is an ultrafilter . Similarly, using $\varepsilon = 1$, $(D \times E) \cup \{t(A')\}$ generates an ultrafilter.

We proceed to the construction of D and E. Let

$$p : 2^{\omega} \to \Gamma$$

be a bijection. (Clearly Γ has cardinality 2^{ω}.) We define filters D_{α} and E_{α} on ω for $\alpha < 2^{\omega}$ inductively, so that

(2) For $\beta < \gamma$, $D_{\beta} \subseteq D_{\gamma}$ and $E_{\beta} \subseteq E_{\gamma}$.

(3) D_{α} and E_{α} have countable bases.

(4) D_{α} and E_{α} are uniform.

(5) $D_{\alpha+1}$ contains an $X_{p(\alpha)}$, and $E_{\alpha+1}$ contains a $Y_{p(\alpha)}$ such that (1) holds for $(R, \epsilon) = p(\alpha)$.

Begin by letting D_0 and E_0 consist of all cofinite subsets of ω; then (4) will hold for all α provided (2) holds. If α is a limit ordinal, obtain D_{α} and E_{α} by applying $FRH(\omega)$ to $\bigcup_{\beta<\alpha} D_{\beta}$ and $\bigcup_{\beta<\alpha} E_{\beta}$ respectively. Now suppose D_{α} and E_{α} are defined; we will let $D_{\alpha+1}$ and $E_{\alpha+1}$ be generated by $D_{\alpha} \cup \{X\}$ and $E_{\alpha} \cup \{Y\}$, where X and Y will serve as $X_{p(\alpha)}$ and $Y_{p(\alpha)}$ respectively. Let $p(\alpha) = (R, \epsilon)$, and suppose $\epsilon = 0$. (The other case is analagous.) We want

$$x \leq y,\ x \in X,\ y \in Y \Rightarrow (x, y) \in R_0 ,$$

where R_0 is R or its complement. Let $\{B_i \mid i < \omega\}$ be a base for D_α, and let F be any ultrafilter containing E_α (so F is uniform, by choice of E_0). Let

$$W = \{x < \omega \mid \tilde{R}(x) = \{y \mid (x, y) \in R\} \in F\} .$$

Suppose W meets every one of the B_i, and set $R_0 = R$. (Otherwise, $\omega - W$ meets every B_i, we set $R_0 = (\omega \times \omega) - R$, and the rest of the proof is exactly the same.) Let $\{C_i \mid i < \omega\} \subseteq F$ be a countable base for E_α. Inductively choose $b_i \in B_i \cap W$ and $c_i \in C_i$ as follows. Choose b_0 to be any element of $B_0 \cap W$, and choose c_0 to be any element of $\tilde{R}_0(b_0) \cap C_0 \in F$. Suppose b_j and c_j are chosen for $j < i$. Let b_i be any element of $B_i \cap W$ which is $>$ all c_j $(j < i)$; such a b_i exists, for $B_i \cap W$ must be infinite because W meets all B_k and every finite set is disjoint from some B_k by choice of D_0. Then let c_i be any element of

$$\tilde{R}_0(b_0) \cap \dots \cap \tilde{R}_0(b_i) \cap C_i \in F .$$

Let $X = \{b_i \mid i < \omega\}$ and $Y = \{c_i \mid i < \omega\}$. Clearly X meets every B_i (at b_i), hence every set of D_α, and Y meets every set of E_α. Suppose $x \le y$, $x \in X$, and $y \in Y$. Say $x = b_i$ and $y = c_j$. Since $b_i \le c_j$, and b_i was defined to be $> c_j$ if $j < i$, we must have $j \ge i$. But then c_j was chosen to be in $\tilde{R}_0(b_i)$. Hence

$(x, y) \in R_0$. This completes the inductive definition of D_α and E_α.

If we let D and E be any ultrafilters containing $\bigcup_{\alpha < 2\omega} D_\alpha$ and $\bigcup_{\alpha < 2\omega} E_\alpha$ respectively, then (1) holds, and the theorem is proved. (Actually the two unions which we extended to form D and E were ultrafilters already.) □

COROLLARY 3 : <u>Assume</u> FRH(ω). <u>The</u> D <u>and</u> E <u>of the theorem may be taken to be minimal.</u>

<u>Proof</u> : Let $\{f_\alpha | \alpha < 2^\omega\}$ be the set of all maps $\omega \to \omega$. In the definition of $D_{\alpha+1}$ in the proof of the theorem, replace X by a subset, still meeting each B_i, on which f_α is one-to-one or constant, and similarly for $E_{\alpha+1}$. Then D and E will be minimal. □

For any permutation σ of n, let $t_\sigma : \omega^n \to \omega^n$ be defined by

$$(t_\sigma(x))_i = x_{\sigma^{-1}(i)} \quad ; \text{ that is } \quad \pi_i t_\sigma = \pi_{\sigma^{-1}(i)} \quad .$$

Hence, for any ultrafilters $D^0, \ldots, D^{n-1}$ on ω,

$$\pi_i t_\sigma (D^{\sigma(0)} \ldots D^{\sigma(n-1)}) = \pi_{\sigma^{-1}(i)} (D^{\sigma(0)} \ldots D^{\sigma(n-1)})$$

$$= D^i \quad ,$$

so

$$t_\sigma (D^{\sigma(0)} \ldots D^{\sigma(n-1)}) \supseteq D^0 \times \ldots \times D^{n-1} .$$

Generalizing the theorem, we have

COROLLARY 4 : Assume FRH(ω) . There exist uniform ultrafilters D^i on ω $(0 \leq i < n)$ such that all ultrafilters containing $D^0 \times \ldots \times D^{n-1}$ are of the form $t_\sigma (D^{\sigma(0)} \ldots D^{\sigma(n-1)})$ for some permutation σ of n .

Proof : The proof is essentially the same as that of the theorem. The major modification will be illustrated sufficiently by the case $n = 3$. In constructing $D^j_{\alpha+1}$ $(j = 0, 1, 2)$, we have D^j_α with countable bases $\{B^j_i \mid i < \omega\}$, and we want to find X^j , meeting every B^j_i , and such that

$$x \leq y \leq z , \quad x \in X^0 , \ y \in X^1 , \ z \in X^2 \Rightarrow (x, y, z) \in R_0$$

where R_0 is R or ω^3-R for a given R . (There are five other cases, depending on the order of x, y, z, but they are analogous.) Let F^1 , F^2 be ultrafilters containing D^1_α , D^2_α respectively . Suppose

$$W = \{x \mid (\forall y \, F^1)(\forall z \, F^2) \ (x, y, z) \in R\}$$

meets every set in D^0_α , and set $R_0 = R$. (Otherwise, $R_0 = \omega^3$-R , and the rest is analogous.) Then choose, by induction on i ,

$$b_i^0 \in B_i^0 \cap W \ ,$$

$$b_i^1 \in B_i^1 \cap \bigcap_{j<i} \{y \mid (\forall z\, F^2)(b_j^0, y, z) \in R_0\} \in F^1 \ ,$$

and

$$b_i^2 \in B_i^2 \cap \bigcap_{k<j<i} \{z \mid (b_k^0 , b_j^1 , z) \in R_0\} \in F^2 \ ,$$

in such a way that each chosen number is larger than all those chosen previously. Then set $X^j = \{b_i^j \mid i < \omega\}$. □

COROLLARY 5 : <u>Assume</u> $FRH(\omega)$. <u>There are</u> 2^{ω} <u>minimal ultrafilters</u> $D^j (j < 2^{\omega})$ <u>such that, for any finite subset</u> $\{\alpha_0, \cdots , \alpha_{n-1}\} \subseteq 2^{\omega}$, <u>every ultrafilter containing</u> $D^{\alpha_0} \times \ldots \times D^{\alpha_{n-1}}$ <u>is</u> $t_{\sigma} (D^{\alpha_{\sigma(0)}} \ldots D^{\alpha_{\sigma(n-1)}})$ <u>for some permutation</u> σ <u>of</u> n .

<u>Proof</u> : Combine the techniques of the theorem and the previous two corollaries. The induction is with respect to triples $(\{\alpha_0, \ldots , \alpha_{n-1}\}, R , \sigma)$ where $R \subseteq \omega^n$ and σ is a permutation of n . □

REMARK 6 : In the situation of Corollary 4, if F is an ultrafilter containing $D^0 \times \ldots \times D^{n-1}$, the permutation σ is determined by the fact that, in F-prod ω,

$$[\pi_{\sigma(0)}] < [\pi_{\sigma(1)}] < \ldots < [\pi_{\sigma(n-1)}] \ .$$

To see this, note first that

$$t_\sigma^*[\pi_{\sigma(i)}]_F = [\pi_i] \in D^{\sigma(0)} \ldots D^{\sigma(n-1)}\text{-prod}\,\omega \ .$$

Then compute that, in any $D \cdot E$-prod ω, $[\pi_1] < [\pi_2]$, and use the fact that t_σ^* is an elementary embedding (in fact, an isomorphism).

Our next goal is to give two equivalent conditions, one model-theoretic and the other topological, for a pair of ultrafilters to satisfy the conclusion of the theorem.

DEFINITION 7 : Let D be an ultrafilter on ω , and let $\mathfrak{A}$ be an elementary extension of (the complete model on) ω. An element $a \in |\mathfrak{A}|$ has type D if and only if, for all $S \subseteq \omega$,

$$S \in D \Longleftrightarrow \mathfrak{A} \models \underline{S}\,(a)$$

It is clear that every element of $|\mathfrak{A}|$ has a unique type (see also Proposition 12. 3), and every ultrafilter is the type of an element in some $\mathfrak{A}$ (by the compactness theorem). Indeed, the type of $[f]_D$ in D-prod ω is exactly $f(D)$, so D is the type of $[id]_D$.

PROPOSITION 8 : Let D and E be uniform ultrafilters on ω. The following are equivalent.

(1) *The only ultrafilter containing* $(D \times E) \cup \{A\}$, *where* $A = \{(x,y) \mid x < y\}$, *is* $D \cdot E$.

(2) *Let* $\mathfrak{A}$ *and* $\mathfrak{A}'$ *be elementary extensions of* ω , *let* $a \in |\mathfrak{A}|$, $a' \in |\mathfrak{A}'|$ *have type* D , *let* $b \in |\mathfrak{A}|$, $b' \in |\mathfrak{A}'|$ *have type* E , *and let* $a < b$, $a' < b'$ *(in* $\mathfrak{A}$ *and* $\mathfrak{A}'$). *Then there is an isomorphism, from an elementary submodel* $\mathfrak{B}$ *of* $\mathfrak{A}$ *containing* a *and* b , *to an elementary submodel* $\mathfrak{B}'$ *of* $\mathfrak{A}'$ *containing* a' *and* b' , *mapping* a *to* a' *and* b *to* b' .

(3) *With* $\mathfrak{A}$, $\mathfrak{A}'$, a , a' , b , b' *as in* (2) , *if* $\varphi(x,y)$ *is any formula (of the language of the complete model on* ω) *with just* x *and* y *free, then*

$$\mathfrak{A} \models \varphi(a,b) \Longleftrightarrow \mathfrak{A}' \models \varphi(a',b') \quad .$$

Proof : Let $J : \omega \times \omega \to \omega$ be a bijection , and let $J^{-1}(x) = (K(x), L(x))$.

Assume (1), and let $\mathfrak{A}$, $\mathfrak{A}'$, a , a' , b , b' be as in (2). Let $c = J(a,b)$ in $\mathfrak{A}$ (i. e. $\mathfrak{A} \models c = \underline{J}(a,b)$) and $c' = J(a',b')$ in $\mathfrak{A}'$. Since a and a' have type D and b and b' have type E , one easily computes that the types of c and c' include $J(D \times E)$.

They also contain $J(A)$ because $a < b$ and $a' < b'$. By (1), they must both be $J(D \cdot E)$. Thus, for any formula $\psi(x)$,

$$\mathfrak{A} \models \psi(c) \Longleftrightarrow \mathfrak{A}' \models \psi(c') \ ;$$

if we let $\psi(x)$ be $\varphi(\underline{K}(x), \underline{L}(x))$ we get (3),

Obviously (2) implies (3). Assuming (3), we prove (2) by letting $|\mathfrak{B}|$ be the set of all $e \in |\mathfrak{A}|$ such that $\mathfrak{A} \models e = \underline{f}(a, b)$ for some $f : \omega \times \omega \to \omega$, and $\mathfrak{B}'$ similarly. The only non-trivial thing to check is that $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$. Let $\varphi(x, e_i)$ be a formula with one free variable x and various parameters $e_i = f_i(a, b) \in |\mathfrak{B}|$. Suppose for some $\alpha \in |\mathfrak{A}|$, $\mathfrak{A} \models \varphi(\alpha, e_i)$; then we must find $\beta \in |\mathfrak{B}|$ such that $\mathfrak{A} \models \varphi(\beta, e_i)$. (It is well-known that then $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$.) Define $g : \omega \times \omega \to \omega$ by

$$\begin{aligned} g(x, y) &= \text{the least } z \text{ such that } \varphi(z, f_i(x, y)), \text{ if there is such a } z, \\ &= 0 \text{ otherwise}. \end{aligned}$$

Then

$$(\forall x)(\forall y)\ ((\exists z)\varphi(z, \underline{f}_i(x, y)) \Longrightarrow \varphi(\underline{g}(x, y), \underline{f}_i(x, y)))$$

is true in ω, hence in $\mathfrak{A}$. Therefore, $\mathfrak{A} \models \varphi(\underline{g}(a, b), e_i)$, and we may take $\beta = g(a, b)$.

Finally we prove that (3) implies (1). Suppose F and G are ultrafilters containing $(D \times E) \cup \{A\}$, and suppose $R \in F$. We shall show $R \in G$, so $F = G$, and (1) follows . Take

$$\mathfrak{A} = F\text{-prod}\ \omega \quad , \quad \mathfrak{A}' = G\text{-prod}\ \omega \quad ,$$
$$a = [\pi_1]_F \quad , \quad a' = [\pi_1]_G \quad ,$$
$$b = [\pi_2]_F \quad , \quad b' = [\pi_2]_G \quad .$$

Then a and a' have type $\pi_1(F) = \pi_1(G) = D$, and b and b' have type E , because $D \times E \subseteq F, G$. $a < b$ and $a' < b'$ because $A \in F, G$. Also, as $R \in F$, $\mathfrak{A} \models \underline{R}(a,b)$. By (3) , $\mathfrak{A}' \models \underline{R}(a,b)$, which means that $R \in G$. □

REMARK 9 : In (2) of the proposition, the models $\mathfrak{B}$ and $\mathfrak{B}'$ obtained in the above proof are isomorphic to $D \cdot E$-prod ω, with $f(a,b)$ and $f(a',b')$ corresponding to $[f]_{D \cdot E}$.

Using Theorem 2, and (1)⟹(3) of the last proposition, we find

COROLLARY 10 : _Assuming_ FRH(ω) , _there are uniform ultrafilters_ D _and_ E on ω _such that all first-order properties of any two elements_ a, b _of any elementary extension of_ ω _are completely determined by the following information_ : a _has type_ D , b _has type_ E , _and the_

the relative order of a and b . □

REMARK 11 : Of course all first-order properties of a single element are determined by its type. The types of two elements determine all their first-order properties only if one of the types is principal, for otherwise the relative order of the two elements is not determined. (This follows easily from the compactness theorem.) Corollary 10 then says that, in certain cases, this relative order is the only additional information needed to determine everything. Extensions to more than two elements can be obtained by appealing to Corollary 4 .

Now we consider the topological interpretation of the statement that $D \cdot E$ and $E \cdot D$ are the only ultrafilters containing $D \times E$. The natural inclusion of $\omega \times \omega$ into the compact space $\beta\omega \times \beta\omega$ factors uniquely through $\beta(\omega \times \omega)$ (by definition of Stone-Čech compactification). One can easily compute that the map

$$p : \beta(\omega \times \omega) \to \beta\omega \times \beta\omega$$

maps an ultrafilter F to $(\pi_1(F), \pi_2(F))$. Thus, for $D, E \in \beta\omega$, $p^{-1}\{(D,E)\}$ consists of all ultrafilters F containing $D \times E$ (by Lemma 3. 2). From what we already know, we can immediately deduce

COROLLARY 12 : (1) $p^{-1}(D, E)$ has only one element if and only if D or E is principal.

(2) $p^{-1}(D, D)$ has at least three elements unless D is principal; it has exactly three elements if and only if D is minimal.

(3) Assuming FRH(ω), there are points of $\beta\omega \times \beta\omega$ whose inverse image under the map p consists of exactly two points. □

REMARK 13 : The first part of (2) can be slightly strengthened. If Card $p^{-1}(D, D)$ is finite, then it is odd, for the map t takes $p^{-1}(D, D)$ to itself, has order 2, and fixes the single element $\delta(D)$.

The question naturally arises of determining all possible cardinalities for $p^{-1}(D, E)$. We know that 1, 2, and 3 are possible if FRH(ω). In fact,

COROLLARY 14 : If FRH(ω), then, given $n < \omega$, $n \neq 0$, there is a point of $\beta\omega \times \beta\omega$ which is the image of exactly n points of $\beta(\omega \times \omega)$.

Proof : Let $D^0, \ldots D^{n-1}$ be as in Corollary 4, and set $D = D^0$, $E = D^1 \ldots D^{n-1}$.

Since $E \subseteq D^1 \times \ldots \times D^{n-1}$, any ultrafilter $F \supseteq D \times E$ contains $D^0 \times \cdots \times D^{n-1}$, so $F = t_\sigma(D^{\sigma(0)} \ldots D^{\sigma(n-1)})$. Using Remark 6 and the fact that F projects to E, we find that σ must leave the numbers $1, 2 \ldots, n-1$ in their correct order; the only freedom in the choice of σ is where to insert the 0. Thus, there are n choices for σ, hence n possible F's. □

It is also possible for $p^{-1}(D, E)$ to be infinite. To obtain an example, start with any uniform ultrafilter D on ω. Let X be the set of functions $\omega \to \omega$ which have the value 0 at all but finitely many arguments and which do not take the same non-zero value twice. Let $e_n : X \to \omega$ be evaluation at n. It is easy to see that the family of sets

$$\{e_n^{-1}(A) \mid A \in D,\ n < \omega\}$$

has the finite intersection property, so it is contained in an ultrafilter F. As X is countable, F is isomorphic to an ultrafilter E on ω. Further, all the $[e_n]_F$ are distinct (for

$$\{x \in X \mid e_n(x) = e_m(x)\} \cap e_n^{-1}(\omega - \{0\}) \cap e_m^{-1}(\omega - \{0\}) = \emptyset$$

when $n \neq m$) and map F to D, so E has infinitely many morphisms f_n to D. Let

$$g_n : \omega \to \omega \times \omega : x \to (f_n(x), x) .$$

Then

$$\pi_1 g_n(E) = f_n(E) = D$$

and

$$\pi_2 g_n(E) = E$$

so

$$g_n(E) \supseteq D \times E \quad .$$

As the g_n are all one-to-one and distinct modulo E, the $g_n(E)$ are all distinct by Corollary 2.6 .

It is known that any closed infinite subset of $\beta\omega$ (or the homeomorphic $\beta(\omega \times \omega)$) has cardinality 2^{2^ω}. (See [6, p. 134].) Hence,

COROLLARY 15 : Assume FRH(ω) . The inverse images of points under the natural mapping $\beta(\omega \times \omega) \to \beta\omega \times \beta\omega$ can have the following cardinalities and no others : All finite numbers except 0, and 2^{2^ω}. □

Corollary 3 told us that the D and E of Theorem 2 can be taken to the minimal. The following proposition implies that they are necessarily P-points.

PROPOSITION 16 : Let D and E be uniform ultrafilters on ω. The following are equivalent.

(1) E is a P-point.

(2) For any $f : \omega \to \omega$, let

$$A_f = \{(x, y) \mid f(x) < y\} .$$

$(D \times E) \cup \{A_f \mid f : \omega \to \omega\}$ generates an ultrafilter F. (F must be $D \cdot E$, for all A_f are in $D \cdot E$.)

Proof : First, suppose E is a P-point, and let $R \subseteq \omega \times \omega$. We shall show that, if $R \in D \cdot E$, then R is in the filter F generated by $(D \cdot E) \cup \{A_f \mid f : \omega \to \omega\}$; this clearly implies $F = D \cdot E$ and thereby proves (2). As $R \in D \cdot E$, we have $\tilde{R}(i) \in E$ for all i in some set $X \in D$. As E is a P-point, it contains a set Y such that, for all $i \in X$, $Y - \tilde{R}(i)$ is finite. (See Proposition 9.1.) For $i \in X$, let

$$f(i) = \max (Y - \tilde{R}(i)) ;$$

for $i \notin X$, let $f(i) = 0$. Then, if $(x, y) \in (X \times Y) \cap A_f$, we have

$$x \in X , \ y \in Y , \ y > f(x) = \max (Y - \tilde{R}(x)) ,$$

so $y \in \tilde{R}(x)$, and $(x,y) \in R$. As $(X \times Y) \cap A_f \in F$, $R \in F$.

Conversely, suppose (2) holds, and let $f : \omega \to \omega$. Let

$$R = \{(i,j) \mid f(j) > i\} \quad .$$

If $R \notin D \cdot E$, then, for some i (in fact for D-most i)

$$\tilde{R}(i) = \{j \mid f(j) > i\} \notin E \quad ,$$

so f is bounded on a set of E (namely $\omega - \tilde{R}(i)$) , and therefore f is constant on a set of E . On the other hand, suppose $R \in D \cdot E = F$. Then there exist $X \in D$, $Y \in E$, and $g : \omega \to \omega$ such that $(X \times Y) \cap A_g \subseteq R$. Given any $n \in \omega$, choose $i \in X$, $i > n$ (as D is uniform). Then, for $j \in Y$, $(i,j) \in X \times Y$. So

$$(i,j) \in A_g \implies (i,j) \in R \quad ;$$

that is,

$$j > g(i) \implies f(j) > i > n \quad .$$

So f assumes the value n at most $g(i) + 1$ times on Y . Therefore, f is either constant or finite-to-one on a set of E , so E is a P-point. □

§ 21. Products of minimal ultrafilters In this section we shall use minimal ultrafilters and their products to get new information about the structure (or lack thereof) of $RK(\leq \omega)$.

THEOREM 1 : Let $D_1, \ldots, D_n$ be minimal ultrafilters on ω. Any $F \leq D_1 \cdot D_2 \ldots D_n$ is isomorphic to $D_{i_1} \ldots D_{i_s}$ for some $1 \leq i_1 < i_2 < \ldots < i_s \leq n$, provided we agree that the empty product $(s = 0)$ is a principal ultrafilter.

Proof : The case $n = 1$ is true by definition of minimality, and we proceed by induction on n. So let $D_1, \ldots, D_n$ be given $(n \geq 2)$ and assume the theorem for $n-1$. Suppose $f : \omega^n \to \omega$ maps $D_1 \cdot D_2 \ldots D_n$ to F. For simplicity, let $E = D_n$ and $D = D_1 \ldots D_{n-1}$, so $f : D \cdot E \to F$. For each $i \in \omega^{n-1}$, $\tilde{f}(i) : \omega \to \omega$ is one-to-one or constant on a set of E, as E is minimal. If, for D-most i, $\tilde{f}(i)$ is constant on a set of E, then f is equal modulo $D \cdot E$ to a map that factors through $\pi : \omega^n \to \omega^{n-1}$, the first projection. But then $F \leq \pi(D \cdot E) = D$, and the required conclusion follows from the induction hypothesis. So suppose, from now on, that, for D-most i, $\tilde{f}(i)$ is one-to-one on a set of E ; replacing f by a map equal to it modulo $D \cdot E$, we may suppose that $\tilde{f}(i)$ is one-to-one on all of

ω for all i. Let $g : \omega^{n-1} \to \omega$ be a function such that

$$g(i) = g(j) \iff \tilde{f}(i)(E) = \tilde{f}(j)(E)$$

$$(\iff \tilde{f}(i) = \tilde{f}(j) \bmod E \text{ , by Corollary 2.6}) .$$

Let $\quad G_n = \tilde{f}(i)(E)$ for any i with $g(i) = n$. Then

$$\begin{aligned} F &= f(D \cdot E) \\ &= D\text{-}\lim_i \tilde{f}(i)(E) \\ &= D\text{-}\lim_i G_{g(i)} \\ &= g(D)\text{-}\lim_n G_n . \end{aligned}$$

As the G_n are distinct P-points (being minimal), Propositions 15.9 and 15.11 give that

$$F \tilde{=} g(D) \text{-} \Sigma_n G_n \tilde{=} g(D) \cdot E .$$

Applying the induction hypothesis to $g(D)$, we get the required conclusion. □

LEMMA 2 : (1) If, for D-most i, F_i is non-principal, then $D < D - \Sigma_i F_i$.

(2) If D and F_i are non-principal for D-most i, then $D\text{-}\Sigma_i F_i$ is neither principal nor minimal.

(3) If D and E are non-principal, then $D \cdot E$ is neither principal nor minimal.

(4) If $D\text{-}\Sigma_i E_i \cong D\text{-}\Sigma_i F_i$ with D, E_i, F_i of size ω, then for D-most i, $E_i \cong F_i$.

(5) If $D \cdot E \cong D' \cdot E'$, where D, D', E, E' have size ω, and if either E and E' are minimal or D and D' are minimal, then $D \cong D'$ and $E \cong E'$.

Proof: (1) follows from Lemma 15.7(5) and Corollary 2.6. (2) and (3) follow from (1). (4) follows from Theorem 19.2 and (1). (5) follows from Corollary 19.4 and (2). □

THEOREM 3: Let $D_1, \ldots, D_n, D_1', \ldots, D_m'$ be minimal ultrafilters on ω. If $D_1 \ldots D_n \cong D_1' \ldots D_m'$, then $m = n$, and, for $1 \leq i \leq n$, $D_i \cong D_i'$.

Proof: If $n = 1$, then then $m = 1$ by (3) of the lemma, and the assertion of the theorem holds. Proceeding by induction on n, suppose that $n \geq 2$, that the assertion holds for $n-1$, and that $D_1 \ldots D_n \cong D_1' \ldots D_m'$. By (5) of the lemma, $D_n \cong D_m'$, and $D_1 \ldots D_{n-1} \cong D_1' \ldots D_{m-1}'$. By induction hypothesis, the assertion of the theorem follows. □

As an application, we have

PROPOSITION 4 : If FRH(ω), then RK($\leq \omega$) is neither an upper nor a lower semi-lattice.

Proof : By Corollary 20. 3 , let D and E be minimal ultrafilters on ω such that $D \cdot E$ and $t(E \cdot D)$ are the only ultrafilters containing $D \times E$. First, we shall show that $D \not\geq E$. If $g : D \to E$, then the map

$$f : \omega \to \omega \times \omega : x \to (x, g(x))$$

takes D to an ultrafilter which contains $D \times E$ (by direct computation using Lemma 3. 2) but which, being isomorphic to D (via f) cannot be $D \cdot E$ or $t(E \cdot D)$ by Lemma 2(1). This contradicts the choice of D and E, so there can be no $g : D \to E$.

Now, by Theorem 1, the only elements of RK($\leq\omega$) below $\overline{D \cdot E}$ are $\overline{D \cdot E}$, $\overline{E}$, $\overline{D}$, and $\overline{0}$. By Lemma 2(3), none of these except possibly $\overline{D \cdot E}$ can equal $\overline{E \cdot D}$. But $D \cdot E \cong E \cdot D$ implies $D \cong E$ (by Lemma 2(5)) which is not the case. Hence $E \cdot D \not\leq D \cdot E$, and symmetrically $D \cdot E \not\leq E \cdot D$.

Hence, the only common lower bounds of $\overline{D \cdot E}$ and $\overline{E \cdot D}$ are $\overline{D}$, $\overline{E}$, and $\overline{0}$. As none of these is $\geq$ the others, $\overline{D \cdot E}$ and

$\overline{E \cdot D}$ have no greatest lower bound .

By Proposition 3. 3, any common upper bound of $\overline{D}$ and $\overline{E}$ is above either $\overline{D \cdot E}$ or $\overline{E \cdot D}$. As these two products are incomparable, $\overline{D}$ and $\overline{E}$ have no least upper bound. □

REMARKS 5 : With D and E as in the preceding proof , $\overline{D \cdot E}$ and $\overline{E \cdot D}$ have no least upper bound either. For, the only elements of RK that are below both of the upper bounds $\overline{D \cdot E \cdot D}$ and $\overline{E \cdot D \cdot E}$ are $\overline{0}$, $\overline{D}$, $\overline{E}$, $\overline{D \cdot E}$, and $\overline{E \cdot D}$, none of which is an upper bound of $\overline{D \cdot E}$ and $\overline{E \cdot D}$. Thus, as promised in Section 5, we have two elements of $RK(\omega)$ which have neither a least upper bound nor a greatest lower bound in RK .

Combining the ideas in the proofs of Proposition 4 and Corollary 20. 14, we can obtain two ultrafilters D and E such that the set of upper bounds of $\overline{D}$ and $\overline{E}$ has exactly n minimal elements and every common upper bound is above one of these n , for any prescribed $n \neq 0$ $(n < \omega)$. (Proposition 4 was the case $n = 2$.)

LEMMA 6 : <u>Let</u> D <u>be a minimal ultrafilter on</u> ω, E <u>and</u> F_i $(i < \omega)$ <u>arbitrary ultrafilters on</u> ω . <u>If</u> $D \cdot E \leq D\text{-}\Sigma_i F_i$, <u>then, for</u> D-<u>most</u> i , $E \leq F_i$.

Proof : Let $f : \omega \times \omega \to \omega \times \omega$ map $D\text{-}\Sigma_i F_i$ to $D \cdot E$. The function $i \to \tilde{f}(i)(F_i)$ is one-to-one or constant on a set A of D .

Case 1 : All $\tilde{f}(i)(F_i)$ for $i \in A$ are the same F . Then

$$
\begin{aligned}
E < D \cdot E &= f(D\text{-}\Sigma_i F_i) \\
&= D\text{-}\lim_i \tilde{f}(i)(F_i) \\
&= F \\
&\leq F_i
\end{aligned}
$$

for all $i \in A \in D$.

Case 2 : All $\tilde{f}(i)(F_i)$ for $i \in A$ are distinct. Then, by minimality of D and Propositions 15. 3 and 15. 14,

$$
\begin{aligned}
D \cdot E &= D\text{-}\lim_i \tilde{f}(i)(F_i) \\
&\cong D\text{-}\Sigma_i \tilde{f}(i)(F_i) \quad .
\end{aligned}
$$

By Theorem 19. 2 , for D-most i ,

$$E \cong \tilde{f}(i)(F_i) \leq F_i \quad ,$$

cases (2) and (3) of that theorem being ruled out by Lemma 2(1). □

THEOREM 7 : Assume $FRH(\omega)$. Then $P(\omega)$, partially ordered by inclusion, can be order-isomorphically embedded into $RK(\leq \omega)$.

Proof : Using Corollary 8.9, let D and E_n $(n < \omega)$ be countably many pairwise non-isomorphic minimal ultrafilters on ω. For any $A \subseteq \omega$ and any $i < \omega$, let F_i^A be an ultrafilter on ω isomorphic to $E_{n_1} \cdots E_{n_s}$ where the n_j are the elements of $A \cap i$ in increasing order. If $A \subseteq B$, then, for any i, $F_i^A \leq F_i^B$, because the product of E_n's for $n \in A \cap i$ is the image of the corresponding product for B under a projection map. Define

$$G^A = D\text{-}\Sigma_i F_i^A \quad .$$

By Lemma 15.7(7), if $A \subseteq B$, then $G^A \leq G^B$. Thus, $P(\omega)$ is mapped into $RK(\leq \omega)$ in an order-preserving way by $A \to G^A$.

We must still show that $G^A \leq G^B$ implies $A \subseteq B$. Suppose not ; let $G^A \leq G^B$ but $A \nsubseteq B$. Let $p \in A - B$. As $\{p\} \subseteq A$, $G^{\{p\}} \leq G^A \leq G^B$. For D-most i (namely, all $i > p$), $F_i^{\{p\}} \cong E_p$, so $G^{\{p\}} \cong D \cdot E_p$. Hence,

$$D \cdot E_p \leq G^B = D\text{-}\Sigma_i F_i^B \quad .$$

By the lemma, $E_p \leq F_i^B$ for D-most i. By definition of F_i^B and Theorem 1, E_p is isomorphic to a product of certain E_m's with $n \in B$. By Theorem 3, $E_p \cong E_n$ for some $n \in B$, which is impossible because $p \notin B$ and the various E_n's are non-isomorphic. □

Notice that Corollary 9.10 is an immediate consequence of Theorem 7(except that $FRH(\omega)$ is used), for $\mathbb{R}$ can be isomorphically embedded into $P(\omega)$.

§ 22. <u>External ultrafilters</u> We have three partial orderings, $\leq$, $\leq_\omega$, and $\leq_{RF}$ on RK(ω). (If $\kappa > \omega$, then $\leq_\kappa$ is trivial on RK(ω) by Corollary 13. 3) We have seen (Corollary 15. 18) that $\leq_{RF}$ implies $\leq_\omega$, which in turn implies $\leq$. The latter implication is not reversible, for $\leq$ is directed upward (Proposition 5. 10) while $\leq_\omega$ is a tree ordering (Corollary 13. 8). A tree is directed only if it is linearly ordered, which RK(ω) is not by a result of Kunen [12] (or of we assume FRH(ω), by Corollary 8. 9). The possibility that $\leq_{RF}$ and $\leq_\omega$ agree seems more plausible ; at least they are both trees. But we shall show in this section that they do not agree. In fact, we prove

THEOREM 1 : <u>Let</u> D <u>be any uniform ultrafilter on</u> ω. <u>There is an</u> E <u>on</u> ω <u>such that</u> $D \leq_\omega E$ <u>but</u> $D \not\leq_{RF} E$, <u>assuming</u> CH .

The proof is quite long and involves several intermediate propositions. We begin by observing that the content of the theorem is unchanged if we require E to be on $\omega \times \omega$ rather than ω and if we assume that the IS(ω)-morphism from E to D is the projection to the first factor $\pi : \omega \times \omega \to \omega$. E will be obtained by constructing the non-standard ultrafilter $F = E/D$ in ${}^*P(\omega)$ in D-prod V . By Corollary 14. 2 the requirement that π be an IS(ω)-morphism means

(1) If f is an internal function on ${}^*\omega$ such that in D-prod V , Card $f''\,{}^*\omega < {}^*\omega$, then there is an internal $A \subseteq {}^*\omega$ such that $A \in F$ and $f \restriction A$ is constant in D-prod V.

(In other words, given an internal partition of ${}^*\omega$ into *-finitely many internal pieces, one of the pieces lies in F .) In order that E not be D-$\Sigma_i G_i$ (which would imply $D \leq_{RF} E$), we must have, by Corollary 15.17,

(2) F is external .

A priori , it appears that we must require more, for $D \nleq_{RF} E$ means not only that E cannot equal D-$\Sigma_i G_i$ but also that they cannot even be isomorphic. Hence we prove

LEMMA 2 : If π is an IS(ω)-morphism from E on $\omega \times \omega$ to D on ω, and if E is not of the form D-$\Sigma_i G_i$, then E is not isomorphic to any E′ of that form.

Proof : Suppose f : D-$\Sigma_i G_i$ = E′ → E is an isomorphism . Without loss of generality, suppose the G_i are on ω, so E′ is on $\omega \times \omega$. Both π and $\pi \circ f$ are IS(ω)-morphisms from E′ to D (see Corollary 15.18). By Corollary 13.6, $\pi = \pi \circ f$ mod E′. By modifying f on the complement of a set of E′, we may assume $\pi = \pi \circ f$. But then

$$E = f(D\text{-}\Sigma_i G_i) = D\text{-}\Sigma_i \pi' \tilde{f}(i)(G_i) \quad ,$$

where $\pi' : \omega \times \omega \to \omega$ is the second projection, by Lemma 15.7(7), contrary to the hypothesis . □

Thus, to prove Theorem 1, all we need to do is construct an ultrafilter F in ${}^*P(\omega)$ satisfying (1) and (2) . The following proposition , besides being an important step in that construction, is of interest in its own right.

PROPOSITION 3 : Let D be a uniform ultrafilter on ω. There is an external subset A of ${}^*\aleph_1$ such that, for each $x \in {}^*\aleph_1$, $A \cap x$ is internal .

Proof : If $x \leq y$, then, for any A , $A \cap x = (A \cap y) \cap x$, so $A \cap x$ will be internal provided $A \cap y$ is . Thus, it will suffice to check that A is internal for cofinally many $x \in {}^*\aleph_1$. Since D is on ω, the standard ordinals ${}^*\alpha\,(\alpha \in \aleph_1)$ are cofinal in ${}^*\aleph_1$; any $[f]_D \in {}^*\aleph_1$, where $f : \omega \to \aleph_1$, is majorized by ${}^*\alpha$, where $\alpha = \mathrm{Un}(\mathrm{Ra}(f)) \in \aleph_1$. So we need only make sure that $A \cap {}^*\alpha$ is internal for all (standard) $\alpha \in \aleph_1$.

We shall define functions $A^{\alpha}: \omega \to P(\alpha)$, one for each countable ordinal α, so that, for $\alpha < \beta < \aleph_1$,

$$\text{(3)} \quad C\{\alpha, \beta\} = \{n \mid A^{\alpha}_n = A^{\beta}_n \cap \alpha\} \in D \quad .$$

Then $[A^{\alpha}]_D = [A^{\beta}]_D \cap {}^*\alpha$ in D-prod V. We shall let

$$A = \bigcup_{\alpha < \aleph_1} [A^{\alpha}] \subseteq {}^*\aleph_1$$

Then $A \cap {}^*\alpha = [A^{\alpha}]$ is internal. We shall also make the A^{α} sufficiently complicated that A itself will not be internal. This will prove the proposition.

We define A^{γ} by induction on γ. For finite γ, set $A^{\gamma}(n) = \emptyset$ for all n ; clearly (3) holds. Now suppose $\gamma \geq \omega$, A^{α} is defined for $\alpha < \gamma$, and (3) holds for $\alpha < \beta < \gamma$. Let $g : \omega \to \gamma$ be a bijection. For $n < \omega$, let

$$H(n) = (\omega - n) \cap \bigcap_{i,j < n} C\{g(i), g(j)\} \in D \quad ,$$

and let h(k) be the first n such that $k \notin H(n)$; h(k) exists because $k \notin H(k+1)$. Among the ordinals g(i) for $i < h(k)$, let $\xi(k)$ be the largest ; by definition of h, H, and C,

$$A^{g(i)}(k) = A^{\xi(k)}(k) \cap g(i) \quad \text{for all } i < h(k) \quad .$$

Thus, (3) will continue to hold for $\beta = \gamma$, provided we choose A^{γ} so that, for all $k < \omega$,

(4) $A^{\gamma}(k) \cap \xi(k) = A^{\xi(k)}(k)$,

for, given any $\alpha = g(i) < \gamma$, we have

$$\begin{aligned} k \in H(i+1) &\Rightarrow i < h(k) \\ &\Rightarrow A^{\alpha}(k) = A^{g(i)}(k) \\ &\qquad = A^{\xi(k)}(k) \cap g(i) \\ &\qquad = A^{\gamma}(k) \cap g(i) \quad , \end{aligned}$$

and $H(i+1) \in D$.

We must still make sure that A is external ; all we have said so far does not rule out the possibility that all $A^{\alpha}(k)$ are $\emptyset$, which we clearly do not want. Every ordinal $\alpha \geq \omega$ can be uniquely written in the form $\lambda + n$ where λ is a limit ordinal and $n < \omega$; let us write $\lambda(\alpha)$ and $n(\alpha)$ for the λ and n whose sum is α. Let $R : \aleph_1 \to P(\omega)$ be a function such that, if $\alpha \neq \beta$, then $R(\alpha)$ and $R(\beta)$ have infinite symmetric difference. (For example, let R' be any one-to-one map $\aleph_1 \to P(\omega)$, let $f : \omega \to \omega \times \omega$ be a bijection and let $R(\alpha) = f^{-1}\pi^{-1}(R'(\alpha))$.) Define, for any $\theta < \gamma$,

$$(5)\quad \theta \in A^{\gamma}(k) \iff \theta < \xi(k) \quad \text{and} \quad \theta \in A^{\xi(k)}(k)\,, \quad \text{or}$$

$$\theta \geq \xi(k) \quad \text{and} \quad n(\theta) \in R(\gamma) \quad .$$

The first clause of (5) gives (4); the second will give that A is external.

The functions g, H, h, and ξ are all dependent on γ ; when necessary, we shall write them with γ as a subscript .

Temporarily fix $k < \omega$, and suppose X is a set of countable limit ordinals such that

$$\alpha < \beta \quad \text{and} \quad \alpha, \beta \in X \implies k \in C\,\{\alpha, \beta\}$$

$$(\iff A^{\alpha}(k) = A^{\beta}(k) \cap \alpha) \quad .$$

I claim that no element of X can be the limit of a sequence of smaller elements of X . Suppose not ; say $\alpha_1 < \alpha_2 < \ldots$ is a sequence $\subseteq X$ with limit $\beta \in X$. As $\xi_{\beta}(k) < \beta$, we must have $\xi_{\beta}(k)$ less than one of the α's ; omitting an initial segment of the α-sequence, we may suppose $\xi_{\beta}(k) < \alpha = \alpha_1$. Let γ be the larger of $\xi_{\beta}(k)$ and $\xi_{\alpha}(k)$, so $\gamma < \alpha$. If ζ is an ordinal such that $\gamma \leq \zeta < \alpha$, then the definitions of A^{α} and A^{β}, together with the fact that $A^{\alpha}(k) = A^{\beta}(k) \cap \alpha$, give

$$n(\zeta) \in R(\alpha) \iff \zeta \in A^{\alpha}(k)$$

$$\iff \zeta \in A^{\beta}(k)$$

$$\iff n(\zeta) \in R(\beta) \quad .$$

Applying this to $\zeta = \gamma$, $\gamma + 1$, $\gamma + 2$, ..., all of which are $< \alpha$ because α is a limit ordinal, we find that the symmetric difference of $R(\alpha)$ and $R(\beta)$ contains only numbers less than $n(\gamma)$, contrary to the definition of R. This proves that X contains none of its own limit points.

Now, let $B : \omega \to P(\aleph_1)$ be any function. For each k, let X_k be the set of limit ordinals α such that $A^{\alpha}(k) = B(k) \cap \alpha$. By the preceding paragraph, X_k contains none of its limit points. Let

$$L = \{k \in \omega \mid X_k \text{ is countable}\}$$

and

$$M = \omega - L = \{k \in \omega \mid X_k \text{ is cofinal in } \aleph_1\} \quad .$$

Let α_0 be an ordinal $< \aleph_1$, but greater than all elements of X_k for all $k \in L$. Define α_n by induction on n as follows. If $n = 2^a(2b+1)$, let α_n be any ordinal in X_a which is $> \alpha_{n-1}$, provided $a \in M$ so such an α_n exists; if $a \in L$, let $\alpha_n > \alpha_{n-1}$

be arbitrary but $< \aleph_1$. Let β be the limit of the increasing sequence α_n . Thus, $\beta < \aleph_1$, and, by choice of α_0 , $\beta \notin X_k$ for $k \in L$. On the other hand, if $k \in M$, then β is the limit of the subsequence

$$\alpha_{2^k(2b+1)} \qquad b = 1, 2, \ldots < \omega$$

all of whose members are in X_k ; as X_k contains none of its own limit points, $\beta \notin X_k$. Thus $\beta \notin X_k$ for any k . Since β is clearly a limit ordinal, we conclude from the definition of X_k that

$$\{k \mid A^{\beta}(k) = B(k) \cap \beta\} = \emptyset \quad ,$$

so

$$[B] \cap {}^*\beta \neq [A^{\beta}] = A \cap {}^*\beta \quad ,$$

and, a fortiori, $A \neq [B]$. As B was arbitrary , A is external, and the proposition is proved . □

Before we complete the proof of Theorem 1 by constructing the required F , we make a few heuristic remarks to clarify the idea behind the construction. Recall the standard method of constructing ultrafilters (see Tarski [17]). You well-order all subsets of ω, and, starting with any family $\subseteq P(\omega)$ with the finite intersection property, you consider in turn each subset of ω , throwing it into the family if this can be done without destroying the finite intersection property ; otherwise you throw in its complement. This method gives preferential treatment to the set under consideration as opposed to its complement ; you could just as well throw in the complement whenever possible. More generally, if $A \subseteq 2^{\omega}$, you can use the following procedure. The αth time you have to make a choice (i.e., either the set or its complement can safely be thrown in), choose the set if $\alpha \in A$, the complement if $\alpha \notin A$. The ultrafilter you get will "encode" A (unless you get an ultrafilter with a basis of cardinality $< 2^{\omega}$; in the theorem, we are assuming CH, so this is no problem). The idea is, in the non-standard world, to get F to encode the A of the proposition. F cannot be constructed in the non-standard world, for A doesn't exist there ; indeed, we want F to be external. But F cannot be constructed directly in the real world either, for here ${}^*\aleph_1$ is not well-ordered. The solution of this difficulty is a division of labor between the two worlds. The residents of D-prod V can construct approximations F_{α} to the required F using the internal sets $A \cap {}^*\alpha$. (Each F_{α} is internal, but the sequence of all of them is not.) Then we, in the real world, use these approximations to define F.

Using CH, let $S : \aleph_1 \to P(\omega)$ be a bijection. Let $B \subseteq \aleph_1$.

Inductively define a nested sequence of filters $G^B(\alpha)$ and a non-decreasing sequence of ordinals $\gamma^B(\alpha)$ $(\alpha < \aleph_1)$ as follows. $G^B(0)$ consists of the cofinite subsets of ω, and $\gamma^B(0) = 0$. If λ is a limit, $G^B(\lambda) = \bigcup_{\alpha<\lambda} G^B(\alpha)$ and $\gamma^B(\lambda) = \sup_{\alpha<\lambda} \gamma^B(\alpha)$. For successors, let $G^B(\alpha+1)$ be

(a) $G^B(\alpha)$ if $S(\alpha)$ or $\omega - S(\alpha)$ is in $G^B(\alpha)$,

(b) The filter generated by $G^B(\alpha) \cup \{S(\alpha)\}$ if case (a) doesn't apply and $\gamma^B(\alpha) \in B$,

(c) The filter generated by $G^B(\alpha) \cup \{\omega - S(\alpha)\}$ otherwise;

in case (a) set $\gamma^B(\alpha+1) = \gamma^B(\alpha)$, and in the other cases set

$$\gamma^B(\alpha+1) = \gamma^B(\alpha) + 1 .$$

Let

$$F^B = \bigcup_{\alpha<\aleph_1} G^B(\alpha) .$$

One sees (by induction on β) that if some ordinal $\beta < \aleph_1$ were not of the form $\gamma^B(\alpha)$, then the sequence $G^B(\alpha)$ would eventually become constant. This is impossible, because $G^B(\alpha)$ has a countable base, while F^B, being an ultrafilter, does not. Hence γ^B maps onto $\aleph_1$. Let $\delta^B(\beta)$ be the first α such that $\delta^B(\alpha) = \beta + 1$; thus δ^B is a strictly increasing map $\aleph_1 \to \aleph_1$, and $\delta^B(\beta) > \beta$.

If $B \neq C$ and β is the first ordinal in their symmetric difference, then $\delta^B(\xi) = \delta^C(\xi)$ for all $\xi \leq \beta$, and, for $\alpha < \delta^B(\beta)$, $G^B(\alpha) = G^C(\alpha)$ and $\gamma^B(\alpha) = \gamma^C(\alpha)$. $\delta^B(\beta)$ is the successor $\eta + 1$ of an ordinal η, and $G^B(\eta+1) \neq G^C(\eta+1)$, for one of these contains $S(\eta)$ while the other contains $\omega - S(\eta)$. Thus, $F^B \neq F^C$. For $\sigma < \eta$, $S(\sigma) \in F^B \Longleftrightarrow S(\sigma) \in F^C$. As $\eta \geq \beta$, this holds in particular for all $\sigma < \beta$.

If F is any uniform ultrafilter on ω, then $F = F^B$ for some $B \subseteq \aleph_1$. For we can let

$G(\alpha)$ = the filter generated by all cofinite sets plus those sets in F of the form $S(\beta)$ or $\omega - S(\beta)$ with $\beta < \alpha$,

and then the inductive conditions used to define G^B and γ^B above can be used "in reverse" to define γ^B and B given G.

Hence, we have a bijection $\Phi : P(\aleph_1) \to \text{unif}(\omega)$. With $A \subseteq {}^*\aleph_1$ as in the proposition, let

$$F_\alpha = ({}^*\Phi)(A \cap {}^*\alpha) .$$

(Actually we mean $({}^*\Phi)(B)$, where $B \in {}^*P(\aleph_1)$ represents $A \cap {}^*\alpha$.)

Thus, F_α is an internal uniform ultrafilter on $^*\omega$. If $\alpha < \beta$, then $A \cap {}^*\alpha$ and $A \cap {}^*\beta$ agree on ordinals $< {}^*\alpha$, so, by the remarks two paragraphs ago, F_α and F_β agree as far as the sets $^*S(x)$, $x < {}^*\alpha$, are concerned. Thus, we may define F to be the set of those $^*S(x)$ which are in F_α for one, hence for every, $\alpha < \aleph_1$ with $x < {}^*\alpha$.

If $X \in {}^*P(\omega)$, there are $x, y \in {}^*\aleph_1$ with $X = {}^*S(x)$ and $^*\omega - X = {}^*S(y)$. ($^*S$ is a bijection because S is one.) Choose an $\alpha < \aleph_1$ so that $x, y, < {}^*\alpha$. Then, as F_α is an ultrafilter on $^*P(\omega)$,

$$\begin{aligned} X \in F &\iff X \in F_\alpha \\ &\iff {}^*\omega - X \notin F_\alpha \\ &\iff {}^*\omega - X \notin F \quad . \end{aligned}$$

Similarly, F is closed under intersection, so F is an ultrafilter in $^*P(\omega)$. We now verify that it has properties (1) and (2).

There is a function Γ assigning to each function g on ω an upper bound $\Gamma(g) < \aleph_1$ for the countable set of ordinals of the form $S^{-1}(g^{-1}\{n\})$, $n \in g''\omega$. Let f be an internal function on $^*\omega$ such that Card $f''{}^*\omega < {}^*\omega$ in D-prod V. Let α be an ordinal $< \aleph_1$ with $({}^*\Gamma)(f) < {}^*\alpha$. Then, by definition of Γ, for all $\nu \in f''\,{}^*\omega$,

$f^{-1}\{\nu\}$ is ${}^*S(x)$ for some $x \leq {}^*\Gamma(f) < {}^*\alpha$, so $f^{-1}\{\nu\} \in F \Longleftrightarrow f^{-1}\{\nu\} \in F_\alpha$. It is true in V, hence in D-prod V, that:

A function taking fewer than $\underline{\omega}$ values is constant on some set of any given ultrafilter on its domain.

Hence, there is a $\nu \in f\,''\,{}^*\omega$ with $f^{-1}\{\nu\} \in F_\alpha$. Therefore $f^{-1}\{\nu\} \in F$, and (1) is proved.

If F were internal, it would be $({}^*\Phi)(B)$ for some $B \in {}^*P(\aleph_1)$. As A is external, it cannot be represented by B, so choose an $x_0 \in B \,\Delta\, A \subseteq {}^*\aleph_1$, where Δ denotes symmetric difference. Choose $\alpha < \aleph_1$ with ${}^*\alpha > ({}^*\delta)^B(x_0) > x_0$. The internal set $(A \cap {}^*\alpha) \,\Delta\, B$ is nonempty, so let x be its least element. As $x \leq x_0$, $({}^*\delta)^B(x) \leq ({}^*\delta)^B(x_0) < {}^*\alpha$. As δ maps to successor ordinals, there is a $y < {}^*\alpha$ such that ${}^*\delta^B(x) = y+1$. By the discussion following the definition of δ,

$${}^*S(y) \in F = ({}^*\Phi)(B) \Longleftrightarrow {}^*S(y) \notin ({}^*\Phi)(A \cap {}^*\alpha) = F_\alpha ,$$

contradicting the definition of F. Therefore (2) holds, and Theorem 1 is proved. □

The theorem can be improved as follows. In defining F^B, change the inductive conditions on G^B so that $G^B(\alpha+1)$ contains, in addition to $S(\alpha)$ or $\omega - S(\alpha)$, some canonically selected (i.e., depending only on $G^B(\alpha)$, not on B directly) set on which $Q(\alpha)$ is constant or one-to-one, where Q is a fixed bijection from $\aleph_1$ to $\mathrm{Hom}(\omega, \omega)$; call that set in $G^B(\alpha+1)$ $T^B(\alpha)$. If B and C first differ at β and $\alpha < \beta$, then $T^B(\alpha) = T^C(\alpha)$. Let F then be defined as in the proof just given, and consider any internal $f : {}^*\omega \to {}^*\omega$. f is ${}^*Q(x)$ for some $x \in {}^*\aleph_1$, and we choose α so that $x < {}^*\alpha$. For $\beta > \alpha$, $A \cap {}^*\alpha$ and $A \cap {}^*\beta$ first differ at α or later, so F_α and F_β contain the same set $T_x = ({}^*T)^{A \cap {}^*\alpha}(x)$ on which f is constant or one-to-one; therefore $T_x \in F$.

Let E be the ultrafilter on $\omega \times \omega$ determined by F; $F = E/D$ Then, as in the theorem, $\pi : E \to D$ is an $IS(\omega)$-morphism, and $D \not\leq_{RF} E$. (F is external because it still codes A; the proof of this is a bit more complicated than the corresponding part of the proof of the theorem.) Furthermore, if $f : \omega \times \omega \to \omega$, then, by what has just been shown, the internal map $[\tilde{f}]_D$ is constant or one-to-one on a set of F. Thus there is an $A \in E$ such that

$$\text{D-prod } V \models [\tilde{f}]_D \text{ is one-to-one or constant on } [\tilde{A}]_D ,$$

which means that, for D-most i, $\tilde{f}(i)$ is one-to-one or constant on $\tilde{A}(i)$. Replacing f be a map equal to it mod E, we may suppose that either f factors through π or $\tilde{f}(i)$ is one-to-one for all i.

Until now, D has been quite arbitrary. Let us now consider the special case that D is minimal. I claim that then any $IS(\omega)$-morphism f with domain E (where E is as in the preceding discussion) is either an isomorphism, or π followed by an isomorphism, or a constant map. If f factors through π, then we have one of the last possibilities, because D is minimal. So assume that $\tilde{f}(i)$ is one-to-one for all i. By Proposition 13.5 applied to f and π, we find that one factors through the other. Since we have disposed of the case that f factors through π, we assume π factors through f. This, together with the fact that all $\tilde{f}(i)$ are one-to-one, implies that f is one-to-one on a set of E, hence is an isomorphism. This proves the claim.

PROPOSITION 4 : <u>There is an</u> E <u>on</u> ω <u>which is minimal in</u> $RF(\omega)$ <u>but not in</u> $IS(\omega)$.

<u>Proof</u> : Let D and E be as in the preceding discussion. As $D <_{\omega} E$, is not minimal in $IS(\omega)$. Now suppose that $\overline{G} \leq_{RF} \overline{E}$; say

$$g ; E \xrightarrow{\cong} G - \Sigma_i H_i \quad .$$

Apply the preceding discussion to $f = \pi \circ g$. In the first case, πg is an isomorphism, so $\overline{G} = \overline{E}$. In the second case $\overline{G} = \overline{D}$, which is impossible because $\overline{D} \not\leq_{RF} \overline{E}$. In the last case, $\overline{G} = \overline{0}$. Thus, no element of $RF(\omega)$ is $<_{RF} \overline{E}$. □

Since all P-points are $IS(\omega)$-minimal, we have as a corollary the theorem of Kunen (quoted in [2]; also see [12]) that minimality in the RF-ordering is a strictly weaker condition than being a P-point.

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