ON THE VOLUMES OF BALLS

Andreas Blass and Stephen Schanuel

Abstract. We prove the formula for the volume of an even-dimensional ball by exhibiting an explicit, locally volume-preserving (in fact symplectic) map from a polydisk.

The volume of a ball $B_d(r)$ of radius r in d-dimensional Euclidean space is given by the well-known formula

$$\operatorname{vol}(B_d(r)) = \frac{\pi^{d/2} r^d}{(d/2)!}.$$

If d is odd, the fractional factorial in the denominator is to be interpreted using the gamma function, but we shall be concerned only with the case of even d. In this case, we have, setting $d = 2m$,

$$\operatorname{vol}(B_{2m}(r)) = \frac{(\pi r^2)^m}{m!} = \frac{(\operatorname{vol}(B_2(r))^m}{m!}.$$

Since the symmetric group of degree m acts on $B_2(r)^m$ by permuting the factors, and since almost all of the orbits of this action have size $m!$, the volume formula above strongly suggests the conjecture that there is a "nice," volume preserving correspondence between the orbit space of $B_2(r)^m$ and $B_{2m}(r)$, or at least between subsets of full measure in these spaces. Equivalently, one expects a "nice," locally volume preserving map $f : B_2(r)^m \to B_{2m}(r)$, such that $f(x_1, \dots, x_m)$ is unchanged by permutations of the x_i's.

The purpose of this paper is to exhibit two such maps f. The first is the simplest one we know; the second, though slightly more complicated, enjoys additional interesting properties. Both are "nice" at least in the senses of being independent

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of r (they are each the restriction of a single map $\mathbb{C}^m \to \mathbb{C}^m$) and of being well-defined and smooth away from a singular locus of dimension $< 2m$. For additional comments about "niceness" see the last paragraph of the paper.

The existence of such maps provides a proof, more explicit than the usual computational ones, of the volume formula for even-dimensional spheres. Here "more explicit" has a meaning similar to that arising in finite combinatorics, where bijective proofs are regarded as more explicit than computational ones. Just as in finite combinatorics, a proof based on an explicit bijection has the virtue that the bijection may preserve more structure than originally envisaged; indeed, this occurs for the second of our two maps, as we shall explain after the construction.

To describe our maps, it is convenient to regard $B_{2m}(r)$ as a ball centered at the origin in $\mathbb{C}^m$. In particular, $B_2(r)$ is a disk in $\mathbb{C}$, and so $B_2(r)^m$ is identified with $\{(z_1, \dots, z_m) \mid \max_j |z_j| < r\} \subseteq \mathbb{C}^m$. We write $z_j = r_j e^{i\alpha_j}$ for the jth coordinate in $\mathbb{C}^m$.

The first of our maps f is defined as follows. Given $(z_1, \dots, z_m)$, first re-order the components so that $r_1 \geq r_2 \geq \dots \geq r_m$, and then produce as output the m-tuple

$$\left(\sqrt{r_1^2 - r_2^2}\, e^{i\alpha_1}, \sqrt{r_2^2 - r_3^2}\, e^{i\alpha_2}, \dots, \sqrt{r_{m-1}^2 - r_m^2}\, e^{i\alpha_{m-1}}, r_m e^{i\alpha_m}\right).$$

The re-ordering instruction is ambiguous when two of the z_j have the same modulus r_j; for example, $(1, i)$ can be re-ordered as either $(1, i)$ or $(i, 1)$, resulting in an f-value of either $(0, i)$ or $(0, 1)$. So our function f is well-defined only where all r_j are distinct. This is almost all of $\mathbb{C}^n$, so the exceptional locus where f is undefined doesn't affect any volumes.

We note that f maps $D = \{(z_1, \dots, z_m) \mid \text{all } |z_j| \text{ distinct}\}$ onto $\{(z_1, \dots, z_m) \mid z_j \neq 0 \text{ for } j < m\}$, and that two points in D have the same image under f if and only if they differ by a permutation of the coordinates. Furthermore, the Euclidean norm of $f(z_1, \dots, z_n)$ is simply r_1, the largest of the norms of the z_j's. Thus, except for sets of measure zero, the polydisk $B_2(r)^m$ is mapped onto $B_{2m}(r)$.

It remains to check that f locally preserves volumes. For this purpose, it is convenient to use the coordinates $s_j = \frac{1}{2} r_j^2$ and α_j. In these coordinates, the volume form on $\mathbb{C}^m$ is simply $\prod_{j=1}^m ds_j\, d\alpha_j$. Furthermore, the transformation f, when expressed in these coordinates, is linear on each connected component of its domain D. Its matrix is the product of an even permutation matrix (for the initial re-ordering of coordinates) and a triangular matrix with all diagonal entries equal to 1. So its determinant is 1 and the volume is preserved.

A slight modification of the definition of f makes its behavior even nicer. Begin by re-ordering the input tuple as above, but then define the output of f to be

$$\left(\sqrt{r_1^2 - r_2^2}\, e^{i\alpha_1}, \sqrt{r_2^2 - r_3^2}\, e^{i(\alpha_1 + \alpha_2)}, \dots, \sqrt{r_{m-1}^2 - r_m^2}\, e^{i\sum_{j=1}^{m-1} \alpha_j}, r_m e^{i\sum_{j=1}^{m} \alpha_j}\right).$$

The proofs above concerning the old f apply equally well to the new. Unlike the old f, the new one is well-defined and continuous everywhere (but differentiable only on D). For example, the input $(1, i)$ now produces output $(0, i)$ no matter which of the two re-orderings is chosen.

In addition, the new f preserves the symplectic structure in the regions where it is smooth. That is, the 2-form $\sum_j dx_j\, dy_j = \sum_j ds_j\, d\alpha_j$ (where $x_j + iy_j = z_j$) is unchanged when pulled back along f. This property of f is stronger than preserving the volume; the volume form is, up to a constant factor, the mth exterior power of the symplectic 2-form. (We thank André Joyal for suggesting that we try to improve our result from "volume-preserving" to "symplectic.")

Finally, let us indicate a larger picture into which our example fits as one piece of a puzzle. As in [1], we call a subset of Euclidean space *semi-algebraic* if it can be defined by a finite combination, using propositional connectives, of polynomial equations and inequalities over $\mathbb{R}$. (By the well-known quantifier elimination for real-closed fields, this is equivalent to requiring the set to be definable by a first-order formula (with parameters) over the structure $(\mathbb{R}, +, \cdot)$.) Call a function semi-algebraic if its graph is. The problem of classifying semi-algebraic sets up to semi-algebraic isomorphism was treated in [1], along with the analogous problem for other categories like the piecewise linear category. If we are interested in volumes, then it is natural to consider only volume-preserving maps and to ignore lower-dimensional pieces of their domains and codomains. Thus, we arrive at the natural question: Given two semi-algebraic sets A and B with the same dimension d and the same d-dimensional volume, when do there exist lower-dimensional semi-algebraic sets A_0 and B_0 and a measure-preserving, semi-algebraic bijection from $A - A_0$ to $B - B_0$? The answer to the analogous question for piecewise linear sets and maps is "always," by a triangulation and subdivision argument. In the semi-algebraic case, the answer is certainly not "always"; consider the circumference of the unit circle and a straight line segment of the same length 2π. On the other hand, when A is the surface of a sphere in $\mathbb{R}^3$ and B is the curved surface of a circumscribed cylinder, the existence of such an area-preserving map, namely the projection perpendicular to the cylinder's axis, has been known since antiquity. Our example provides such a map when A is a fundamental domain $\{(z_1, \ldots, z_m) \mid r \geq |z_1| > \cdots > |z_m|\}$ for the action of the symmetric group on a polydisk and B is the ball $B_{2m}(r)$. It seems to be unknown whether such a map exists when A is the disk $B_2(1)$ and B is a $1 \times \pi$ rectangle (semi-algebraic circle-squaring). The general problem, to classify semi-algebraic sets up to semi-algebraic volume-preserving bijections modulo lower-dimensional sets, seems quite intractable.

References

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Mathematics Dept., University of Michigan, Ann Arbor, MI 48109, U.S.A.
E-mail address: `ablass@umich.edu`

Mathematics Dept., SUNY at Buffalo, 106 Diefendorf Hall, Buffalo, NY 14214
E-mail address: `schanuel@ubvms.cc.buffalo.edu`