

Quasivelocities and Symmetries in Nonholonomic Systems

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Abstract

This paper is concerned with the theory of quasivelocities for nonholonomic systems. The equations of nonholonomic mechanics are derived using the Lagrange–d’Alembert principle written in an arbitrary configuration-dependent frame. The paper also shows how quasivelocities may be used in the formulation of nonholonomic systems with symmetry. In particular, the use of quasivelocities in the analysis of symmetry that leads to unusual momentum conservation laws is investigated, as is the applications of these conservation laws and discrete symmetries to the qualitative analysis of nonholonomic dynamics. The relationship between asymptotic dynamics and discrete symmetries of the system is also elucidated.

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1 Introduction

Quasivelocities are the velocities of a mechanical system expressed relative to a configuration-dependent frame. The mathematical underpinnings of quasivelocities in texts such as [Neimark and Fufaev \[1972\]](#) and [Greenwood \[2003\]](#) have not been as developed as is desirable, although their usefulness was clearly demonstrated. The additional clarity provided by the present paper should expand the scope of applications and uses of this important concept.

The goal of this paper is to develop the use of quasivelocities in the dynamics of nonholonomic systems with symmetry. Key forms of the equations of dynamics including the Hamel equations and the Euler–Lagrange–Poincaré equations are derived, and their relationship to quasivelocities is spelled out. The equations of motion for nonholonomic systems are obtained, and conditions under which an appropriate choice of frame evokes otherwise hidden momentum conservation laws are found. The resulting formalism is utilized in the analysis of the dynamics of some instructive nonholonomic systems including the Chaplygin sleigh and the sleigh coupled to an oscillator.

Said from a slightly different perspective, quasivelocities are the components of a mechanical system’s velocity relative to a set of vector fields that span the fibers of the tangent bundle of the configuration space. These vector fields need not be associated with (local) configuration coordinates, so in fact, this paper does not mention “quasicoordinates”, a vacuous concept that has led to some confusion in the literature. A good example of quasivelocities is the set of components of the body angular velocity of a rigid body rotating about a fixed point.

One of the reasons for using quasivelocities is that the Euler–Lagrange equations written in generalized coordinates are not always effective for analyzing the dynamics of a mechanical system of interest. For example, it is difficult to study the motion of the Euler top if the Euler–Lagrange equations (either intrinsically or in generalized coordinates) are used to represent the dynamics. On the other hand, the use of the angular velocity components relative to a body frame pioneered by [Euler \[1752\]](#) results in a much simpler representation of dynamics. Euler’s approach was further developed by [Lagrange \[1788\]](#) for reasonably general Lagrangians on the rotation group and by [Poincaré \[1901\]](#) for arbitrary Lie groups (see [Marsden and Ratiu \[1999\]](#) for details and history). Other examples include the use of velocity and angular velocity components relative to a moving frame in the study of dynamics of a rigid body moving on a surface as discussed in [Routh \[1860\]](#) and [Markeev \[1992\]](#).

Quasivelocities used in [Lagrange \[1788\]](#) and [Poincaré \[1901\]](#) are associated with a group action. [Hamel \[1904\]](#) obtained the equations of motion in terms of quasivelocities that were unrelated to a group action on the configuration space. The Hamel equations include both the Euler–Lagrange and Euler–Poincaré equations (for the rigid body for example) as special cases.

This paper derives the Hamel equations from a variational point of view; that is, it develops the form of the principle of critical action that is equivalent to the Hamel equations. It is then shown how the quasivelocity approach is useful in the treatment of nonholonomic systems with symmetry. In particular, the selection of

suitable quasivelocities appropriate to the presence of discrete symmetries is developed along with the application to the qualitative analysis of dynamics.

Nonholonomic systems with symmetry are studied in Bloch, Krishnaprasad, Marsden, and Murray [1996] and improved in related references, such as Cendra, Marsden, and Ratiu [2001a]. According to these papers, symmetry and constraints define certain subbundles of the tangent bundle of the configuration manifold. As will be shown, it is beneficial to select frames that respect this subbundle structure, that is, frames obtained by concatenation of the bases of these subbundles. These frames elucidate the momentum equations and conservation laws in the system.

The dynamics of a nonholonomic system with symmetry on the subbundle whose fibers are tangent to the group orbits is governed by the *momentum equation*, an equation that was introduced in Bloch, Krishnaprasad, Marsden, and Murray [1996]. As shown in Zenkov, Bloch, and Marsden [1998], the momentum equation is important in the energy-based analysis of stability of relative equilibria. In the absence of external dissipation these relative equilibria are often orbitally asymptotically stable in certain directions in the phase system. Related work on this subject includes Ruina [1998], Bloch [2000], and Zenkov and Bloch [2003]. Additional background and references may be found in Bloch [2003].

The definition of nonholonomic momentum and the derivation of the momentum equation in Bloch, Krishnaprasad, Marsden, and Murray [1996] relies on the formalism that uses spatial frames. This paper utilizes the flexibility of Hamel equations and gives a derivation of the momentum equation in a body frame. In a sense, this is a more natural and straightforward approach since the spatial momentum is almost never conserved in the nonholonomic setting.

The structure of the momentum equation is then used to find conditions under which one has frames that reveal otherwise hidden momentum conservation laws. In some instances the construction of such a frame is algorithmic, although it often leads to implicitly defined frames. In other instances a frame that depends on the group variables in a nontrivial way is necessary for uncovering momentum conservation laws. An example of this latter case is the unbalanced Chaplygin sleigh.

It is not unusual to have points in the configuration space where the fields that define these frames become linearly dependent. This feature is studied and its utilization in the qualitative analysis of a system's dynamics is discussed.

The paper is organized as follows. In §2 we review the Hamel equations. In §3 nonholonomic systems are briefly discussed and their dynamics is written in the form of constrained Hamel equations. In §4 the equations of motion of systems with symmetry are derived using quasivelocities. This derivation differs from that in Bloch, Krishnaprasad, Marsden, and Murray [1996] and in Cendra, Marsden, and Ratiu [2001a] in interesting ways and illustrates the influence of symmetry on the structure of quasivelocities. The nonholonomic momentum conservation and the corresponding frame selection are discussed. In §5 nonholonomic systems on Lie groups and their measure-preserving properties are treated. Examples are given in §6, where, in particular, momentum conservation in the unbalanced Chaplygin sleigh and integrability of the coupled sleigh-oscillator system are established. Finally, the asymptotic behavior of momentum conservation laws is studied.

2 Lagrangian Mechanics

In this section we review the equations of motion for *holonomic systems* from the Lagrangian viewpoint.

2.1 Equations of Motion

The Euler–Lagrange Equations. A Lagrangian mechanical system is specified by a smooth manifold Q called the *configuration space* and a function $L : TQ \rightarrow \mathbb{R}$ called the *Lagrangian*. In many cases, the Lagrangian is the kinetic minus potential energy of the system with the kinetic energy defined by a Riemannian metric on the configuration manifold and the potential energy being a smooth function on Q . If necessary, non-conservative forces can be introduced (e.g., gyroscopic forces that are represented by terms in L that are linear in the velocity), but this is not discussed in this paper.

In local coordinates $q = (q^1, \dots, q^n)$ on the configuration space Q we write $L = L(q, \dot{q})$. The dynamics is given by the Euler–Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}, \quad i = 1, \dots, n. \quad (2.1)$$

These equations were originally derived by Lagrange in 1788 by requiring that simple force balance $\mathbf{F} = m\mathbf{a}$ be *covariant*, *i.e.* expressible in arbitrary generalized coordinates. A variational derivation of the Euler–Lagrange equations, namely Hamilton’s principle (also called the principle of critical action), came later in the work of Hamilton in 1834/35. For more details see Marsden and Ratiu [1999] and Bloch [2003].

The Hamel Equations. In this paragraph we introduce the Hamel’s equations. In §2.3 we derive these equations from a global variational principle, generalizing the reduced principle of critical action for systems with symmetry.

In many cases the Lagrangian and the equations of motion have a simpler structure when written using so-called *noncommuting variables*. An example of such a system is the *rigid body*. Below we develop a general approach that allows one to obtain the Euler–Lagrange equations in such noncommuting variables.

Let $q = (q^1, \dots, q^n)$ be local coordinates on the configuration space Q and $u_i \in TQ$, $i = 1, \dots, n$, be smooth independent *local* vector fields defined in the the same coordinate neighborhood.¹ The components of u_i relative to the basis $\partial/\partial q^j$ will be denoted ψ_i^j ; that is,

$$u_i(q) = \psi_i^j(q) \frac{\partial}{\partial q^j}, \quad i, j = 1, \dots, n, \quad (2.2)$$

where a sum on j is understood.

¹In certain cases, some or all of u_i can be chosen to be *global* vector fields on Q .

Let $v = (v^1, \dots, v^n) \in \mathbb{R}^n$ be the components of the velocity vector $\dot{q} \in TQ$ relative to the basis u_1, \dots, u_n , *i.e.*,

$$\dot{q} = v^i u_i(q); \quad (2.3)$$

then

$$l(q, v) := L(q, v^i u_i(q)) \quad (2.4)$$

is the Lagrangian of the system written in the local coordinates (q, v) on the tangent bundle TQ . The coordinates (q, v) are the Lagrangian analogues of non-canonical variables in Hamiltonian dynamics.

Given two elements $v, w \in \mathbb{R}^n$, define the antisymmetric bracket operation $[\cdot, \cdot]_q : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$[v, w]_q = [v^i u_i, w^j u_j](q)$$

where $[\cdot, \cdot]$ is the Jacobi–Lie bracket of vector fields on TQ . Therefore, each tangent space $T_q Q$ is isomorphic to the Lie algebra $V_q := (\mathbb{R}^n, [\cdot, \cdot]_q)$. Thus, if the fields u_1, \dots, u_n are independent in $U \subset Q$, the tangent bundle TU is diffeomorphic to a Lie algebra bundle over U .

The **dual** of $[\cdot, \cdot]_q$ is, by definition, the operation $[\cdot, \cdot]_q^* : V_q \times V_q^* \rightarrow V_q^*$ given by

$$\langle [v, \alpha]_q^*, w \rangle \equiv \langle \text{ad}_v^* \alpha, w \rangle := \langle \alpha, [v, w]_q \rangle.$$

Here ad^* is the dual of the usual ad operator in a Lie algebra; note that in general this operation need not be associated with a Lie group.

Let $u = (u_1, \dots, u_n) \in TQ \times \dots \times TQ$. For a function $f : Q \rightarrow \mathbb{R}$, define $u[f] \in V_q^*$ by $u[f] = (u_1[f], \dots, u_n[f])$, where $u_i[f] = \psi_i^j \partial_j f$ is the usual directional derivative of f along the vector field u_i . Viewing u_i as vector fields on TQ whose fiber components equal 0 (that is, taking the vertical lift of these vector fields), one defines the directional derivatives $u_i[l]$ for a function $l : TQ \rightarrow \mathbb{R}$ by the formula

$$u_i[l] = \psi_i^j \frac{\partial l}{\partial q^j}.$$

The evolution of the variables (q, v) is governed by the **Hamel equations**

$$\frac{d}{dt} \frac{\partial l}{\partial v} = \left[v, \frac{\partial l}{\partial v} \right]_q^* + u[l] \equiv \text{ad}_v^* \frac{\partial l}{\partial v} + u[l] \quad (2.5)$$

coupled with (2.3). In (2.5), $u[l] = (u_1[l], \dots, u_n[l])$. If $u_i = \partial/\partial q^i$, equations (2.5) become the Euler–Lagrange equations (2.1). Below, we will derive equations (2.5) using the principle of critical action. A different approach that makes connections with algebroids is studied in [Cariñena, Nunes da Costa, and Santos \[2007\]](#).

2.2 The Virtual Displacement Principle

A **virtual displacement** at $q \in Q$ is an element of the tangent space $T_q Q$ denoted $\delta q = (\delta q^1, \dots, \delta q^n)$. The **principle of virtual displacements** states that the equations of motion are determined by the requirement

$$\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0. \quad (2.6)$$

The quantities $\delta q^1, \dots, \delta q^n$ for a Lagrangian system are independent. Therefore, equation (2.6) is equivalent to the Euler–Lagrange equations (2.1)

We now rewrite the virtual displacement principle using the frame u_i , $i = 1, \dots, n$. From equations (2.2), (2.3) and (2.4) we obtain

$$\begin{aligned} u_i[l] &= u_i(L(q^k, v^j \psi_j^k)) = \frac{\partial L}{\partial q^j} \psi_i^j + \frac{\partial L}{\partial \dot{q}^k} \frac{\partial(v^l \psi_l^k)}{\partial q^j} \psi_i^j, \\ \frac{\partial l}{\partial v^k} &= \frac{\partial}{\partial v^k} L(q^i, v^j \psi_j^i) = \frac{\partial L}{\partial \dot{q}^i} \psi_k^i. \end{aligned}$$

Denote the components of the virtual displacement δq relative to the basis $u_1(q), \dots, u_n(q)$ by w^i and define the quantities $c_{il}^m(q)$ by

$$[u_i, u_l] = c_{il}^m u_m. \quad (2.7)$$

One finds that

$$c_{il}^m = (\psi^{-1})_k^m \left[\frac{\partial \psi_l^k}{\partial q^j} \psi_i^j - \frac{\partial \psi_i^k}{\partial q^j} \psi_l^j \right].$$

Thus, from (2.7),

$$([v_1, v_2]_q)^m = c_{ij}^m v_1^i v_2^j \quad \text{and} \quad ([v, \alpha]_q^*)_j = c_{ij}^m v^i \alpha_m.$$

The principle of virtual displacements becomes

$$\begin{aligned} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} \right) \delta q^j &= \left(\left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} \right] \psi_i^j - \frac{\partial L}{\partial q^j} \psi_i^j \right) w^i \\ &= \left(\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}^j} \psi_i^j \right] - \frac{\partial L}{\partial \dot{q}^j} \frac{d \psi_i^j}{dt} - u_i[l] + \frac{\partial L}{\partial \dot{q}^k} \frac{\partial(v^l \psi_l^k)}{\partial q^j} \psi_i^j \right) w^i \\ &= \left(\frac{d}{dt} \frac{\partial l}{\partial v^i} - u_i[l] + \frac{\partial L}{\partial \dot{q}^k} \left[\frac{\partial \psi_l^k}{\partial q^j} \psi_i^j - \frac{\partial \psi_i^k}{\partial q^j} \psi_l^j \right] v^l \right) w^i \\ &= \left(\frac{d}{dt} \frac{\partial l}{\partial v^i} - u_i[l] + \frac{\partial l}{\partial v^m} (\psi^{-1})_k^m \left[\frac{\partial \psi_l^k}{\partial q^j} \psi_i^j - \frac{\partial \psi_i^k}{\partial q^j} \psi_l^j \right] v^l \right) w^i \\ &= \left(\frac{d}{dt} \frac{\partial l}{\partial v^i} - u_i[l] + \frac{\partial l}{\partial v^m} c_{il}^m v^l \right) w^i. \end{aligned}$$

Since the components w^i of δq are independent, we conclude that the equations of motion are

$$\frac{d}{dt} \frac{\partial l}{\partial v^i} = c_{ji}^m \frac{\partial l}{\partial v^m} v^j + u_i[l], \quad (2.8)$$

which is the coordinate form of equations (2.5).

Equations (2.8) were introduced in Hamel [1904] (see also Neimark and Fufaev [1972] for details and some history).

2.3 The Principle of Critical Action

Let $\gamma : [a, b] \rightarrow Q$ be a smooth curve in the configuration space. A *variation* of the curve $\gamma(t)$ is a smooth map $\beta : [a, b] \times [-\varepsilon, \varepsilon] \rightarrow Q$ that satisfies the condition $\beta(t, 0) = \gamma(t)$. This variation defines the vector field

$$\delta\gamma(t) = \left. \frac{\partial\beta(t, s)}{\partial s} \right|_{s=0}$$

along the curve $\gamma(t)$.

Theorem 2.1. *Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian and $l : TQ \rightarrow \mathbb{R}$ be its representation in local coordinates (q, v) . Then, the following statements are equivalent:*

- (i) *The curve $q(t)$, where $a \leq t \leq b$, is a critical point of the action functional*

$$\int_a^b L(q, \dot{q}) dt \quad (2.9)$$

on the space of curves $\Omega(Q; q_a, q_b)$ in Q connecting q_a to q_b on the interval $[a, b]$, where we choose variations of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$.

- (ii) *The curve $q(t)$ satisfies the Euler–Lagrange equations*

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

- (iii) *The curve $(q(t), v(t))$ is a critical point of the functional*

$$\int_a^b l(q, v) dt \quad (2.10)$$

with respect to variations δv , induced by the variations $\delta q = w^i u_i(q)$, and given by

$$\delta v = \dot{w} + [v, w]_q.$$

- (iv) *The curve $(q(t), v(t))$ satisfies the Hamel equations*

$$\frac{d}{dt} \frac{\partial l}{\partial v} = \left[v, \frac{\partial l}{\partial v} \right]_q^* + u[l]$$

coupled with the equations $\dot{q} = \langle u(q), v \rangle \equiv v^i u_i(q)$.

For the early development of these equations see [Poincaré \[1901\]](#) and [Hamel \[1904\]](#).

Proof. The equivalence of (i) and (ii) is proved by computing the variational derivative of the action functional (2.9),

$$\delta \int_a^b L(q, \dot{q}) dt = \int_a^b \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = \int_a^b \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt.$$

Denote the components of $\delta q(t)$ relative to the basis $u_1(q(t)), \dots, u_n(q(t))$ by $w(t) = (w^1(t), \dots, w^n(t))$, that is,

$$\delta q(t) = \langle u, w \rangle \equiv w^i(t)u_i(q(t)).$$

To prove the equivalence of (i) and (iii), we first compute the quantities $\delta \dot{q}$ and $d(\delta q)/dt$:

$$\begin{aligned} \delta \dot{q} &= \delta [v^i(t)u_i(q(t))] = \delta v^i(t)u_i(q(t)) + v^i(t)\frac{\partial u_i}{\partial q^j}\delta q^j, \\ \frac{d(\delta q)}{dt} &= \frac{d}{dt}(w^i(t)u_i(q(t))) = \dot{w}^i(t)u_i(q(t)) + w^i(t)\frac{\partial u_i}{\partial q^j}\dot{q}^j. \end{aligned}$$

Since $\delta \dot{q} = d(\delta q)/dt$, we obtain

$$\begin{aligned} \delta v^i(t)u_i(q(t)) &= \dot{w}^i(t)u_i(q(t)) + v^k(t)w^l(t)\left(\frac{\partial u_l}{\partial q^j}\psi_k^j - \frac{\partial u_k}{\partial q^j}\psi_l^j\right)(q(t)) \\ &= \left(\dot{w}^i(t) + c_{ki}^j(q(t))v^k(t)w^l(t)\right)u_i(q(t)); \end{aligned}$$

that is,²

$$\delta v(t) = \dot{w}(t) + [v(t), w(t)]_{q(t)}.$$

To prove the equivalence of (iii) and (iv), we use the above formula and compute the variational derivative of the functional (2.10):

$$\begin{aligned} \delta \int_a^b l(q, v) dt &= \int_a^b \left(\frac{\partial l}{\partial q} \delta q + \frac{\partial l}{\partial v} \delta v \right) dt \\ &= \int_a^b \left(\frac{\partial l}{\partial q} w^i u_i + \frac{\partial l}{\partial v} (\dot{w} + [v, w]_{q(t)}) \right) dt \\ &= \int_a^b \left(u[l]w + \frac{\partial l}{\partial v} [v, w]_{q(t)} - \frac{d}{dt} \left(\frac{\partial l}{\partial v} \right) w \right) dt \\ &= \int_a^b \left(u[l] + \left[v, \frac{\partial l}{\partial v} \right]_{q(t)}^* - \frac{d}{dt} \frac{\partial l}{\partial v} \right) w dt. \end{aligned}$$

This variational derivative vanishes if and only if the Hamel equations are satisfied. ■

2.4 Systems with Symmetry

Assume now that a Lie group G acts on the configuration space Q . This action is denoted $q \mapsto gq = \Phi_g(q)$. Throughout this paper we make the assumption that the action of G on Q is free and proper. The quotient space Q/G , whose points are the group orbits, is called the *shape space*. It is known that if the group action is free and proper then shape space is a smooth manifold and the projection map

²If Q is a Lie group, this formula is derived in Bloch, Krishnaprasad, Marsden, and Ratiu [1996].

$\pi : Q \rightarrow Q/G$ is a smooth surjective map with a surjective derivative $T_q\pi$ at each point. The configuration space thus has the structure of a principal fiber bundle. We denote the bundle coordinates (r, g) where r is a local coordinate in the base, or *shape space* Q/G , and g is a group coordinate. Such a local trivialization is characterized by the fact that in such coordinates the group does not act on the factor r but acts on the group coordinate by *left* translations. Thus, locally in the base, the space Q is isomorphic to the product $Q/G \times G$ and in this local trivialization, the map π becomes the projection onto the first factor.

Let \mathfrak{g} denote the Lie algebra of the group G . When the configuration space is a principle fiber bundle, some of the vector fields u_1, \dots, u_n can be defined globally, as shown below.

Definition 2.2. *We say that the Lagrangian is G -invariant if L is invariant under the induced action of G on TQ .*

2.5 The Euler–Lagrange–Poincaré Equations

Assuming that the vectors $e_a(r)$, $a = 1, \dots, k$, form a basis of $\mathfrak{g} = T_eG$ for each $r \in Q/G$, we define the frame $e(r)$ by

$$e(r) = (e_1(r), \dots, e_k(r)) \in \mathfrak{g}^k. \quad (2.11)$$

Let \mathcal{A}_s be a principal connection on the bundle $\pi : Q \rightarrow Q/G$ (see Bloch [2003] for a review of principal connections). Below we will discuss how the structure of the Lagrangian can be used for selecting a connection. Tangent vectors in a local trivialization $Q = Q/G \times G$ at the point (r, g) are denoted (v, w) . We write the action of \mathcal{A}_s on this vector as $\mathcal{A}_s(v, w)$. Using this notation, we can write the connection form in the local trivialization as $\mathcal{A}_s(v, w) = \text{Ad}_g(w_b + \mathcal{A}v)$, where w_b is the left translation of w to the identity and, since we are working locally in shape space, we can regard \mathcal{A} as a \mathfrak{g} -valued one-form on Q/G . The connection components \mathcal{A}_α^a are defined by writing $\mathcal{A}v = \mathcal{A}_\alpha^a v^\alpha e_a$.

Define the vector fields u_i by

$$u_\alpha = \frac{\partial}{\partial r^\alpha} - \mathcal{A}_\alpha^a(r) L_{g*} e_a(r), \quad u_{\sigma+a} = L_{g*} e_a(r), \quad \alpha = 1, \dots, \sigma, \quad a = 1, \dots, k. \quad (2.12)$$

Thus the fields u_α span the horizontal space and the remaining fields $u_{\sigma+a}$ span the vertical space (tangent space to the group orbit) at $q \in Q$. The components of the velocity vector \dot{q} relative to basis (2.12) are

$$\dot{r}^\alpha \quad \text{and} \quad \Omega^a = \xi^a + \mathcal{A}_\alpha^a \dot{r}^\alpha,$$

where the ξ^a are Lie algebra variables. The Lie algebra element $\xi = \xi^a e_a(r)$ and the group element g are related by the equation $\xi = L_{g^{-1}*} \dot{g} \equiv g^{-1} \dot{g}$.

Theorem 2.3. *The equations of motion of a system with a G -invariant Lagrangian $L : TQ \rightarrow \mathbb{R}$ are*

$$\frac{d}{dt} \frac{\partial l}{\partial \dot{r}} - \frac{\partial l}{\partial r} = - \left\langle \frac{\partial l}{\partial \Omega}, \mathbf{i}_r \mathcal{B} + \langle \mathbf{i}_r \gamma, \mathcal{A} \rangle - \langle \gamma, \mathbf{i}_r \mathcal{A} \rangle + \langle \mathcal{E}, \Omega \rangle \right\rangle, \quad (2.13)$$

$$\frac{d}{dt} \frac{\partial l}{\partial \Omega} = \text{ad}_\Omega^* \frac{\partial l}{\partial \Omega} + \left\langle \frac{\partial l}{\partial \Omega}, \mathbf{i}_r \mathcal{E} \right\rangle, \quad (2.14)$$

$$\dot{g} = g(\Omega - \mathbf{i}_r \mathcal{A}). \quad (2.15)$$

Equations (2.13) and (2.14) are independent of the group element g and govern the reduced dynamics. The *reconstruction equation* (2.15) is used to obtain the group dynamics. The terms that appear on the right-hand sides of equations (2.13)–(2.15) are defined below.

Proof. Using definition (2.11), we see that

$$de = \langle \gamma(r), e \rangle, \quad (2.16)$$

where γ is a $\mathfrak{g} \otimes \mathfrak{g}^*$ -valued one-form on Q/G . In coordinates we have

$$\frac{\partial e_a}{\partial r^\alpha} = \gamma_{a\alpha}^b(r) e_b(r).$$

Let C_{ab}^c be the structure constants of the Lie algebra \mathfrak{g} and let \mathcal{B}_s be the curvature of the connection \mathcal{A}_s ; that is $\mathcal{B}_s(X, Y) = d\mathcal{A}_s(\text{hor } X, \text{hor } Y)$, where X and Y are two vector fields. The curvature can be written in the local representation as

$$\mathcal{B}_s((v_1, w_1), (v_2, w_2)) = \text{Ad}_g(\mathcal{B}(v_1, v_2)). \quad (2.17)$$

The coordinate representation of \mathcal{B} is $\mathcal{B}(v_1, v_2) = \mathcal{B}_{\alpha\beta}^a v_1^\alpha v_2^\beta$, with $\mathcal{B}_{\alpha\beta}^a$ given by the formula

$$\mathcal{B}_{\alpha\beta}^c = \frac{\partial \mathcal{A}_\alpha^c}{\partial r^\beta} - \frac{\partial \mathcal{A}_\beta^c}{\partial r^\alpha} - C_{ab}^c \mathcal{A}_\alpha^a \mathcal{A}_\beta^b.$$

Define the form \mathcal{E} by

$$\mathcal{E} = \gamma - \text{ad}_\mathcal{A}, \quad (2.18)$$

which, in coordinates, reads $\mathcal{E}_{ab}^c = \gamma_{b\alpha}^c - C_{ab}^c \mathcal{A}_\alpha^a$. Straightforward calculation shows that

$$[u_\alpha, u_\beta] = \left(\mathcal{B}_{\alpha\beta}^c + \gamma_{b\beta}^c \mathcal{A}_\alpha^b - \gamma_{b\alpha}^c \mathcal{A}_\beta^b \right) u_{\sigma+c}, \quad (2.19)$$

$$[u_\alpha, u_{\sigma+b}] = \mathcal{E}_{\alpha b}^c u_{\sigma+c}, \quad (2.20)$$

$$[u_{\sigma+a}, u_{\sigma+b}] = C_{ab}^c u_{\sigma+c}. \quad (2.21)$$

Thus, from the definitions of \mathcal{B} , \mathcal{E} , and formulae (2.19)–(2.21) one obtains

$$\begin{aligned} [(\dot{r}_1, 0), (\dot{r}_2, 0)]_q &= \mathcal{B}(\dot{r}_1, \dot{r}_2) - \langle \mathbf{i}_{\dot{r}_1} \gamma, \mathbf{i}_{\dot{r}_2} \mathcal{A} \rangle + \langle \mathbf{i}_{\dot{r}_2} \gamma, \mathbf{i}_{\dot{r}_1} \mathcal{A} \rangle, \\ [(\dot{r}, 0), (0, \Omega)]_q &= \langle \mathbf{i}_r \mathcal{E}, \Omega \rangle, \quad [(0, \Omega_1), (0, \Omega_2)]_q = \text{ad}_{\Omega_1} \Omega_2. \end{aligned} \quad (2.22)$$

Using (2.22) and evaluating $[\cdot, \cdot]_q^*$, equations (2.5) become (2.13)–(2.15) ■

Now assume that the G -invariant Lagrangian equals the kinetic minus potential energy of the system and that the kinetic energy is given by a Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ on the configuration space Q .

Definition 2.4. The *mechanical connection* $\mathcal{A}^{\text{mech}}$ is, by definition, the connection on Q regarded as a bundle over shape space Q/G that is defined by declaring its horizontal space at a point $q \in Q$ to be the subspace that is the orthogonal complement to the tangent space to the group orbit through $q \in Q$ using the kinetic energy metric. The *locked inertia tensor* $\mathbb{I}(q) : \mathfrak{g} \rightarrow \mathfrak{g}^*$ is defined by $\langle\mathbb{I}(q)\xi, \eta\rangle = \langle\langle \xi_Q(q), \eta_Q(q) \rangle\rangle$, where ξ_Q is the infinitesimal generator of $\xi \in \mathfrak{g}$ and where $\langle\langle \cdot, \cdot \rangle\rangle$ is the kinetic energy inner product.

Given a system with symmetry, one may use the mechanical connection to set up equations (2.13) and (2.14). This choice of connection changes the Lie algebra variables from ξ to the local version of the locked angular velocity Ω , which has the physical interpretation of the body angular velocity. This choice also is such that the kinetic energy metric becomes *block-diagonal*, that is,

$$\langle\langle \dot{q}, \dot{q} \rangle\rangle = \langle\langle \dot{r}, \dot{r} \rangle\rangle + \langle\langle \Omega, \Omega \rangle\rangle.$$

When the momentum $p = \partial_\Omega l$ is used as an independent variable, equations (2.13) and (2.14) are called the **Euler–Lagrange–Poincaré equations**. In coordinates, these equations read:

$$\begin{aligned} \frac{d}{dt} \frac{\partial l}{\partial \dot{r}^\alpha} - \frac{\partial l}{\partial r^\alpha} &= - \left(\mathcal{B}_{\alpha\beta}^c + (\gamma_{b\beta}^c \mathcal{A}_\alpha^b - \gamma_{b\alpha}^c \mathcal{A}_\beta^b) \right) p_c \dot{r}^\beta - \mathcal{E}_{\alpha b}^c p_c \Omega^b, \\ \dot{p}_b &= C_{ab}^c p_c \Omega^a + \mathcal{E}_{\alpha b}^c p_c \dot{r}^\alpha \end{aligned}$$

(see also Cendra, Marsden, and Ratiu [2001a]).

Example: The Generalized Rigid Body. If the configuration space is a Lie group G and the Lagrangian is G -invariant, equations (2.13) and (2.14) become

$$\frac{d}{dt} \frac{\partial l}{\partial \xi} = \text{ad}_\xi^* \frac{\partial l}{\partial \xi}. \quad (2.23)$$

These equations are called the **Euler–Poincaré equations** (see Euler [1752]³ and Lagrange [1788], Vol. II, for the case $G = \text{SO}(3)$, and Poincaré [1901] for the general case); they describe the momentum dynamics of a *generalized rigid body*. For more on the history of these equations see Marsden and Ratiu [1999].

3 Nonholonomic Systems

3.1 The Lagrange–d’Alembert Principle

Assume now that there are *velocity constraints* imposed on the system. We confine our attention to constraints that are homogeneous in the velocity. Accordingly, we

³Euler wrote the three components of the equations $\frac{d}{dt} \frac{\partial l}{\partial \xi} = \frac{\partial l}{\partial \xi} \times \xi$, with explicit formulae substituted for $\frac{\partial l}{\partial \xi}$, where l is the reduced rigid body Lagrangian.

consider a configuration space Q and a distribution \mathcal{D} on Q that describes these constraints. Recall that a distribution \mathcal{D} is a collection of linear subspaces of the tangent spaces of Q ; we denote these spaces by $\mathcal{D}_q \subset T_qQ$, one for each $q \in Q$. A curve $q(t) \in Q$ will be said to **satisfy the constraints** if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t . This distribution will, in general, be nonintegrable; *i.e.*, the constraints are, in general, nonholonomic.⁴

Consider a Lagrangian $L : TQ \rightarrow \mathbb{R}$. In coordinates $q^i, i = 1, \dots, n$, on Q with induced coordinates (q^i, \dot{q}^i) for the tangent bundle, we write $L(q^i, \dot{q}^i)$. The equations of motion are given by the following Lagrange–d’Alembert principle.

Definition 3.1. *The Lagrange–d’Alembert equations of motion for the system are those determined by*

$$\delta \int_a^b L(q, \dot{q}) dt = 0,$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$ and $\delta q(t) \in \mathcal{D}_{q(t)}$ for each t where $a \leq t \leq b$.

This principle is supplemented by the condition that the curve $q(t)$ itself satisfies the constraints. Note that we take the variation *before* imposing the constraints; that is, we do not impose the constraints on the family of curves defining the variation. This is well known to be important to obtain the correct mechanical equations (see Bloch, Krishnaprasad, Marsden, and Murray [1996] for a discussion and references).

One way to write the dynamics is to make use of the *Euler–Lagrange equations with multipliers*. This is done below in coordinates.

The distribution \mathcal{D} can be locally written as

$$\mathcal{D} = \{\dot{q} \in TQ \mid A_i^s(q)\dot{q}^i = 0, \quad s = 1, \dots, p\}.$$

The **constrained variations** $\delta q(t) \in TQ$ satisfy the equations

$$A_i^s(q)\delta q^i = 0, \quad s = 1, \dots, p. \quad (3.1)$$

Using (2.6), and (3.1), one writes the equations of motion with *Lagrange multipliers* as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} + \lambda_s A_i^s(q), \quad A_i^s(q)\dot{q}^i = 0.$$

3.2 The Constrained Hamel Equations

Given a nonholonomic system, that is, a Lagrangian $L : TQ \rightarrow \mathbb{R}$ and constraint distribution \mathcal{D} , select the independent (local) vector fields

$$u_i : Q \rightarrow TQ, \quad i = 1, \dots, n,$$

such that $\mathcal{D}_q = \text{span}\{u_1(q), \dots, u_{n-p}(q)\}$. Each $\dot{q} \in TQ$ can be uniquely written as

$$\dot{q} = \langle u(q), v^{\mathcal{D}} \rangle + \langle u(q), v^{\mathcal{U}} \rangle, \quad \text{where} \quad \langle u(q), v^{\mathcal{D}} \rangle \in \mathcal{D}_q, \quad (3.2)$$

⁴Constraints are nonholonomic if and only if they cannot be rewritten as *position* constraints.

i.e., $\langle u(q), v^{\mathcal{D}} \rangle$ is the component of \dot{q} along \mathcal{D}_q . Similarly, each $a \in T^*Q$ can be uniquely decomposed as

$$a = \langle a_{\mathcal{D}}, u^*(q) \rangle + \langle a_{\mathcal{U}}, u^*(q) \rangle,$$

where $\langle a_{\mathcal{D}}, u^*(q) \rangle$ is the component of a along the dual of \mathcal{D}_q , and where $u^*(q) \in T^*Q \times \cdots \times T^*Q$ denotes the dual frame of $u(q)$. Using (3.2), the constraints read

$$v = v^{\mathcal{D}} \quad \text{or} \quad v^{\mathcal{U}} = 0.$$

This implies

$$\delta v = \delta v^{\mathcal{D}} \quad \text{or} \quad \delta v^{\mathcal{U}} = 0. \quad (3.3)$$

Using the principle of critical action and (3.3) proves the following theorem:

Theorem 3.2. *The dynamics of a nonholonomic system is represented by the **constrained Hamel equations***

$$\left(\frac{d}{dt} \frac{\partial l}{\partial v} - \left[v^{\mathcal{D}}, \frac{\partial l}{\partial v} \right]_q^* - u[l] \right)_{\mathcal{D}} = 0, \quad v^{\mathcal{U}} = 0, \quad \dot{q} = \langle u(q), v^{\mathcal{D}} \rangle. \quad (3.4)$$

In coordinate notation, equations (3.4) read as follows:

$$\frac{d}{dt} \frac{\partial l}{\partial v^i} = c_{ji}^m \frac{\partial l}{\partial v^m} v^j + u_i[l], \quad \dot{q} = v^i u_i(q), \quad i, j = 1, \dots, n-p.$$

3.3 The Hamel Equations with Lagrange Multipliers

Assuming that the distribution \mathcal{D} is locally written as

$$\mathcal{D}_q = \{v \in \mathbb{R}^n \mid \langle a^s(q), v \rangle = a_i^s(q) v^i = 0, \quad s = 1, \dots, p\},$$

and taking into account that variations are written as $\delta q(t) = \langle w(t), u(q(t)) \rangle$, we obtain

$$\langle A(q(t)), \delta q(t) \rangle = \langle a(q(t)), w(t) \rangle. \quad (3.5)$$

In the above, $a(q) = (a^1(q), \dots, a^p(q)) \in T_q^*Q \times \cdots \times T_q^*Q$. Equation (3.5) implies that the constrained variations of the curve $v(t) \in \mathbb{R}^n$ are given by

$$\delta v(t) = \dot{w}(t) + [v(t), w(t)]_{q(t)} \quad \text{with} \quad \langle a(q(t)), w(t) \rangle = 0.$$

Therefore,

$$\delta \int_a^b l(q, v) dt = \int_a^b \left(u[l] + \left[v, \frac{\partial l}{\partial v} \right]_q^* - \frac{d}{dt} \frac{\partial l}{\partial v} + \lambda_s a^s \right) w dt,$$

which implies

$$\frac{d}{dt} \frac{\partial l}{\partial v} - \left[v^{\mathcal{D}}, \frac{\partial l}{\partial v} \right]_q^* - u[l] = \lambda_s a^s, \quad \langle a(q), v \rangle = 0, \quad \dot{q} = \langle u(q), v \rangle. \quad (3.6)$$

If the fields $u_1(q), \dots, u_{n-p}(q)$ span the subspace \mathcal{D}_q , equations (3.6) are equivalent to (3.4). We will see in §6 that equations (3.6) are useful when studying the dynamics of the spatial momentum.

Below we evaluate the Lagrange multipliers for systems whose Lagrangian $L : TQ \rightarrow \mathbb{R}$ equals the kinetic minus potential energy, and the kinetic energy is given by a Riemannian metric on Q :

$$L(q, \dot{q}) = \frac{1}{2} \langle \dot{q}, \dot{q} \rangle - U(q).$$

Let $G_q : T_q Q \rightarrow T_q^* Q$ be the kinetic energy metric; that is, $\langle \dot{q}, \dot{q} \rangle = \langle G_q v, v \rangle$. Let $\mu = \partial l / \partial v = G_q v \in T_q^* Q$ be the conjugate momentum. The constraints in the momentum representation are given by

$$\langle a(q), G_q^{-1} \mu \rangle = \langle \mu, b(q) \rangle = 0,$$

where $b(q) = G_q^{-1} a(q) \in T_q Q$. Using (3.6), we get

$$\dot{\mu} = [v^{\mathcal{D}}, \mu]_q^* + u[l] + \lambda_s a^s. \quad (3.7)$$

Let $\lambda = (\lambda_1, \dots, \lambda_p)$. Taking the time derivative of $\langle \mu, b(q) \rangle = 0$ and replacing $\dot{\mu}$ with the right-hand side of (3.7), we obtain

$$\langle \mu, \dot{b}(q) \rangle + \langle [v^{\mathcal{D}}, \mu]_q^* + u[l], b(q) \rangle + \lambda \langle a(q), b(q) \rangle = 0,$$

which implies

$$\begin{aligned} \lambda &= -(\langle \mu, \dot{b}(q) \rangle + \langle [v^{\mathcal{D}}, \mu]_q^* + u[l], b(q) \rangle) \cdot \langle a(q), b(q) \rangle^{-1} \\ &= -\left(\left\langle \frac{\partial l}{\partial v}, \dot{b}(q) \right\rangle + \left\langle \left[v^{\mathcal{D}}, \frac{\partial l}{\partial v} \right]_q^* + u[l], b(q) \right\rangle \right) \cdot \langle a(q), b(q) \rangle^{-1}. \end{aligned} \quad (3.8)$$

4 Dynamics of Nonholonomic Systems with Symmetry

4.1 Reduced Dynamics

Assume that a Lie group G acts freely and properly on the configuration space Q and that the Lagrangian L and constraint distribution \mathcal{D} are invariant with respect to the induced action of G on TQ .

Although it is not needed for everything that we will be doing, the examples and the theory are somewhat simplified if we make the following assumption:

Dimension Assumption. *The constraints and the orbit directions span the entire tangent space to the configuration space:*

$$\mathcal{D}_q + T_q \text{Orb}(q) = T_q Q.$$

If this condition is satisfied, we say that the **principal case** holds.

Let \mathcal{S} be the subbundle of \mathcal{D} whose fiber at q is $\mathcal{S}_q = \mathcal{D}_q \cap T_q \text{Orb}(q)$. We assume in this paper that $\mathcal{S}_q \neq \{0\}$.⁵ The subbundle \mathcal{S} is invariant with respect

⁵If $\mathcal{S}_q = \{0\}$, a set of nonholonomic constraints is said to be *purely kinematic*.

to the action of G on TQ induced by the left action of G on Q . Choose subspaces $\mathcal{U}_q \subset \text{Orb}(q)$ such that $\text{Orb}(q) = \mathcal{S}_q \oplus \mathcal{U}_q$ and the subbundle \mathcal{U} is G -invariant. Since the distributions \mathcal{S} and \mathcal{U} are left-invariant, there exist subspaces $\mathfrak{b}_q^{\mathcal{S}}$ and $\mathfrak{b}_q^{\mathcal{U}}$ of the Lie algebra \mathfrak{g} such that in a local trivialization $\mathcal{S}_q = L_{g*} \mathfrak{b}_q^{\mathcal{S}}$ and $\mathcal{U}_q = L_{g*} \mathfrak{b}_q^{\mathcal{U}}$.

Let $\mathfrak{b}^{\mathcal{S}}$ and $\mathfrak{b}^{\mathcal{U}}$ be the bundles over Q/G whose fibers are the subspaces $\mathfrak{b}_q^{\mathcal{S}}$ and $\mathfrak{b}_q^{\mathcal{U}}$ of the Lie algebra \mathfrak{g} . Given $\xi \in \mathfrak{g}$, we write its components along these subspaces as $\xi^{\mathcal{S}}$ and $\xi^{\mathcal{U}}$. Let \mathcal{A}_s be a connection defined in a local trivialization by the formula

$$\mathcal{A}_s = \text{Ad}_g(\xi + \mathcal{A}\dot{r}),$$

where $\xi \in \mathfrak{g}$ and where \mathcal{A} is a \mathfrak{g} -valued form on Q/G . This form is such that the constraints in a local trivialization read

$$\xi^{\mathcal{U}} + \mathcal{A}^{\mathcal{U}}\dot{r} = 0. \quad (4.1)$$

That is, the \mathcal{U} -component of the form \mathcal{A} is defined by the constraints. The \mathcal{S} -component of \mathcal{A} is arbitrary at the moment; later on we will see how the structure of the Lagrangian affects the choice of this component.

Let $r = (r^1, \dots, r^\sigma)$ be local coordinates in the shape space Q/G , and let $e_1(r), \dots, e_k(r)$ be a basis of $\mathfrak{g} = T_e G$ for each $r \in Q/G$ such that $e_1(r), \dots, e_m(r)$ span $\mathfrak{b}^{\mathcal{S}}$ and $e_{m+1}(r), \dots, e_k(r)$ span $\mathfrak{b}^{\mathcal{U}}$.

As before, we define the vector fields $u_1(q), \dots, u_n(q)$ by formulae (2.12). Thus, the fields u_α , $\alpha = 1, \dots, \sigma$, span the horizontal space, the fields $u_{\sigma+a}$, $a = 1, \dots, m$, span the fibers of \mathcal{S} , and the remaining fields span the fibers of \mathcal{U} . The components of the velocity vector \dot{q} relative to these vector fields are \dot{r} and $\Omega = \xi + \mathcal{A}\dot{r}$. The constraints in this representation are $\Omega^{\mathcal{U}} = 0$.

Using the nonholonomic Hamel equations (3.4) and the G -invariance of the constraint distribution \mathcal{D} , one obtains the reduced nonholonomic equations of motion from (2.13) and (2.14) by projecting equation (2.14) onto the fibers of the bundle \mathfrak{b}_S^* , and imposing constraints, *i.e.*, setting $\Omega = \Omega^{\mathcal{S}}$.

Summarizing, we have:

Theorem 4.1. *The reduced nonholonomic dynamics is given by the equations*

$$\frac{d}{dt} \frac{\partial l}{\partial \dot{r}} - \frac{\partial l}{\partial r} = - \left\langle \frac{\partial l}{\partial \Omega}, \mathbf{i}_r \mathcal{B} + \langle \mathbf{i}_r \gamma, \mathcal{A} \rangle - \langle \gamma, \mathbf{i}_r \mathcal{A} \rangle + \langle \mathcal{E}, \Omega^{\mathcal{S}} \rangle \right\rangle, \quad (4.2)$$

$$\left[\frac{d}{dt} \frac{\partial l}{\partial \Omega} \right]_{\mathcal{S}} = \left[\text{ad}_{\Omega^{\mathcal{S}}}^* \frac{\partial l}{\partial \Omega} + \left\langle \frac{\partial l}{\partial \Omega}, \mathbf{i}_r \mathcal{E} \right\rangle \right]_{\mathcal{S}}. \quad (4.3)$$

In the above, the Lagrangian l is written as a function of (r, \dot{r}, Ω) , \mathcal{B} is the curvature of \mathcal{A} (see (2.17)), and the quantities γ and \mathcal{E} are defined by equations (2.16) and (2.18), respectively. Note that the partial derivatives of l in (4.2) and (4.3) are computed *before* setting $\Omega = \Omega^{\mathcal{S}}$.

Equations (4.2) and (4.3) can be rewritten as

$$\begin{aligned} \frac{d}{dt} \frac{\partial l_c}{\partial \dot{r}} - \frac{\partial l_c}{\partial r} &= - \left\langle \frac{\partial l}{\partial \Omega}, \mathbf{i}_r \mathcal{B} + \langle \mathbf{i}_r \gamma, \mathcal{A} \rangle - \langle \gamma, \mathbf{i}_r \mathcal{A} \rangle + \langle \mathcal{E}, \Omega^{\mathcal{S}} \rangle \right\rangle, \\ \frac{d}{dt} \frac{\partial l_c}{\partial \Omega^{\mathcal{S}}} &= \left[\text{ad}_{\Omega^{\mathcal{S}}}^* \frac{\partial l}{\partial \Omega} + \left\langle \frac{\partial l}{\partial \Omega}, \mathbf{i}_r \mathcal{E} \right\rangle \right]_{\mathcal{S}}, \end{aligned}$$

where $l_c(r, \dot{r}, \Omega^S) := l(r, \dot{r}, \Omega^S)$ is the **constrained reduced Lagrangian**. These equations follow directly from (4.2) and (4.3) because

$$\frac{\partial l_c}{\partial \dot{r}} = \frac{\partial l}{\partial \dot{r}} \Big|_{\Omega=\Omega^S}, \quad \frac{d}{dt} \frac{\partial l_c}{\partial \dot{r}} = \frac{d}{dt} \frac{\partial l}{\partial \dot{r}} \Big|_{\Omega=\Omega^S}, \quad \frac{\partial l_c}{\partial r} = \frac{\partial l}{\partial r} \Big|_{\Omega=\Omega^S}, \quad \frac{\partial l_c}{\partial \Omega^S} = \frac{\partial l}{\partial \Omega^S} \Big|_{\Omega=\Omega^S}.$$

Note that in general

$$\frac{\partial l_c}{\partial \Omega^S} \neq \frac{\partial l}{\partial \Omega} \Big|_{\Omega=\Omega^S}.$$

4.2 The Moving Body Frame and Nonholonomic Connection

In the rest of the paper we assume that the Lagrangian equals the kinetic minus potential energy and that the kinetic energy is given by a Riemannian metric on the configuration space Q .

Definition 4.2. *Under the dimension assumption and the assumption that the Lagrangian is of the form kinetic minus potential energies, the **nonholonomic connection** \mathcal{A}^{nhc} is, by definition, the connection on the principal bundle $Q \rightarrow Q/G$ whose horizontal space at $q \in Q$ is given by the orthogonal complement to the space $\mathcal{S}_q = \mathcal{D}_q \cap T_q \text{Orb}(q)$ within the space \mathcal{D}_q .*

Under the assumption that the distribution \mathcal{D} is invariant and from the fact that the group action preserves orthogonality (since it is assumed to preserve the Lagrangian and hence the kinetic energy metric), it follows that the distribution and the horizontal spaces transform to themselves under the group action. Therefore, the nonholonomic connection in a local trivialization is defined by the formula

$$\mathcal{A}^{\text{nhc}} = \text{Ad}_g(\xi + \mathcal{A}\dot{r}), \quad (4.4)$$

where $\xi \in \mathfrak{g}$ and where \mathcal{A} is a \mathfrak{g} -valued one-form on Q/G . Given $\dot{q} = (\dot{r}, \xi) \in T_q Q$, the vertical and horizontal components of \dot{q} in a local trivialization are

$$(0, \xi + \mathcal{A}\dot{r}) \quad \text{and} \quad (\dot{r}, -\mathcal{A}\dot{r}).$$

Using Definition 4.2, we rewrite $\xi + \mathcal{A}\dot{r}$ in (4.4) as

$$(\xi^S + \mathcal{A}^S \dot{r}) + (\xi^U + \mathcal{A}^U \dot{r}), \quad \xi^S + \mathcal{A}^S \dot{r} \in \mathfrak{b}^S, \quad \xi^U + \mathcal{A}^U \dot{r} \in \mathfrak{b}^U.$$

Using the nonholonomic connection, define the **body angular velocity** $\Omega \in \mathfrak{g}$ by the formula

$$\Omega = \xi + \mathcal{A}\dot{r}.$$

If the body angular velocity is used, the constraints (4.1) read

$$\Omega^U = 0.$$

Let the kinetic energy metric in a local trivialization be written as

$$\langle\langle \dot{q}, \dot{q} \rangle\rangle = \langle G(r) \dot{r}, \dot{r} \rangle + 2\langle \mathcal{K}(r) \dot{r}, \xi \rangle + \langle \mathcal{I}(r) \xi, \xi \rangle. \quad (4.5)$$

The *constrained locked inertia tensor* $\mathcal{I}_S : \mathfrak{b}^S \rightarrow (\mathfrak{b}^S)^*$ is given in a local trivialization by

$$\langle \mathcal{I}_S(r)\xi, \eta \rangle = \langle \langle L_{g^*}\xi, L_{g^*}\eta \rangle \rangle, \quad \xi, \eta \in \mathfrak{b}^S.$$

Similarly, define $\mathcal{I}_U(r) : \mathfrak{b}^U \rightarrow (\mathfrak{b}^U)^*$, $\mathcal{I}_{SU}(r) : \mathfrak{b}^U \rightarrow (\mathfrak{b}^S)^*$, and $\mathcal{I}_{US}(r) : \mathfrak{b}^S \rightarrow (\mathfrak{b}^U)^*$ by

$$\begin{aligned} \langle \mathcal{I}_U(r)\xi, \eta \rangle &= \langle \langle L_{g^*}\xi, L_{g^*}\eta \rangle \rangle, \quad \xi, \eta \in \mathfrak{b}^U, \\ \langle \mathcal{I}_{SU}(r)\xi, \eta \rangle &= \langle \langle L_{g^*}\xi, L_{g^*}\eta \rangle \rangle, \quad \xi \in \mathfrak{b}^U, \quad \eta \in \mathfrak{b}^S, \\ \langle \mathcal{I}_{US}(r)\xi, \eta \rangle &= \langle \langle L_{g^*}\xi, L_{g^*}\eta \rangle \rangle, \quad \xi \in \mathfrak{b}^S, \quad \eta \in \mathfrak{b}^U, \end{aligned}$$

respectively.

Definition 4.2 implies that the constrained kinetic energy metric written as a function of (\dot{r}, Ω^S) is *block-diagonal*, that is, it reads

$$\langle G_c(r)\dot{r}, \dot{r} \rangle + \langle \mathcal{I}_S(r)\Omega^S, \Omega^S \rangle.$$

Substituting $\xi = \Omega^S - \mathcal{A}\dot{r}$ in (4.5), one obtains

$$\langle G_c(r)\dot{r}, \dot{r} \rangle + 2\langle \mathcal{K}_S(r)\dot{r}, \Omega^S \rangle + \langle \mathcal{I}_S(r)\Omega^S, \Omega^S \rangle - 2\langle \mathcal{I}_S(r)\mathcal{A}^S\dot{r}, \Omega^S \rangle - 2\langle \mathcal{I}_{SU}(r)\mathcal{A}^U\dot{r}, \Omega^S \rangle,$$

and therefore

$$\mathcal{A}^S = \mathcal{I}_S^{-1}(r)\mathcal{K}_S(r) - \mathcal{I}_S^{-1}(r)\mathcal{I}_{SU}(r)\mathcal{A}^U.$$

In the rest of the paper, we follow the following index conventions:

1. The first batch of indices range from 1 to m corresponding to the symmetry directions along the constraint space. These indices will be denoted a, b, c, \dots
2. The second batch of indices range from $m + 1$ to k corresponding to the symmetry directions not aligned with the constraints. Indices for this range will be denoted by a', b', c', \dots
3. The indices A, B, C, \dots on the Lie algebra \mathfrak{g} range from 1 to k .
4. The indices α, β, \dots on the shape variables r range from 1 to σ . Thus, σ is the dimension of the shape space Q/G and so $\sigma = n - k$.

The summation convention for all of these indices will be understood.

In the basis $\partial/\partial r^1, \dots, \partial/\partial r^\sigma, e_1(r), \dots, e_k(r)$, with $e_1(r), \dots, e_m(r)$ spanning \mathfrak{b}^S and $e_{m+1}(r), \dots, e_k(r)$ spanning \mathfrak{b}^U , the components of the nonholonomic connection are

$$\mathcal{A}_\alpha^a = \mathcal{I}_S^{ab}\mathcal{K}_{\alpha b} - \mathcal{I}_S^{ab}\mathcal{I}_{ba'}\mathcal{A}_\alpha^{a'} \quad \text{and} \quad \mathcal{A}_\alpha^{a'}.^6$$

Note that the \mathcal{I}_S^{ab} need *not* equal \mathcal{I}^{ab} .

⁶Recall that the components $\mathcal{A}_\alpha^{a'}$ are defined by the constraints.

One often uses the *nonholonomic momentum relative to the body frame* $p = \partial l_c / \partial \Omega^S$ and writes the reduced dynamics as

$$\begin{aligned} \frac{d}{dt} \frac{\partial l_c}{\partial \dot{r}} - \frac{\partial l_c}{\partial r} = & - \left\langle p, \frac{\partial \mathcal{I}_S^{-1}}{\partial r} p \right\rangle \\ & - \left\langle p + \mathcal{I}_{\mathcal{U}S} \mathcal{I}_S^{-1} p + \mathbf{i}_r \Lambda, \mathbf{i}_r \mathcal{B} + \langle \mathbf{i}_r \gamma, \mathcal{A} \rangle - \langle \gamma, \mathbf{i}_r \mathcal{A} \rangle + \langle \mathcal{E}, \mathcal{I}_S^{-1} p \rangle \right\rangle, \end{aligned} \quad (4.6)$$

$$\dot{p} = \left[\text{ad}_{\mathcal{I}_S^{-1} p}^* p + \langle p, \mathbf{i}_r \tau \rangle + \langle \mathbf{i}_r \Lambda, \mathbf{i}_r \mathcal{E} \rangle \right]_S. \quad (4.7)$$

Equation (4.7) is called the *momentum equation in body representation*.

In these equations the reduced Lagrangian is represented as a function of (r, \dot{r}, p) , Λ is a $(\mathfrak{b}^{\mathcal{U}})^*$ -valued one-form on Q/G given by

$$\Lambda = (\mathcal{K}_{\mathcal{U}} - \mathcal{I}_{\mathcal{U}S} \mathcal{A}^S - \mathcal{I}_{\mathcal{U}} \mathcal{A}^{\mathcal{U}}) dr,$$

and τ is a $\mathfrak{b}^S \otimes (\mathfrak{b}^S)^*$ -valued one-forms on Q/G defined by the formula

$$\langle p, \mathbf{i}_r \tau \rangle = \left[\text{ad}_{\mathcal{I}_S^{-1} p}^* \mathbf{i}_r \Lambda + \langle p + \mathcal{I}_{\mathcal{U}S} \mathcal{I}_S^{-1} p, \mathbf{i}_r \mathcal{E} \rangle \right]_S.$$

Remark. The first term in the right-hand side of the shape equation (4.6) appears for the following reason. The term $\frac{1}{2} \langle \mathcal{I}_S \Omega^S, \Omega^S \rangle$ in the constrained reduced Lagrangian $l_c(r, \dot{r}, \Omega^S)$ produces the term $\frac{1}{2} \langle \partial_r \mathcal{I}_S \Omega^S, \Omega^S \rangle$ in the right-hand side of the shape equation. When p is used instead of Ω , this term becomes $\frac{1}{2} \langle \partial_r \mathcal{I}_S \mathcal{I}_S^{-1} p, \mathcal{I}_S^{-1} p \rangle = \frac{1}{2} \langle \mathcal{I}_S^{-1} \partial_r \mathcal{I}_S \mathcal{I}_S^{-1} p, p \rangle = -\frac{1}{2} \langle p, \partial_r \mathcal{I}_S^{-1} p \rangle$. On the other hand, the contribution of the term $\frac{1}{2} \langle p, \mathcal{I}_S^{-1} p \rangle$ in $l_c(r, \dot{r}, p)$ is $\frac{1}{2} \langle p, \partial_r \mathcal{I}_S^{-1} p \rangle$. Thus, in order to obtain the correct term, $-\frac{1}{2} \langle p, \partial_r \mathcal{I}_S^{-1} p \rangle$, one needs to subtract $\langle p, \partial_r \mathcal{I}_S^{-1} p \rangle$ from the right-hand side of the shape equation.

Sometimes it is desirable to use a slightly different representation of equations (4.2), (4.3) and (4.6), (4.7). Recall that the inertia tensor is an invertible operator $\mathcal{I} : \mathfrak{g} \rightarrow \mathfrak{g}^*$. Its inverse \mathcal{I}^{-1} maps \mathfrak{g}^* to \mathfrak{g} , and defines the operators

$$\mathcal{I}^S : \mathfrak{b}_S^* \rightarrow \mathfrak{b}^S, \quad \mathcal{I}^{\mathcal{U}} : \mathfrak{b}_{\mathcal{U}}^* \rightarrow \mathfrak{b}^{\mathcal{U}}, \quad \mathcal{I}^{S\mathcal{U}} : \mathfrak{b}_{\mathcal{U}}^* \rightarrow \mathfrak{b}^S, \quad \mathcal{I}^{\mathcal{U}S} : \mathfrak{b}_S^* \rightarrow \mathfrak{b}^{\mathcal{U}} \quad (4.8)$$

by the formulae

$$\begin{aligned} \mathcal{I}^{-1} \alpha &= \mathcal{I}^S \alpha + \mathcal{I}^{\mathcal{U}S} \alpha, \quad \mathcal{I}^S \alpha \in \mathfrak{b}^S, \quad \mathcal{I}^{\mathcal{U}S} \alpha \in \mathfrak{b}^{\mathcal{U}}, \quad \alpha \in \mathfrak{b}_S^*, \\ \mathcal{I}^{-1} \alpha &= \mathcal{I}^{S\mathcal{U}} \alpha + \mathcal{I}^{\mathcal{U}} \alpha, \quad \mathcal{I}^{S\mathcal{U}} \alpha \in \mathfrak{b}^S, \quad \mathcal{I}^{\mathcal{U}} \alpha \in \mathfrak{b}^{\mathcal{U}}, \quad \alpha \in \mathfrak{b}_{\mathcal{U}}^*. \end{aligned}$$

The operators (4.8) are uniquely defined by these formulae since $\mathfrak{g} = \mathfrak{b}^S \oplus \mathfrak{b}^{\mathcal{U}}$. Note that $\mathcal{I}^{S\mathcal{U}}$ and $\mathcal{I}^{\mathcal{U}S}$ normally are not zero operators. However, it is straightforward to see that

$$\mathcal{I}^{\mathcal{U}S} \mathcal{I}_S + \mathcal{I}^{\mathcal{U}} \mathcal{I}_{\mathcal{U}S} : \mathfrak{b}^S \rightarrow \mathfrak{b}^{\mathcal{U}}$$

maps every $\xi \in \mathfrak{b}^S$ to 0. Therefore,

$$(\mathcal{I}^{\mathcal{U}})^{-1} (\mathcal{I}^{\mathcal{U}S} \mathcal{I}_S + \mathcal{I}^{\mathcal{U}} \mathcal{I}_{\mathcal{U}S}) \mathcal{I}_S^{-1} = (\mathcal{I}^{\mathcal{U}})^{-1} \mathcal{I}^{\mathcal{U}S} + \mathcal{I}_{\mathcal{U}S} \mathcal{I}_S^{-1} = 0,$$

and one can replace the terms $\mathcal{I}_{\mathcal{U}S}\mathcal{I}_S^{-1}p$ with $-(\mathcal{I}^{\mathcal{U}})^{-1}\mathcal{I}^{\mathcal{U}S}p$.

In coordinate notation, the reduced equations of motion become

$$\begin{aligned} \frac{d}{dt} \frac{\partial l_c}{\partial \dot{r}^\alpha} - \frac{\partial l_c}{\partial r^\alpha} &= -\mathcal{D}_{b\alpha}^c \mathcal{I}_S^{bd} p_c p_d - \mathcal{K}_{\alpha\beta\gamma} \dot{r}^\beta \dot{r}^\gamma \\ &\quad - (\mathcal{B}_{\alpha\beta}^c - \mathcal{I}_{c'a'}^{\mathcal{U}} \mathcal{I}^{a'c} \mathcal{B}_{\alpha\beta}^{c'} + \mathcal{D}_{b\beta\alpha} \mathcal{I}_S^{bc}) p_c \dot{r}^\beta, \end{aligned} \quad (4.9)$$

$$\dot{p}_a = (C_{ba}^c - C_{ba}^{c'} \mathcal{I}_{c'a'}^{\mathcal{U}} \mathcal{I}^{a'c}) \mathcal{I}_S^{bd} p_c p_d + \mathcal{D}_{a\alpha}^c p_c \dot{r}^\alpha + \mathcal{D}_{a\alpha\beta} \dot{r}^\alpha \dot{r}^\beta. \quad (4.10)$$

Here and below $l_c(r^\alpha, \dot{r}^\alpha, \Omega^a)$ is the constrained Lagrangian, and \mathcal{I}_S^{bd} and $\mathcal{I}_{a'c'}^{\mathcal{U}}$ are the components of the tensors \mathcal{I}_D^{-1} and $(\mathcal{I}^{\mathcal{U}})^{-1}$, respectively. We stress that in general $\mathcal{I}_S^{bd} \neq \mathcal{I}^{bd}$ and $\mathcal{I}_{a'c'}^{\mathcal{U}} \neq \mathcal{I}_{a'c'}$. The coefficients $\mathcal{B}_{\alpha\beta}^c$, $\mathcal{D}_{b\alpha}^c$, $\mathcal{D}_{b\alpha\beta}$, and $\mathcal{K}_{\alpha\beta\gamma}$ are given by the formulae

$$\begin{aligned} \mathcal{B}_{\alpha\beta}^c &= \frac{\partial \mathcal{A}_\alpha^C}{\partial r^\beta} - \frac{\partial \mathcal{A}_\beta^C}{\partial r^\alpha} - C_{BA}^C \mathcal{A}_\alpha^A \mathcal{A}_\beta^B + \gamma_{A\beta}^C \mathcal{A}_\alpha^A - \gamma_{A\alpha}^C \mathcal{A}_\beta^A, \\ \mathcal{D}_{b\alpha}^c &= -(C_{Ab}^c - C_{Ab}^{c'} \mathcal{I}_{c'a'}^{\mathcal{U}} \mathcal{I}^{a'c}) \mathcal{A}_\alpha^A + C_{ab}^{c'} \Lambda_{c'\alpha} \mathcal{I}_S^{ac} + \gamma_{b\alpha}^c - \gamma_{b\alpha}^{c'} \mathcal{I}_{c'a'}^{\mathcal{U}} \mathcal{I}^{a'c}, \\ \mathcal{D}_{b\alpha\beta} &= \Lambda_{c'\beta} (\gamma_{b\alpha}^{c'} - C_{Ab}^{c'} \mathcal{A}_\alpha^A), \\ \mathcal{K}_{\alpha\beta\gamma} &= \Lambda_{c'\gamma} \mathcal{B}_{\alpha\beta}^{c'}, \end{aligned} \quad (4.11)$$

and $\gamma_{b\alpha}^c$ are the components of the form γ introduced in (2.16). Equations (4.9) and (4.10) originated in Bloch, Krishnaprasad, Marsden, and Murray [1996] and Cendra, Marsden, and Ratiu [2001b].

The keys to the qualitative behavior of this system and stability in particular are the terms on the right hand side of the momentum equation (4.10). One case of interest is where the the matrix $C_{ba}^c - C_{ba}^{c'} \mathcal{I}_{c'a'}^{\mathcal{U}} \mathcal{I}^{a'c}$ is skew. This is discussed in Zenkov, Bloch, and Marsden [1998]. This again divides into two cases: the terms quadratic in \dot{r} are present or not. There are cases where these terms vanish and one does not obtain asymptotic stability—for example the rolling penny problem. If the terms quadratic in \dot{r} are present one often obtains asymptotic stability, as in the rattleback top.

A key case of interest to us in this paper are the Euler–Poincaré–Suslov equations discussed below where there are no internal or shape degrees of freedom, *i.e.* no coordinates r^α . We discuss in detail in this case when one does or does not obtain partial asymptotic stability.

Whether the nonholonomic systems exhibit asymptotic behavior or not it is striking that we have

Proposition 4.3. *The nonholonomic equations (4.9), (4.10), in the case that l_c is quadratic in p and \dot{r} , are time reversible.*

Proof. The equations are verified to be invariant under the discrete \mathbb{Z}_2 symmetry

$$(t \rightarrow -t, p \rightarrow -p, \dot{r} \rightarrow -\dot{r}),$$

giving the stated result. ■

Furthermore, in this setting it is easy to check that energy is always preserved. This is in contrast to the type of asymptotic stability exhibited by, say a harmonic oscillator with external dissipation (friction), which is not reversible.

4.3 Momentum Conservation Relative to the Body Frame

Here we discuss how one can select the basis in $\mathcal{D}_q \cap T_q \text{Orb}(q)$ in order to observe momentum conservation in the nonholonomic setting. More details can be found in [Zenkov \[2003\]](#).

Theorem 4.4. *For $\eta(r) \in \mathfrak{b}^S$, the quantity $\langle p, \eta(r) \rangle$ is a conservation law of the reduced nonholonomic dynamics (4.6) and (4.7) if*

- (i) *The form τ is exact;*
- (ii) *The component of $[\text{ad}_{T_S^{-1}p}^* p + \langle \mathbf{i}_r \Lambda, \mathbf{i}_r \mathcal{E} \rangle]_S$ along $\eta(r)$ equals zero, i.e.,*

$$\left\langle [\text{ad}_{T_S^{-1}p}^* p + \langle \mathbf{i}_r \Lambda, \mathbf{i}_r \mathcal{E} \rangle]_S, \eta(r) \right\rangle = 0.$$

Proof. Taking the flow derivative of $\langle p, \eta(r) \rangle$, we obtain

$$\frac{d}{dt} \langle p, \eta(r) \rangle = \langle \dot{p}, \eta(r) \rangle + \langle p, \dot{\eta} \rangle = \left\langle [\text{ad}_{T_S^{-1}p}^* p + \langle p, \mathbf{i}_r \tau \rangle + \langle \mathbf{i}_r \Lambda, \mathbf{i}_r \mathcal{E} \rangle]_S, \eta(r) \right\rangle + \langle p, \dot{\eta} \rangle.$$

Condition (ii) implies

$$\frac{d}{dt} \langle p, \eta(r) \rangle = \langle [\langle p, \mathbf{i}_r \tau \rangle]_S, \eta(r) \rangle + \langle p, \dot{\eta} \rangle = \langle p, \langle \mathbf{i}_r \tau, \eta \rangle + \dot{\eta} \rangle.$$

Equation

$$d\eta = -\langle \tau, \eta \rangle \tag{4.12}$$

implies that the flow derivative of $\langle p, \eta(r) \rangle$ vanishes. Condition (i) of the theorem ensures the existence of $\eta(r) \in \mathfrak{b}^S$. ■

Corollary 4.5. *Assume that the form τ is exact and $[\text{ad}_{T_S^{-1}p}^* p + \langle \mathbf{i}_r \Lambda, \mathbf{i}_r \mathcal{E} \rangle]_S = 0$. Then there exists a basis $\eta_1(r), \dots, \eta_m(r)$ of \mathfrak{b}^S such that the momentum equation written relative to this basis becomes*

$$\dot{p} = 0.$$

This basis, if it exists, is often nontrivial even for simple (e.g. commutative) symmetry groups.

Example. Consider a system whose Lagrangian $L : T\mathbb{R}^3 \rightarrow \mathbb{R}$ and constraint are

$$L = \frac{1}{2} (\dot{r}^2 + (1 - b^2(r))(\dot{s}^1)^2 + (\dot{s}^2)^2) - V(r)$$

and

$$\dot{s}^2 = b(r)\dot{s}^1,$$

respectively. This system is \mathbb{R}^2 -invariant. The group elements are written as $s = (s^1, s^2) \in \mathbb{R}^2$. The potential energy $V(r)$ and the constraint coefficient $b(r)$ are

smooth functions of the shape variable r . The subspace \mathfrak{b}^S of the Lie algebra \mathbb{R}^2 for this system is one-dimensional; it is spanned by

$$\frac{\partial}{\partial s^1} + b(r) \frac{\partial}{\partial s^2}.$$

The momentum equation for this system is computed to be

$$\dot{p} = b(r)b'(r)pr.$$

The right-hand side of the momentum equation satisfies the conditions of Corollary 4.5.

Using (4.12), $d\eta = -b(r)b'(r)\eta dr$, which implies

$$\eta(r) = e^{-\frac{1}{2}b^2(r)} (\partial_{s^1} + b(r)\partial_{s^2}).$$

Thus, if one defines the vector fields $u_1, u_2, u_3 \in T\mathbb{R}^3$ by the formulae

$$u_1 = \frac{\partial}{\partial r}, \quad u_2 = e^{-\frac{1}{2}b^2(r)} \left(\frac{\partial}{\partial s^1} + b(r) \frac{\partial}{\partial s^2} \right), \quad u_3 = \frac{\partial}{\partial s^2},$$

nonholonomic momentum conservation is observed.

5 Euler–Poincaré–Suslov Equations

An important special case of the reduced nonholonomic equations is the following: The configuration space is a Lie group G and the system is characterized by a left-invariant Lagrangian $l = \frac{1}{2} \langle \mathcal{I}\Omega, \Omega \rangle = \frac{1}{2} \mathcal{I}_{AB} \Omega^A \Omega^B$, where $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$, \mathcal{I}_{AB} are the components of the inertia tensor $\mathcal{I} : \mathfrak{g} \mapsto \mathfrak{g}^*$, and by the left-invariant constraint

$$\langle a, \Omega \rangle = a_A \Omega^A = 0, \tag{5.1}$$

where a lies in the dual space \mathfrak{g}^* and $\langle \cdot, \cdot \rangle$ denotes the natural pairing between the Lie algebra and its dual. Multiple constraints may be imposed as well.

In the absence of constraints the reduced dynamics is governed by the Euler–Poincaré equations (2.23). Here we represent these equations as

$$\dot{p} = \text{ad}_\Omega^* p, \tag{5.2}$$

where $p = \mathcal{I}\Omega \in \mathfrak{g}^*$ is the body momentum. In components, these equations read

$$\dot{p}_B = C_{AB}^C \mathcal{I}^{AD} p_C p_D = C_{AB}^C p_C \Omega^A. \tag{5.3}$$

It is of interest that while the unconstrained dynamics on TG preserves the phase volume, the reduced dynamics (5.2) may fail to be measure preserving and thus may exhibit asymptotic behavior, as was shown by Kozlov [1988].

Theorem 5.1. *The Euler–Poincaré equations (5.2) have an integral invariant with positive C^1 density if and only if the group G is unimodular.⁷*

⁷A Lie group is said to be unimodular if it has a bilaterally invariant measure.

We only outline the proof here, for the complete exposition refer to [Kozlov \[1988\]](#). Sufficiency can be seen as follows: A criterion for unimodularity (see [Kozlov \[1988\]](#)) is $C_{AC}^C = 0$ (with the summation convention). Since the divergence of the right hand side of equation (5.3) is $C_{AC}^C \mathcal{I}^{AD} p_D = 0$, the corresponding flow preserves the volume form $dp_1 \wedge \cdots \wedge dp_k$ in \mathfrak{g}^* . Recall that the flow of a vector field f is volume preserving if and only if $\operatorname{div} f = 0$. The necessity is derived from the following theorem of [Kozlov \[1988\]](#): *A flow of a homogeneous vector field in \mathbb{R}^n has an integral invariant with positive C^1 density if and only if this flow preserves the standard volume in \mathbb{R}^n .*

Now, turning to the case where we have the constraint (5.1) we obtain the Euler–Poincaré–Suslov equations

$$\dot{p} = \operatorname{ad}_\Omega^* p + \lambda a \quad (5.4)$$

together with the constraint (5.1), where λ is the Lagrange multiplier. Using (3.8), we obtain

$$\lambda = -\langle \operatorname{ad}_\Omega^* p, \mathcal{I}^{-1} a \rangle / \langle a, \mathcal{I}^{-1} a \rangle = -\langle p, \operatorname{ad}_\Omega \mathcal{I}^{-1} a \rangle / \langle a, \mathcal{I}^{-1} a \rangle.$$

One can then formulate a condition for the existence of an invariant measure of the Euler–Poincaré–Suslov equations. The following result was proved by [Kozlov \[1988\]](#) for compact algebras and for arbitrary algebras by [Jovanović \[1998\]](#).

Theorem 5.2. *The Euler–Poincaré–Suslov equations (5.4) have an integral invariant with positive C^1 density if and only if*

$$K \operatorname{ad}_{\mathcal{I}^{-1} a}^* a + T = \gamma a \quad (5.5)$$

for some $\gamma \in \mathbb{R}$, where $K = 1 / \langle a, \mathcal{I}^{-1} a \rangle$ and $T \in \mathfrak{g}^*$ is defined by $\langle T, \xi \rangle = \operatorname{Tr}(\operatorname{ad}_\xi)$.

In coordinates condition (5.5) reads

$$K C_{AB}^C \mathcal{I}^{AD} a_C a_D + C_{BC}^C = \gamma a_B. \quad (5.6)$$

The proof is done by solving for the Lagrange multiplier λ and computing the divergence of the right hand side.

For a unimodular Lie group the quantities $\sum_{B=1}^k C_{AB}^B$, $A = 1, \dots, k$, always vanish. In particular if the group is compact or semisimple, then the group is unimodular.

Further, in the semisimple case we can use the Killing form to identify \mathfrak{g}^* with \mathfrak{g} and condition (5.6) can be written as

$$[\mathbb{I}^{-1} a, a] = \gamma a, \quad \gamma \in \mathbb{R}.$$

For a matrix Lie algebra the Killing form is a multiple of the trace and so pairing a with itself (via the Killing form or multiple of the trace) we have

$$\langle [\mathcal{I}^{-1} a, a], a \rangle = \langle \gamma a, a \rangle$$

and, since the left hand side is zero, we conclude that γ must be zero. Thus in this case only constraint vectors a which commute with $\mathcal{I}^{-1}a$ allow volume in \mathfrak{g}^* to be preserved.

This means that a and $\mathcal{I}^{-1}a$ must lie in the same maximal commuting subalgebra. In particular if a is an eigenstate of the inertia tensor, the flow is measure-preserving. In the case that the maximal commuting subalgebra is one-dimensional this is a necessary condition. This is the case for groups such as $\text{SO}(3)$ (see below). One can introduce several constraints of this type.

We can thus formulate the following result as a symmetry requirement on the constraints:

Theorem 5.3. *A compact Euler–Poincaré–Suslov system is volume-preserving (and therefore generically cannot exhibit asymptotic dynamics in the momentum space) if the constraint vectors a are eigenvectors of the inertia tensor, or the constrained system is \mathbb{Z}_2 symmetric about all principal axes. If the maximal commuting subalgebra is one-dimensional this condition is necessary.*

In the case the group is not compact a little more freedom is allowed as shall see in the case of the Chaplygin sleigh below.

Spatial Momentum Dynamics. Let e_A be a basis of \mathfrak{g} , and let $u_A = R_{g*}e_A$. Then $\dot{g} = \langle u(g), \eta \rangle$, where $\eta = \dot{g}g^{-1}$. If $G = \text{SO}(3)$, η is the spatial angular velocity.

The Lagrangian and the constraints are

$$l^R(g, \eta) = \frac{1}{2} \langle \mathbb{I}(g)\eta, \eta \rangle$$

and

$$\langle a^R(g), \eta \rangle = 0,$$

respectively. In the above, $\mathbb{I}(g) = \text{Ad}_{g^{-1}}^* \mathcal{I} \text{Ad}_{g^{-1}}$ and $a^R(g) = \text{Ad}_{g^{-1}}^* a \in \mathfrak{g}^*$.

Define the *spatial momentum* μ by the formula

$$\mu = \frac{\partial l^R}{\partial \eta} = \text{Ad}_{g^{-1}}^* p.$$

Then, using (3.6), we obtain

$$\dot{\mu} = \lambda a^R(g), \tag{5.7}$$

where, according to (3.8)

$$\lambda = -\langle \mu, \dot{b}^R(g) \rangle \cdot \langle a^R(g), b^R(g) \rangle^{-1}.$$

Equation (5.7) implies that the spatial momentum is not generically conserved in the nonholonomic setting.

6 Examples

In this section we illustrate the above theory by discussing a number of key nonholonomic examples where the dynamics is conveniently described using quasivelocities. Perhaps the simplest nonholonomic system is the following:

6.1 Euler–Poincaré–Suslov Problem on SO(3)

In this case one can formulate the problem as the standard Euler equations

$$\mathcal{I}\dot{\Omega} = \mathcal{I}\Omega \times \Omega,$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the system angular velocities in a frame where the inertia matrix is of the form $\mathcal{I} = \text{diag}(I_1, I_2, I_3)$ and the system is subject to the constraint

$$a \cdot \Omega = 0,$$

where $a = (a_1, a_2, a_3)$.

The nonholonomic equations of motion are then given by

$$\mathcal{I}\dot{\Omega} = \mathcal{I}\Omega \times \Omega + \lambda a, \quad a \cdot \Omega = 0,$$

and the Lagrange multiplier is given by

$$\lambda = -\frac{\mathcal{I}^{-1}a \cdot (\mathcal{I}\Omega \times \Omega)}{\mathcal{I}^{-1}a \cdot a}.$$

One can then check that if $a_2 = a_3 = 0$ (a constraint that is an eigenstate of the moment of inertia operator) for example, the (reduced) phase volume is preserved.

6.2 Chaplygin Sleigh

Here we describe the Chaplygin sleigh, one of the simplest mechanical system which illustrates the possible dissipative nature of nonholonomic systems. Note that this dissipation is “nonphysical” in the sense that the energy is preserved while the volume on the level surfaces of energy is not.

The Chaplygin sleigh is discussed for example in [Neimark and Fufaev \[1972\]](#) (see also the paper [Ruina \[1998\]](#), where an interesting connection with systems with impacts is made).

The sleigh is essentially a flat rigid body in the plane supported at three points, two of which slide freely without friction while the third is a knife edge constraint which allows no motion perpendicular to its edge.

We will derive the equations from the Euler–Poincaré–Suslov equations (5.4) on SE(2). Let θ be the angular orientation of the sleigh, (x, y) be the coordinates of the contact point as shown in [Figure 6.1](#). The blade is depicted as a bold segment in the Figure. The body frame is

$$e_1 = \frac{\partial}{\partial \theta}, \quad e_2 = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, \quad e_3 = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}$$

(see [Figure 6.1](#)); the vector e_1 may be visualized of as a vector orthogonal to the body and directed towards the reader. Let $\Omega = g^{-1}\dot{g} \in \mathfrak{se}(2)$, the components of Ω relative to the body frame are $(\Omega^1, \Omega^2, \Omega^3)$, where $\Omega^1 = \dot{\theta}$ is the angular velocity of the sleigh relative to the vertical line through the contact point, and Ω^2 and Ω^3 are the components of linear velocity of the contact point in the directions along and

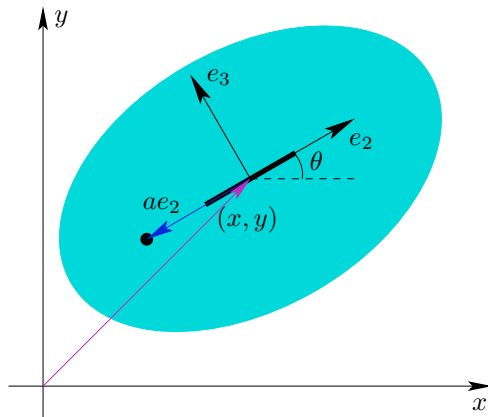


Figure 6.1: The Chaplygin sleigh.

orthogonal to the blade, respectively. Let the center of mass be located on the line through the blade at the point ae_2 relative to the body frame. See Figure 6.1 for details. Denote the mass and the moment of inertia of the sleigh by m and J . The constraint reads $\Omega^3 = 0$.

The velocity of the center of mass relative to the body frame is $\Omega^2 e_2 + (\Omega^3 + a\Omega^1) e_3$. The Lagrangian is just the kinetic energy of the sleigh, which is the sum of the kinetic energies of the linear and rotational modes of the body. Therefore the reduced Lagrangian and the reduced constrained Lagrangian of the Chaplygin sleigh are (see Bloch [2003])

$$l(\Omega) = \frac{1}{2} ((J + ma^2)(\Omega^1)^2 + m((\Omega^2)^2 + (\Omega^3)^2 + 2a\Omega^1\Omega^3)),$$

and

$$l_c(\Omega) = \frac{1}{2} ((J + ma^2)(\Omega^1)^2 + m(\Omega^2)^2),$$

respectively. The constrained space \mathfrak{b}^S is defined by

$$\mathfrak{b}^S = \{\Omega^3 = 0\}.$$

The dual of \mathfrak{b}^S is

$$(\mathfrak{b}^S)^* = \{p_3 = 0\}.$$

Equations of Motion in the Body Frame. The reduced dynamics of the Chaplygin sleigh is given by the Euler–Poincaré–Suslov equations on the algebra $\mathfrak{se}(2)$. The two independent equations are

$$\dot{p}_1 = -\frac{a p_1 p_2}{J + ma^2}, \quad \dot{p}_2 = \frac{ma p_1^2}{(J + ma^2)^2}, \quad (6.1)$$

where

$$p_1 = \frac{\partial l_c}{\partial \Omega^1} = (J + ma^2)\Omega^1, \quad p_2 = \frac{\partial l_c}{\partial \Omega^2} = m\Omega^2$$

are the components of nonholonomic momentum relative to the body frame. If $a \neq 0$, equations (6.1) have a family of relative equilibria given by $p_1 = 0, p_2 = \text{const.}$ Linearizing at any of these equilibria, we find that there is one zero eigenvalue and one real eigenvalue. The trajectories of (6.1) are either equilibria situated on the line $p_1 = 0$, or elliptic arcs, as shown in Figure 6.2. Assuming $a > 0$, the equilibria located in the upper half plane are asymptotically stable (filled dots in Figure 6.2) whereas the equilibria in the lower half plane are unstable (empty dots). The elliptic arcs form heteroclinic connections between the pairs of equilibria.

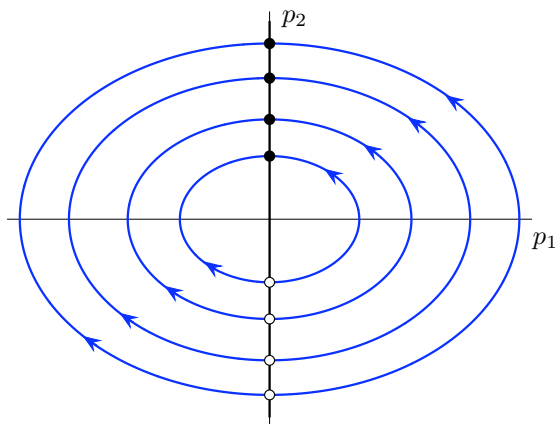


Figure 6.2: The momentum dynamics of the unbalanced Chaplygin sleigh.

If $a = 0$, momentum dynamics is trivial. Taken together with the analysis of equations (6.1), this means that we do not observe asymptotic stability if and only if $a = 0$.

The condition $a = 0$ means that the center of mass is situated at the contact point of the sleigh and the plane. The inertia tensor \mathcal{I} becomes block-diagonal in this case, *i.e.*, the direction defined by the constraint is an eigendirection of the inertia tensor.

The Dynamics of the Spatial Momentum. Recall that the spatial angular velocity is defined as $\eta = R_{g^{-1}}^* \dot{g} = \dot{g}g^{-1} \in \mathfrak{se}(2)$. For the group $\text{SE}(2)$ the components of η are computed to be

$$\eta^1 = \dot{\theta}, \quad \eta^2 = \dot{x} + y\dot{\theta}, \quad \eta^3 = \dot{y} - x\dot{\theta}.$$

The corresponding frame consists of the generators of the left action of $\text{SE}(2)$ on itself. The latter are given by the formulae

$$u_1 = \frac{\partial}{\partial \theta} - y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \quad u_2 = \frac{\partial}{\partial x}, \quad u_3 = \frac{\partial}{\partial y},$$

and the corresponding brackets are computed to be

$$[u_1, u_2] = -u_3, \quad [u_1, u_3] = u_2, \quad [u_2, u_3] = 0.$$

The Lagrangian becomes

$$l^R(g, \theta) = \frac{J + ma^2}{2}(\eta^1)^2 + \frac{m}{2} \left((\eta^2 - y\eta^1)^2 + (\eta^3 + x\eta^1)^2 + 2a\eta^1 [-(\eta^2 - y\eta^1) \sin \theta + (\eta^3 + x\eta^1) \cos \theta] \right),$$

and the constraint becomes

$$-(\eta^2 - y\eta^1) \sin \theta + (\eta^3 + x\eta^1) \cos \theta = 0.$$

The spatial components of the nonholonomic momentum are

$$\begin{aligned} \mu_1 &= (J + ma^2)\eta^1 - my(\eta^2 - y\eta^1) + mx(\eta^3 + x\eta^1) + ma\eta^1(x \cos \theta + y \sin \theta), \\ \mu_2 &= m[(\eta^2 - y\eta^1) - a\eta^1 \sin \theta], \\ \mu_3 &= m[(\eta^2 - y\eta^1) + a\eta^1 \cos \theta]. \end{aligned}$$

The momentum belongs to the subspace of $\mathfrak{se}^*(2)$ defined by the equation

$$-\mu_2 \sin \theta + \mu_3 \cos \theta - \frac{ma}{J + ma^2}(\mu_1 + y\mu_2 - x\mu_3) = 0.$$

Using (5.7), the dynamics of spatial momentum for the Chaplygin sleigh is

$$\dot{\mu}_1 = \lambda(x \cos \theta + y \sin \theta), \quad \dot{\mu}_2 = -\lambda \sin \theta, \quad \dot{\mu}_3 = \lambda \cos \theta,$$

where λ is given by the formula

$$\lambda = \frac{J(\mu_1 + y\mu_2 - x\mu_3)(\mu_2 \cos \theta + \mu_3 \sin \theta)}{(J + ma^2)^2}.$$

Regardless of the value of a , the spatial momentum of the Chaplygin sleigh is conserved if and only if $\theta = \text{const}$, *i.e.*, when the contact point of the sleigh is moving along a straight line.

Momentum Conservation for the Unbalanced Chaplygin Sleigh. We now show that the components of momentum are conserved if one uses the vector fields $u_1, u_2, u_3 \in T\text{SE}(2)$ given by

$$u_1 = \cos k\theta \frac{\partial}{\partial \theta} + \frac{a}{k} \sin k\theta \left(\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right), \quad (6.2)$$

$$u_2 = -\frac{k}{a} \sin k\theta \frac{\partial}{\partial \theta} + \cos k\theta \left(\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right), \quad (6.3)$$

$$u_3 = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}, \quad (6.4)$$

where $k^2 := ma^2/(J + ma^2)$.

The structure of the momentum trajectories in Figure 6.2 in combination with translational symmetry of the sleigh suggests that there may exist vector fields

$u = u^1(\theta)e_1 + u^2(\theta)e_2$ such that $\langle p, u \rangle = \text{const}$. That is, the components of the nonholonomic momentum along a field whose direction and magnitude relative to the body frame depends on the angle θ may be conserved. Note that θ is not a shape variable as in Theorem 4.4.

Differentiating the quantity $\langle p, u(\theta) \rangle$ and taking into account equations (6.1) yields

$$\dot{u}^1 = -\frac{map_1}{(J + ma^2)^2} u^2, \quad \dot{u}^2 = \frac{ap_1}{J + ma^2} u^1.$$

Using the formula $p_1 = (J + ma^2)\Omega^1 = (J + ma^2)\dot{\theta}$, we obtain

$$\frac{du^1}{d\theta} = -\frac{ma}{J + ma^2} u^2, \quad \frac{du^2}{d\theta} = au^1.$$

This system defines two independent vector fields (6.2) and (6.3). The field u_3 is defined by formula (6.4) in order to have the sleigh constraint to be written as $\xi^3 = 0$, where (ξ^1, ξ^2, ξ^3) are the components of \dot{g} relative to the frame u_1, u_2, u_3 .

Conservation of momentum can be confirmed by the Hamel equations for the sleigh. Computing the Jacobi–Lie brackets for the fields u_1, u_2, u_3 , we obtain

$$\begin{aligned} [u_1, u_2] &= -\frac{k^2}{a} \cos k\theta u_1 + k \sin k\theta u_2 + u_3, \\ [u_1, u_3] &= -\frac{k}{a} \cos k\theta \sin k\theta u_1 - \cos^2 k\theta u_2, \\ [u_2, u_3] &= \frac{k^2}{a^2} \sin^2 k\theta u_1 + \frac{k}{a} \cos k\theta \sin k\theta u_2, \end{aligned}$$

which implies

$$\begin{aligned} C_{12}^1 &= -\frac{k^2}{a} \cos k\theta, & C_{12}^2 &= k \sin k\theta, & C_{12}^3 &= 1, \\ C_{13}^1 &= -\frac{k}{a} \cos k\theta \sin k\theta, & C_{13}^2 &= -\cos^2 k\theta, & C_{13}^3 &= 0, \\ C_{23}^1 &= \frac{k^2}{a^2} \sin^2 k\theta, & C_{23}^2 &= \frac{k}{a} \cos k\theta \sin k\theta, & C_{23}^3 &= 0. \end{aligned}$$

Written relative to the frame u_1, u_2, u_3 , the Lagrangian and the constrained Lagrangian become

$$l(\xi) = \frac{1}{2}((J + ma^2)(\xi^1)^2 + m(\xi^2)^2 + 2ma \cos k\theta \xi^1 \xi^3 - 2ma \sin k\theta \xi^2 \xi^3 + m(\xi^3)^2)$$

and

$$l_c(\xi) = \frac{1}{2}((J + ma^2)(\xi^1)^2 + m(\xi^2)^2).$$

The constrained Hamel equations are computed to be

$$\frac{d}{dt} \frac{\partial l_c}{\partial \xi^1} = 0, \quad \frac{d}{dt} \frac{\partial l_c}{\partial \xi^2} = 0.$$

Thus, the components of momentum relative to the frame u_1, u_2, u_3 are conserved.

6.3 Chaplygin Sleigh with an Oscillator

Here we analyze the dynamics of the Chaplygin sleigh coupled to an oscillator. We show that the phase flow is integrable, and generic invariant manifolds are two-dimensional tori.

The Lagrangian, Nonholonomic Connection, and Reduced Dynamics.

Consider the Chaplygin sleigh with a mass sliding along the direction of the blade. The mass is coupled to the sleigh through a spring. One end of the spring is attached to the sleigh at the contact point, the other end is attached to the mass. The spring force is zero when the mass is positioned above the contact point. See Figure 6.1 where the sliding mass is represented by a black dot and the blade is shown as a bold black segment.

The configuration space for this system is $\mathbb{R} \times \text{SE}(2)$. This system has one shape (the distance from the mass to the contact point, r) and three group degrees of freedom.

The reduced Lagrangian $l : T\mathbb{R} \times \mathfrak{se}(2) \rightarrow \mathbb{R}$ is given by the formula

$$l(r, \dot{r}, \xi) = \frac{1}{2}m\dot{r}^2 + m\dot{r}\xi^2 + \frac{1}{2}((J + mr^2)(\xi^1)^2 + 2mr\xi^1\xi^3 + (M + m)((\xi^2)^2 + (\xi^3)^2)) - \frac{1}{2}kr^2,$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{se}(2)$ and k is the spring constant. The constrained reduced Lagrangian is

$$l_c(r, \dot{r}, \xi^1, \xi^2) = \frac{1}{2}m\dot{r}^2 + m\dot{r}\xi^2 + \frac{1}{2}((J + mr^2)(\xi^1)^2 + (M + m)(\xi^2)^2) - \frac{1}{2}kr^2.$$

The constrained reduced energy

$$\frac{1}{2}m\dot{r}^2 + m\dot{r}\xi^2 + \frac{1}{2}((J + mr^2)(\xi^1)^2 + (M + m)(\xi^2)^2) + \frac{1}{2}kr^2$$

is positive-definite, and thus the mass cannot move infinitely far from the sleigh throughout the motion.

The nonholonomic connection is

$$\xi + \mathcal{A}\dot{r},$$

where

$$\mathcal{A}^1 = 0, \quad \mathcal{A}^2 = \frac{m}{M + m}, \quad \mathcal{A}^3 = 0.$$

The constraint is given by the formula

$$\Omega^3 = 0.$$

The reduced Lagrangian written as a function of (r, \dot{r}, Ω) becomes

$$l(r, \dot{r}, \Omega) = \frac{1}{2}\frac{Mm}{M + m}\dot{r}^2 + \frac{1}{2}((J + mr^2)(\Omega^1)^2 + 2mr\Omega^1\Omega^3 + (M + m)((\Omega^2)^2 + (\Omega^3)^2)) - \frac{1}{2}kr^2.$$

The constrained reduced Lagrangian written as a function of (r, \dot{r}, p) is

$$l_c(r, \dot{r}, p) = \frac{1}{2} \frac{Mm\dot{r}^2}{M+m} + \frac{1}{2} \left(\frac{p_1^2}{J+mr^2} + \frac{p_2^2}{M+m} \right) - \frac{kr^2}{2}.$$

The reduced dynamics for the sleigh-mass system is computed to be

$$\begin{aligned} \frac{Mm}{M+m} \ddot{r} &= \frac{Mmr}{(M+m)(J+mr^2)^2} p_1^2 - kr, \\ \dot{p}_1 &= -\frac{mr}{(M+m)(J+mr^2)} p_1 p_2 + \frac{m^2 r}{(M+m)(J+mr^2)} p_1 \dot{r}, \\ \dot{p}_2 &= \frac{mr}{(J+mr^2)^2} p_1^2. \end{aligned} \quad (6.5)$$

We now select a new frame in the Lie algebra $\mathfrak{se}(2)$ in order to eliminate the second term in equation (6.5). Put

$$e_1 = ((J+mr^2)^{-\frac{m}{2(M+m)}}, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$

The reduced Lagrangian written in this frame becomes

$$\begin{aligned} l(r, \dot{r}, \Omega) &= \frac{1}{2} \frac{Mm}{M+m} \dot{r}^2 + \frac{1}{2} \left((J+mr^2)^{\frac{M}{M+m}} (\Omega^1)^2 + (M+m)((\Omega^2)^2 + (\Omega^3)^2) \right. \\ &\quad \left. + 2mr(J+mr^2)^{-\frac{M}{2(M+m)}} \Omega^1 \Omega^3 \right) - \frac{1}{2} kr^2. \end{aligned}$$

Using equations (4.2) and (4.3), the reduced dynamics becomes

$$\frac{Mm}{M+m} \ddot{r} = \frac{Mmr}{(M+m)(J+mr^2)^{\frac{M}{M+m}+1}} p_1^2 - kr, \quad (6.6)$$

$$\dot{p}_1 = -\frac{mr}{(M+m)(J+mr^2)} p_1 p_2, \quad (6.7)$$

$$\dot{p}_2 = \frac{mr}{(J+mr^2)^{\frac{M}{M+m}+1}} p_1^2. \quad (6.8)$$

Relative Equilibria of the Sleigh-Mass System. Assuming that $(r, p) = (r_0, p_0)$ is a relative equilibrium, equation (6.8) implies $r_0 p_1^0 = 0$. Thus, either $r_0 = 0$ and p_1^0 is an arbitrary constant, or, using (6.6), $p_1^0 = 0$ and $r_0 = 0$. Thus, the only relative equilibria of the sleigh-mass system are

$$r = 0, \quad p = p_0.$$

The Discrete Symmetries and Integrability. It is straightforward to see that equations (6.6)–(6.8) are invariant with respect to the following transformations:

- (i) $(r, p_1, p_2) \rightarrow (r, -p_1, p_2)$,
- (ii) $(r, p_1, p_2) \rightarrow (-r, p_1, -p_2)$,

$$(iii) \quad (t, r, p) \rightarrow (-t, -r, p),$$

$$(iv) \quad (t, r, p_1, p_2) \rightarrow (-t, r, p_1, -p_2).$$

We now use these transformations to study some of the solutions of (6.6)–(6.8). Consider an initial condition $(r, \dot{r}, p) = (0, \dot{r}_0, p_0)$. Then the r -component of the solution subject to this initial condition is odd, and the p -component is even. Indeed, let

$$(r(t), p(t)), \quad t > 0, \quad (6.9)$$

be the part of this solution for $t > 0$. Then

$$(-r(-t), p(-t)), \quad t < 0, \quad (6.10)$$

is also a solution. This follows from the invariance of equations (6.6)–(6.8) with respect to transformation (iii). Using the formula

$$\left. \frac{dr(t)}{dt} \right|_{t=0} = \left. \frac{d(-r(-t))}{dt} \right|_{t=0},$$

we conclude that (6.9) and (6.10) satisfy the same initial condition and thus represent the forward in time and the backward in time branches of a the same solution. Thus, $r(-t) = -r(t)$ and $p(-t) = p(t)$.

Next, $p_1(t) = 0$ implies that $p_2(t) = \text{const}$ and $r(t)$ satisfies the equation

$$\frac{Mm}{M+m} \ddot{r} = -kr,$$

and thus equations (6.6)–(6.8) have periodic solutions

$$r(t) = A \cos \omega t + B \sin \omega t, \quad p_1 = 0, \quad p_2 = C,$$

where A , B , and C are arbitrary constants and $\omega = \sqrt{k(M+m)/Mm}$.

Without loss of generality, we set $A = 0$ and consider periodic solutions

$$r(t) = \dot{r}_0/\omega \sin \omega t, \quad p_1 = 0, \quad p_2 = p_2^0, \quad (6.11)$$

which correspond to the initial conditions

$$r(0) = 0, \quad \dot{r}(0) = \dot{r}_0, \quad p_1(0) = 0, \quad p_2(0) = p_2^0.$$

We now perturb solutions (6.11) by setting

$$r(0) = 0, \quad \dot{r}(0) = \dot{r}_0, \quad p_1(0) = p_1^0, \quad p_2(0) = p_2^0. \quad (6.12)$$

Assuming that p_1^0 is small and using a continuity argument, there exists $\tau = \tau_{p, \dot{r}_0} > 0$ such that

$$r(\tau_{p, \dot{r}_0}) = 0$$

for solutions subject to initial conditions (6.12). That is, the r -component is 2τ -periodic if p_1^0 is sufficiently small.

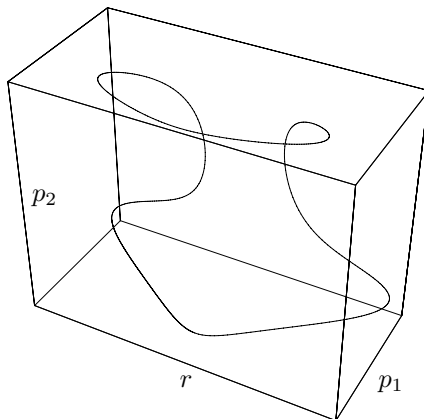


Figure 6.3: A reduced trajectory of the sleigh-mass system.

Using equation (6.6) and periodicity of $r(t)$, we conclude that p_1 is 2τ -periodic as well. Equation (6.7) then implies that $p_2(t)$ is also 2τ -periodic. Thus, *the reduced dynamics is integrable in an open subset of the reduced phase space*. The invariant tori are one-dimensional, and the reduced flow is periodic. A generic periodic trajectory in the direct product of the shape and momentum spaces is shown in Figure 6.3.

Using the quasi-periodic reconstruction theorem (see Ashwin and Melbourne [1997] and Field [1980]), we obtain the following theorem:

Theorem 6.1. *Generic trajectories of the coupled sleigh-oscillator system in the full phase space are quasi-periodic motions on two-dimensional invariant tori.*

Typical trajectories of the contact point of the sleigh with the plane are shown in Figure 6.4. The symmetry observed in these trajectories follows from the existence,

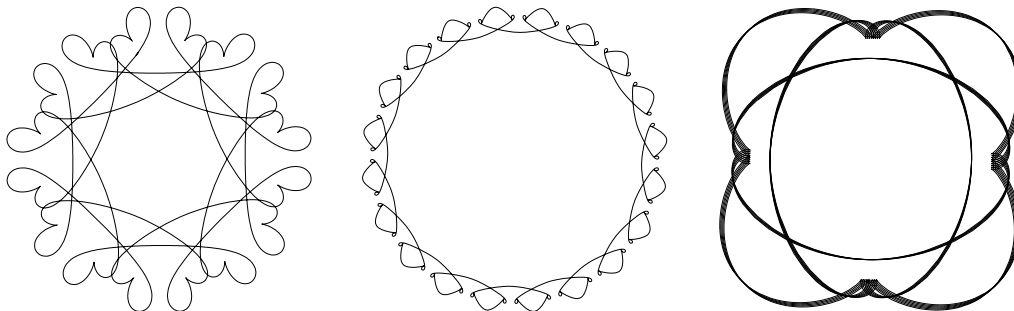


Figure 6.4: Trajectories of the contact point of the blade for various initial states.

for each group trajectory $g(t)$, of a group element h such that $g(t+2\tau) \equiv hg(t)$, where 2τ is the period of the corresponding reduced dynamics. It would be interesting to see if the technique of pattern evocation (see Marsden and Scheurle [1995] and Marsden, Scheurle, and Wendlandt [1996]) extends to the current circumstances.

6.4 Asymptotic Behavior of Momentum Integrals

Assuming that the only non-zero term in the right-hand side of the momentum equation (4.7) is $\langle p, \mathbf{i}_r \tau \rangle$, Theorem 4.4 and Corollary 4.5 give sufficient conditions for existence of m independent momentum integrals $\langle p, \eta_a(r) \rangle$, $a = 1, \dots, m$. The fields $\eta_a(r) \in \mathfrak{b}^S$ satisfy equation (4.12), their existence follows from the integrability of the distribution associated with the form τ (see Theorem 4.4 for details). In the coordinate form, equation (4.12) reads

$$d\eta^b = -\mathcal{D}_{a\alpha}^b(r)\eta^a dr^\alpha, \quad (6.13)$$

where the quantities $\mathcal{D}_{a\alpha}^b(r)$ are defined by formula (4.11).

The properties of distribution (6.13) resemble the properties of linear homogeneous systems of differential equations as the coefficients of the differential forms in (6.13) are linear functions in the components of η . It is straightforward to show that the space of integral manifolds of distribution (6.13) has the structure of an m -dimensional vector space.

Definition 6.2. *The **Wronskian** of the m integral manifolds $\eta = \eta_a(r)$, $a = 1, \dots, m$, is the determinant*

$$W(r) = \det \begin{bmatrix} \eta_1^1(r) & \dots & \eta_m^1(r) \\ \vdots & & \vdots \\ \eta_1^m(r) & \dots & \eta_m^m(r) \end{bmatrix}. \quad (6.14)$$

The properties of the Wronskian of the system of invariant manifolds of (6.13) are similar to those of the Wronskian of the solutions of a system of linear ordinary differential equations.

Theorem 6.3. *The Wronskian satisfies the equation*

$$d \ln W = -\text{Tr } \tau := -\mathcal{D}_{a\alpha}^a(r) dr^\alpha. \quad (6.15)$$

Proof. Taking into account (6.13) and (6.14), we obtain

$$\frac{\partial W}{\partial r^\alpha} = -W \mathcal{D}_{a\alpha}^a.$$

Therefore,

$$dW = \frac{\partial W}{\partial r^\alpha} dr^\alpha = -W \mathcal{D}_{a\alpha}^a dr^\alpha = -W \text{Tr } \tau,$$

which is equivalent to formula (6.15). ■

Corollary 6.4. *The Wronskian can be evaluated by the formula*

$$W(r) = W(r_0) \exp \int_{r_0}^r \text{Tr } \tau. \quad (6.16)$$

The integral in (6.16) is independent of path from r_0 to r .

Formula (6.16) allows one to obtain the Wronskian explicitly in the multidimensional shape space setting. This result can be used to study the reduced energy levels even if the momentum integrals cannot be effectively computed. We demonstrate this in the following section for the rolling disk.

6.5 The Rolling Falling Disk

To illustrate the results of §4.3 and §6.4, consider a disk rolling without sliding on a horizontal plane—see Figure 6.5. The configuration space for this system is $(-\pi/2, \pi/2) \times \text{SO}(2) \times \text{SE}(2)$. The coordinates on the configuration space are denoted $(\theta, \psi, \phi, x, y)$. As the figure indicates, we denote the coordinates of contact of the disk in the xy -plane by (x, y) , and let θ , ϕ , and ψ represent the angle between the plane of the disk and the vertical axis, the “heading angle” of the disk, and “self-rotation” angle of the disk, respectively. Denote the mass, the radius, and the

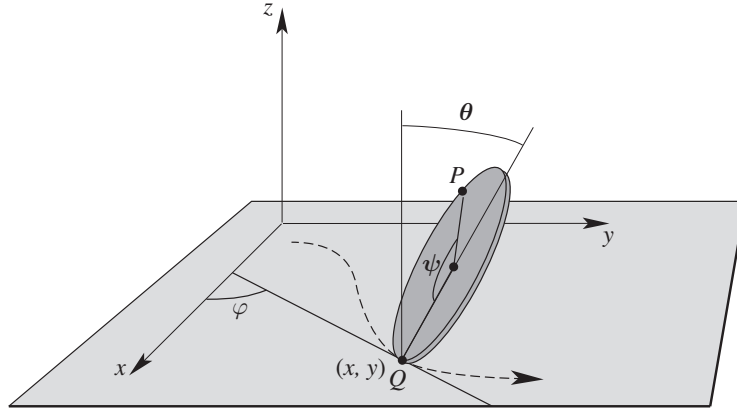


Figure 6.5: The geometry for the rolling disk.

moments of inertia of the disk by m , R , A , and B , respectively. The Lagrangian is given by the kinetic minus potential energies:

$$L = \frac{m}{2} \left[(\zeta^1 - R(\dot{\phi} \sin \theta + \dot{\psi}))^2 + (\zeta^2)^2 \sin^2 \theta + (\zeta^2 \cos \theta + R\dot{\theta})^2 \right] \\ + \frac{1}{2} \left[A(\dot{\theta}^2 + \dot{\phi}^2 \cos^2 \theta) + B(\dot{\phi} \sin \theta + \dot{\psi})^2 \right] - mgR \cos \theta,$$

where $\zeta^1 = \dot{x} \cos \phi + \dot{y} \sin \phi + R\dot{\psi}$ and $\zeta^2 = -\dot{x} \sin \phi + \dot{y} \cos \phi$, while the constraints are given by

$$\dot{x} = -\dot{\psi}R \cos \phi, \quad \dot{y} = -\dot{\psi}R \sin \phi.$$

Note that the constraints may also be written as $\zeta^1 = 0$, $\zeta^2 = 0$.

This system is invariant under the action of the group $G = \text{SO}(2) \times \text{SE}(2)$; the action by the group element $(\alpha, \beta, a, b) \in \text{SO}(2) \times \text{SE}(2)$ is given by

$$(\theta, \psi, \phi, x, y) \mapsto (\theta, \psi + \alpha, \phi + \beta, x \cos \beta - y \sin \beta + a, x \sin \beta + y \cos \beta + b).$$

The fibers of the constraint distribution \mathcal{D}_q are spanned by the vector fields ∂_θ , ∂_ϕ , and $-R \cos \phi \partial_x - R \sin \phi \partial_y + \partial_\psi$. Therefore,

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q \text{Orb}(q) = \text{span}(\partial_\phi, -R \cos \phi \partial_x - R \sin \phi \partial_y + \partial_\psi).$$

We select the basis of the subspace $\mathfrak{b}^S \subset \mathfrak{so}(2) \times \mathfrak{se}(2)$ to be

$$e_1 = (1, 0, R, 0) \quad \text{and} \quad e_2 = (-\tan \theta, \cos^{-1} \theta, -R \tan \theta, 0).$$

The two components of the nonholonomic momentum relative to e_1 and e_2 are

$$\begin{aligned} p_1 &= A\dot{\phi} \cos^2 \theta + (mR^2 + B)(\dot{\phi} \sin \theta + \dot{\psi}) \sin \theta, \\ p_2 &= -A\dot{\phi} \cos^2 \theta \sin \theta + (mR^2 + B)(\dot{\phi} \sin \theta + \dot{\psi}) \cos^2 \theta. \end{aligned}$$

The momentum equations for the falling disk are computed to be

$$\dot{p}_1 = \left(\tan \theta p_1 - \frac{B}{mR^2 + B} p_2 \right) \dot{\theta}, \quad \dot{p}_2 = -\frac{mR^2}{A} p_1 \dot{\theta}.$$

According to Corollary 4.5, there exists a basis $\eta_1(\theta), \eta_2(\theta)$ of the subspace \mathfrak{b}^S such that the components $\langle p, \eta_1(\theta) \rangle$ and $\langle p, \eta_2(\theta) \rangle$ of nonholonomic momentum are constants of motion.

We write the momentum levels as c_1 and c_2 , *i.e.*,

$$\langle p, \eta_1(\theta) \rangle = c_1, \quad \langle p, \eta_2(\theta) \rangle = c_2. \quad (6.17)$$

Given the momentum levels c_1 and c_2 , equations (6.17) define p_1 and p_2 as functions of θ . The vector fields $\eta_1(\theta)$ and $\eta_2(\theta)$ form the fundamental solution of the linear system

$$\frac{d\eta^1}{d\theta} = -\tan \theta \eta^1 + \frac{mR^2}{A} \eta^2, \quad \frac{d\eta^2}{d\theta} = \frac{B}{mR^2 + B} \eta^2. \quad (6.18)$$

It is possible to write the solutions of system (6.18) in terms of the hypergeometric function. See Appel [1900], Chaplygin [1897], and Korteweg [1899] for details. For additional work and history of the falling disk see Bloch [2003], Vierkandt [1892] and Cushman, Hermans, and Kemppainen [1996].

Kolesnikov [1985] proves that the disk does not fall, *i.e.*, the absolute value of the tilt does not reach the value of $\pi/2$ in finite time, for all but a codimension one set of initial conditions. It is interesting that the proof does not use the explicit formulae for the fields $\eta_1(\theta)$ and $\eta_2(\theta)$.

Indeed, the boundaries for the tilt of the disk θ are determined by the inequality $U_c(\theta) \leq h$, where h is the energy level and

$$U_c(\theta) = \frac{1}{2}(p_1^2(\theta)/A + p_2^2(\theta)/(mR^2 + B)) + mgR \cos \theta$$

is the amended potential restricted to the momentum level (6.17). The crucial step in the proof is to show that $\lim_{\theta \rightarrow \pm\pi/2} U_c(\theta) = \infty$ for most initial conditions. As in §6.4, consider the Wronskian

$$W(\theta) = \det \begin{bmatrix} \eta_1^1(\theta) & \eta_2^1(\theta) \\ \eta_1^2(\theta) & \eta_2^2(\theta) \end{bmatrix}.$$

By Liouville's formula (which is a special case of (6.16))

$$W(\theta) = W(0) \exp \int_0^\theta -\tan \theta d\theta = W(0) \cos \theta.$$

Therefore, $W(\theta) \rightarrow 0$ as $\theta \rightarrow \pm\pi/2$. Formulae (6.17) thus imply that the absolute value of at least one of the momentum components $p_1(\theta), p_2(\theta)$ approaches infinity as $\theta \rightarrow \pm\pi/2$, and so does $U_c(\theta)$ for all but a codimension one subset of $c = (c_1, c_2)$.

7 Conclusions

In this paper we studied the use of quasivelocities and Hamel equations in the dynamics of nonholonomic systems with symmetry. In particular, we discussed the importance of the choice of a basis of the tangent bundle of the configuration space in studying momentum conservation and integrability of nonholonomic systems. We showed that these ideas could be very helpful in analyzing specific examples such as the sleigh-mass system and the falling disk.

In future work we intend to extend some of these ideas to infinite-dimensional systems, to the analysis of control of nonholonomic systems, and to the dynamics of discrete nonholonomic systems, both free and controlled.

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