Dynamic Coverage Optimal Control for Interferometric Imaging Spacecraft Formations

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***Abstract*— In this paper, we study dynamic coverage optimal control, which is a new class of optimal control problems motivated by multi-spacecraft interferometric imaging applications. The dynamics is composed of N second order differential equations representing N fully actuated particles. To be minimized is a cost functional that is a weighted sum of the total fuel expenditure, the relative speeds between the particles and the measure of a given set whose size is a function of the particles' trajectories. We derive optimality conditions and give a simple three spacecraft example to illustrate the concepts and ideas presented in the paper.**

I. Introduction

The use of geometric control methods for spacecraft formation flying has received little attention, whereas extensive investigations have been conducted in the field of robotic path planning (for more on this issue, see Section (IV) in [1]). This work is an attempt to use geometric optimal control theory for spacecraft formation motion planning for imaging applications.

Let $\mathcal{C}^r$ be the set of all r-fold continuously differentiable curves $\mathbf{c} : [0,T] \to M$, where T is a fixed terminal maneuver time and M is an n-dimensional ($n = 1, 2$ or 3) Riemannian manifold equipped with the metric $\langle\cdot,\cdot\rangle$. In our research, we consider the general class of problems described by a system of N particles satisfying dynamics of the form:

$$\begin{aligned}\frac{\mathrm{D}\mathbf{c}_i}{\mathrm{d}t}(t) &= \mathbf{v}_i(t)\\ \frac{\mathrm{D}\mathbf{v}_i}{\mathrm{d}t}(t) &= \mathbf{u}_i(t).\end{aligned} \tag{1.1}$$

Let $\mathbf{u}_i(t) \in TT_{\mathbf{c}_i(t)}M$ be given by

$$\mathbf{u}_i(t) = \sum_{j=1}^{m} u_i^j(t) Y_j(\mathbf{c}_i(t)), \tag{1.2}$$

where $m \leq n$ and Y_j, $j = 1, \ldots, n$, satisfy $\langle Y_j, Y_k\rangle = \delta_{jk}$. In other words, Y_j is an orthonormal set of vector fields on $T_{\mathbf{c}_i(t)}M$. Mathematically, this assumption limits the class of manifolds we consider (to parallelizable manifolds) for the general problem formulation, but is satisfied for the special case where we deal with systems of particles in space. $m = n$ corresponds to the fully actuated system, whereas $m < n$ corresponds to the under-actuated situation. Here we only consider fully actuated systems.

Assumption I.1. Each particle is fully actuated in all n directions. That is to say $m = n$.

II. Imaging and the Coverage Problem

Equations (1.1) represent the spacecraft dynamics, treating each spacecraft as a point particle. Hence, we ignore attitude dynamics and assume all spacecraft are perfectly aligned and are pointing towards the target. Results presented in this paper can be extended to include rigid body dynamics, which is the main reason for using language and tools from geometric control theory. This, however, is the subject of current research. In interferometric imaging, we are interested in the relative position dynamics as projected onto a plane perpendicular to the line of sight. This plane is called the observation plane, denoted by $O \subset \mathbb{R}^2$. Hence, we are interested in the *projected relative curves*:

$$\tilde{\mathbf{c}}_{ij}(t) = \frac{1}{\lambda}\mathbb{P}_O\left(\mathbf{c}_j(t) - \mathbf{c}_i(t)\right), \tag{2.1}$$

where λ is the optical wavelength and $\tilde{\mathbf{c}}_{ij} : [0,T] \to \tilde{O}$ are curves on $\tilde{O}$, the frequency (or, u-v) plane, and $\mathbb{P}_O$ is the operator that projects relative trajectories in M onto the observation plane O. Hence, O is the plane on which motion is projected and $\tilde{O}$ is the u-v frequency plane.

In multi-aperture interferometry, there are two main imaging goals. The first is simply referred to as frequency domain (or u-v plane) coverage. Here, we only state the coverage goal and refer the reader to [2] for a more detailed discussion. We are interested in having the resolution disc as defined by the set $\mathcal{D}_R = \left\{(u,v) : \sqrt{u^2+v^2} \leq 1/\theta_r\right\}$ be completely covered by some ball of radius r_p centered at $\tilde{\mathbf{c}}_{ij}(t)$, for some $t \in [0,T]$, i and j, where θ_r is the angular resolution. An image is said to be successfully completed if a maneuver $\mathcal{M}$ satisfies the following condition.

Definition II.1. (**Successful Imaging Maneuver**) An imaging maneuver $\mathcal{M}$ is said to be successful if, for each $(u,v) \in \mathcal{D}_R$, there exists a time $t \in [0,T]$ and some $i,j = 1,\ldots,N$ such that $(u,v) \in \bar{B}_{r_p}(\tilde{\mathbf{c}}_{ij}(t))$, where $\bar{B}_x(\mathbf{y})$ is a *closed* ball in $\mathbb{R}^2$ of radius x centered at $\mathbf{y}$. r_p is proportional to the size of the telescope's airy disc.

The second objective is that all frequencies in $\mathcal{D}_R$ must be sampled while maximizing the signal-to-noise ratio (SNR). SNR can be controlled by controlling the relative speeds between the spacecraft in the formation [2]. As the projected relative speed between a spacecraft pair is minimized, so

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is the achievable SNR. Intuitively, as a spacecraft moves slower, it has more time spent in the neighborhood of a relative position state in space. This leads to more photon (that is, image information) collection from that neighborhood, resulting in a stronger signal. This is analogous to the notion of exposure time in photography, where the longer the shutter time is, the more photons get deposited on the photographic film and the better the image gets.

III. Dynamic Coverage Optimal Control

Based on the above discussion, we wish to minimize three quantities: (1) the fuel expended by each spacecraft in the constellation, (2) the projected relative speeds between the spacecraft of the system and (3) the number of uncovered points in $\mathcal{D}_R$. The constraints we have are the dynamics (1.1) and boundary conditions on the position and velocity vectors of each spacecraft. Motion constraints (as defined in [3]) are not treated in this paper, though they can be easily incorporated in the analysis. We now state the coverage optimal control problem considered in this paper.

Problem III.1. ***Coverage Optimal Control Problem:*** *Minimize*

$$\mathcal{J}(\mathbf{c}_i, \mathbf{u}_i, t; i = 1, \ldots, N) = \int_0^T \frac{1}{2} \left\{ \sum_{j=1}^{N} \left[\langle \mathbf{u}_j, \mathbf{u}_j \rangle \right.\right.$$

$$\left.\left. + \tau^2 \sum_{k=1}^{N} \left\langle \frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t}, \frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t} \right\rangle \right] \right\} + \kappa^2 \,\mathrm{meas}\,(\Psi)\,\mathrm{d}t, \tag{3.1}$$

where Ψ is the mapping that returns the set of uncovered frequency points in $\mathcal{D}_R$ up to time t; $\Psi : (t, \tilde{\mathbf{c}}_{ij}; i, j = 1, \ldots, N) \to \{(u, v) \in \mathcal{D}_R : \forall \sigma \in [0, t]$ and $\forall i, j \in 1, \ldots, N, (u, v) \notin \bar{B}_{r_p}(\tilde{\mathbf{c}}_{ij}(\sigma))\}$ and the function $\mathrm{meas}\,(\Lambda)$ is a measure function of some set Λ. The constraints are the dynamics (1.1), the boundary conditions

$$\mathbf{c}_i(0) = \mathbf{c}_i^0,\ \mathbf{c}_i(T) = \mathbf{c}_i^T,\ \mathbf{v}_i(0) = \mathbf{v}_i^0,\ \mathbf{c}_i(T) = \mathbf{v}_i^T, \tag{3.2}$$

$i = 1, \ldots, N$, and the relationship in Equation (2.1).

Note that if $\kappa = 0$, then the problem reduces to that discussed in [3] for a two-spacecraft formation. In this case, the terminal boundary conditions alone drive the system. On the other hand if $\kappa \neq 0$, then the system is driven to also minimize the set of uncovered points in $\mathcal{D}_R$. Whenever $\mathrm{meas}\,(\Psi)$ becomes zero, the only drive is to meet the terminal conditions in (3.2).

The measure function $\mathrm{meas}(\cdot)$ is simply the area covered by the set $\Psi(t, \tilde{\mathbf{c}}_{ij})$. Firstly, note that as the curves $\tilde{\mathbf{c}}_{ij} : [0, T] \to \tilde{O}$ change, the measure of Ψ at time T changes. However, if the curves $\tilde{\mathbf{c}}_{ij}$ correspond to trajectories of successful maneuvers as defined in Definition (II.1), then $\mathrm{meas}(\Psi(T, \tilde{\mathbf{c}}_{ij}^*))$ is zero at time T.

Secondly, note that $\mathrm{meas}(\Psi)$ is a monotonically decreasing function in time t. The reason for this is illustrated in Figure (1), which is the situation in the frequency domain for a two spacecraft system (hence, two coverage discs), and is explained as follows. Maximum decrease rate for $\mathrm{meas}(\Psi)$ is when *all* balls $\bar{B}_{r_p}(\tilde{\mathbf{c}}_{ij}(t))$, $i, j = 1, \ldots, N$, are moving into uncovered territory *inside* $\mathcal{D}_R$. In Figure (1), this happens at time t_0 and t_3 since both coverage balls move in previously uncovered territory. The other extreme is when *all* balls $\bar{B}_{r_p}(\tilde{\mathbf{c}}_{ij}(t))$, $i, j = 1, \ldots, N$, are moving in previously covered regions or have wandered outside $\mathcal{D}_R$, which corresponds to a constant value of $\mathrm{meas}(\Psi)$. In Figure (1), this happens instantaneously at time t_2 since both balls cover previously covered territory. Intermediate decrease rates vary between these two extremes (for example, at time t_1 as shown in the figure.) Note that the two coverage balls traverse symmetric curves. Symmetry holds for an arbitrary number of spacecraft by virtue of the condition (2.1).

Assumption III.1. The function meas is differentiable with respect to both arguments t and $\tilde{\mathbf{c}}$.

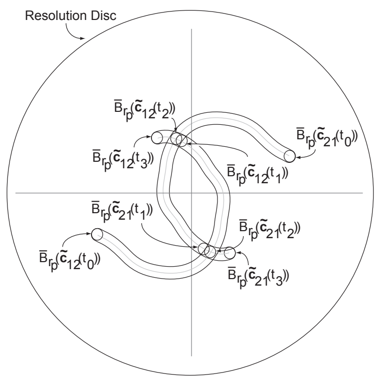

Fig. 1. A two spacecraft illustration of motion in the frequency domain ($t_0 < t_1 < t_2 < t_3$).

Finally, note that according to the definition of the coverage optimal control problem stated above, solutions to this problem do not necessary result in successful maneuvers. As the weight κ approaches infinity, the resulting solutions will tend to be successful maneuvers. Removing the term $\kappa^2\ \mathrm{meas}\,(\Psi(t))$ from the integrand and posing it as the terminal constraint $\mathrm{meas}\,(\Psi\,(T)) = 0$ is another strategy that results in successful maneuvers. This is not pursued here and will be the subject of future research.

IV. Necessary Conditions for Optimality

To obtain necessary optimality conditions we first append the dynamic constraints in Equations (1.1) to the Lagrangian of the cost functional (3.1) by introducing the terms

$$\lambda_1^j \left(\frac{\mathrm{D}\mathbf{c}_j}{\mathrm{d}t} - \mathbf{v}_j \right) + \lambda_2^j \left(\frac{\mathrm{D}\mathbf{v}_j}{\mathrm{d}t} - \mathbf{u}_j \right) \tag{4.1}$$

into the cost functional $\mathcal{J}$, where λ_1^j and λ_2^j, $j = 1, \ldots, N$, are Lagrange multipliers. Collecting terms with the same indexes, Equation (3.1) becomes:

$$\mathcal{J}(\mathbf{c}_i,\mathbf{u}_i)=\int_0^T\sum_{j=1}^N\left[\frac{1}{2}\langle\mathbf{u}_j,\mathbf{u}_j\rangle+\lambda_1^j\left(\frac{\mathrm{D}\mathbf{c}_j}{\mathrm{d}t}-\mathbf{v}_j\right)\right.$$
$$+\lambda_2^j\left(\frac{\mathrm{D}\mathbf{v}_j}{\mathrm{d}t}-\mathbf{u}_j\right)+\frac{\tau^2}{2}\sum_{k=1}^N\left.\left\langle\frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},\frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t}\right\rangle\right]$$
$$+\kappa^2\ \mathrm{meas}\left[\Psi\left(\tilde{\mathbf{c}}_{jk}(t);j,k=1,\ldots,N\right)\right]\mathrm{d}t \qquad (4.2)$$

We then introduce the following one-parameter variations for the curves $\mathbf{c}_i$:

$$\begin{aligned}\mathbf{c}_i(t,0)&=\mathbf{c}_i(t),\\ \frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(t,0)&=\mathbf{W}_i(t)\\ \frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(0,0)&=\frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(T,0)=0,\\ \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(t,0)&=\frac{\mathrm{D}}{\mathrm{d}t}\mathbf{W}_i(t)\text{ is continuous on }[0,T]\\ \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(0,0)&=\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\mathbf{c}_i}{\partial\epsilon}(T,0)=0,\end{aligned}\qquad(4.3)$$

$i=1,\ldots,N$. Likewise, we may define variations in $\mathbf{v}_i(t)$, $\mathbf{u}_i(t)$ and $\lambda_k^i(t)$, $k=1,2$, $i=1,\ldots,N$, by $\mathbf{v}_i(t,\epsilon)$, $\mathbf{u}_i(t,\epsilon)$ and $\lambda_k^i(t,\epsilon)$, $k=1,2$, $i=1,\ldots,N$, as follows:

$$\begin{aligned}\mathbf{u}_i(t,\epsilon)&=\sum_{j=1}^m u_i^j(t,\epsilon)Y_j(\mathbf{c}_i(t,\epsilon))\in T_{\mathbf{c}_i(t,\epsilon)}M\\ \mathbf{v}_i(t,\epsilon)&=\sum_{j=1}^n v_i^j(t,\epsilon)Y_j(\mathbf{c}_i(t,\epsilon))\in T_{\mathbf{c}_i(t,\epsilon)}M\\ \lambda_k^i(t,\epsilon)&=\sum_{j=1}^n \lambda_k^{ij}(t,\epsilon)\omega_j(\mathbf{c}_i(t,\epsilon))\in T^*_{\mathbf{c}_i(t,\epsilon)}M\end{aligned}$$

where ω_j, $j=1,\ldots,n$, are co-vector fields such that $\omega_l(Y_j)=\delta_{lj}$. Taking variations in $\mathbf{u}_i$ and $\mathbf{v}_i$, we have:

$$\begin{aligned}\left.\frac{\mathrm{D}\mathbf{u}_i}{\partial\epsilon}(t,\epsilon)\right|_{\epsilon=0}&=\delta\mathbf{u}_i(t)+\left(\mathbf{B}\left(\mathbf{W}_i,\mathbf{u}_i\right)\right)(\mathbf{c}_i(t))\in TTM\\ \left.\frac{\mathrm{D}\mathbf{v}_i}{\partial\epsilon}(t,\epsilon)\right|_{\epsilon=0}&=\delta\mathbf{v}_i(t)+\left(\mathbf{B}\left(\mathbf{W}_i,\mathbf{u}_i\right)\right)(\mathbf{c}_i(t))\in TTM\end{aligned}$$

where, for instance,

$$\delta\mathbf{u}_i(t)=\sum_{j=1}^m\frac{\partial u_i^j}{\partial\epsilon}(t,0)Y_j(\mathbf{c}_i(t))$$

and

$$\left(\mathbf{B}\left(\mathbf{W}_i,\mathbf{u}_i\right)\right)(\mathbf{c}_i(t))=\sum_{j=1}^m u_i^j(t)\left(\nabla_{\mathbf{W}_i}Y_j\right)(\mathbf{c}_i(t)).$$

Similar expressions can be obtained for $\frac{\mathrm{D}\mathbf{v}_i}{\partial\epsilon}$ and $\frac{\mathrm{D}\lambda_i^j}{\partial\epsilon}$, $j=1,2$, $i=1,\ldots,N$. $\mathbf{B}(\cdot,\cdot)$ is a bilinear form that we introduce in order to be able to separate variations in the components of $\mathbf{u}_i$, $\mathbf{v}_i$ and λ_i^j, $i=1,\ldots,N$, $j=1,2$, from variations in the basis vector fields. It is important to separate these terms since the variations $\delta\mathbf{u}_i$, $\delta\mathbf{v}_i$ and $\delta\lambda_j^i$, $i=1,\ldots,N$, $j=1,2$, are independent from each other as well as from $\mathbf{W}_i$ –a fact which has significant importance in deriving necessary conditions.

For variations in $\tilde{\mathbf{c}}_{ij}(t)$, let

$$\tilde{\mathbf{c}}_{ij}(t,\epsilon)=\sum_{k=1}^2\tilde{c}_{ij}^k(t,\epsilon)\mathbf{Z}_k(\tilde{\mathbf{c}}_{ij}(t,\epsilon))\in T_{\tilde{\mathbf{c}}_{ij}(t,\epsilon)}\tilde{O},$$

where $\mathbf{Z}_k$, $k=1,2$, is an orthonormal set of vector fields on $T_{\tilde{\mathbf{c}}_{ij}(t,\epsilon)}\tilde{O}$. The set $\mathbf{Z}_k$, $k=1,2$, may be taken to be the standard set of vector fields spanning $\mathbb{R}^2$.

Assumption IV.1. Let $M=O\subset\mathbb{R}^2$. In other words, the surfaces M and O coincide.

Under Assumption (IV.1), the projection $\mathbb{P}_O$ is the identity operator. This implies that all the derivatives $\mathrm{D}/\mathrm{d}t$ could be replaced by regular derivatives in $\mathbb{R}^2$. However, we will retain the former notation because future research will not make Assumption (IV.1) and will consider rigid body dynamics, where most of the results obtained below will only be slightly modified. Moreover, as long as one chooses non-rectangular coordinates the former notation becomes very convenient. Thus, for $i,j=1,\ldots,N$ we have

$$\begin{aligned}\tilde{\mathbf{c}}_{ij}(t,0)&=\tilde{\mathbf{c}}_{ij}(t)=\frac{\mathbf{c}_j(t)-\mathbf{c}_i(t)}{\lambda},\\ \frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(t,0)&=\frac{1}{\lambda}\left[W_j(t)-W_i(t)\right]\\ \frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(0,0)&=\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(T,0)=0,\\ \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(t,0)&=\frac{1}{\lambda}\frac{\mathrm{D}}{\mathrm{d}t}\left[W_j(t)-W_i(t)\right]\text{ cont. on }[0,T]\\ \frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(0,0)&=\frac{\mathrm{D}}{\mathrm{d}t}\frac{\mathrm{D}\tilde{\mathbf{c}}_{ij}}{\partial\epsilon}(T,0)=0.\end{aligned}\qquad(4.4)$$

Theorem IV.1. *Under Assumptions (I.1), (III.1) and (IV.1), taking first order variations of the expression in Equation (4.2) leads to the following relationship:*

$$\left.\frac{\partial\mathcal{J}}{\partial\epsilon}(\mathbf{c}_i(t,\epsilon),\mathbf{u}_i(t,\epsilon),t;i=1,\ldots,N)\right|_{\epsilon=0}=$$
$$=\int_0^T\sum_{j=1}^N\langle\mathbf{u}_j,\mathbf{B}\left(\mathbf{W}_j,\mathbf{u}_j\right)\rangle-\frac{\mathrm{D}\lambda_1^j}{\mathrm{d}t}\left(W_j\right)$$
$$-\lambda_1^j\left(\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)-\lambda_2^j\left(\mathbf{B}\left(\mathbf{W}_j,\mathbf{u}_j\right)\right)$$
$$+\lambda_2^j\left(R\left(\mathbf{W}_j,\mathbf{v}_j\right)\mathbf{v}_j\right)-\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t}\left(\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)$$
$$+\sum_{k=1}^N\frac{\tau^2}{\lambda^2}\langle\mathbf{u}_k-\mathbf{u}_j,W_j\rangle-\frac{\kappa^2}{\lambda}\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(W_j\right)\mathrm{d}t\qquad(4.5)$$
$$+\int_0^T\sum_{j,k=1}^N-\frac{\tau^2}{\lambda^2}\langle\mathbf{u}_k-\mathbf{u}_j,W_k\rangle+\frac{\kappa^2}{\lambda}\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(W_k\right)\mathrm{d}t$$
$$+\int_0^T\sum_{j=1}^N-\lambda_2^j(\delta\mathbf{u}_j)+\langle\mathbf{u}_j,\delta\mathbf{u}_j\rangle\,\mathrm{d}t$$
$$+\int_0^T\sum_{j=1}^N-\lambda_1^j\left(\delta\mathbf{v}_j\right)-\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t}\left(\delta\mathbf{v}_j\right)\mathrm{d}t.$$

Proof In Equation (4.2), we replace $\tilde{\mathbf{c}}_{jk}(t)$, $\mathbf{u}_j(t)$ and $\mathbf{v}_j(t)$

with the perturbed variables $\tilde{\mathbf{c}}_{jk}(t,\epsilon)$, $\mathbf{u}_j(t,\epsilon)$ and $\mathbf{v}_j(t,\epsilon)$, respectively. To prove the theorem, we compute $\partial\mathcal{J}/\partial\epsilon$ on a term by term basis as follows. First, we have:

$$\frac{\partial}{\partial\epsilon}\int_0^T \frac{1}{2}\left\langle \mathbf{u}_j(t,\epsilon),\mathbf{u}_j(t,\epsilon)\right\rangle \mathrm{d}t\bigg|_{\epsilon=0} = \int_0^T \left\langle \mathbf{u}_j, \delta\mathbf{u}_j + \mathbf{B}\left(\mathbf{W}_j,\mathbf{u}_j\right)\right\rangle \mathrm{d}t, \qquad (4.6)$$

where a summation over j is understood. For the fourth term in Equation (4.2), we use the fourth identity in Equations (4.4) and integrate by parts to obtain

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}\int_0^T \frac{\tau^2}{2}\sum_{j=1}^N\sum_{k=1}^N\left\langle\frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},\frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t}\right\rangle\mathrm{d}t\bigg|_{\epsilon=0}\\ &=\int_0^T\frac{\tau^2}{\lambda}\sum_{j=1}^N\sum_{k=1}^N\left\langle\frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},\frac{\mathrm{D}}{\mathrm{d}t}\left[W_k-W_j\right]\right\rangle\mathrm{d}t\\ &=-\int_0^T\frac{\tau^2}{\lambda}\sum_{j=1}^N\sum_{k=1}^N\left\langle\frac{\mathrm{D}^2\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t^2},W_k-W_j\right\rangle\mathrm{d}t\\ &+\sum_{j=1}^N\sum_{m=1}^N\frac{\tau^2}{\lambda}\left\langle\frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},W_k-W_j\right\rangle\bigg|_0^T.\end{aligned}$$

The second term vanishes due to the fixed boundary conditions (4.3). Under Assumption (IV.1), we have

$$\frac{\mathrm{D}^2\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t^2}=\frac{1}{\lambda}\left(\mathbf{u}_k-\mathbf{u}_j\right).$$

Thus, for the fourth term in Equation (4.2) we have

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}\int_0^T\frac{\tau^2}{2}\sum_{j,k=1}^N\left\langle\frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t},\frac{\mathrm{D}\tilde{\mathbf{c}}_{jk}}{\mathrm{d}t}\right\rangle\mathrm{d}t\bigg|_{\epsilon=0}\\ &=-\int_0^T\frac{\tau^2}{\lambda^2}\sum_{j,k=1}^N\left\langle\mathbf{u}_k-\mathbf{u}_j,W_k-W_j\right\rangle\mathrm{d}t. \qquad (4.7)\end{aligned}$$

For the second term, we have

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}\int_0^T\lambda_1^j\left(\frac{\mathrm{D}\mathbf{c}_j}{\mathrm{d}t}-\mathbf{v}_j\right)\mathrm{d}t\\ &=\int_0^T\lambda_1^j\left(\frac{\mathrm{D}}{\mathrm{d}t}\mathbf{W}_j-\delta\mathbf{v}_j-\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)\mathrm{d}t.\end{aligned}$$

For the first term in the parenthesis, we integrate by parts to obtain

$$\int_0^T\lambda_1^j\left(\frac{\mathrm{D}}{\mathrm{d}t}\mathbf{W}_j\right)\mathrm{d}t=\lambda_1^j\left(\mathbf{W}_j\right)\bigg|_0^T-\int_0^T\frac{\mathrm{D}\lambda_1^j}{\mathrm{d}t}\left(\mathbf{W}_j\right)\mathrm{d}t.$$

The first term on the right hand side vanishes by virtue of the boundary conditions (4.3). We then obtain

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\int_0^T\sum_{j=1}^N\lambda_1^j\left(\frac{\mathrm{D}\mathbf{c}_j}{\mathrm{d}t}-\mathbf{v}_j\right)\mathrm{d}t&=\int_0^T\sum_{j=1}^N-\frac{\mathrm{D}\lambda_1^j}{\mathrm{d}t}\left(\mathbf{W}_j\right)\\ &-\lambda_1^j\left(\delta\mathbf{v}_j+\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)\mathrm{d}t. \qquad (4.8)\end{aligned}$$

For the third term in Equation (4.2), first recall the identity (page 52 in [4]):

$$\frac{\mathrm{D}}{\partial\epsilon}\frac{\mathrm{D}}{\partial t}\mathbf{Y}-\frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}}{\partial\epsilon}\mathbf{Y}=R\left(\frac{\mathrm{D}\mathbf{c}}{\partial\epsilon},\frac{\mathrm{D}\mathbf{c}}{\partial t}\right)\mathbf{Y}.$$

Then, we have

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}\int_0^T\sum_{j=1}^N\lambda_2^j\left(\frac{\mathrm{D}\mathbf{v}_j}{\mathrm{d}t}-\mathbf{u}_j\right)\mathrm{d}t\\ &=\int_0^T\sum_{j=1}^N\lambda_2^j\Bigg(R(\mathbf{W}_j,\mathbf{v}_j)\mathbf{v}_j+\frac{\mathrm{D}^2\mathbf{v}_j}{\partial t\partial\epsilon}-\delta\mathbf{u}_j\\ &-\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\Bigg)\mathrm{d}t\\ &=\int_0^T\sum_{j=1}^N\lambda_2^j\Big(R(\mathbf{W}_j,\mathbf{v}_j)\mathbf{v}_j-\delta\mathbf{u}_j-\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\Big)\\ &-\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t}\left(\delta\mathbf{v}_j+\mathbf{B}\left(\mathbf{W}_j,\mathbf{v}_j\right)\right)\mathrm{d}t \qquad (4.9)\end{aligned}$$

where integration by parts has been used to arrive at the last equation. Finally, under Assumption III.1, for the last term we have

$$\begin{aligned}\frac{\partial}{\partial\epsilon}\int_0^T\kappa^2\mathrm{meas}\left[\Psi\right]\mathrm{d}t&=\int_0^T\sum_{j,k=1}^N\kappa^2\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\frac{\partial\tilde{\mathbf{c}}_{jk}}{\partial\epsilon}\mathrm{d}t\\ &=\int_0^T\sum_{j,k=1}^N\frac{\kappa^2}{\lambda}\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(\mathbf{W}_k-\mathbf{W}_j\right)\mathrm{d}t, \qquad (4.10)\end{aligned}$$

where it is understood that the meas function is applied to the set $\Psi\left(\tilde{\mathbf{c}}_{jk}\right)$ for all $j,k=1,\ldots,N$. Finally, from equations (4.6-4.10), by separating terms involving the coefficients $\mathbf{W}_j$, $\mathbf{W}_k$, $\delta\mathbf{v}_j$ and $\delta\mathbf{u}_j$, we obtain the expression (4.5) and, hance, proving the theorem. ■

Theorem IV.2. *Under Assumptions (I.1), (III.1) and (IV.1), a set of optimal trajectories $\tilde{\mathbf{c}}_i$, $i=1,\ldots,N$, that minimize $\mathcal{J}$ while satisfying the dynamic constraints (1.1) and the boundary conditions (3.2) satisfies the following necessary conditions for an arbitrary vector field $\mathbf{X}$:*

$$\begin{aligned}&\frac{\mathrm{D}\mathbf{c}_i}{\mathrm{d}t}=\mathbf{v}_i\\ &\frac{\mathrm{D}\mathbf{v}_i}{\mathrm{d}t}=\left(\lambda_2^j\right)^\#,\\ &\frac{\mathrm{D}\lambda_1^j}{\mathrm{d}t}\left(\mathbf{X}\right)=\left(R\left(\mathbf{u}_j,\mathbf{v}_j\right)\mathbf{v}_j\right)^\flat\left(\mathbf{X}\right) \qquad (4.11)\\ &\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t}\left(\mathbf{X}\right)=-\lambda_1^j\left(\mathbf{X}\right)\\ &\mathbf{u}_j=\left(\lambda_2^j\right)^\#\\ &0=\sum_{k=1}^N\frac{\tau^2}{\lambda^2}\left(\lambda_2^k\left(\mathbf{X}\right)-\lambda_2^j\left(\mathbf{X}\right)\right)-\frac{\kappa^2}{\lambda}\frac{\partial\mathrm{meas}}{\partial\tilde{\mathbf{c}}_{jk}}\left(\mathbf{X}\right)\end{aligned}$$

for $j=1,\ldots,N$ and where $\mathbf{Y}^\flat\left(\mathbf{X}\right)=\left\langle\mathbf{Y},\mathbf{X}\right\rangle$, with $\flat$ denoting the flat operator [5].

Proof The first equation follows immediately from Equation (1.1). For an optimal solution, the first order necessary condition is that

$$\frac{\partial \mathcal{J}}{\partial \epsilon}(\mathbf{c}_i(t,\epsilon), \mathbf{u}_i(t,\epsilon), t; i=1,\dots,N)\Big|_{\epsilon=0} = 0. \quad (4.12)$$

The rest of the proof relies on this condition and the fact that $\mathbf{W}_j$, $\mathbf{W}_k$, $\delta\mathbf{u}_j$ and $\delta\mathbf{v}_j$ are independent for all $j,k = 1,\dots,N$. The fourth equation follows immediately from the last integral in Equation (4.5) and the independence of $\delta\mathbf{v}_j$, $j=1,\dots,N$. The fifth equation follows immediately from condition (4.12), the third integral in Equation (4.5) and the independence of $\delta\mathbf{u}_j$, $j = 1,\dots,N$. The last (algebraic) equation in (4.11) is obtained by studying the second integral in Equation (4.5). Since $\mathbf{W}_k$, $k=1,\dots,N$, are independent, we then have

$$\sum_{j=1}^{N} -\frac{\tau^2}{\lambda^2}(\mathbf{u}_k - \mathbf{u}_j) + \frac{\kappa^2}{\lambda}\frac{\partial \text{meas}(\mathbf{\Psi})}{\partial \tilde{\mathbf{c}}_{jk}} = 0, \ \forall k = 1,\dots,N.$$

Since $\mathbf{u}_j = \left(\lambda_2^j\right)^{\#}$ and by interchanging indices ($j \to k$ and $k \to j$), we obtain the last (algebraic) condition in (4.11). Hence, the last term under the first integral in Equation (4.5) is zero. This, the fact that $\frac{\mathrm{D}\lambda_2^j}{\mathrm{d}t} = -\lambda_1^j$ and the independence of $\mathbf{W}_j$, $j=1,\dots,N$, in the first integral in Equation (4.5) give the third equation in the theorem. The second equation follows from Equation (1.1) and the fifth condition in equation (4.11). ■

Studying the last (algebraic) necessary condition gives further insight into the optimal trajectory. Note that one can write these N conditions in a matrix form:

$$\mathbf{A}\mathbf{U} = \frac{\kappa^2\lambda}{\tau^2}\mathbf{M}, \quad (4.13)$$

where

$$\mathbf{A} = \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & & -1 \\ \vdots & & \ddots & \vdots \\ -1 & \cdots & -1 & N-1 \end{bmatrix}$$

is an $N\times N$ matrix, $\mathbf{U}$ is the $N\times 1$ column matrix whose j^{th} entry is $\mathbf{u}_j$ and $\mathbf{M}$ is the $N \times 1$ column matrix whose j^{th} entry is $\sum_{k=1}^{N}\frac{\partial \text{meas}(\mathbf{\Psi})}{\partial \tilde{\mathbf{c}}_{jk}}$. Let a_{ij} be the ij^{th} element of $\mathbf{A}$. Note that $a_{Nj} = -\sum_{i=1}^{N-1} a_{ij}$. Hence, the last row is dependent on the first $N-1$ rows. In fact, one can show that $\mathbf{A}$ has rank exactly equal to $N-1$. The homogenous solution to the above equation is found to be $\mathbf{u}_1^h = \mathbf{u}_2^h = \cdots = \mathbf{u}_N^h$. The homogeneous solution corresponds to the motion of the center of mass of the formation in the plane. Since it is desired to minimize fuel, then we may set the homogeneous solution to zero: $\mathbf{u}_1^h = \mathbf{u}_2^h = \cdots = \mathbf{u}_N^h = \mathbf{0}$.

What really matters in this situation is the particular solution, if one exists. Indeed, we now show that the matrix $\mathbf{M}$ lies in the range space of the matrix $\mathbf{A}$ and, hence, a particular solution exists. First, append $\mathbf{M}$ to $\mathbf{A}$ to form the new matrix $\tilde{\mathbf{A}} = [\mathbf{A}\ \mathbf{M}]$. Recall that $\mathbf{A}$ has rank equal to $N-1$. If we can show that $\tilde{\mathbf{A}}$ also has rank $N-1$, then $\mathbf{M}$ lies in the range space of $\mathbf{A}$. Let $\tilde{\mathbf{M}}$ be the matrix whose elements are given by $\tilde{M}_{ij} = \frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{ij}}$. With $\tilde{\mathbf{c}}_{ij} = -\tilde{\mathbf{c}}_{ji}$ and $\tilde{\mathbf{c}}_{ii} = \mathbf{0}$ is fixed at the origin, then $\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{ii}} = 0$ and $\tilde{\mathbf{M}}$ is skew symmetric. Next, note that the N^{th} element of $\mathbf{M}$ is given by

$$\begin{aligned} \sum_{j=1}^{N-1}\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{Nj}} &= \sum_{j=1}^{N-1}\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{Nj}} + \sum_{j,k=1}^{N-1}\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{kj}} \\ &= \sum_{j=1}^{N-1}\sum_{k=1}^{N}\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{kj}} = -\sum_{j=1}^{N-1}\sum_{k=1}^{N}\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{jk}}, \end{aligned}$$

where the second term after the first equality sign is zero since $\tilde{\mathbf{M}}$ is skew symmetric. The term after the last equality sign is nothing but the sum of all the first $N-1$ elements of the matrix $\mathbf{M}$. This and the fact that $a_{Nj} = -\sum_{i=1}^{N-1} a_{ij}$ show that the last row of $\tilde{\mathbf{A}}$ is equal to the sum of the first $N-1$ rows of $\tilde{\mathbf{A}}$. Since $\mathbf{A}$ has rank $N-1$, then so must $\tilde{\mathbf{A}}$. Hence, $\mathbf{M}$ must in fact be in the range space of $\mathbf{A}$ and a particular solution must exist ([6], pp. 116-121).

For $N=2$, the condition (4.13) is equivalent to $\mathbf{u}_1 - \mathbf{u}_2 = \frac{\kappa^2\lambda}{\tau^2}\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{12}}$. Hence, for a two spacecraft formation, a necessary optimality condition is that the relative thrusting between the two spacecraft is in the direction of descent of the measure of the uncovered set of u-v points. For $N=3$, the condition (4.13) is equivalent to:

$$\begin{aligned} \mathbf{u}_1 - \mathbf{u}_2 &= \frac{1}{3}\frac{\kappa^2\lambda}{\tau^2}\left[\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{13}} + 2\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{12}} + \frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{32}}\right] \\ \mathbf{u}_1 - \mathbf{u}_3 &= \frac{1}{3}\frac{\kappa^2\lambda}{\tau^2}\left[\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{12}} + 2\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{13}} + \frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{23}}\right], \end{aligned}$$

where now a necessary optimality condition is that the relative thrusting between the three spacecraft is a weighted sum of the direction of descent of the measure of the uncovered set of u-v points.

Hence, each spacecraft's motion and control effort is affected by the amount of area of $\mathcal{D}_R$ that has not been covered by the formation as it involves summations over motions of all the other spacecraft. Therefore, the resulting control law is in some sense decentralized: Given knowledge of the motions of the other spacecraft, the above necessary conditions command each spacecraft to move in directions that attempt to minimize the cost function $\mathcal{J}$.

Remarks:

1) Note that $\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{jk}}$ constitute the components of the differential form $\mathrm{d}(\text{meas})$. Hence, the notation $\frac{\partial \text{meas}}{\partial \tilde{\mathbf{c}}_{jk}}(\mathbf{X})$ denotes this form operating on $\mathbf{X}$.
2) In the proof for Theorem (IV.1) we have not taken variations in the multipliers λ_i^j, $i = 1,2$, $j = 1,\dots,N$. This is standard practice and the justification can found in Section 2 of [7].

V. Example

In this section we state the necessary conditions for a three-spacecraft, rigidly-connected, co-planar formation. The necessary conditions for a general one degree of freedom system are slightly different from those of Theorem

(IV.2). Since we only have a single degree of freedom, a single control vector field suffices to drive the system. Hence, the condition (4.13) will vanish. Instead, the effect of the measure function $\mathrm{meas}\,(\mathbf{\Psi})$ on the closed loop system appears in the dynamics governing the Lagrange multipliers. This result governs other single degree of freedom systems with different numbers of spacecraft and configurations.

The formation we study is shown in Figure (2). The formation assumes the shape of an equilateral triangle. Formations such as this one appear in previous literature. See for example the formation used in [8]. Let the side of the triangle be given by a and each spacecraft is at a distance r from the center of mass CM, where $r = a/\sqrt{3}$. To guarantee that the resulting motion results in a successful maneuver, we impose the condition that $a = 2r_p\lambda$. Moreover, assume the resolution disc D_R has a radius of $1/\theta_r = 3r_p$. These conditions and the rigidity of the formation guarantee that the resulting six picture frame discs (as defined in Section II) are centered such that each scans an annulus about the central disc. After the formation rotates by an angle of $60°$, the maneuver is completed, resulting in a successful maneuver. The motion in the u-v plane is shown in Figure (2) (right).

Since this is a single degree of freedom system evolving on the unit circle S^1, let the angular position, $\theta(t)$, describe the state of the system as shown in Figure (2). Hence, $\theta(0) = 0$ and $\theta(T) = \pi/3$. For this example, an approximation of the measure function is given by

$$\mathrm{meas}\,(\Psi(\theta(t))) = -24r_p^2\theta(t) + 8\pi r_p^2. \qquad (5.1)$$

One can also check that $\mathrm{meas}\,(\Psi(\theta = 0)) = 8\pi r_p^2$ (that is, the area of the initial uncovered annulus) and that $\mathrm{meas}(\Psi\,(\theta = \pi/3)) = 0$ as one expects at the end of a successful maneuver.

If we let the mass of each spacecraft be given by m_s and the torque applied to each spacecraft be given by F, then the equations of motion are given by the equation

$$\frac{\mathrm{D}\theta}{\mathrm{d}t} = \omega, \ \frac{\mathrm{D}\omega}{\mathrm{d}t} = u, \qquad (5.2)$$

where $u = \frac{F}{m_s}$. The cost function to be minimized is

$$\mathcal{J} = \int_0^T \frac{1}{2}\langle u, u\rangle + \frac{\tau'^2 a^2}{2\lambda^2}\langle \omega, \omega\rangle + \kappa^2 \mathrm{meas}(\theta)\mathrm{d}t, \qquad (5.3)$$

where the absolute linear velocity of the discs is given by $\frac{a}{\lambda}\frac{\mathrm{D}\theta}{\mathrm{d}t} = 2r_p\frac{\mathrm{D}\theta}{\mathrm{d}t}$. Appending $\mathcal{J}$ by the terms $\lambda_1\left(\frac{\mathrm{D}\theta}{\mathrm{d}t} - \omega\right)$ and $\lambda_2\left(\frac{\mathrm{D}\omega}{\mathrm{d}t} - u\right)$, and following a procedure similar to that used to derive Theorems (IV.1) and (IV.2), we obtain Equations (5.2) and

$$\begin{aligned}
\frac{\mathrm{D}\lambda_1}{\mathrm{d}t}(\mathbf{X}) &= \kappa^2\frac{\partial \mathrm{meas}}{\partial\theta}(\mathbf{X}) + \langle R(u,\omega)\omega, \mathbf{X}\rangle \\
\frac{\mathrm{D}\lambda_2}{\mathrm{d}t}(\mathbf{X}) &= -\lambda_1(\mathbf{X}) + \tau^2\langle\omega, \mathbf{X}\rangle \\
u &= \lambda_2^{\#},
\end{aligned}$$

as necessary conditions, where $\partial\mathrm{meas}/\partial\theta = -24r_p^2$ and $\mathbf{X} \in TM$ is any arbitrary vector field. We note here that the algebraic condition on λ_2 vanishes since the system has only one degree of freedom. Instead, an additional term is added to the $\frac{\mathrm{D}\lambda_1}{\mathrm{d}t}$ equation.

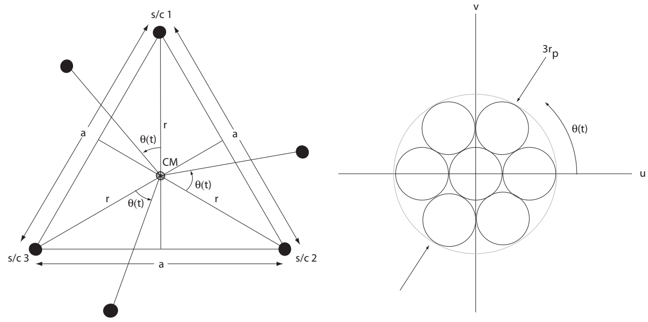

Fig. 2. A three-spacecraft, rigidly-connected, co-planar formation (left) and the motion in the u-v plane (right).

VI. Conclusion

In this paper we studied the dynamic coverage optimal control problem. The problem is motivated by interferometric imaging spacecraft formations. An optimal control problem is defined to achieve maneuvers optimal in both imaging and fuel senses. Optimality conditions were derived and a simple three spacecraft example was given to illustrate our results. Future work will aim at eliminating some of the assumptions made in this paper. Specifically, we aim at studying systems evolving in gravitational fields and those evolving on non-planar Riemannian manifolds.

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