ENERGY in CIRCUITS and MECHANICS

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Workshop ***The Work of Peter Crouch*** **CDC, Orlando, Florida, December 11, 2011**

In honor of Peter Crouch
on the occasion of his 60th birthday

Edzell, Scotland September 1980

Theme

How are open systems formalized?

How are systems interconnected?

How is energy transferred between systems?

Are energy transfer and interconnection related?

A view from afar with almost 50 years of hindsight,

befitting a 60th birthday celebration.

Motivation

The ever-increasing computing power allows to model complex interconnected dynamical systems accurately by tearing, zooming, and linking.

$\leadsto$ **Simulation, model based design, ...**

Requires the right mathematical concepts

- **for dynamical system**
- **for interconnection,**
- **for interconnection architecture**

SYSTEMS with TERMINALS

In order to keep the discussion simple and concrete, we discuss only systems that interact with the environment through terminals, and interact among each other via terminals.

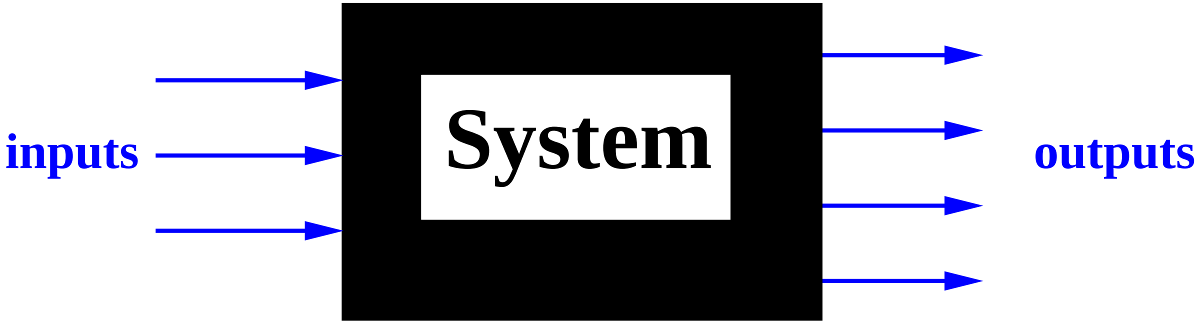
- **Electrical circuits**
- **Mechanical devices**
- **Thermal systems**
- **Hydraulic systems**
- **Multidomain systems as motors, pumps, loudspeakers, ...**
- **etc., etc.**

The why and how

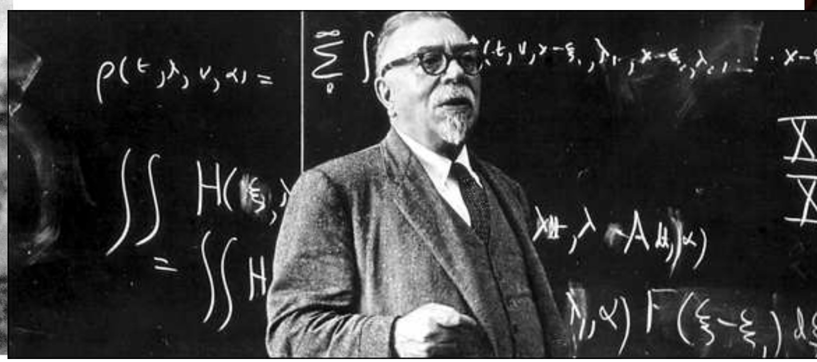
If concepts and methods from system theory do not fit these simple examples seamlessly, what is the sense to pursue, with these concepts and methods, complex systems, as those encountered in cyberphysics, hi-tech, economics, biology, ...?

CLASSICAL VIEW

Input/output systems



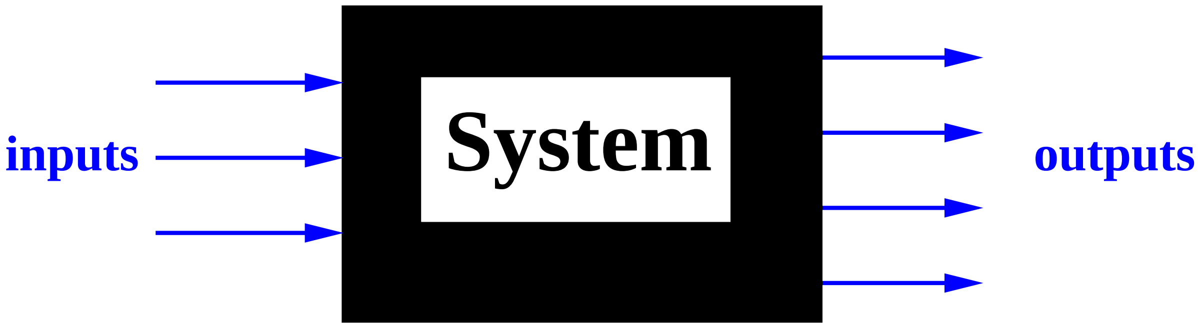
Oliver Heaviside



Norbert Wiener

Rudy Kalman

Input/output systems



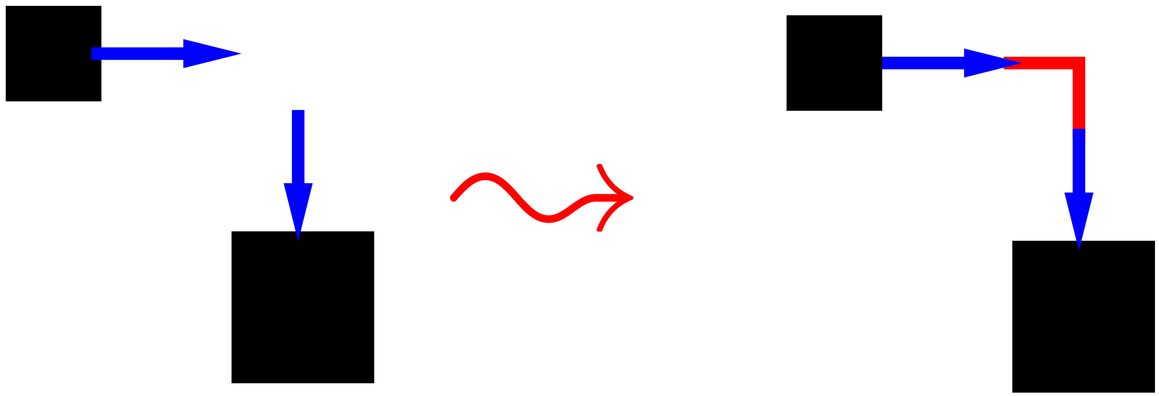
Input/output thinking is *inappropriate* for describing the functioning of open physical systems.

A physical system is not a signal processor.

Better concept: a behavior

Interconnection

Interconnection as output-to-input assignment.

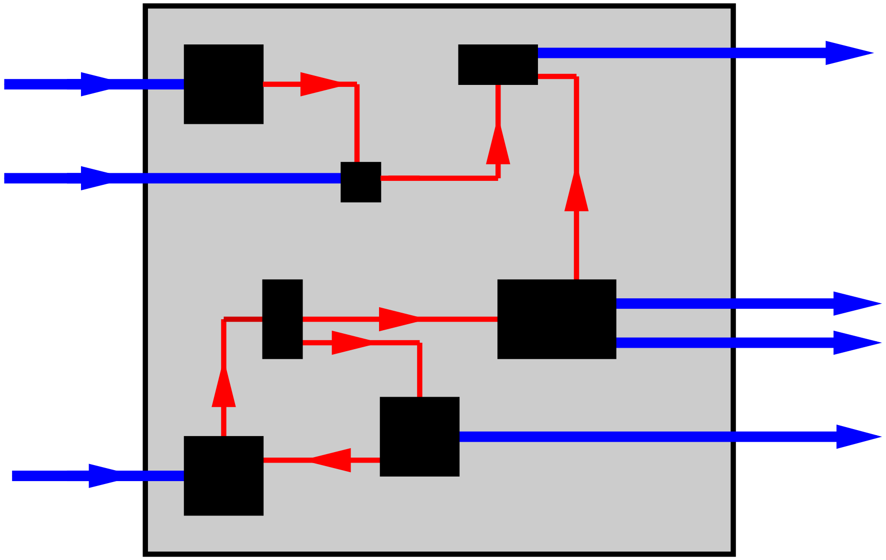


Output-to-input assignment is *inappropriate* for describing the interconnection of physical systems.

A physical system is not a signal processor.

Better concept: variable sharing

Signal flow graphs



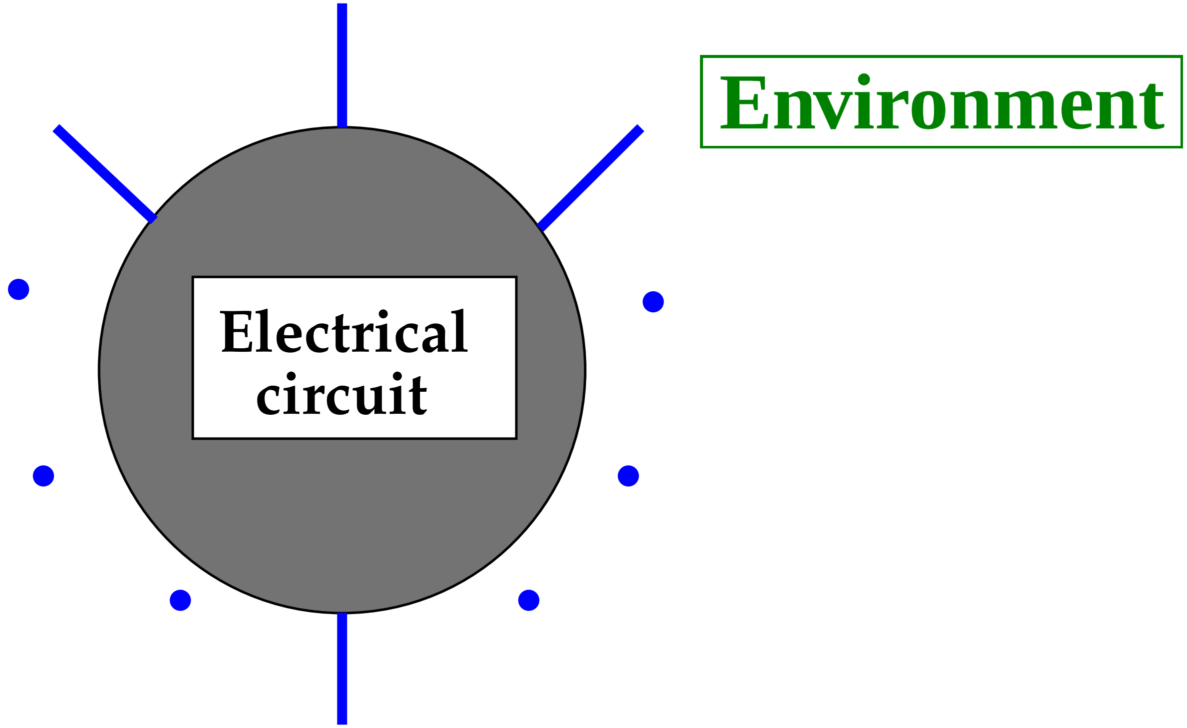
Signal flow graphs are *inappropriate* for describing the interaction architecture of physical systems.

A physical system is not a signal processor.

Better concept: a graph with leaves

ELECTRICAL CIRCUITS

A circuit with external terminals

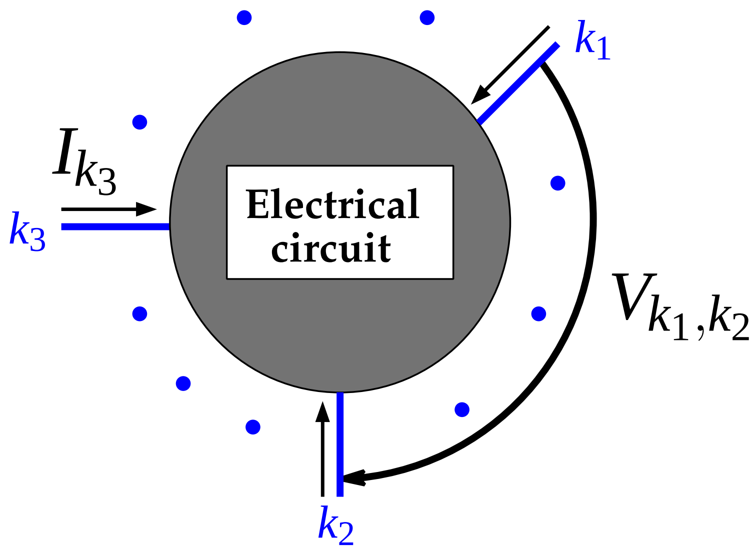


Describe the dynamic terminal behavior!

As seen from the environment.

What are the interaction variables?

Currents and voltages



Interaction variables: currents in & voltages across.

$$I = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}, \quad V = \begin{bmatrix} V_{1,1} & V_{1,2} & \cdots & V_{1,N} \\ V_{2,1} & V_{2,2} & \cdots & V_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ V_{N,1} & V_{N,2} & \cdots & V_{N,N} \end{bmatrix}$$

The behavior

Interaction variables: **currents in & voltages across.**

$$I = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}, \quad V = \begin{bmatrix} V_{1,1} & V_{1,2} & \cdots & V_{1,N} \\ V_{2,1} & V_{2,2} & \cdots & V_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ V_{N,1} & V_{N,2} & \cdots & V_{N,N} \end{bmatrix}$$

Model $:\Leftrightarrow \mathscr{B}_{IV} \subseteq \left(\mathbb{R}^N \times \mathbb{R}^{N\times N}\right)^{\mathbb{R}}$

$(I,V) \in \mathscr{B}_{IV}$ **means**

$$\left(I_1, \ldots, I_k, \ldots, I_N, V_{1,1}, \ldots, V_{k_1,k_2}, \ldots, V_{N,N}\right) : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^{N\times N}$$

is compatible with the circuit architecture and element values.

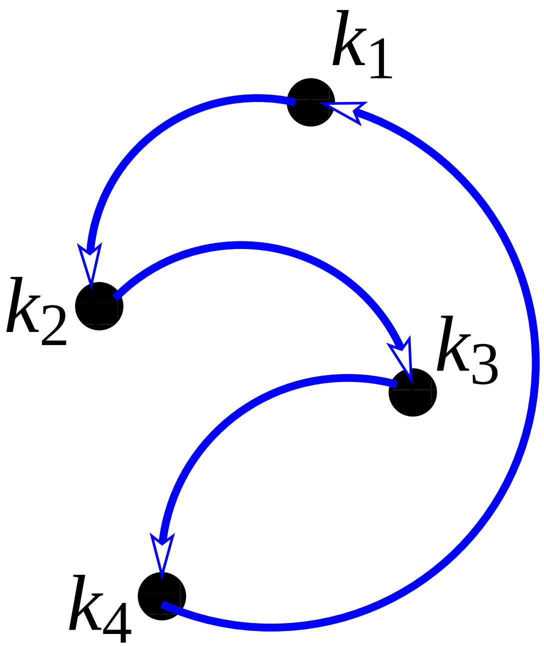
Trajectories $(I,V) \in \mathscr{B}_{IV}$ **are those that can conceivably occur.**

KVL

Kirchhoff voltage law:

$$[\![(I,V)\in\mathscr{B}_{IV}]\!] \;\Rightarrow\; [\![V_{k_1,k_2}+V_{k_2,k_3}+\cdots+V_{k_{n-1},k_n}+V_{k_n,k_1}=0$$

for all $k_1,k_2,\ldots,k_n\in\{1,2,\ldots,N\}]\!]$.



Physically, KVL is evident

(no EM fields outside the wires).

We henceforth assume it.

Potentials

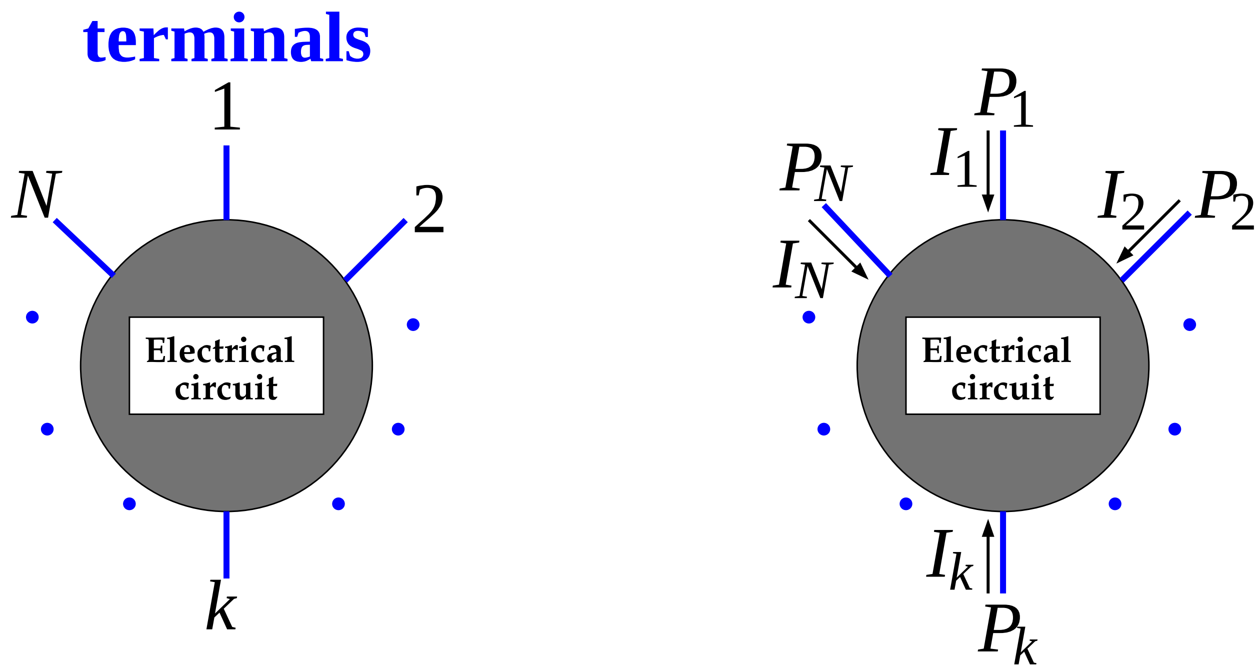
Thm: $V : \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$ **satisfies KVL** $\Leftrightarrow$

$$\exists\, P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} : \mathbb{R} \rightarrow \mathbb{R}^N \textbf{ such that } V_{k_1,k_2} = P_{k_1} - P_{k_2}.$$

$$P \textbf{ `potential'} \Rightarrow \begin{bmatrix} P_1 + \alpha \\ P_2 + \alpha \\ \vdots \\ P_N + \alpha \end{bmatrix} \textbf{ potential } \forall\, \alpha : \mathbb{R} \rightarrow \mathbb{R}.$$

P **is 'unobservable'.**

Electrical circuit

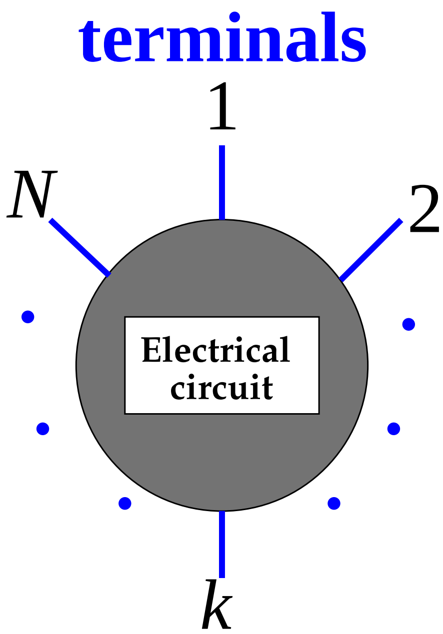


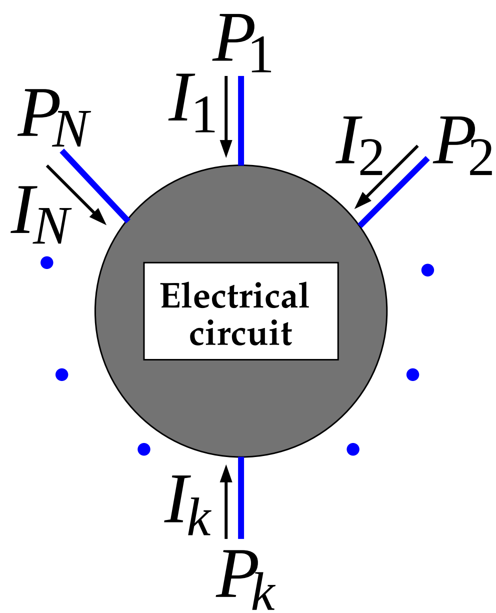
<u>At each terminal</u>:

a current (> 0 into the circuit) and a potential

$\rightsquigarrow$ **behavior** $\mathscr{B}_{IP} \subseteq \left(\mathbb{R}^N \times \mathbb{R}^N\right)^{\mathbb{R}}$.

Electrical circuit



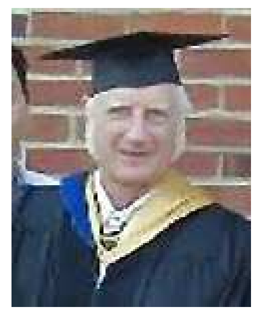


$(I_1, I_2, \ldots, I_N, P_1, P_2, \ldots, P_N) \in \mathscr{B}_{IP}$ **means:**

this current/potential trajectory is compatible with the circuit architecture and its element values.

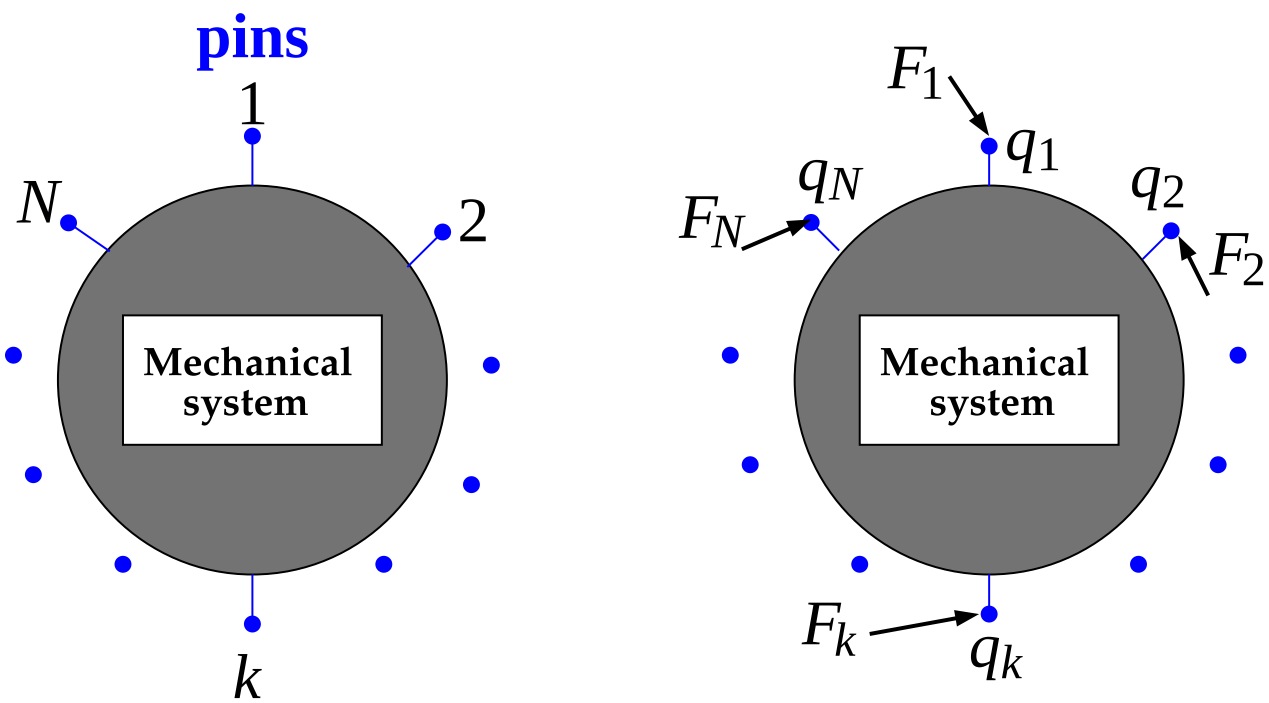
Early sources:

Brockway McMillan



Robert Newcomb

Mechanical device



At each terminal: a position and a force.

$\rightsquigarrow$ **position/force trajectories** $(q, F) \in \mathscr{B} \subseteq ((\mathbb{R}^{\bullet})^{2N})^{\mathbb{R}}$.

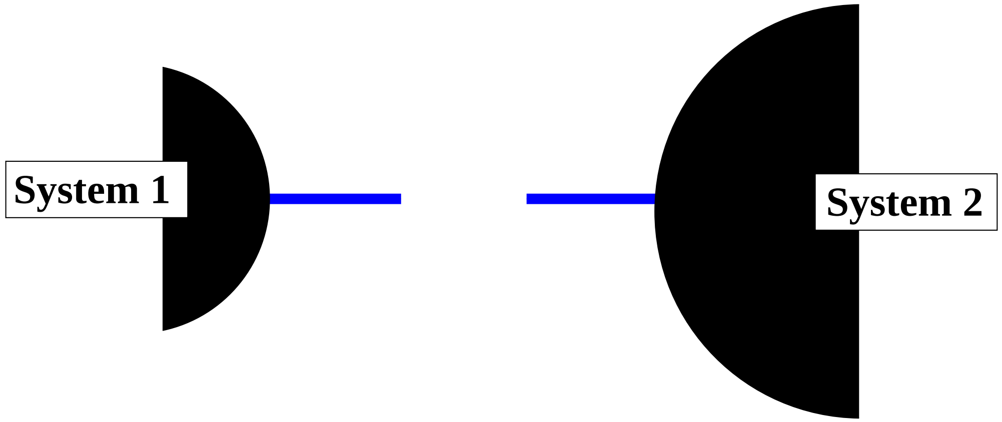
More generally, position, force, angle, torque.

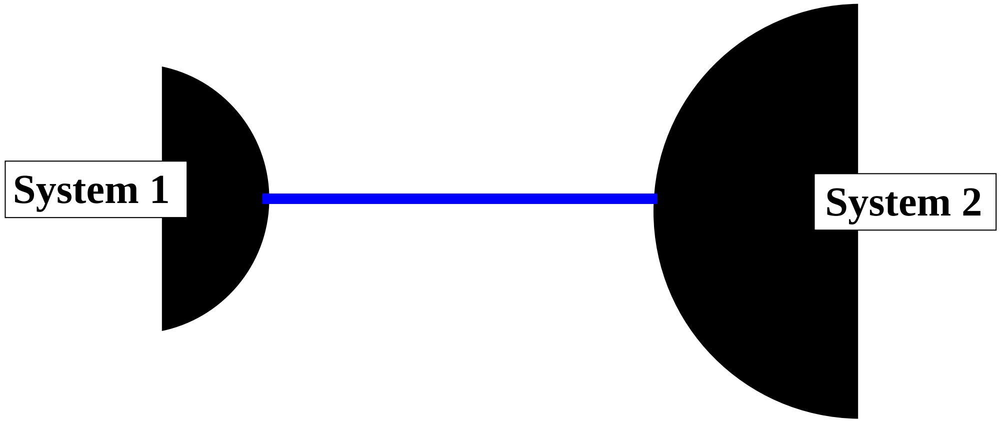
Other domains

- <u>Thermal systems:</u> At each terminal:
  a temperature and a heat flow.
- <u>Hydraulic systems:</u> At each terminal:
  a pressure and a mass flow.
- <u>Multidomain systems:</u>
  Systems with terminals of different types,
  as motors, pumps, loudspeakers, etc.
- ...

INTERCONNECTION

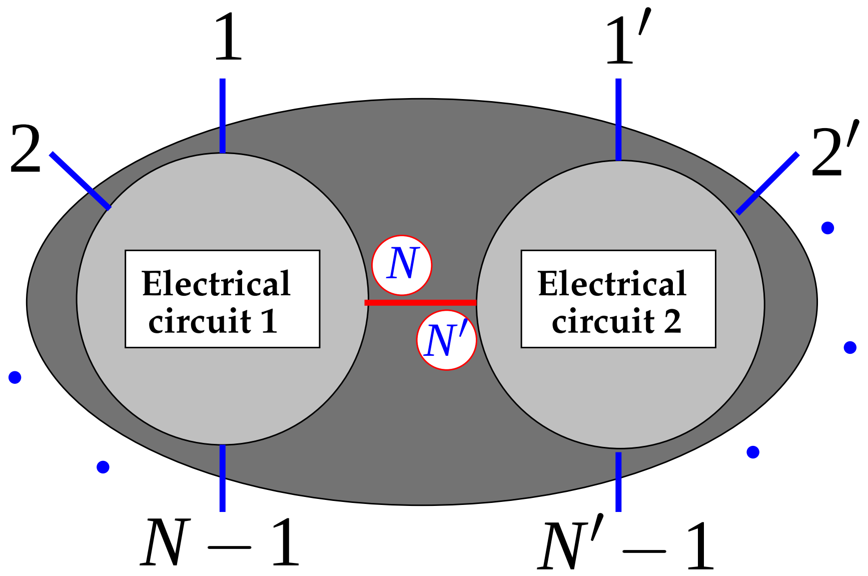
Connection of terminals





By interconnecting, the terminal variables are equated.

Electrical interconnection



$$I_N + I_{N'} = 0 \quad \textbf{and} \quad P_N = P_{N'}.$$

Behavior after interconnection:

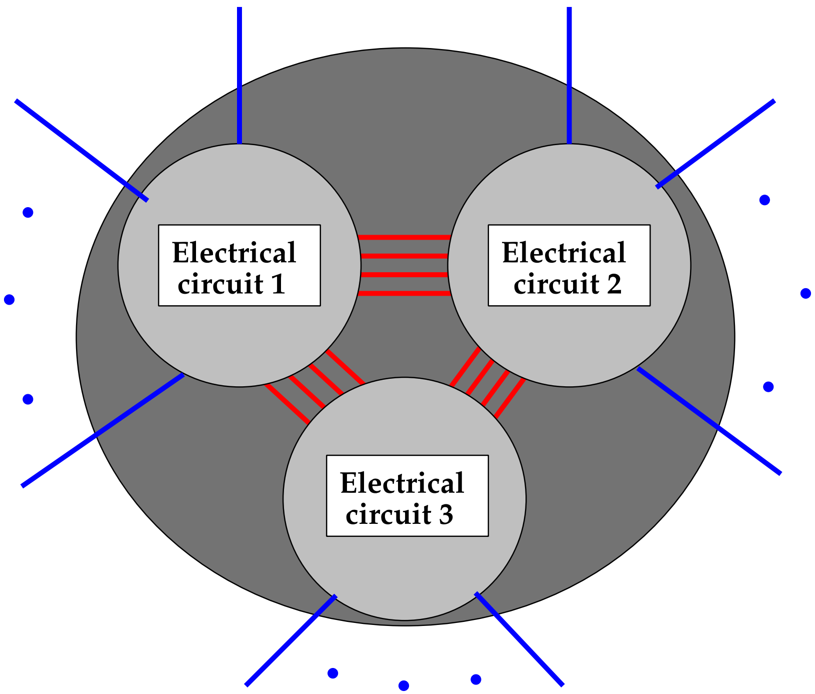
$$\mathscr{B}_1 \sqcap \mathscr{B}_2 := \{(I_1, \ldots, I_{N-1}, I_{1'}, \ldots, I_{N'-1}, P_1, \ldots, P_{N-1}, P_{1'}, \ldots, P_{N'-1}) \mid$$

$$\exists\, I, P \text{ such that } (I_1, \ldots, I_{N-1}, I, P_1, \ldots, P_{N-1}, P) \in \mathscr{B}_1$$

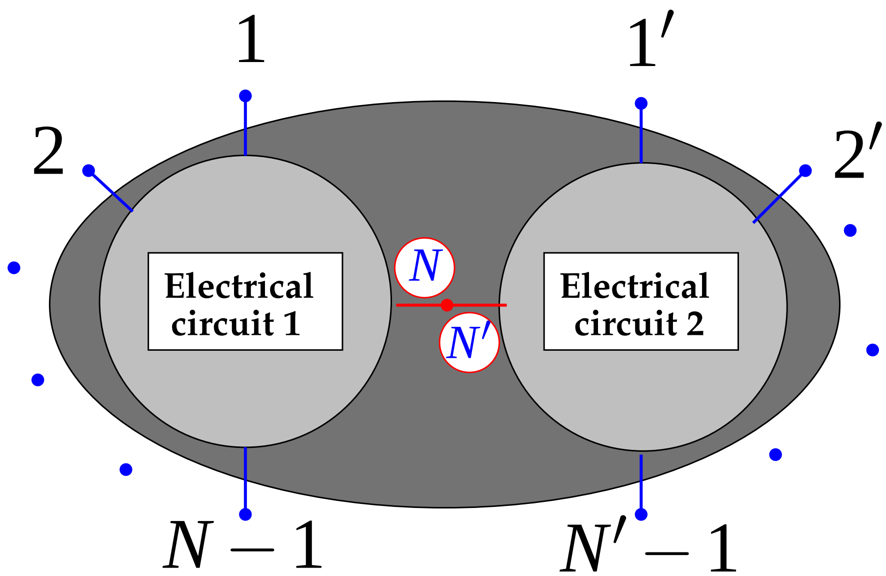
$$(I_{1'}, \ldots, I_{N'-1}, -I, P_{1'}, \ldots, P_{N'-1}, P) \in \mathscr{B}_2\}.$$

Electrical interconnection

⇝ **more terminals and more circuits connected**



Interconnection of 1-D mechanical systems



$$q_N = q_{N'} \quad \textbf{and} \quad F_N + F_{N'} = 0.$$

Variable sharing

- **Thermal systems:**

  At each terminal: a temperature and a heat flow.

  $$T_N = T_{N'} \quad \textbf{and} \quad Q_N + Q_{N'} = 0.$$

- **Hydraulic systems:**

  At each terminal: a pressure and a mass flow.

  $$p_N = p_{N'} \quad \textbf{and} \quad f_N + f_{N'} = 0.$$

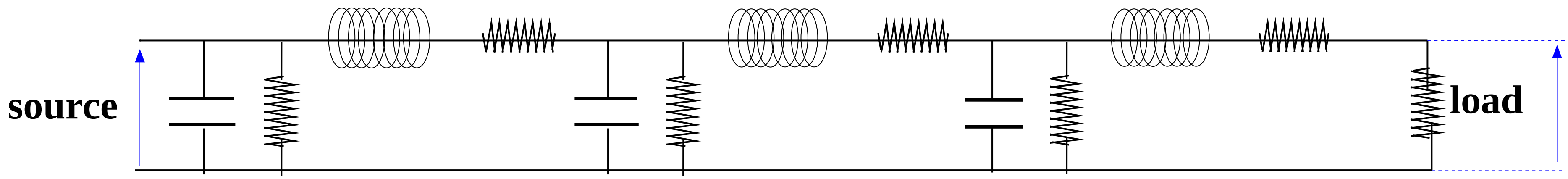
- **...**

Interconnection means variable sharing.

INTERCONNECTION ARCHITECTURE

A transmission line

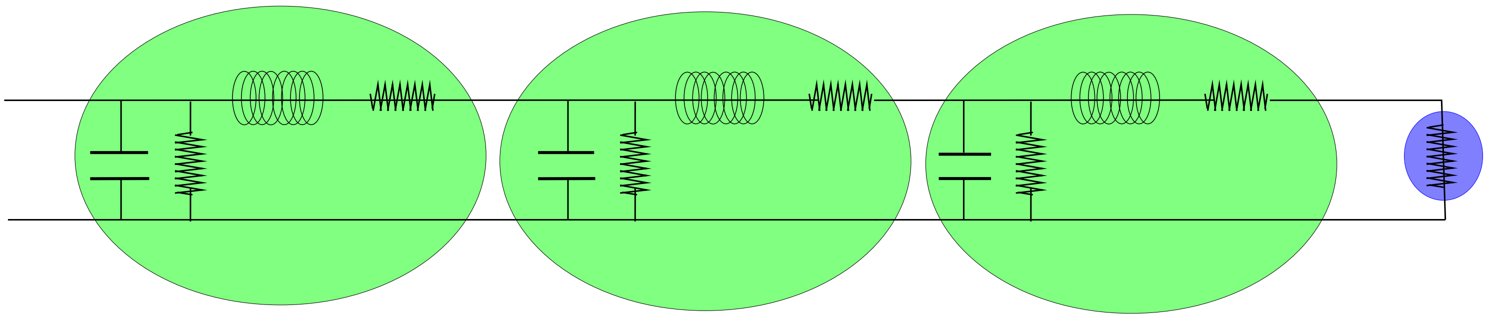
Consider the transmission line shown below.



The aim is to model the relation between the voltage of the source and the voltage across the load.

A transmission line

View as an interconnection of 4 subsystems.



A transmission line

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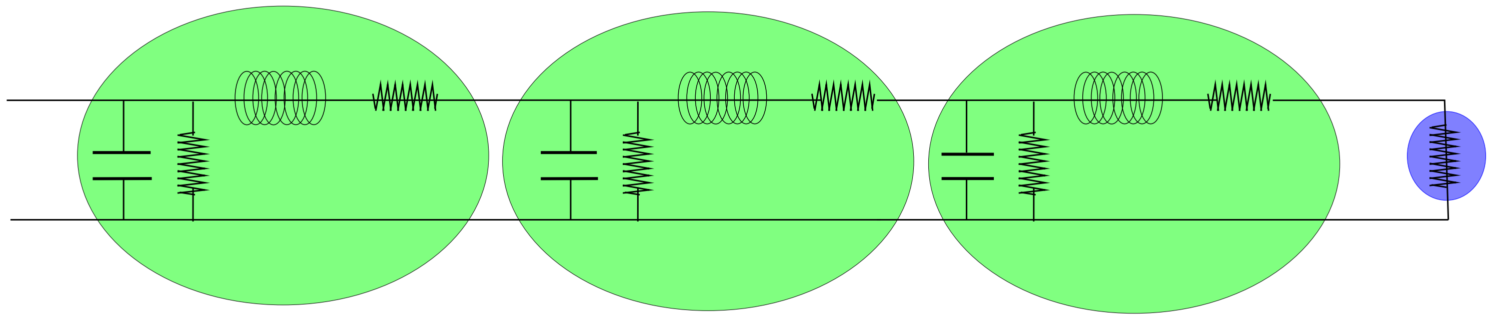
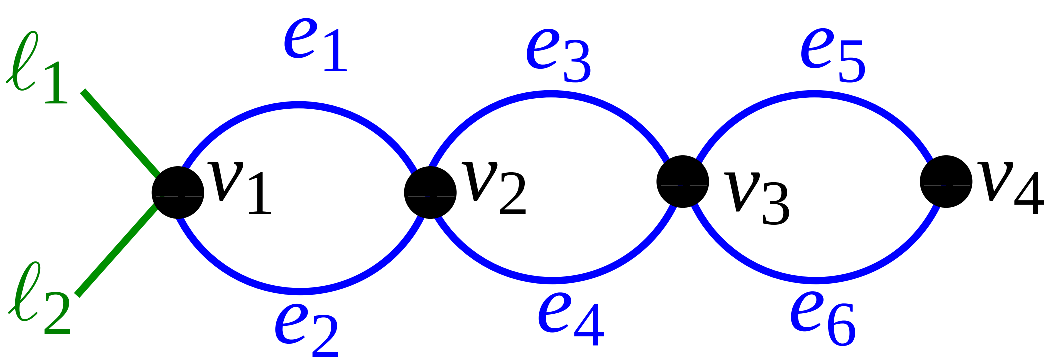


The architecture $\rightsquigarrow$ a graph with leaves.



A transmission line

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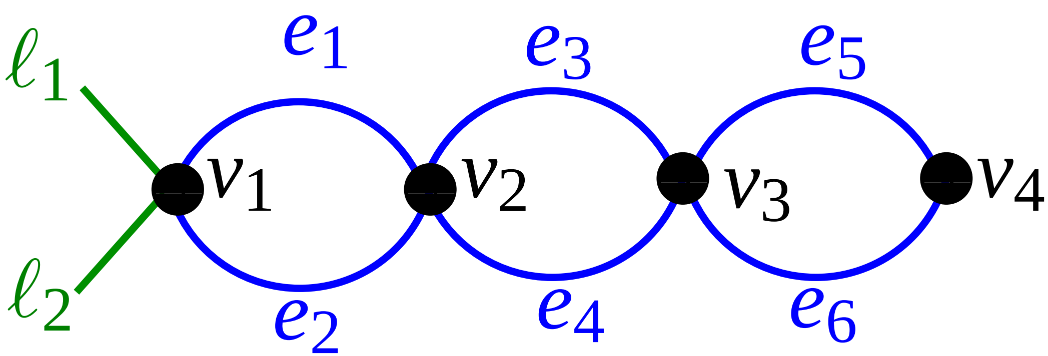
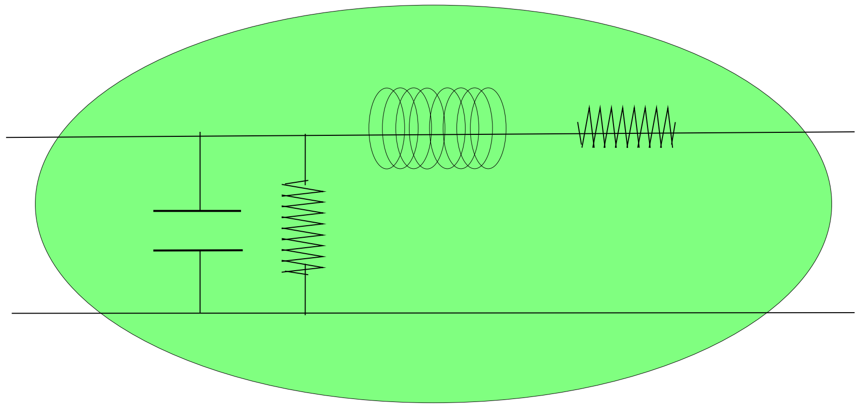


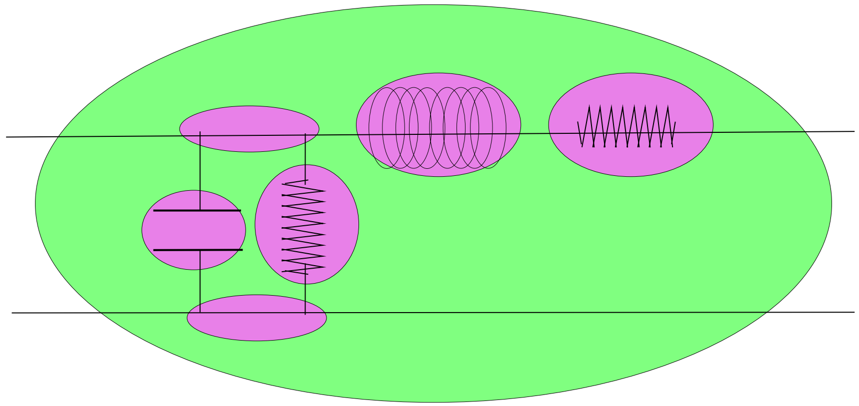
- **Elements in the vertices**
- **Interconnections in the edges**
- **External terminals in the leaves**

A transmission line section

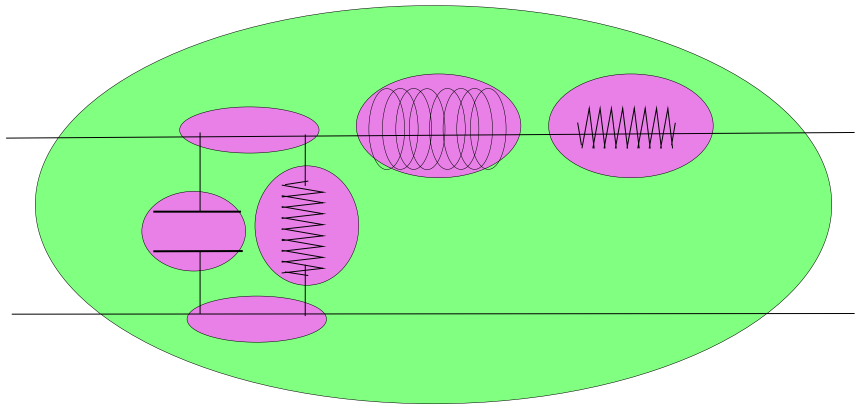
In each of the vertices v_1, v_2, v_3 we have:



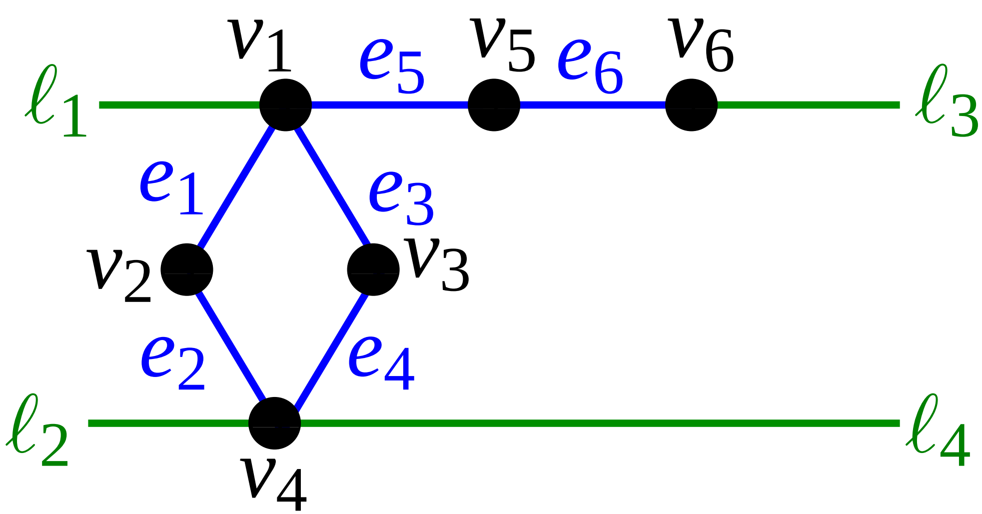
View as the interconnection of 6 subsystems:



A transmission line section



The associated interconnection architecture is



Modeling the transmission line section

$\rightsquigarrow$ **a LTIDS with 4 ODEs in the variables**

$$P_{\ell_1}, I_{\ell_1}, P_{\ell_2}, I_{\ell_2}, P_{\ell_3}, I_{\ell_3}, P_{\ell_4}, I_{\ell_4}.$$

Denote these equations as

$$R\left(\tfrac{d}{dt}\right)\begin{bmatrix} P_{\ell_1} \\ I_{\ell_1} \\ P_{\ell_2} \\ I_{\ell_2} \\ P_{\ell_3} \\ I_{\ell_3} \\ P_{\ell_4} \\ I_{\ell_4} \end{bmatrix} = 0.$$

The transmission line yields the subsystem equations

$$R\left(\frac{d}{dt}\right)\begin{bmatrix}\begin{bmatrix}P_{\ell_1}\\ I_{\ell_1}\\ P_{v_1,2}\\ I_{v_1,2}\\ P_{v_1,3}\\ I_{v_1,3}\\ P_{v_1,4}\\ I_{v_1,4}\end{bmatrix} & \begin{bmatrix}P_{v_2,1}\\ I_{v_2,2}\\ P_{v_2,2}\\ I_{v_2,2}\\ P_{v_2,3}\\ I_{v_2,3}\\ P_{v_2,4}\\ I_{v_2,4}\end{bmatrix} & \begin{bmatrix}P_{v_3,1}\\ I_{v_3,2}\\ P_{v_3,2}\\ I_{v_3,2}\\ P_{v_3,3}\\ I_{v_3,3}\\ P_{v_3,4}\\ I_{v_3,4}\end{bmatrix}\end{bmatrix} = 0,$$

$$P_{v_4,1} - P_{v_4,2} = R\, I_{v_4,1}, \quad I_{v_4,1} + I_{v_4,2} = 0,$$

The transmission line

the interconnection equations

$$\begin{aligned}
P_{v_1,3} &= P_{v_2,1}, \quad I_{v_1,3} + I_{v_2,1} = 0,\\
P_{v_1,4} &= P_{v_2,2}, \quad I_{v_1,4} + I_{v_2,2} = 0,\\
P_{v_2,3} &= P_{v_3,1}, \quad I_{v_2,3} + I_{v_3,1} = 0,\\
P_{v_2,4} &= P_{v_3,2}, \quad I_{v_2,4} + I_{v_3,2} = 0,\\
P_{v_3,3} &= P_{v_4,1}, \quad I_{v_4,3} + I_{v_4,1} = 0,\\
P_{v_3,4} &= P_{v_4,2}, \quad I_{v_3,4} + I_{v_4,2} = 0.
\end{aligned}$$

the interconnection equations

$$
\begin{aligned}
P_{v_1,3} = P_{v_2,1}, &\quad I_{v_1,3} + I_{v_2,1} = 0,\\
P_{v_1,4} = P_{v_2,2}, &\quad I_{v_1,4} + I_{v_2,2} = 0,\\
P_{v_2,3} = P_{v_3,1}, &\quad I_{v_2,3} + I_{v_3,1} = 0,\\
P_{v_2,4} = P_{v_3,2}, &\quad I_{v_2,4} + I_{v_3,2} = 0,\\
P_{v_3,3} = P_{v_4,1}, &\quad I_{v_4,3} + I_{v_4,1} = 0,\\
P_{v_3,4} = P_{v_4,2}, &\quad I_{v_3,4} + I_{v_4,2} = 0.
\end{aligned}
$$

Finally, there is the manifest variable assignment

$$w_1 = P_{\ell_1} - P_{\ell_2}, \quad w_2 = P_{v_4,1} - P_{v_4,2}.$$

The transmission line

After elimination of the latent variables, we obtain the desired differential equation that describes the behavior of (w_1, w_2)

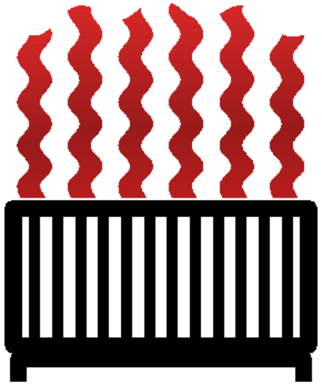
$$r_1\left(\frac{d}{dt}\right)w_1 = r_2\left(\frac{d}{dt}\right)w_2.$$

In practice, all these steps need to be carried out with the help of a toolbox.

ENERGY

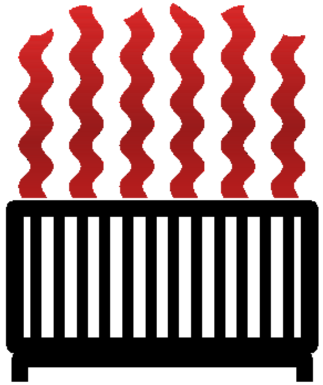
Energy

Energy := a physical quantity transformable into heat.



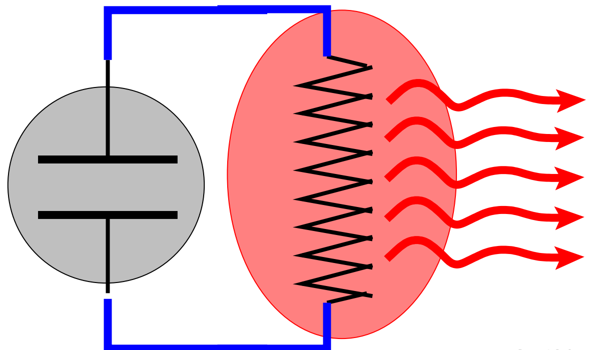
Energy

Energy **:= a physical quantity transformable into heat.**



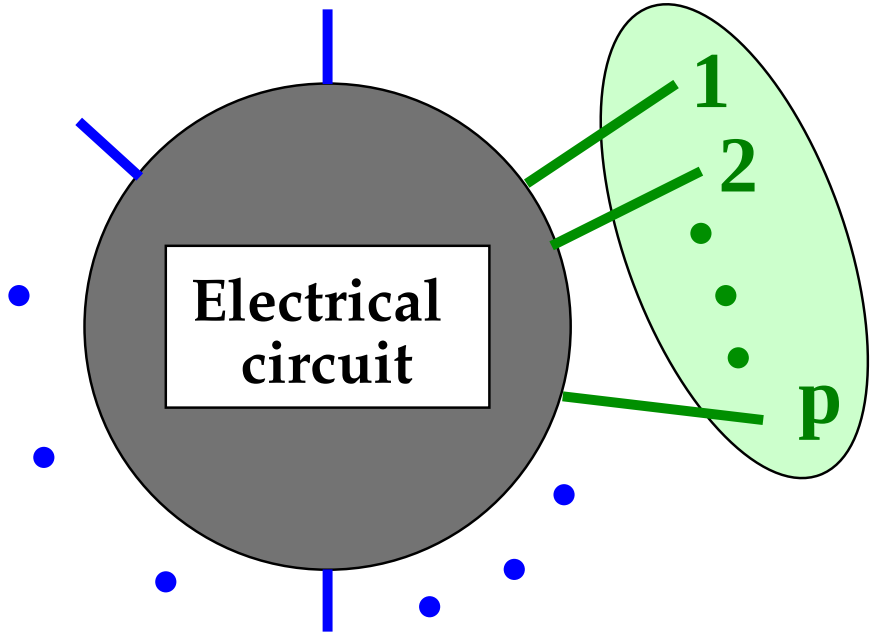
For example capacitor $\rightarrow$ resistor $\rightarrow$ heat.

Energy on capacitor = $\frac{1}{2}CV^2$



PORTS

Energy transfer



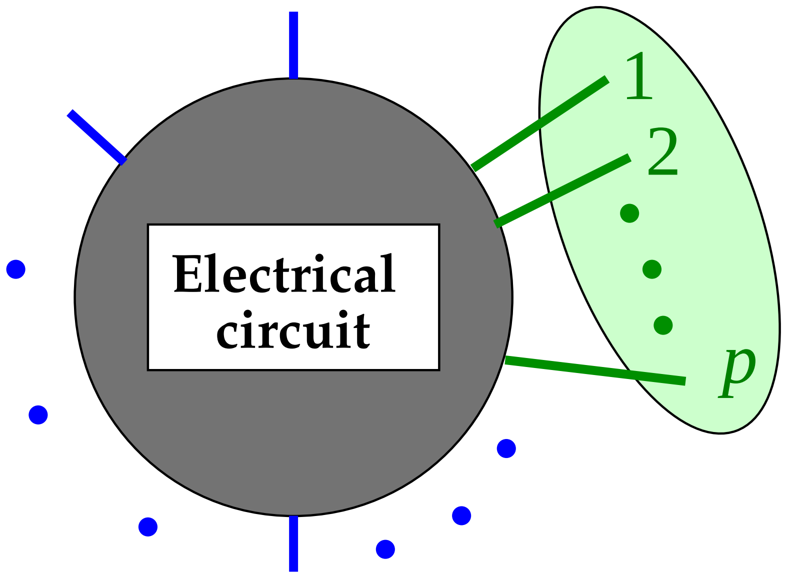
Environment

Can we speak about

the energy transferred from the environment to the circuit along these terminals?

Electrical ports

Assume KVL.

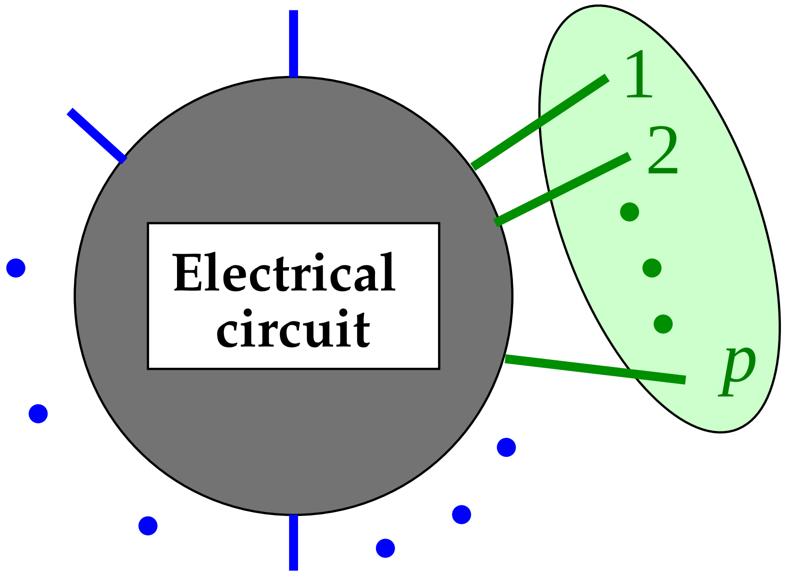


Terminals $\{1,2,\ldots,p\}$ **form a port** $:\Leftrightarrow$

$$[\![\left(I_1,\ldots,I_p,I_{p+1},\ldots,I_N,V_{1,1},\ldots,V_{k_1,k_2},\ldots,V_{N,N}\right) \in \mathscr{B}_{IV}]\!]$$

$$\Rightarrow \;\; [\![I_1+I_2+\cdots+I_p=0]\!].$$ ***'port KCL'***

Energy



If terminals $\{1, 2, \ldots, p\}$ **form a port, then**

power in = $I_1(t)P_1(t) + \cdots + I_p(t)P_p(t)$

energy in = $\int_{t_1}^{t_2} [I_1(t)P_1(t) + \cdots + I_p(t)P_p(t)]\, dt$

This interpretation in terms of power and energy is not valid unless these terminals form a port !

Internal ports

Analogous definition for internal terminals

$\leadsto$ **internal ports,**

combinations of external and internal terminals

$\leadsto$ **mixed ports.**

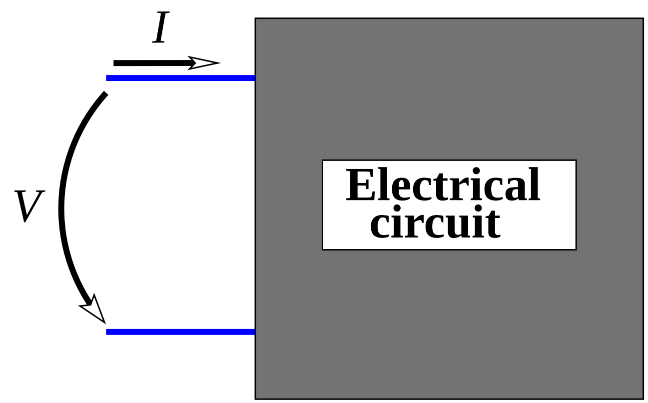
EXAMPLES

2-terminal circuits

2-terminal 1-port devices:

resistors, inductors, capacitors, memristors, etc.,

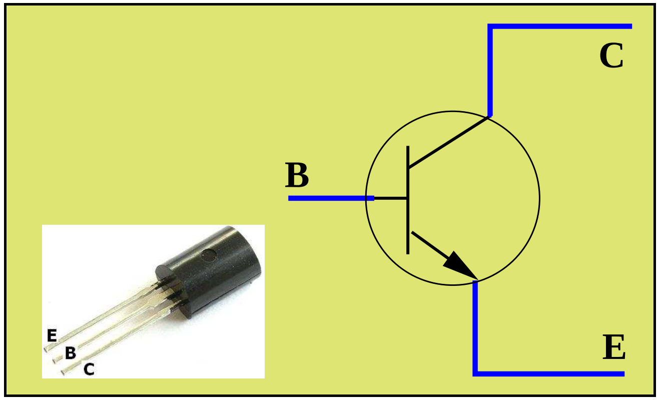
any 2-terminal circuit composed of these.

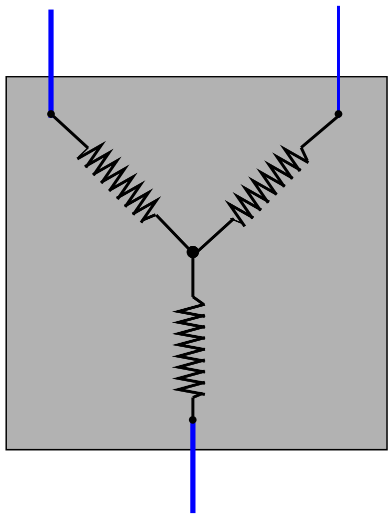


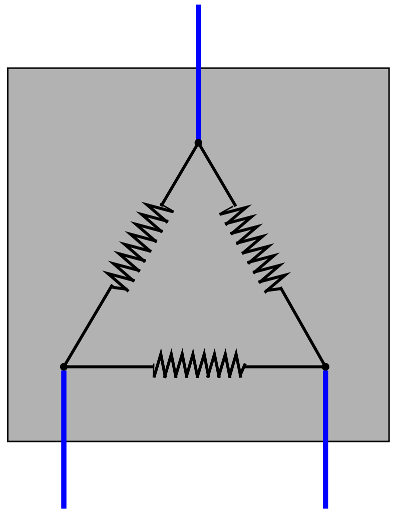
KVL $\Rightarrow$ **only** $V_{1,2} := V$ **matters,**

KCL $\Rightarrow$ $I_1 = -I_2 =: I$**.**

3-terminal circuits

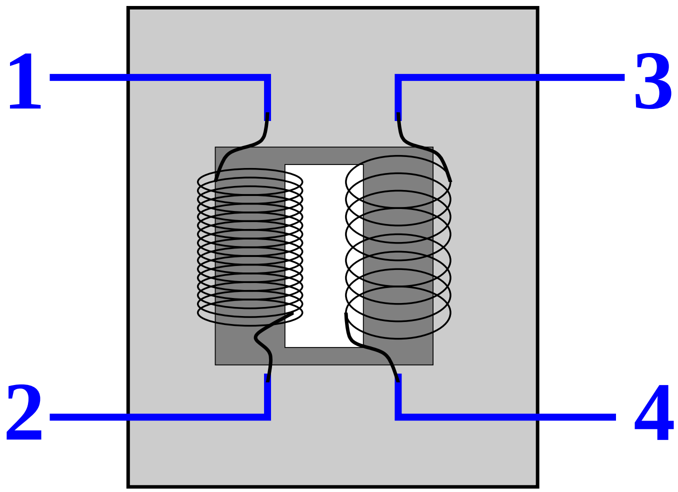






3-terminal 1-ports.

Transformer

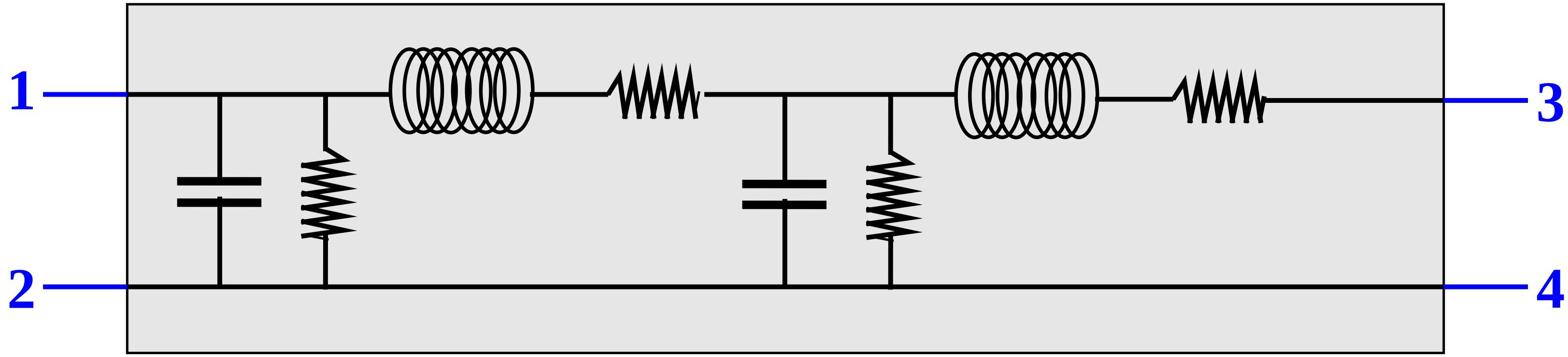


$$P_3 - P_4 = n\left(P_1 - P_2\right),$$
$$I_1 = -nI_3,$$
$$I_1 + I_2 = 0,\ I_3 + I_4 = 0.$$

$\{1,2\}$ **and** $\{3,4\}$ **form ports.**

A transformer = a 2-port with two 2-terminal ports.

Transmission line



Terminals $\{1, 2, 3, 4\}$ **form a port;**

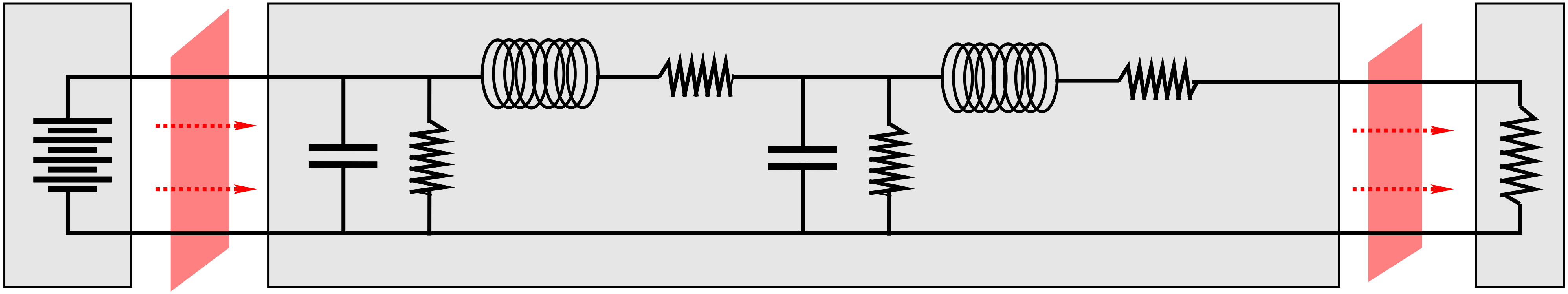
$\{1, 2\}$ **and** $\{3, 4\}$ **do not.**

We cannot speak about

"the energy transferred from terminals $\{1, 2\}$ ***to*** $\{3, 4\}$***"***,

or ***"from the environment to the circuit through*** $\{1, 2\}$***"***.

Transmission line

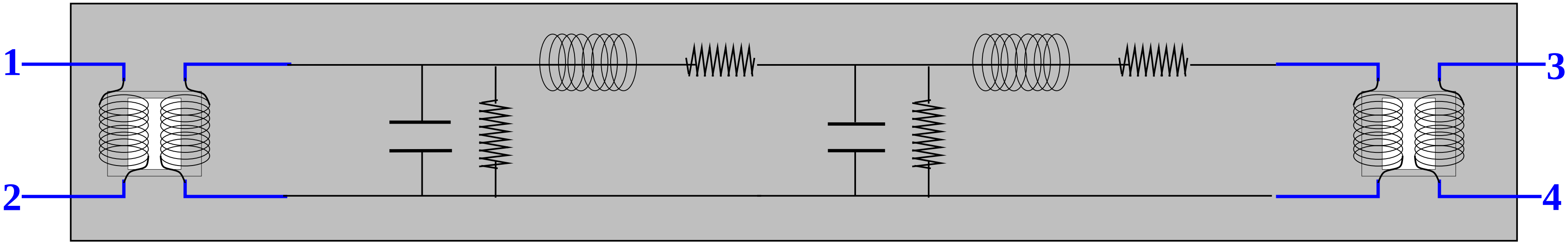


The energy flows from the source and to the load are well-defined, since the terminals form internal ports.

Therefore we can speak about

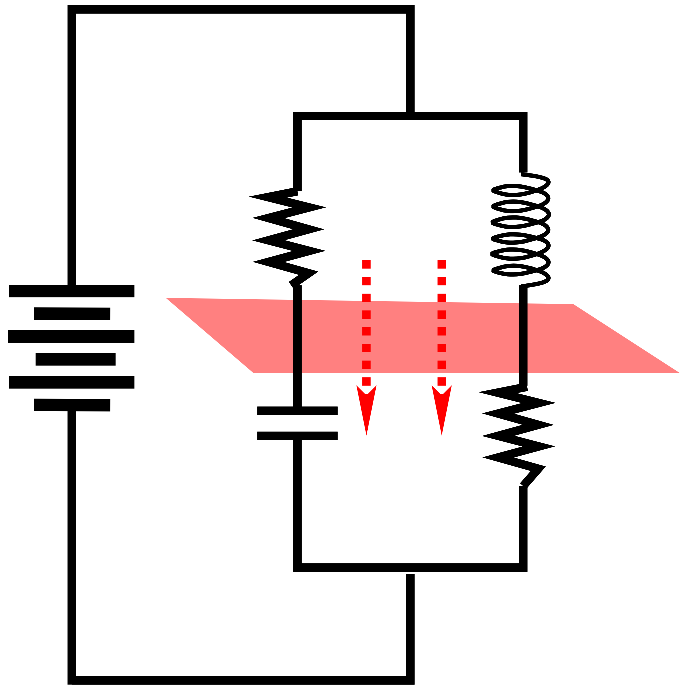
“the energy transferred from the source to the load”.

Transmission line



Terminals $\{1,2\}$ **and** $\{3,4\}$ **now form a port.**

RLC circuit

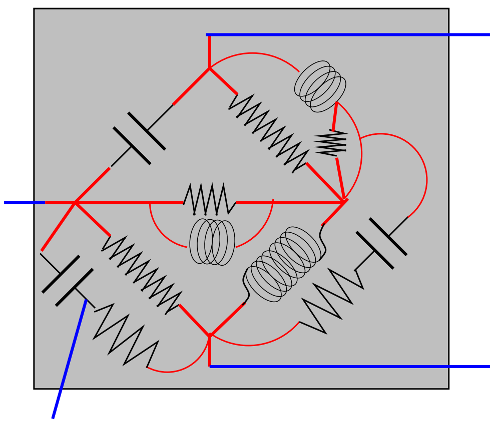


Not an internal port: energy flow not well-defined.

Are ports common?

<u>Theorem</u>: Consider an electrical circuit consisting of an interconnection of (linear passive) R's, L's, C's. Assume that every pair of external terminals is connected by the circuit graph. Then

the only port is the one that consists of all the terminals.



Are ports common?

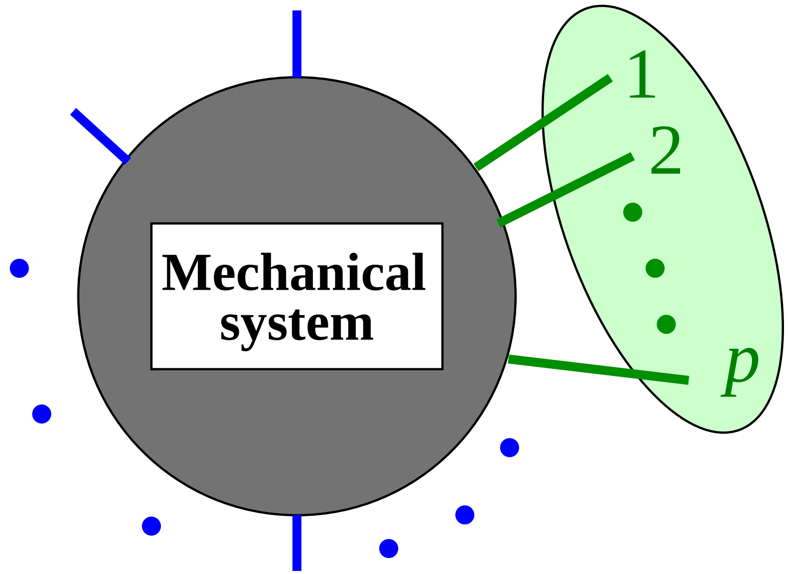
Theorem: Consider an electrical circuit consisting of an interconnection of (linear passive) R's, L's, C's. Assume that every pair of external terminals is connected by the circuit graph. Then

the only port is the one that consists of all the terminals.

For non-trivial ports, we need multi-port elements, as transformers.

MECHANICAL PORTS

Mechanical ports



Environment

Terminals $\{1, 2, \ldots, p\}$ **form a (mechanical) port** $:\Leftrightarrow$

$$(q_1, \ldots, q_p, q_{p+1}, \ldots, q_N, F_1, \ldots, F_p, F_{p+1}, \ldots, F_N) \in \mathscr{B},$$

$$\Rightarrow \quad F_1 + F_2 + \cdots + F_p = 0.$$

'port KFL'

Power and energy

If terminals $\{1, 2, \ldots, p\}$ form a port, then

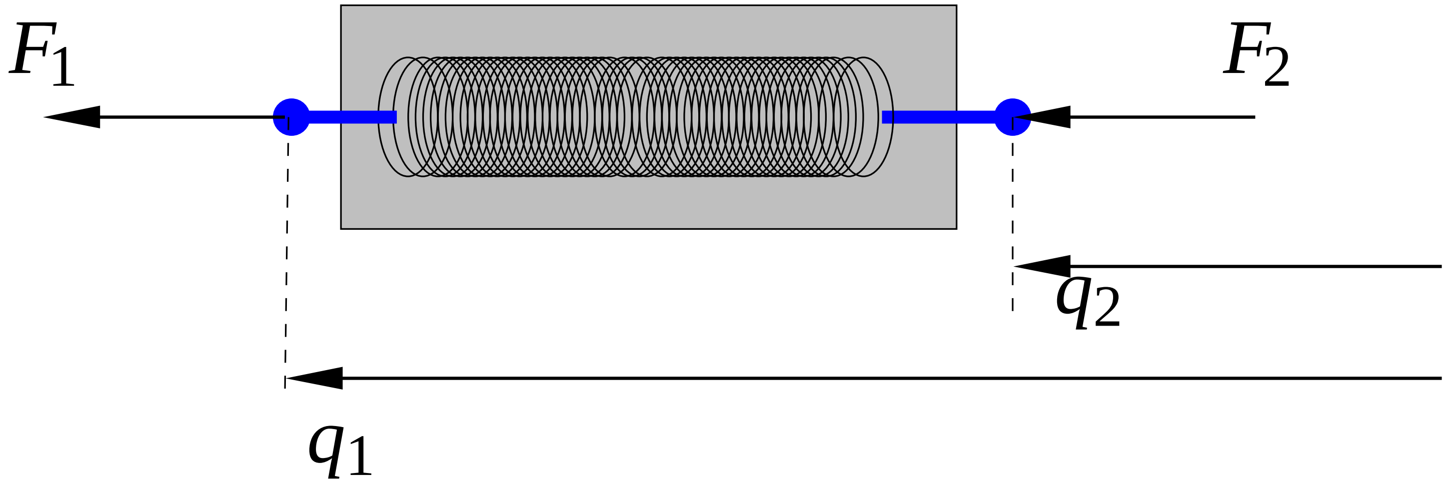
$$\textbf{power in} \;=\; F_1(t)^\top \tfrac{d}{dt} q_1(t) + \cdots + F_p(t)^\top \tfrac{d}{dt} q_p(t),$$

$$\textbf{energy in} \;=\; \int_{t_1}^{t_2} \left(F_1(t)^\top \frac{d}{dt} q_1(t) + \cdots + F_p(t)^\top \frac{d}{dt} q_p(t) \right) dt.$$

This interpretation in terms of power and energy is not valid unless these terminals form a port !

Example

Spring



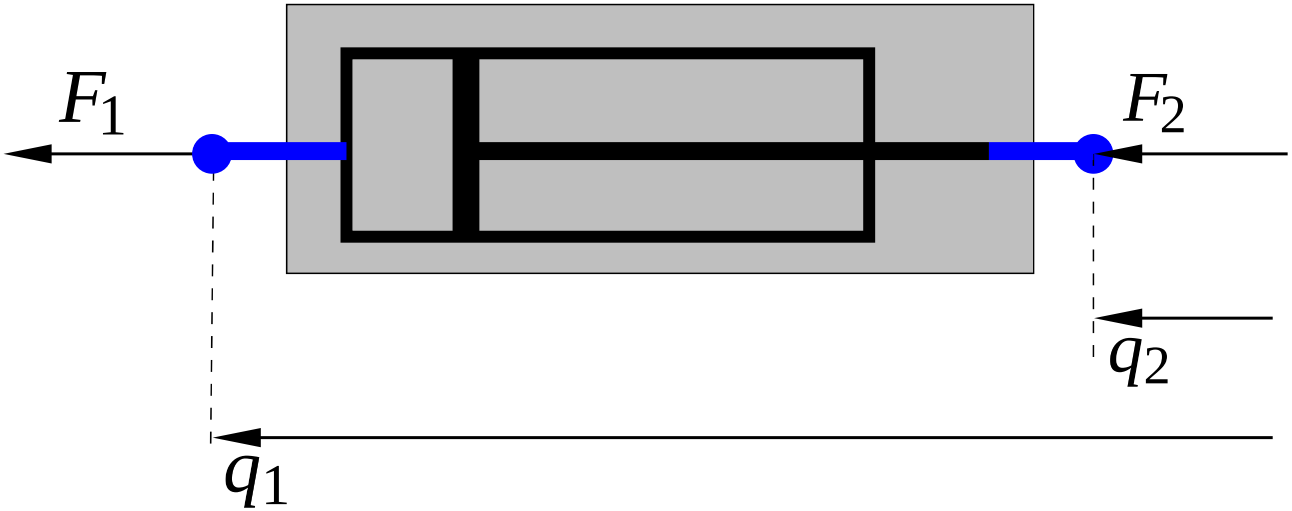
$$F_1 + F_2 = 0, \quad K(q_1 - q_2) = F_1,$$

satisfies KFL.

$\Rightarrow$ **A port.**

Example

Damper



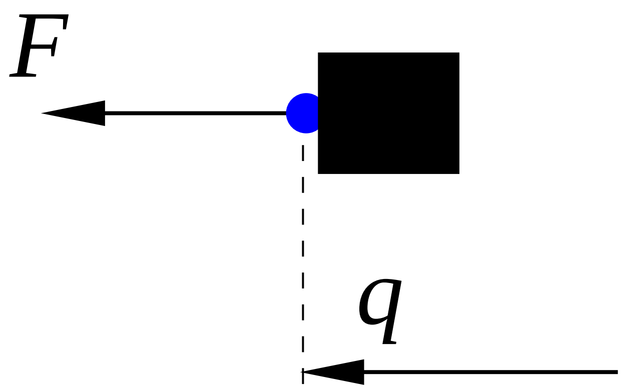
$$F_1 + F_2 = 0, \quad D\frac{d}{dt}(q_1 - q_2) = F_1$$

satisfies KFL.

$\Rightarrow$ A port.

Springs, dampers, & their interconnection $\rightsquigarrow$ ports.

Example



$$M\frac{d^2}{dt^2}q = F$$

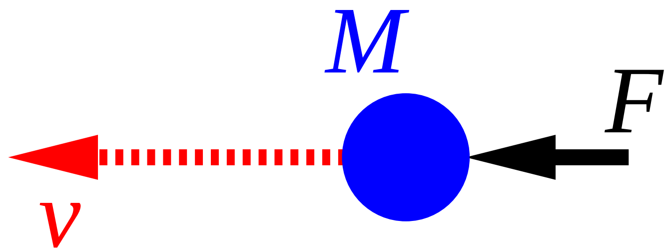
does not satisfy KFL

Not a port!!!

Interconnections of springs, dampers, and masses do not necessarily form a port.

MOTION ENERGY

Conservation law

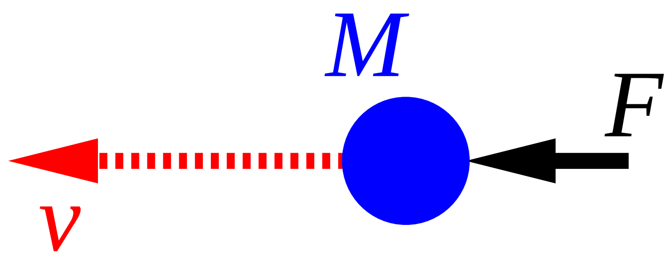


$$M\frac{d^2}{dt^2}q = F \quad \Rightarrow \quad \frac{d}{dt}\frac{1}{2}M||\frac{d}{dt}q||^2 = F^\top \frac{d}{dt}q$$

If $F^\top v$ is not power,

is $\frac{1}{2}M||v||^2$ not stored (kinetic, motion) energy ???

Kinetic energy and invariance under uniform motions



What is the kinetic energy?

$$\mathcal{E}_{\textbf{kinetic}} = \frac{1}{2} M \, ||v||^2$$

Willem ’s Gravesande

1688–1742

Émilie du Châtelet

1706–1749

This expression is not invariant under uniform motion.

Motion energy

What is the motion energy?

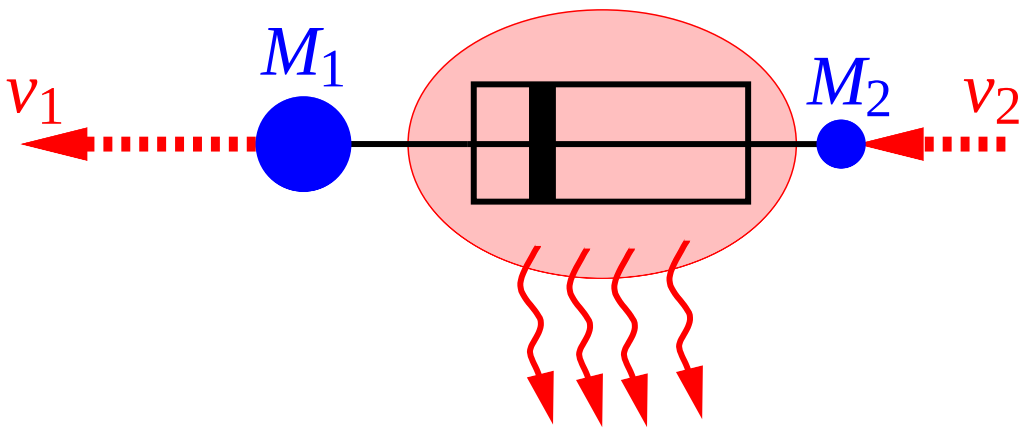
What quantity is transformable into heat?

$$\mathscr{E}_{\textbf{motion}} = \frac{1}{2}\,\frac{M_1\,M_2}{M_1 + M_2}\,||v_1 - v_2||^2$$

Invariant under uniform motion.

Dissipation into heat

Can be justified by mounting a damper between the masses.

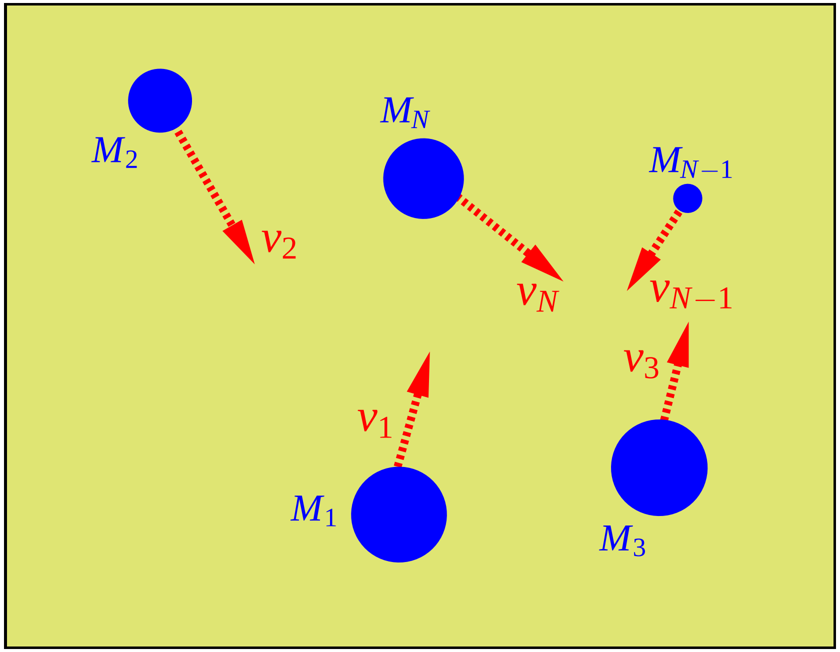


$$\frac{1}{2}\,\frac{M_1\,M_2}{M_1+M_2}\,\|v_1-v_2\|^2$$

is the heat dissipated in the damper.

Motion energy

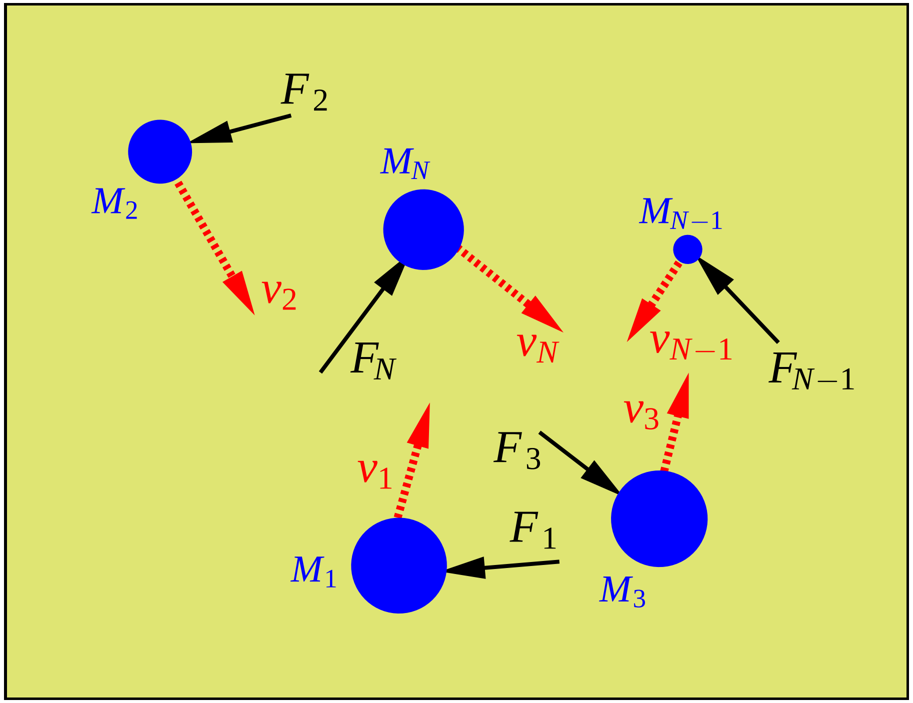
Generalization to N masses.



$$\mathscr{E}_{\textbf{motion}} = \frac{1}{4} \sum_{i,j \in \{1,2,\ldots,N\}} \frac{M_i\, M_j}{M_1 + M_2 + \cdots + M_N} ||v_i - v_j||^2.$$

Motion energy

With external forces.



$$\mathscr{E}_{\mathbf{motion}} = \frac{1}{4} \sum_{i,j \in \{1,2,\ldots,N\}} \frac{M_i \, M_j}{M_1 + M_2 + \cdots + M_N} ||v_i - v_j||^2.$$

(KFL) $\sum_{i \in \{1,2,\ldots,N\}} F_i = 0 \quad \Rightarrow \quad \frac{d}{dt}\mathscr{E}_{\mathbf{motion}} = \sum_{i \in \{1,2,\ldots,N\}} F_i^\top v_i.$

Motion energy

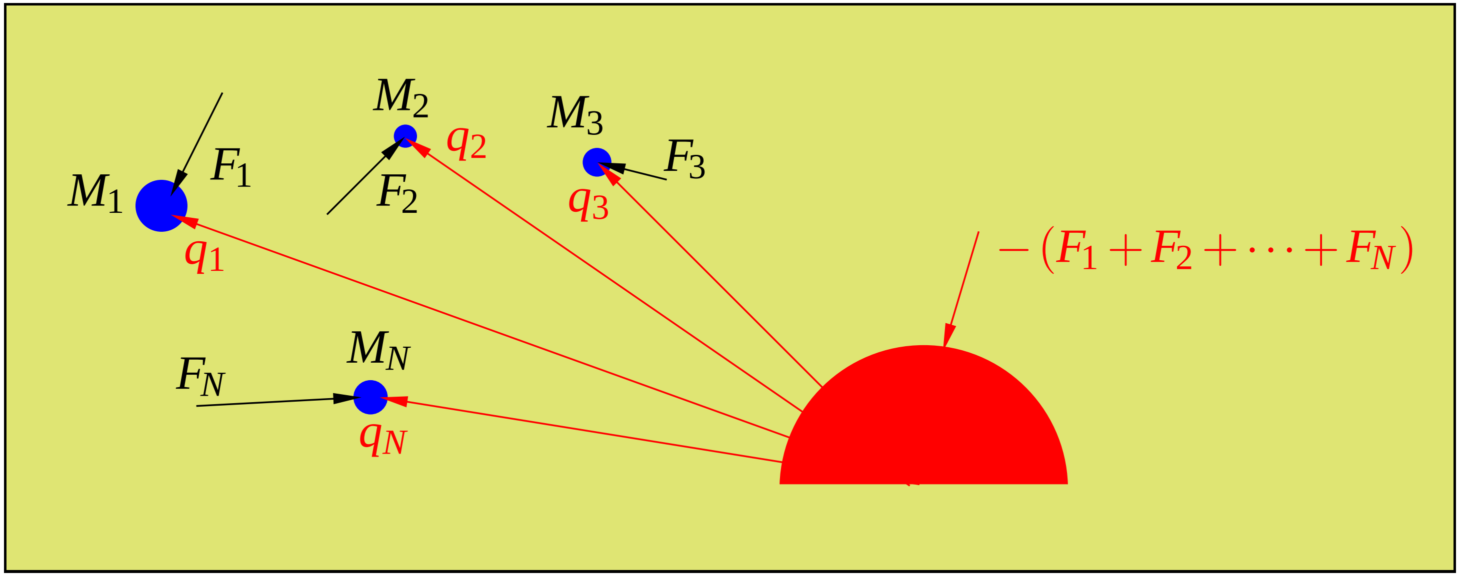
$$\mathscr{E}_{\textbf{motion}} = \frac{1}{4} \sum_{i,j \in \{1,2,\ldots,N\}} \frac{M_i\, M_j}{M_1 + M_2 + \cdots + M_N} ||v_i - v_j||^2.$$

Distinct from the classical expression of the kinetic energy,

$$\mathscr{E}_{\textbf{kinetic}} = \frac{1}{2} \sum_{i \in \{1,2,\ldots,N\}} M_i\, ||v_i||^2.$$

Motion energy

Reconciliation: $M_{N+1} = \infty, F_{N+1} = -(F_1 + F_2 + \cdots + F_N)$,



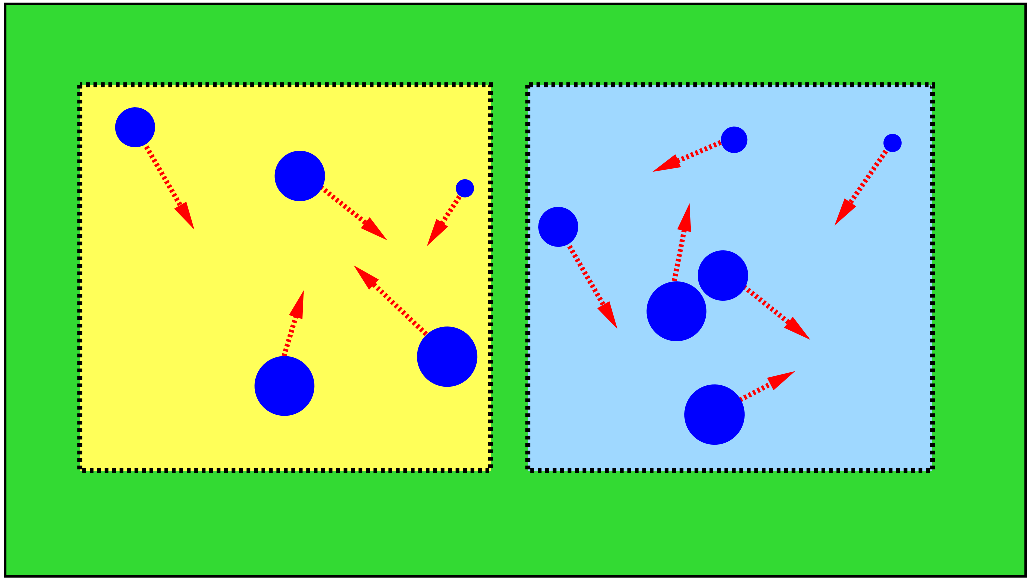
measure velocities w.r.t. this infinite mass ('ground'), then

$$\frac{1}{4} \sum_{i,j \in \{1,2,\ldots,N,N+1\}} \frac{M_i\, M_j}{M_1 + M_2 + \cdots + M_N + M_{N+1}} \|v_i - v_j\|^2$$

$$\xrightarrow[M_N \to \infty]{} \quad \tfrac{1}{2} \textstyle\sum_{i \in \{1,2,\ldots,N\}} M_i \; \|v_i\|^2.$$

Motion energy

Motion energy is not an extensive quantity, it is not additive.

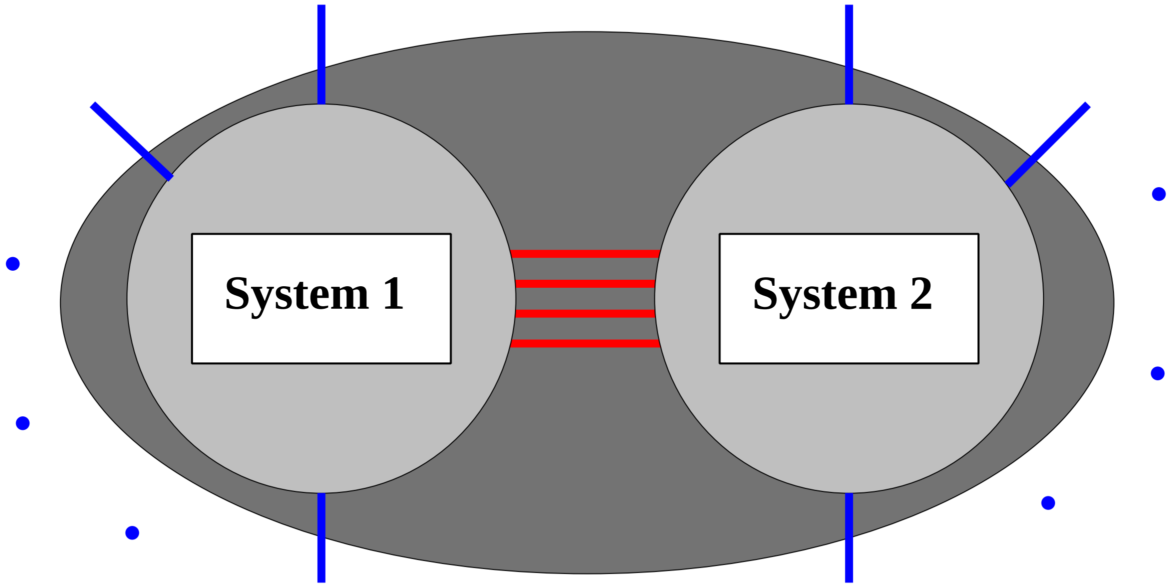


Total motion energy $\neq$ sum of the parts.

Power and energy involve ‘action at a distance’.

ENERGY TRANSFER

Energy transfer



One cannot speak about

"the energy transferred from system 1 to system 2"

or *"from the environment to system 1"*,

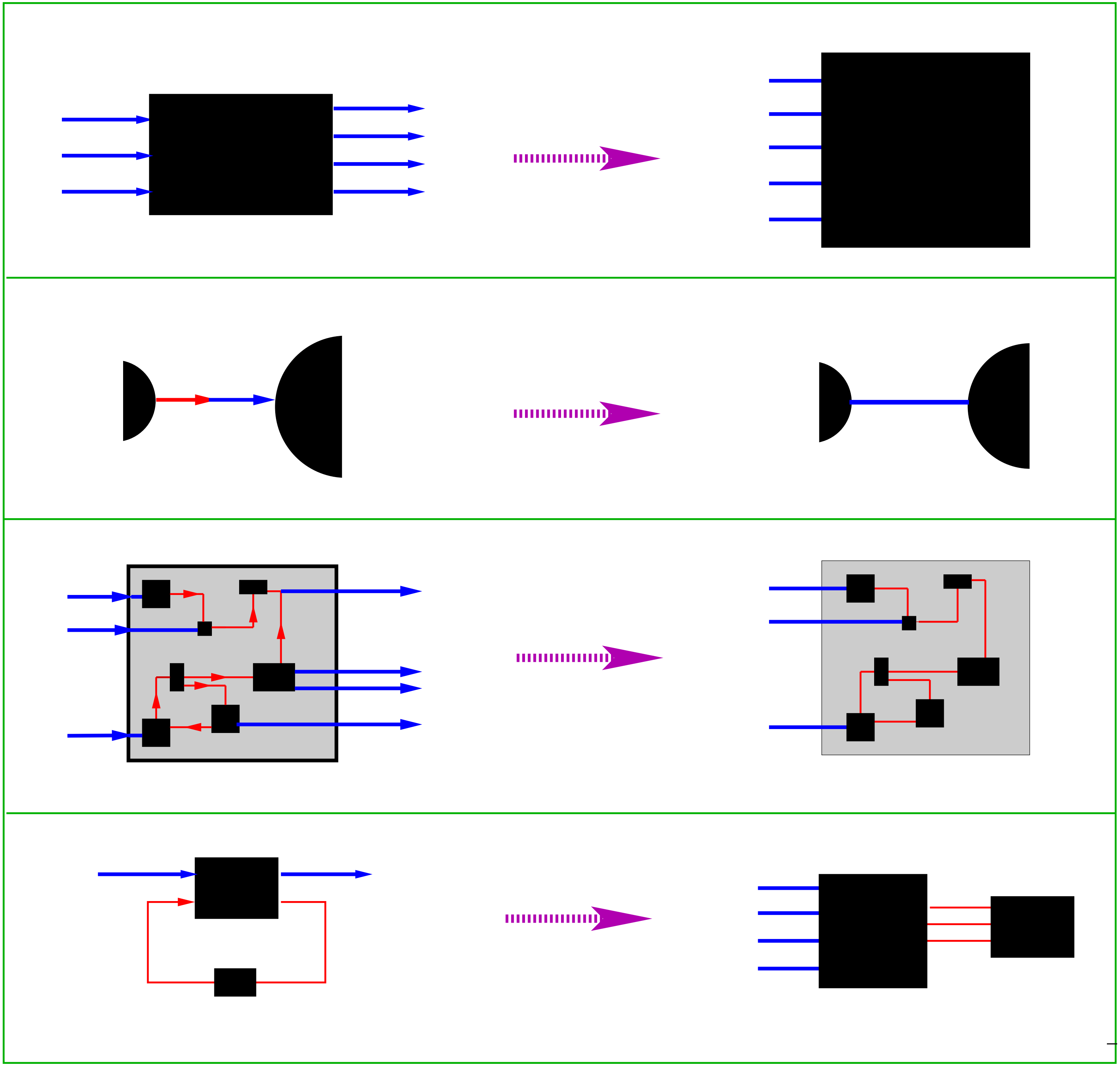
unless the relevant terminals form a port.

Ports and terminals

Terminals are for interconnection,

ports are for energy transfer.

CONCLUSION



Energy transfer

- **Energy transfer is associated with ports.**
- **One cannot in general speak about the energy transferred from system 1 to system 2.**
- **Energy is not an local quantity. It involves action at a distance.**
- **Energy is not an extensive quantity.**

Happy birthday, Peter!

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you