Projector-based Control of Orbit Dynamics in Quantum Lindblad Systems

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***Abstract*—In this paper, we demonstrate that the dynamics of an n-dimensional Lindblad control system can be separated into its inter- and intra-orbit dynamics when there is fast controllability. This can be viewed as a control system on the simplex of density operator spectra, where projectors representing the eigenspaces are viewed as control variables. The local controllability properties of this control system can be analyzed when the control-set of projectors is limited to a finite subset. In particular, there is a natural finite subset of $n!$ projector-tuples that are effective for low-purity orbits.**

***Index Terms*—Quantum control, open systems, Lindblad equation, decoherence, dissipation.**

I. Introduction

Advances in quantum technologies, such as the nascent progress in quantum computation [1][2][3][4], as well as the developments of coherent control of chemical reactions [5][6] and NMR [7], have resulted in great effort to apply mathematical control theory [8] to quantum mechanical systems [9]. The interaction of a system with its environment is a major obstacle in quantum control, and as a result quantum control theory has expanded from closed systems [10] to open systems (see [11], [12], [13] and [14] for surveys; some important papers are [15][16][17][18][19]).

A common method of modeling open systems is to assume they are Markovian and time-independent, in which cases the dynamics are described by a quantum dynamical semi-group and the Lindblad master equation [20][21][22]. Typically, the control functions appear in the system Hamiltonian (although there has been progress in engineering Lindblad dynamics [23][24][25]). This means that, absent the interaction with the environment, the controls are only capable of steering the system within a given unitary orbit [26][27][28]. The motion between orbits depends on the Lindblad super-operator. Consequently, the Hamiltonian cannot directly affect the eigenvalues, or the purity $Tr(\rho^2)$, since the eigenvalues of the density operator are constant on any orbit. If the optimal time [29] between two unitarily equivalent density operators is much smaller than the time-scale characterized by the Lindblad dynamics, it becomes an interesting question as to how best position the system on any given orbit.

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The aim of this paper is to formally consider an approach to control of open quantum systems in which the space of density matrices is decomposed into spectra (the set of possible orbits) and eigenprojectors (the positions along a given orbit). If one has sufficiently fast and complete Hamiltonian control, the intra-orbit dynamics can be made arbitrarily faster than the inter-orbit dynamics, since the Lindblad super-operator is bounded. After separating the dynamics, we want to view the trajectories of the eigenprojectors as control functions, and the spectrum as the state variables. We refer to this viewpoint as projector-based control. After a desired projector trajectory has been determined, we can consequently reconstruct the necessary Hamiltonian, which contains the true control functions. We are building on previous work on two-dimensional systems [30][31]. The $n = 2$ case is easier to study from a control perspective as the set of orbits is isomorphic to a closed line segment, and all orbits but one are isomorphic to a sphere. In order to generalize to $2 < n < \infty$, one must address the delicacies of dealing with more complicated orbit sets, as well as cope with the difficulties that come with non-trivial control sets. Chapter 8 in reference [32] discusses the geometry of density matrices, and, in particular, their orbit sets. Our approach contrasts with the generalized Bloch vector representation approach [28][33], which yields an affine differential equation on the vector space of density operators. This representation has little to do with the orbit structure however.

One obstacle that arises in our approach is the non-linearity of the control space. This space consists of all n-tuples of orthogonal projector operators, which is the quotient manifold $U(n)/(U(m_1) \times \cdots \times U(m_{n_d}))$, where m_α is the multiplicity of the αth eigenvalue of the density operator and n_d is the number of distinct eigenvalues. It is therefore non-trivial to apply standard control theory results to a projector-based control system. In this paper, we demonstrate that a local controllability result can be applied when one limits the projector-controls to a finite subset of the overall control space. In particular, the behavior of the Lindblad operators at the completely mixed state yields a natural set of $n!$ projector-tuples that are particularly useful for low-purity orbits.

Infinite-dimensional quantum systems [34] present many technical difficulties. In particular, the Lindblad super-operator is not necessarily bounded, which means it has no characteristic time-scale, and we cannot assume our unitary control is faster than the Lindblad dynamics. For this reason, we consider only finite-dimensional systems.

In section II, we decompose the Lindblad master equation into its spectral and projector components, and in section III,

we re-interpret the spectral ODE as a control equation. In section IV, we analyze the local controllability of finite projector control-sets, and in section V we show some examples.

II. Separation of Spectral and Projector Dynamics

A state in an n-dimensional open quantum system is described by an operator ρ on the n-dimensional Hilbert space, called the density operator. It must be positive semi-definite with unit trace. It can be written in terms of its eigenvalues:

$$\rho = \sum_{k=1}^{n} \lambda_k \pi_k,$$

where the λ_k's are the (possibly repeated) eigenvalues of ρ, and the π_k's are orthogonal projectors onto the corresponding eigenspaces. The properties of ρ demand that all eigenvalues lie on the interval $[0,1]$ and that $\sum_k \lambda_k = 1$.

The dynamics of a system with Lindblad dissipation is described by the Hamiltonian $H(t)$, which is a (possibly time-dependent) Hermitian operator, and a set of N Lindblad operators $\{L_k\}$ with the Lindblad equation:

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)) := [-iH(t), \rho(t)] + \mathcal{L}_D(\rho(t)) \tag{1}$$
$$\mathcal{L}_D(\rho) := \sum_{k=1}^{N} \left(L_k \rho L_k^\dagger - \frac{1}{2}\left(L_k^\dagger L_k \rho + \rho L_k^\dagger L_k \right) \right).$$

We are interested in investigating and controlling how a system moves between unitary orbits. In the absence of Lindblad dissipation, the solution to (1) can be written $\rho(t) = U(t)\rho(0)U(t)^\dagger$ where $U(t)$ is a trajectory on the unitary group $U(n)$ obeying $\frac{d}{dt}U(t) = -iH(t)U(t)$. Since $U(t)$ is unitary, the eigenvalues of $\rho(t)$ are invariant under the Hamiltonian evolution. That is, if we define the unitary orbit $\mathcal{O}(\rho) := \{U\rho U^\dagger : U \in U(n)\}$, the system does not leave the orbit without the influence of $\mathcal{L}_D$. For simplicity, we will assume fast controllability on the orbit: we can write

$$H(t) = H_0 + \sum_{i=1}^{n^2-1} u_i(t) H_i,$$

where $\{H_i : i = 1, 2, \ldots, n^2-1\}$ is a basis of $\mathfrak{su}(n)$, and the $\{u_j(t)\}$ are real-valued control functions that are unbounded and piecewise-continuous. The unboundedness is a key property: since $\mathcal{L}_D()$ is a bounded super-operator, motion along a unitary orbit can be made arbitrarily faster than motion between orbits. And because $\{H_i\}$ span the Lie algebra, any point on the orbit is reachable from any other.

We want to separate the dynamics of the eigenvalues from that of the projectors. We want to write down a linear ODE for the eigenvalues, which necessitates using vector notation. Let $\Lambda \in \mathbb{R}^n$ be the vector of eigenvalues written in non-increasing order, and let π be the n-tuple of corresponding projectors. Λ lives on an n-simplex $\mathcal{T} \subset \mathbb{R}^n$, with vertices $(1, 0, \cdots, 0)$, $(\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0)$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \cdots, 0)$, $\ldots$, $(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$. Note that we have made a choice of convention here: there are actually $n!$ different simplices we could have chosen, corresponding to the different re-orderings of the eigenvalues (see [32] for a discussion of the geometry of the eigenvalues of ρ).

$\mathcal{T}$ is an $(n-1)$-dimensional subset of $\mathbb{R}^n$. It can be useful to project it onto $\mathbb{R}^{n-1}$. We consider a map $\mathcal{P}$:

$$\bar{\mathcal{T}} := \mathcal{P}(\mathcal{T}) \subset \mathbb{R}^{n-1}$$
$$x := \mathcal{P}(\Lambda)$$
$$x_j := \frac{1}{\sqrt{j(j+1)}} \left(\sum_{i=1}^{j} \lambda_i - j\lambda_{j+1} \right).$$

$\mathcal{P}$ is a linear map: let Π be its corresponding $(n-1) \times n$ matrix, so that $x = \Pi\Lambda$. Let ι denote the Λ corresponding to the completely mixed state: $\iota = \langle \frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \rangle$. One can check the following identities: $\Pi\iota = 0$, $\Pi\Pi^T = I_{n-1}$, $\Pi^T\Pi = I_n - n\iota\iota^T$ and $\iota^T\Lambda = \frac{1}{n}$. Using these identities we can see that $\Lambda = \iota + \Pi^T x$, and also that $\mathcal{P}$ is an isometry[1]. Therefore, $\bar{\mathcal{T}}$ is an n-simplex with the same side-lengths as $\mathcal{T}$.

Our state space $\mathcal{T}$ is a manifold with boundary. $\partial\mathcal{T}$ consists of n faces. One face corresponds to the lowest eigenvalue vanishing: $\lambda_n = 0$. The remaining $(n-1)$ faces correspond to eigenvalue crossings $\lambda_j = \lambda_{j+1}$, $j = 1, 2, \ldots, (n-1)$. We must treat eigenvalue crossings with some care. Our decomposition of ρ into Λ and π becomes ill-defined there, as our choice of π is no longer unique in the sectors with repeated eigenvalues.

We would like to know what the derivatives of Λ and π are. We will refer to Kato [35] for details. At an eigenvalue crossing Λ with n_d distinct eigenvalues λ_α^d, define the *total projectors* $P_\alpha = \sum_{\{j:\lambda_j = \lambda_\alpha^d\}} \pi_j$. For any operator A, also define $(A)_{\alpha\beta} := P_\alpha A P_\beta$. Theorem 5.4 from chapter two of [35] allows us to write down the derivative of a total projector:

$$\frac{d}{dt}P_\alpha = \sum_{\beta \neq \alpha} \frac{(\rho')_{\alpha\beta} + (\rho')_{\beta\alpha}}{\lambda_\alpha^d - \lambda_\beta^d}. \tag{2}$$

Note that the constituent projectors π_j of a total projector may not be differentiable: see section II.3 of [35] for a counter-example. Away from eigenvalue crossings however, there is no distinction between the elements of π and the total projectors, so the above formula becomes a formula for the derivative $\frac{d}{dt}\pi$.

Now let us write down a formula for the derivative of Λ. The aforementioned theorem from [35] tells us that the eigenvalues of ρ are differentiable if ρ itself is differentiable. The elements of Λ however are only left- and right-differentiable, because crossing eigenvalues must be rearranged to maintain the non-decreasing order. In other words, differentiability of $\rho(t)$ implies there exists a differentiable $\bar{\Lambda}(t)$ where $\bar{\Lambda}$ and Λ are identical up to re-ordering.

Moreover, [35] states that the eigenvalue derivatives are given by the derivatives of $(\rho')_{\alpha\alpha}$. Using this, we can write down the following proposition:

Proposition II.1. *If $\rho(t)$ obeys the Lindblad equation, and $(\Lambda(t), \pi(t))$ is a differentiable decomposition of $\rho(t)$, define $w_{ij}^\pi = \sum_{k=1}^{N} Tr(\pi_i L_k \pi_j L_k^\dagger)$. Then:*

$$\frac{d}{dt}\Lambda(t) = \Omega^{\pi(t)}\Lambda(t), \tag{3}$$

[1] That is, $||\Pi(\Lambda_1 - \Lambda_2)|| = ||\Lambda_1 - \Lambda_2||$ for any Λ_1, Λ_2.

where Ω^π is an n-by-n matrix with:

$$\Omega^\pi = \begin{cases} w^\pi_{ij}, & i \neq j \\ -\sum_{l\neq j} w^\pi_{lj}, & i = j. \end{cases}$$

Proof. If $\Lambda(t)$ and $\pi(t)$ are differentiable, we can use the product rule:

$$\begin{aligned} \rho' &= \sum_j \left(\lambda_j' \pi_j + \lambda_j \pi_j'\right) \\ \lambda_j' &= \mathrm{Tr}\left(\pi_j \rho' \pi_j\right) - \sum_k \mathrm{Tr}\left(\pi_j \lambda_k \pi_k' \pi_j\right) \\ &= \mathrm{Tr}\left(\pi_j \rho' \pi_j\right). \end{aligned}$$

From the first to second line, we have sandwiched both sides with projectors and taken the trace. The second term in the second line vanishes because $\pi_j \pi_k' \pi_j = 0$. This formula can be derived by differentiating $\pi_j \pi_k = \delta_{jk}\pi_j$, where δ_{jk} is the Kronecker delta, and sandwiching the results between π_j's. Now we use the Lindblad equation:

$$\begin{aligned} \lambda_j' &= \mathrm{Tr}(\pi_j \rho' \pi_j) \\ &= \sum_{k=1}^n \mathrm{Tr}(\pi_j[-iH, \lambda_k\pi_k]\pi_j + \pi_j \mathcal{L}_D(\lambda_k\pi_k)\pi_j) \\ &= \sum_{k=1}^n \sum_{l=1}^N \mathrm{Tr}(\pi_j L_l \lambda_k \pi_k L_l^\dagger \pi_j \\ &\qquad - \frac{1}{2}\pi_j(L_l^\dagger L_l \lambda_k \pi_k + \lambda_k \pi_k L_l^\dagger L_l)\pi_j) \\ &= \sum_{k=1}^n \sum_{l=1}^N \mathrm{Tr}(\lambda_k \pi_j L_l \pi_k L_l^\dagger \pi_j - \lambda_j \pi_j L_l^\dagger \pi_k L_l \pi_j) \\ &= \sum_{k=1}^n \lambda_k w^\pi_{jk} - \lambda_j w^\pi_{kj} \\ &= \sum_{k=1}^n \Omega^\pi_{jk} \Lambda_k. \end{aligned}$$

We have made use of the identities $\pi_j\pi_k = \delta_{jk}\pi_j$ and $\sum_{k=1}^n \pi_k = I_n$. □

Corollary II.2. *Ω^π is rank-deficient. On the projected simplex, we have the formula for $x(t) \in \bar{\mathcal{T}}$:*

$$\frac{d}{dt}x(t) = b^{\pi(t)} + A^{\pi(t)}x(t), \tag{4}$$

where $b^\pi = \Pi\Omega^\pi\iota$ and $A^\pi = \Pi\Omega^\pi\Pi^T$.

Proof. Ω^π must be rank-deficient because its column-sums are zero, which is a reflection of the fact that the element-sum of Λ must be one. The ODE is obtained by substituting $\Lambda = \iota + \Pi^T x$ into the ODE in the proposition, and then multiplying by Π. □

III. The Projected Control System

We have decomposed the Lindblad system into its spectrum and projectors, and now we want to define a new control system. Let us clarify the distinction between the old and new control systems:

Definition The *ρ-control system* is the Lindblad equation (1), a complete set of control Hamiltonians $\{H_i\}$ that span the Lie algebra $\mathfrak{su}(n)$, and the control functions $u_i(t)$ that are piecewise-continuous, real-valued and unbounded.

Now let $\mathbb{P} \cong U(n)/U(1)^n$ be the space of projector-tuples π, and $\mathbb{P}^\Lambda \cong U(n)/(U(m_1)\times\cdots\times U(m_{n_d}))$ the space of tuples of total projectors. Then:

Definition The *Λ-control system* is the linear ODE (3), together with control projectors $\pi(t)$ on the control-set $\mathbb{P}$. We consider only functions $\pi(t)$ that are piecewise-differentiable. Additionally, the control functions must meet the following two conditions:

1) At any crossing $\lambda_i = \lambda_j$, π_i and π_j diagonalize $(\rho')_{\alpha\alpha}$, where P_α is the appropriate total projector. More strictly, for some neighborhood of the crossing time, there must exist a $C > 0$ such that $||\pi_i \mathcal{L}_D(\sum_\gamma \lambda_\gamma P_\gamma)\pi_j|| < C||\lambda_i - \lambda_j||$.
2) $\pi(t)$ must satisfy an initial and a final condition: $\pi(t_0) = \pi_0$ and $\pi(t_f) = \pi_f$.

The first condition reflects the fact that the orbit space $\mathbb{P}^\Lambda$ has a lower dimension at crossings. We have the freedom to choose the total projectors P_α, but not their diagonalizations. The dimension of the control set is $n^2 - \sum_\alpha m_\alpha^2$. When all eigenvalues are simple, this dimension is $n^2 - n$. The inequality must be satisfied to ensure the Hamiltonian is continuous at crossings. In Proposition III.1 below, we require this inequality to avoid singularities in the Hamiltonian.

The second condition above is imposed since we typically have an initial and target density matrix in mind, each with their own eigenprojectors that we may not choose. Note that both conditions can be dropped if we are willing to settle for approximate controllability: that is, if it suffices that our final ρ is arbitrarily close to our target ρ. We will expand on this shortly.

We can now write down a formula for the Hamiltonian:

Proposition III.1. *Given a trajectory $\Lambda(t)$ and controls $\pi(t)$ that satisfy the ODE (3) in the Λ-control system, we can recover the density operator $\rho(t) = \sum_j \lambda_j(t)\pi_j(t)$ using the following Hamiltonian:*

$$H^\pi(t) = i\left(-\sum_{j=1}^n \pi_j(t)\pi_j'(t) + \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{n_d} \frac{\left(\mathcal{L}_D(\sum_\gamma \lambda_\gamma P_\gamma)\right)_{\alpha\beta}}{\lambda_\alpha^d(t) - \lambda_\beta^d(t)}\right).$$

This Hamiltonian is piecewise-continuous.

Proof. Firstly, note that the piecewise-continuity follows from condition one in the definition of the Λ-control system. If we write the two terms of the Hamiltonian $H^\pi = H_A^\pi + H_B^\pi$, it is clear that H_A^π is piecewise-continuous due to the piecewise-differentiability of π. H_B^π is piecewise-differentiable because the numerator and denominator are, and condition one demands the numerator always approaches zero at least as fast as the denominator.

We must now show that our re-constructed $\rho(t)$ and $H^\pi(t)$

obey the Lindblad equation (1), which amounts to:

$$\sum_{j=1}^{n}\left(\lambda_j'\pi_j+\lambda_j\pi_j'\right)=\sum_{j=1}^{n}\left([-iH^\pi,\lambda_j\pi_j]+\mathcal{L}_D(\lambda_j\pi_j)\right).$$

We claim that $[-iH_A^\pi,\sum_j\lambda_j\pi_j]=\sum_j\lambda_j\pi_j'$ and that $[-iH_B^\pi,\sum_j\lambda_j\pi_j]+\sum_j\mathcal{L}_D(\lambda_j\pi_j)=\sum_j\lambda_j'\pi_j$, which if true would prove the proposition.

For the first part of the claim:

$$\begin{aligned}
&[-iH_A^\pi,\sum_j\lambda_j\pi_j]=-\sum_{j,k=1}^{n}[\pi_k\pi_k',\lambda_j\pi_j]\\
&=-\sum_{j,k=1}^{n}\lambda_j\pi_k\pi_k'\pi_j+\sum_{j=1}^{n}\lambda_j\pi_j\pi_j'\\
&=-\sum_{j,k=1}^{n}\lambda_j(\pi_k'-\pi_k'\pi_k)\pi_j+\sum_{j=1}^{n}\lambda_j(\pi_j'-\pi_j'\pi_j)\\
&=-\sum_{j,k=1}^{n}\pi_k'\lambda_j\pi_j+\sum_{k=1}^{n}\lambda_k\pi_k'\pi_k+\sum_{j=1}^{n}\lambda_j(\pi_j'-\pi_j'\pi_j)\\
&=\sum_{j=1}^{n}\lambda_j\pi_j',
\end{aligned}$$

where we have used the identities $\sum_k\pi_k'=0$, and $\pi_j'\pi_j+\pi_j\pi_j'=\pi_j'$.

For the second part of the claim:

$$\begin{aligned}
&[-iH_B^\pi,\sum_{j=1}^{n}\lambda_j\pi_j]+\sum_{j=1}^{n}\mathcal{L}_D(\lambda_j\pi_j)=\sum_{j=1}^{n}\Big(\mathcal{L}_D(\lambda_j\pi_j)\\
&\qquad+\sum_{\substack{\alpha,\beta=1\\\alpha\neq\beta}}^{n_d}\sum_{\gamma=1}^{n_d}\frac{[P_\alpha\mathcal{L}_D(\lambda_\gamma^dP_\gamma)P_\beta,\lambda_j\pi_j]}{\lambda_\alpha^d-\lambda_\beta^d}\Big)\\
&=\sum_{j=1}^{n}\mathcal{L}_D(\lambda_j\pi_j)-\sum_{\substack{\alpha,\beta=1\\\alpha\neq\beta}}^{n_d}\sum_{\gamma=1}^{n_d}P_\alpha\mathcal{L}_D(\lambda_\gamma^dP_\gamma)P_\beta\\
&=\sum_{\alpha=1}^{n_d}P_\alpha\mathcal{L}_D(\rho)P_\alpha=\sum_{\alpha=1}^{n_d}P_\alpha\left([-iH^\pi,\rho]+\mathcal{L}_D(\rho)\right)P_\alpha\\
&=\sum_{\alpha=1}^{n_d}P_\alpha\frac{d\rho}{dt}P_\alpha=\sum_{j=1}^{n}\lambda_j'\pi_j.
\end{aligned}$$

In the second to last line, we can insert the commutator because $P_\alpha\rho=\rho P_\alpha=\lambda_\alpha P_\alpha$. We have shown that our construction obeys the Lindblad equation. □

Note that the constructed $H^\pi(t)$ may become very large if two eigenvalues become very close. If the eigenvalues actually cross however, the Hamiltonian does not blow up, due to the first condition. There are only certain $\pi(t)$ that allow an eigenvalue crossing, and trying to approach a crossing with an illegal $\pi(t)$ requires an infinite energy cost. Note that orbits with repeated eigenvalues fall on the boundary of $\mathcal{T}$, so if we only require that we steer arbitrarily close to such an orbit, we can ignore the first condition, since nearby points are in the interior where the condition does not apply.

We now explore the implications of eliminating the second condition. If we construct a trajectory $(\Lambda(t),\pi(t))$ with the desired initial and final Λ, but with an undesired initial and final π, we can book-end the trajectory with fast unitary transformations. Say we have initial and target density operators ρ_0 and ρ_T. We are able to construct $\Lambda(t)$ and $\pi(t)$ on the interval $[0,T]$ that brings ρ_1 to ρ_2, where there are skew-symmetric matrices $-ih_i$ and $-ih_T$ such that $\rho_1=e^{-ih_i}\rho_0e^{ih_i}$ and $\rho_2=e^{-ih_T}\rho_Te^{ih_T}$. Then we can construct the following motion on the interval $[-\Delta,T+\Delta]$:

$$\begin{aligned}
t\in[-\Delta,0]&\begin{cases}\rho_0\to\bar\rho_1\\ H(t)=h_i/\Delta\end{cases}\\
t\in[0,T]&\begin{cases}\bar\rho_1\to\bar\rho_2\\ H(t)=H^\pi(t)\end{cases}\\
t\in[T,T+\Delta]&\begin{cases}\bar\rho_2\to\rho_f\\ H(t)=h_T/\Delta.\end{cases}
\end{aligned}$$

Let $\rho_a(t)$ denote our ideal trajectory $\rho_0\to\rho_1\to\rho_2\to\rho_T$ and $\rho_b(t)$ the actual trajectory $\rho_0\to\bar\rho_1\to\bar\rho_2\to\rho_f$. To measure distance between density operators, we will use the trace distance[2]:

$$d(\rho_a,\rho_b)=\frac{1}{2}Tr(\sqrt{(\rho_a-\rho_b)^2})=\frac{1}{2}\sum_{k=1}^{n}|\lambda_k^\delta|,$$

where λ_k^δ are the (real) eigenvalues of $\rho_a-\rho_b$.

We now show a proposition regarding approximate trajectories.

Proposition III.2. *If there exists a $\pi(t)$ on $[0,T]$ that brings Λ_{ρ_0} to Λ_{ρ_T}, then there is a Hamiltonian $H(t)$ on $[-\Delta,T+\Delta]$ that brings ρ_0 to ρ_f such that $d(\rho_f,\rho_T)\leq C\Delta$. The constant C is universal for all initial and final density operators.*

Proof. To begin, we note that the time-derivative of the distance is $d'(\rho_a,\rho_b)=\sum_{k\in C^\delta(t)}\lambda_k^{\delta\prime}$, where $C^\delta(t)$ is the subset of indices such that $\lambda_k^\delta>0$. If one or more eigenvalues are zero with non-zero derivative, the metric has different left- and right-side derivatives. In this case, define $C^\delta(t-)$ to include indices for zero and decreasing eigenvalues, and $C^\delta(t+)$ to include indices for zero and increasing eigenvalues. We know that the eigenvalues are differentiable, since the aforementioned work of Kato [35] can be applied with minimal modification to $\rho_a-\rho_b$.

Now for the first part of the trajectory:

$$\begin{aligned}
d(\rho_1,\bar\rho_1)&\leq\Delta\cdot\sup_{-\Delta\leq t\leq0}|d'(\rho_a(t),\rho_b(t))|\\
&\leq\frac{\Delta}{2}\cdot\sup_{-\Delta\leq t\leq0}\sum_{k=1}^{n}|\lambda_k^{\delta\prime}(t)|\\
&=\frac{\Delta}{2}\cdot\sup_{-\Delta\leq t\leq0}\sum_{k=1}^{n}|\mu_k^\delta(t)|,
\end{aligned}$$

where the μ_k^δ are eigenvalues of $(\rho_a'-\rho_b')$ projected onto its different eigenspaces. Now $(\rho_a'-\rho_b')=[-ih_i/\Delta,\rho_a-\rho_b]+\mathcal{L}_D(\rho_b)$. The Hamiltonian piece projected onto its eigenspaces vanishes, so we are left with only the dissipative piece. It follows that $\mu_k^\delta\leq\sup_{-\Delta\leq t\leq0}|\mathcal{L}_D(\rho_b(t))|\leq2\sum_{m=1}^{N}|L_m|^2$. So $d(\rho_1,\bar\rho_1)\leq n\Delta\sum_m|L_m|^2$.

[2]See [32] for other distance measures for density matrices.

The middle piece of the trajectory causes no problems, since both ρ_a and ρ_b experience the same dynamics, and the Lindblad equation is known to be contractive [20]. We can adapt equation (3) for $(\rho_a - \rho_b)$ instead of ρ, where Λ^δ and π^δ replace Λ and π (this can be done since the positive semi-definiteness of ρ is not invoked in the proof). On the interval $[0, T]$, we have:

$$\begin{aligned} d'(\rho_a, \rho_b) &= \sum_{k \in C^\delta} \lambda_k^{\delta\prime} = \sum_{k \in C^\delta} \sum_{l=1}^{n} \Omega_{kl}^{\pi^\delta} \lambda_l^\delta \\ &= \left(\sum_{k,l \in C^\delta} + \sum_{k \in C^\delta, l \notin C^\delta} \right) \Omega_{kl}^{\pi^\delta} \lambda_l^\delta \\ &= - \sum_{k \notin C^\delta, l \in C^\delta} w_{kl}^\delta \lambda_l^\delta + \sum_{k \in C^\delta, l \notin C^\delta} w_{kl}^\delta \lambda_l^\delta \\ &= - \sum_{k \notin C^\delta, l \in C^\delta} w_{kl}^\delta |\lambda_l^\delta| - \sum_{k \in C^\delta, l \notin C^\delta} w_{kl}^\delta |\lambda_l^\delta| \\ &\leq 0, \end{aligned}$$

where in the third line, first sum, we have used the fact that the column-sums of Ω^{π^δ} are zero.

So $|\rho_2 - \bar{\rho}_2| \leq |\rho_1 - \bar{\rho}_1|$. To finish, we have:

$$\begin{aligned} d(\rho_f, \rho_T) &\leq d(\rho_2, \bar{\rho}_2) + \Delta \cdot \sup_{T \leq t \leq T+\Delta} |d'(\rho_a(t), \rho_b(t))| \\ &\leq 2n\Delta \sum_{m=1}^{N} |L_m|^2. \end{aligned}$$

The multiplicative constant $2n \sum_{m=1}^{N} |L_m|^2$ is independent of ρ_0 and ρ_f. □

Corollary III.3. *If we expand the Λ-control system to allow piecewise-differentiable $\pi(t)$ with a finite number of discontinuities, the final density operator corresponding to the final Λ can be reached within an arbitrarily small error.*

Proof. This is merely an extension of the previous proposition, where instead of book-ending one continuous trajectory with fast unitary transformations, we are intersplicing a finite number of fast unitary transformations at the discontinuities. □

While the two conditions in the definition of the Λ-control system are necessary for planning trajectories in ρ-space and their corresponding Hamiltonians, they can be disregarded when analyzing controllability. This will be made clearer in the next section; for now, we define the following control system:

Definition The *unconstrained Λ-control system* is the linear ODE (3), together with a piecewise-differentiable projectors $\pi(t)$, with a finite number of possible discontinuities.

"Unconstrained" in this case refers to the absence of the two conditions.

Because the control set of the Λ-control system is a non-Euclidean manifold, it is not trivial to use standard control-theoretic results for the projected system. However, if we view the elements w_{ij}^π as controls, we are left with a bi-linear control system, since Ω^π is linear in these elements. Define the map $w : \mathbb{P} \to \mathbb{R}^{n^2-n}$ that sends π to the corresponding vector of w_{jk}^π. Note that $w(\mathbb{P})$ is a closed and bounded set in $\mathbb{R}^{n^2-n}$. Also define $\Omega(w)$, $w \in \mathbb{R}^{n^2-n}$ to be the matrix with off-diagonal elements equal to w_{jk} and diagonal elements equal to $-\sum_{l \neq k} w_{lk}$. Define the following control system, which is the unconstrained Λ-control system with a transformation:

Definition The w-control system is the bi-linear ODE $\frac{d}{dt}\Lambda = \Omega(w)\Lambda$ on $\mathcal{T}$. The control set is $w(\mathbb{P})$ and control functions must be piecewise-differentiable, with a finite number of discontinuities.

The derivatives of w are, where $h \in T_\pi \mathbb{P} \subset \mathfrak{su}(n)$:

$$\begin{aligned} dw_{jk}(\pi) \cdot h &= \sum_{l=1}^{N} \pi_j [L_l, h] \pi_k L_l^\dagger \pi_j + \pi_j L_l \pi_k [L_l^\dagger, h] \pi_j \\ w'_{jk}(t) &= \sum_{l=1}^{N} \pi_j(t) [L_l, \pi'(t)] \pi_k(t) L_l^\dagger \pi_j(t) \\ &\quad + \pi_j(t) L_l \pi(t)_k [L_l^\dagger, \pi'(t)] \pi_j(t). \end{aligned} \tag{5}$$

Note that $w(\pi)$ is a map from a manifold of dimension $n^2 - n$ to $\mathbb{R}^{n^2-n}$. This means that if dw is full rank along a trajectory $w(t)$, then, by the Inverse Function Theorem, the projectors $\pi(t)$ can be recovered. The difficulty in analyzing this w-control system is understanding the structure of the control set $w(\mathbb{P})$.

IV. Local Controllability Analysis

In the remainder of this paper, we wish to examine the controllability of the Λ-control system. We will restrict ourselves to local controllability, as this simplifies the analysis somewhat:

Definition A control system is *locally controllable (LC)* [8] in time T at a point p if for every neighborhood V of p, V contains another neighborhood W such that $\forall y, z \in W$, y can be controlled to z in time T. The system is *strongly locally controllable (SLC)* if a W can be found for any V such that $\forall y, z \in W$, y can be controlled to z without leaving W.

In plain terms, local controllability guarantees a trajectory between two local points in a given time, while strong local controllability demands this trajectory also be local. We will give a sufficient condition for SLC in both the unconstrained and constrained Λ-control system. First let $\mathbb{P}^{\Lambda *}$ be the subset of $\mathbb{P}$ satisfying the first sentence of condition 1) in the definition of the Λ-control system. Now define $\mathcal{V}_u(\Lambda) = \{\Omega^\pi \Lambda : \pi \in \mathbb{P}\}$ and $\mathcal{V}_c(\Lambda) = \{\Omega^\pi \Lambda : \pi \in \mathbb{P}^{\Lambda *}\}$. These are the sets of possible tangent vectors $\frac{d}{dt}\Lambda$ for the unconstrained and constrained systems, respectively. For Λ with distinct elements, these sets are identical, but when there is eigenvalue multiplicity, $\mathcal{V}_c(\Lambda) \subset \mathcal{V}_u(\Lambda)$.

Here, int denotes "interior" and co "convex hull":

Proposition IV.1. *If $0 \in$ int co $\mathcal{V}_u(\Lambda)$, then both the unconstrained and constrained Λ-systems are SLC at Λ. If $0 \notin$ co $\mathcal{V}_u(\Lambda)$, then there is a τ for which neither are LC at Λ for $T < \tau$.*

Proof. The first part is an application of Lemma 3.8.5 and its corollary from [8], which states that if 0 lies in the interior of the convex hull of the set of available tangent vectors, then

the system is SLC. The wrinkle we must deal with is showing that the SLC extends to the constrained system, despite the smaller control set.

For the constrained system, we claim that co $\mathcal{V}_u(\Lambda)$ = co $\mathcal{V}_c(\Lambda)$, which if true yields the desired result. Our claim follows from the Schur-Horn theorem [36][37], which states that for any Hermitian operator A, $\{diag(UAU^\dagger) : U \in U(n)\}$ = co $\{\sigma.\Gamma_A : \sigma \in S_n\}$. Here $diag()$ denotes the vector of diagonal elements, $\sigma.\Gamma$ denotes Γ with elements permuted with permutation $\sigma \in S_n$, and Γ_A denotes the vector of eigenvalues of A. This can be extended to direct sums: for any set of Hermitian operators A_α, $\{\bigoplus_\alpha diag(U_\alpha A_\alpha U_\alpha^\dagger) : U_\alpha \in U(n_\alpha)\} = co(\{\bigoplus_\alpha \sigma.\Gamma_{A_\alpha} : \sigma \in S_{n_\alpha}\})$. In our case we use $A_\alpha = (\rho')_{\alpha\alpha}$. Then we have:

$$\begin{aligned}
\text{co } \mathcal{V}_c(\Lambda) &= \text{co } \{\bigoplus_\alpha \sigma_\alpha.\Gamma_{\rho'_{\alpha\alpha}} : \sigma_\alpha \in S_{m_\alpha}, P_\alpha \in \mathbb{P}^\Lambda\} \\
&= \{\bigoplus_\alpha diag(U_\alpha P_\alpha \rho' P_\alpha U_\alpha^\dagger) : U_\alpha \in U(m_\alpha), P_\alpha \in \mathbb{P}^\Lambda\} \\
&= \{\bigoplus_j diag(\pi_j \rho' \pi_j) : \pi \in \mathbb{P}\} \\
&= \text{co } \{\sigma.\Gamma_{\rho'} : \sigma \in S_n\} = \text{co } \mathcal{V}_u(\Lambda).
\end{aligned}$$

In the second and fourth lines, we apply the Schur-Horn theorem. In the third line, we recognize the set of all diagonal vectors of $\rho'_{\alpha\alpha}$ is equal to the set of all possible $\bigoplus_{j\in C_\alpha} \pi_j \rho' \pi_j$ for $\sum_{j\in C_\alpha} \pi_j = P_\alpha$.

To show the second part of the proposition, note that co $\mathcal{V}_u(\Lambda)$ is compact, since $w(\mathbb{P})$ and thus $\mathcal{V}_u(\Lambda)$ is compact, and the convex hull of a compact set in $\mathbb{R}^n$ is compact. Suppose at some $\Lambda = \Lambda_0$, $0 \notin$ int co $\mathcal{V}_u(\Lambda_0)$. Due to the compactness and convexity, there is a unique point $v_m \in \partial$ co $\mathcal{V}_u(\Lambda_0) \subset \mathbb{R}^n$ with minimal magnitude, and this fixes a hyperplane passing through Λ_0 that is orthogonal to v_m. The magnitude of this vector as Λ varies cannot vary more than $C_\Omega|\delta\Lambda|$, where $C_\Omega = \sup|\Omega(w(\mathbb{P}))|$. Due to compactness, there is also a point v_M, not necessarily unique, of maximal magnitude. If we define $\tau = \frac{|v_m|}{2|v_M|C_\Omega}$, then the reachable set $R_\tau(\Lambda_0)$ falls entirely on one side of the hyperplane and thus cannot contain zero. This is because:

$$\begin{aligned}
(\Lambda(\tau) - \Lambda_0) \cdot \frac{v_m}{|v_m|} &\geq \tau \inf_{t\in[0,\tau]} (\Lambda'(t) \cdot \frac{v_m}{|v_m|}) \\
&\geq \tau(|v_m| - C_\Omega \sup_{t\in[0,\tau]} |\Lambda'(t)|\tau) \\
&\geq \tau(|v_m| - C_\Omega |v_M|\tau) = \frac{1}{2}\tau|v_m| > 0.
\end{aligned}$$

It follows that LC does not hold at Λ_0 for any $T < \tau$. □

Analyzing the local controllability of the Λ-system requires studying $\mathcal{V}_u(\Lambda)$. For general $\Lambda \in \mathcal{T}$, this is difficult, but at the completely mixed state, its structure simplifies greatly, as it is the convex hull of a finite set of vectors:

Proposition IV.2. *$\mathcal{V}_u(\iota) = co\ \{\sigma.\Gamma_{A_\iota} : \sigma \in S_n\}$, where A_ι is the operator $\sum_k [L_k, L_k^\dagger]$.*

Proof. This is a consequence of the fact that when $\rho = \frac{1}{n}I$, $\rho' = \frac{1}{n}A_\iota$. If one applies the Schur-Horn theorem, the proposition immediately follows. □

In general, $\mathcal{V}_u(\Lambda)$ is not the convex hull of a finite number of vectors, as it is at the completely mixed state. However, it does raise a tractable question: where does SLC hold for the Λ-control system when one is restricted to a finite control-set? To this end, we state a theorem (which is easier to state in terms of $x = \Pi\Lambda \in \mathbb{R}^{n-1}$ rather than $\Lambda \in \mathbb{R}^n$) about the region $\mathcal{A} \subset \bar{\mathcal{T}}$ where the necessary condition for SLC from proposition IV.1 holds. It states that $\mathcal{A}$ is the image under a rational function of an n-simplex of parameters, and that the boundary $\partial\mathcal{A}$ is the image of the parameter-simplex's boundary.

Define the parameter simplex $\mathcal{T}_s := \{s \in \mathbb{R}^n : s_J \geq 0, \sum_J s_J = 1\}$, and the following function:

$$\begin{aligned}
B &: \mathcal{T}_s \to \mathbb{R}^{n-1} \\
B(s) &= -\left(\sum_{J=1}^n s_J A_J\right)^{-1} \left(\sum_{J=1}^n s_J b_J\right).
\end{aligned}$$

We will assume we have a finite set of control projector-tuples π^J. For each J define:

$$\begin{aligned}
A^J &:= \Pi\Omega^{\pi^J}\Pi^T \\
b^J &:= \Pi\Omega^{\pi^J}\iota
\end{aligned}$$

Theorem IV.3. *If one has n control projector-tuples π^J such that A^J is invertible $\forall J$, then SLC must hold on the following set:*

$$\mathcal{A} = int\,(B(\mathcal{T}_s))\,|_{\bar{\mathcal{T}}}\ .$$

The boundary of this region will overlap $\partial\bar{\mathcal{T}}$. The remaining piece is given by:

$$\partial\mathcal{A} - \partial\bar{\mathcal{T}} = \big(int\ \bar{\mathcal{T}}\big) \cap B(\partial\mathcal{T}_s).$$

Proof. The necessary condition for SLC is $0 \in$ int co $\{b_J + A_J x : J = 1, \ldots, n\}$. Either the n points $b_J + A_J x$ lie in a hyperplane, in which case the interior is empty, or they form an n-simplex. We will show shortly that the former case never occurs in the interior of $B(\mathcal{T}_s)$, which we call claim (A). In the latter case, convexity means the condition reduces to $0 = \sum_{J=1}^n s_J(b_J + A_J x)$, $s \in$ int $\mathcal{T}_s$. Re-writing we get:

$$\begin{aligned}
\sum_{J=1}^n s_J A_J x &= -\sum_{J=1}^n s_J b_J \\
x &= -\left(\sum_{J=1}^n s_J A_J\right)^{-1} \left(\sum_{J=1}^n s_J b_J\right) = B(s).
\end{aligned}$$

We can take the inverse because each A_J is invertible, and since Ω is always negative semi-definite[3], each A_J is also negative semi-definite. We have shown that the necessity condition for SLC holds in $B(\text{int } \mathcal{T}_s)$ given claim (A). We will also prove shortly that $B(\partial\mathcal{T}_s) = \partial B(\mathcal{T}_s)$, which we call claim (B). Given this claim, int $B(\mathcal{T}_s) \subseteq B(\text{int } \mathcal{T}_s)$, which means that SLC must hold on int $(B(\mathcal{T}_s))$. Restricting to $\bar{\mathcal{T}}$, we have proven the first part of the theorem, pending proof of claims (A) and (B).

To prove claim (A), we apply Carathéodory's Theorem [38], which says that any point in a convex hull of a set P in an

[3] $v^T\Omega v = \sum_{i<j}(w_{ij}+w_{ji})v_iv_j - w_{ij}v_i^2 - w_{ji}v_j^2 \leq 0$ if $w_{ij}, w_{ji} \leq 0$.

m-dimensional linear space must also lie in the convex hull of a set $P' \subseteq P$ with at most m elements. This means if the $b_J + A_J x$ are co-planar, there is one we can eliminate without changing the convex hull. But this means one element of s is zero, and this only occurs on $\partial\mathcal{T}_s$. So the exceptional case only occurs if $x \in B(\partial\mathcal{T}_s)$. Given claim (B), this means $x \in \partial B(\mathcal{T}_s)$.

Now we prove claim (B). There are three types of points on $\mathcal{T}_s$: boundary points, interior points that are critical points of B and interior points that are regular points of B. Regular points must map to points in int $B(\mathcal{T}_s)$, due to the Inverse Function Theorem. To examine the interior critical points, write $A(s) = \sum_{J=1}^{n} s_J A_J$ and $b(s) = \sum_{J=1}^{n} s_J b_J$. Then the directional derivative of B is:

$$\begin{aligned} dB_s(\delta s) &= -A(s)^{-1}b(\delta s) - A(s)^{-1}A(\delta s)A(s)^{-1}b(s) \\ &= -A(s)^{-1}\left(b(\delta s) + A(\delta s)x(s)\right), \end{aligned}$$

where δs is an arbitrary vector in $\mathcal{T}_s$. We have used the product rule as well as the derivative formula for matrix inverse: $A^{-1\prime} = -A^{-1}A'A^{-1}$. We claim there are no isolated critical points, and that the critical points form disjoint subsimplices of $\mathcal{T}_s$. If the derivative is degenerate at some s^*, there is some non-zero δs^* for which $dB_{s^*}(\delta s^*) = 0$. Since $A(s)^{-1}$ is full-rank, this means $b(\delta s^*) + A(\delta s^*)x(s^*) = 0$. Linearity of b and A in s means that $b(s^* + k\delta s^*) + A(s^* + k\delta s^*)x(s^*) = 0$ for all real k. But this implies that $x(s^* + k\delta s^*) = x(s^*)$ for all real k. It follows that s^* lies in some affine subspace $V_* = s^* + \ker dB_{s^*}$ and that every point in V_* is a critical point. There may be more than one critical subspace, but they must be disjoint: a non-zero intersection could be used to generate a higher-dimensional critical subspace that contained the intersecting subspaces. Now if we restrict a critical subspace to $\mathcal{T}_s$, we are left with a subsimplex $\mathcal{T}_*$. We have seen that any critical subsimplex maps to a single point under B. We claim that this point lies on the boundary of $B(\mathcal{T}_s)$.

To see why $B(\mathcal{T}_*) \in \partial B(\mathcal{T}_s)$, we show that $\bigcup_{s\in\mathcal{T}_*} \text{im } dB_s \neq \mathbb{R}^{n-1}$ which means that there are directions from $B(\mathcal{T}_*)$ that can't be generated by small deviations from $\mathcal{T}_*$. Therefore a neighborhood of $\mathcal{T}_*$ cannot map to a ball in $\mathbb{R}^{n-1}$, which it must if $\mathcal{T}_*$ mapped to the interior of $B(\mathcal{T}_s)$. To determine which direction, let $V_\perp$ be the complementary subspace to V_*, so that im $dB_s = \bigcup_{v\in V_\perp} dB_s(v)$. From the formula for dB_s we get that im $dB_s = -A(s)^{-1}(b(V_\perp + A(V_\perp)x_*)$ where $x_* = x(\mathcal{T}_*)$. Since $V_\perp$ has dimension $m < n-1$, $b(V_\perp) + A(V_\perp)x_*$ is an m-dimensional linear subspace of $\mathbb{R}^{n-1}$, and there is a vector v_{**} orthogonal to it. This is the direction we are looking for, because $-A(s)^{-1}$ is a positive-definite matrix, which can never map a vector in a linear subspace to the complement of that subspace (open half-spaces are invariant under positive-definite linear maps). It follows that a sufficiently small neighborhood of $\mathcal{T}_*$ maps to a set that only intersects span v_{**} at x_*. Therefore $x_* \notin$ int $B(\mathcal{T}_s)$.

What we really want to show is that the boundary points of $\mathcal{T}_s$ map to $\partial B(\mathcal{T}_s)$. If $s \in \partial\mathcal{T}_s$ and $s \in V_*$, then we know that $B(s) \in \partial B(\mathcal{T}_s)$, so let us consider a boundary point s that is regular. s cannot map locally to an interior point, so if it maps to an interior point, some other s' must also map there *i.e.*$B(s) = B(s')$. Note however that the structure of B demands that $B(s) = B(s + k(s' - s))$ for any real k. This means that s is part of an affine space that maps to B. This affine space must be one of the critical subspaces, and so s must map to a boundary point. So we have proven claim (B).

Finally, we must prove the second part of the theorem. Claim (B) states that the boundary of $B(\mathcal{T}_s)$ is $B(\partial\mathcal{T}_s)$. When we restrict to $\bar{\mathcal{T}}$, the boundary becomes a union of some piece of $\bar{\mathcal{T}}$ and the restriction of $B(\partial\mathcal{T}_s)$ to $\bar{\mathcal{T}}$, so we are done. □

The theorem applies only for a control set of n projector-tuples, but it can be extended to a larger set:

Corollary IV.4. *If one uses $n_P > n$ projector-tuples as controls, $\mathcal{A} = \bigcup_K \mathcal{A}_K$, where K is a subset of $\{1, \cdots, n_P\}$ with n elements, and $\mathcal{A}_K$ is the region given by the theorem using $\{b_J, A_J : J \in K\}$. Furthermore, $\partial\mathcal{A} \subseteq \bigcup_K B_K(\partial\mathcal{T}_s)$ where B_K is the map B using $\{b_J, A_J : J \in K\}$.*

Proof. If $0 \in$ int co $\{b_J + A_J x : J = 1, \cdots, n_P\}$, then Carathéodory's Theorem says that there is a subset K of n indices such that $0 \in$ int co $\{b_J + A_J x : J \in K\}$. We can use the theorem to construct $\mathcal{A}_K$ for each K, and Carathéodory implies that $\mathcal{A} = \bigcup_K \mathcal{A}_K$. It also follows that $\partial\mathcal{A} \subseteq \bigcup_K B_K(\partial\mathcal{T}_s)$, but equality will typically not hold (the boundary of a union is not necessarily the union of boundaries). □

The preceding theorem can be used to visualize SLC sets for $n = 3$ and 4. We show some examples of this in the following section.

V. Examples

The requirement that the A_J's be invertible is not terribly restrictive, as it only requires a certain number of w_{ij}^J be non-zero. For $n = 3$, we have:

$$\begin{aligned} \det A_J = w_{12}^J w_{23}^J + w_{13}^J w_{32}^J + w_{12}^J w_{13}^J + w_{21}^J w_{13}^J \\ + w_{23}^J w_{31}^J + w_{21}^J w_{23}^J + w_{31}^J w_{12}^J + w_{32}^J w_{21}^J + w_{31}^J w_{32}^J. \end{aligned}$$

Since the w_{ij}^J's are always non-negative, we only need one of nine pairs to be non-zero.

Theorem IV.3 states that for any triple of projector-tuples, the SLC region is the image of $\mathcal{T}_s$ under $B(s)$, restricted to $\bar{\mathcal{T}}$, which for $n = 3$ is a quotient of two homogeneous quadratic functions. Since the boundary of $\mathcal{T}_s$ consists of three line segments, $B(\partial\mathcal{T}_s)$ consists of three arcs. Now, if we have more than three projector-tuples, say n_c, the unrestricted SLC region is the union of the SLC sets for each triple. It follows that there are $\binom{n_c}{2}$ arcs that may contribute to $B(\partial\mathcal{T}_s)$. If one plots these candidate arcs, we can visualize the SLC region.

For our examples, let π_ι be some tuple of projectors formed out of the eigenbasis of the Hermitian operator A_ι. Define $\pi^1, \cdots, \pi^6$ to be the projector-tuples obtained by permuting the elements of π_ι, so that we have a control-set of six projector-tuples. Call this set $\mathbb{P}_\iota$. If A_ι is simple, it is unique up to re-numbering. This choice of control-set is attractive because all possible tangent vectors at the completely mixed

state are contained in the convex hull generated by $\mathbb{P}_\iota$. We have $n_c = 6$, and therefore there are fifteen candidate arcs.

Figure 1 shows an example for a random Lindblad system. By random, we mean eight Lindblad operators were generated with elements whose real and imaginary parts were uniform on the interval $[0, 100]$. The top panel shows the fifteen arcs generated by π_ι, as well as the boundary of $\bar{\mathcal{T}}$. The SLC set is the interior of the region formed by these arcs, restricted to $\bar{\mathcal{T}}$, and this is the dark region shown in the bottom panel. To get some sense of how "good" our SLC region is we generated twenty random unitary matrices, used them to generate twenty projector-tuples as well as their permutations. With these random projector-tuples, we used corollary (IV.4) to plot a "better" SLC set. This makes for $\binom{6+6\cdot 20}{2} = 7875$ arcs. In the bottom panel of figure 1, we have shown the SLC region for this extended control set as the light region. It is clearly larger, but the original controls cover a good portion.

Instead of examining randomly generated Lindblad systems, we can investigate systems with two specific types of Lindblad operators: jump operators and de-phasing operators. A jump operator relative to a certain orthonormal basis is a Lindblad operator with only one non-zero element, which is off-diagonal. Fix a basis and define, for $j \neq k$, $L^J_{jk} := \sqrt{\gamma_{jk}} e_{jk}$, where e_{jk} is the matrix with a one at the (j, k) position and zeros elsewhere. Such an operator is called a jump operator as it models a stochastic jump from state k to state j. A de-phasing operator meanwhile is a Lindblad operator with only diagonal non-zero elements. It is so-called as any coherent superposition of states will decay to an incoherent mixture so long as the respective diagonal elements are non-zero. In the same basis, define $L^D_l = \sum_{j=1}^n c_{l,j} e_{jj}$, where l indexes the de-phasing operators. Note that with these Lindblad operators, $A_\iota = \sum_{j,k=1,k\neq j}^3 \gamma_{jk} e_{jj}$. Hence π_ι is in fact generated by the projectors e_{jj}.

Figure 2 shows $\mathcal{A}$ for a system with six jump operators (the coefficients are $\sqrt{\gamma_{12}} = 81$, $\sqrt{\gamma_{77}} = 81$, $\sqrt{\gamma_{23}} = 73$, $\sqrt{\gamma_{32}} = 36$, $\sqrt{\gamma_{31}} = 70$ and $\sqrt{\gamma_{13}} = 48$). The SLC region obtained using π_ι, in dark, covers almost the entire SLC region with an extended control set (similar to the preceding example, where there are 7875 controls in total). This is not an accident. When restricted to jump and de-phasing operators in some basis, it is difficult to find projectors other than π_ι and its permutations that enlarge $\mathcal{A}$. The reason for this is that these π are critical points of the map $w(\pi)$, and in fact the derivative of this map vanishes when $\pi \in \mathbb{P}_\iota$.

To see why, consider that the derivative (5) vanishes if, for each Lindblad operator L_m and component dw_{jk}, either $\pi_j L_m \pi_k = 0$, or $\pi_j [L_m, h] \pi_k \ \forall h \in T_\pi \mathbb{P}$. For the de-phasing operators, the first condition is automatically satisfied, since they are diagonal with respect to π_ι. For a jump operator $L^J_{j'k'}$, we have $\pi_j L^J_{j'k'} \pi_k = \delta_{jj'} \delta_{kk'} \sqrt{\gamma_{j'k'}}$, so the first condition is satisfied for all components except for $j = j'$, $k = k'$. And for this component, we claim the second condition is satisfied.

To see why this claim is true, note that $T_\pi \mathbb{P}$ is the subspace of $\mathfrak{su}(n)$ consisting of all off-diagonal matrices (since any projector set is stationary when acted upon by diagonal matrices). For this reason, $\pi_k h \pi_k \neq 0$, which means $L^J_{jk} h \pi_k = 0$. Similarly, $\pi_j h L^J_{jk} = 0$, and therefore $\pi_j [L^J_{jk}, h] \pi_k = 0$.

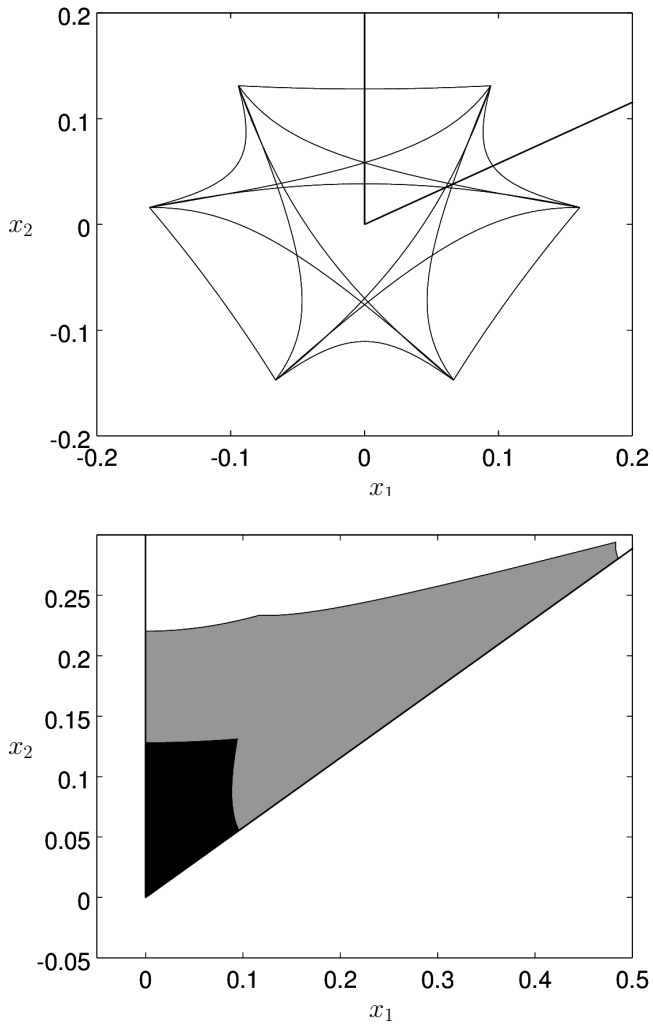

Fig. 1. (Top) Candidate arcs for $\partial\mathcal{A}$ for a random Lindblad system, with the simplex $\bar{\mathcal{T}}$ also shown. (Bottom) $\mathcal{A}$ for the same system when (dark) only π_ι are used and (light) twenty random projector-tuples extend the control set. Axes co-ordinates refer to the components of x.

So we can say that $dw(\pi_\iota) = 0$. It follows that the map $\pi \to \Omega(w(\pi))\Lambda$ has a critical point when $\pi = \pi_\iota$, since $\Omega()$ is linear in w.

The significance of π_ι being a critical point is that proposition IV.1 implies that SLC fails when 0 moves from an interior point of $\mathcal{V}_u(\Lambda)$ to to a boundary point. But a boundary point of $\mathcal{V}_u(\Lambda)$ must be a critical value of $\pi \to \Omega(w(\pi))\Lambda$, or alternatively a critical value of the map $\pi \to \Pi\Omega(w(\pi))\Lambda$. Setting $b(\sigma.\pi_\iota) - A(\sigma.\pi_\iota)x = 0$ yields the six terminal points of the fifteen arcs from which $\partial\mathcal{A}$ is obtained. Note that in principle, the non-terminal points of the arcs are *not* critical points, but in practice, there is not much room between the arcs and any points that fall outside.

We can also visualize $\mathcal{A}$ for $n = 4$. Figures 3 and 4 show candidate surfaces for ∂A for two randomly generated systems consisting of only jump operators. Figure 3 shows a system with four Lindblad operators: $\sqrt{5}e_{12}$, $\sqrt{3}e_{21}$, $\sqrt{4}e_{23}$ and $\sqrt{3}e_{34}$. Straight line-segments in the figures indicate the simplex $\bar{\mathcal{T}}$. In figure 3, we see that $\partial\mathcal{A}$ shares a portion of $\partial\bar{\mathcal{T}}$,

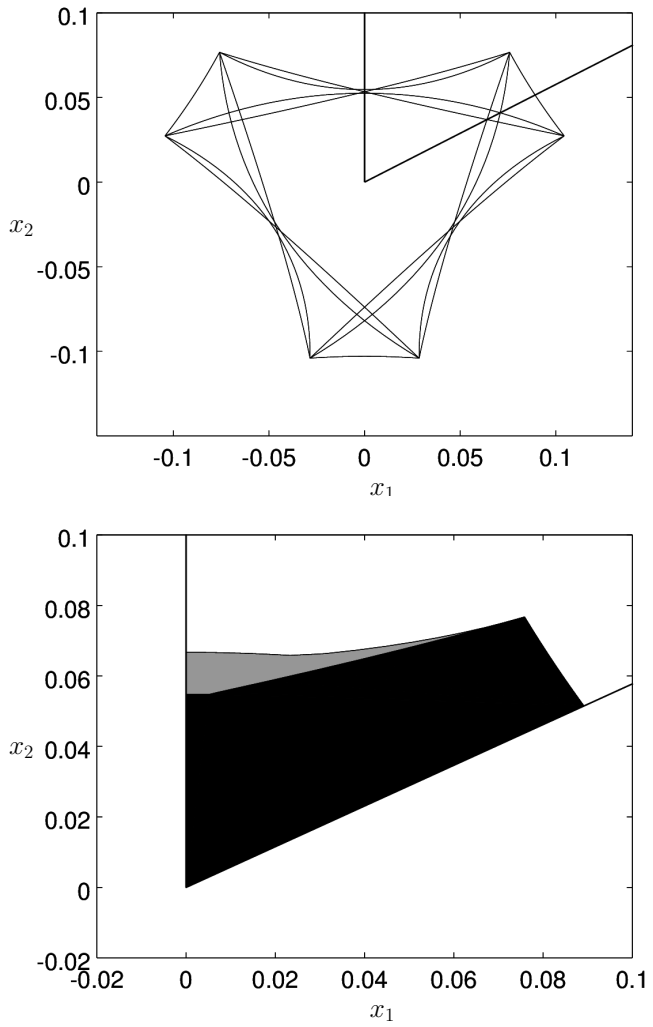

Fig. 2. (Top) Candidate arcs for $\partial\mathcal{A}$ for a Lindblad system with only jump and de-phasing operators (Bottom) $\mathcal{A}$ for the same system when (dark) only π_ι are used and (light) twenty random projector-tuples extend the control set.

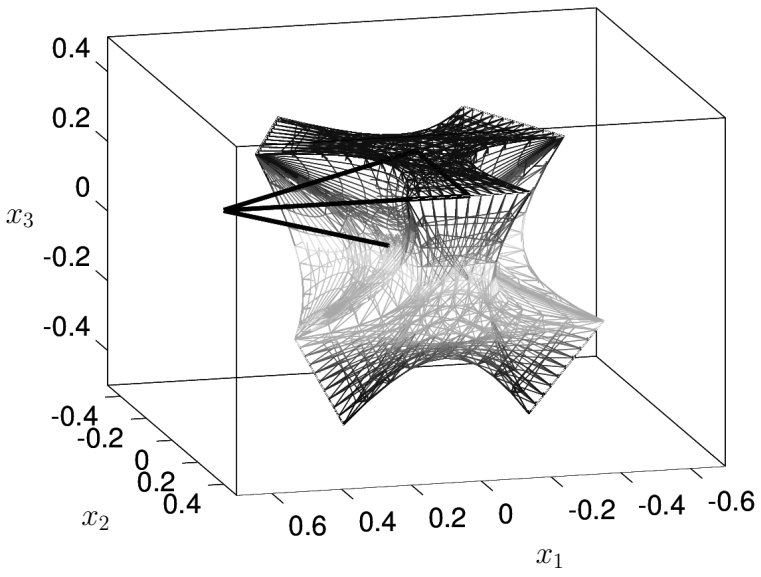

Fig. 3. Candidate surfaces for an $n = 4$ Lindblad system that cannot be purified, but for which mixtures of two pure-states are reachable. Axes coordinates refer to components of x.

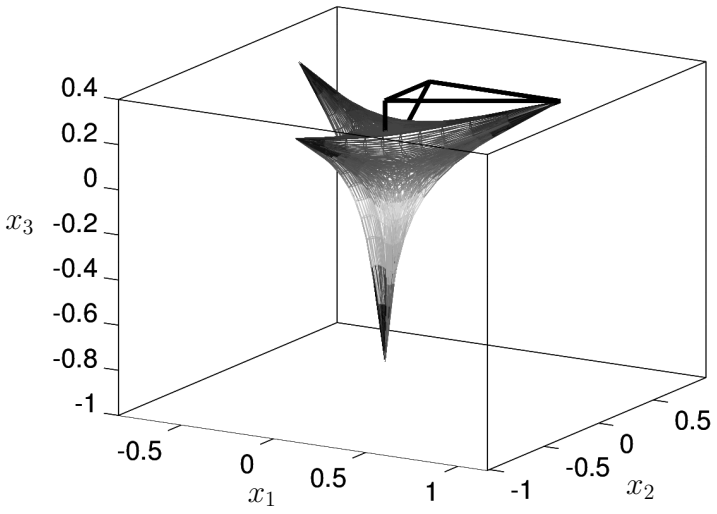

Fig. 4. Candidate surfaces for an $n = 4$ Lindblad system for which the only globally reachable mixtures of three pure states are the pure states.

but does not include the vertices corresponding to the orbit of pure states, so it is not possible to purify this system with the projectors π_ι. However, the closest edge corresponds to states where the two lower eigenvalues are zero. Since one of the surfaces meets this edge, it is possible to obtain states that are a mixture of only two pure states.

Figure 4 has eight Lindblad operators: $\sqrt{4}e_{12}$, $\sqrt{8}e_{13}$, $\sqrt{6}e_{14}$, $\sqrt{13}e_{23}$, $\sqrt{8}e_{32}$, $\sqrt{17}e_{34}$, $\sqrt{4}e_{42}$ and $\sqrt{5}e_{43}$. These have been chosen so that $\partial\mathcal{A}$ includes the orbit of pure states. Interestingly, it is possible to purify this system with π_ι, but it is not possible to obtain arbitrary mixtures of two pure states, or even other mixtures of three pure states.

VI. Conclusions and Future Work

This paper has demonstrated a procedure by which the dynamics of a quantum Lindblad system can be decomposed into its inter- and intra-orbit dynamics. The purpose of this is to investigate how the system moves between orbits depending on how the system moves along the orbit. Since we can construct arbitrary paths along the orbit given sufficiently fast Hamiltonian control, we would like to know which orbits are reachable, and how to construct the necessary Hamiltonians. We have shown that the orbits can be represented by a state vector Λ (technically an equivalence class of such vectors), and the position within the orbit can be represented by a tuple of projectors π. Given this decomposition, we have written down a dynamical equation (3) and a control system (III).We have shown how to reconstruct a Hamiltonian from a desired trajectory along the orbit manifold. Because the orbits are lower-dimensional manifolds at eigenvalue crossings, planning trajectories through crossings require projectors obeying a

technical condition.

If one is only studying local controllability, the technicalities concerning eigenvalue crossings can be safely ignored. The challenge in studying local controllability is the fact the control set is not a linear space, but a compact manifold. We have shown that if one limits the control set to a finite subset, the region of strong local controllability can be calculated analytically. We have shown several examples for $n = 3$ and $n = 4$. While a dramatically smaller control set may appear to be an unnecessary limitation, we have shown for the case where all Lindblad operators are jump and de-phasing operators in a certain basis, almost the entire SLC set can be recovered from a set of $n!$ carefully chosen controls.

The obvious limitation of this approach is that the control set is highly non-linear and thus it is difficult to attain analytic results. Its compactness however is an attractive feature, and so numerical work may pay dividends. A further drawback to using the analytic result for finite control sets is that the number of hypersurfaces that are candidates for $\partial\mathcal{A}$ grow extremely quickly: there are $n!$ possible $\sigma.\pi_\iota$ and thus the number of hypersurfaces is $\binom{n!}{n-1} \sim n!^n$. It is only practical for low-dimensional systems, and even for $n = 4$, we must construct $\binom{24}{3} = 2024$ surfaces (although symmetry makes many of these redundant). Nevertheless, if the Lindblad structure is simple (*i.e.* only one Lindblad operator, or several jump operators), these complications may be mollified.

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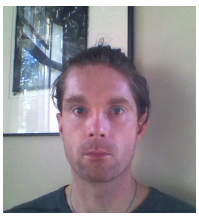

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