The geometric nature of the Flaschka transformation

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January 21, 2017

Abstract

We show that the Flaschka map, originally introduced to analyze the dynamics of the integrable Toda lattice system, is the inverse of a momentum map. We discuss the geometrical setting of the map and apply it to the generalized Toda lattice systems on semisimple Lie algebras, the rigid body system on Toda orbits, and to coadjoint orbits of semidirect products groups. In addition, we develop an infinite-dimensional generalization for the group of area preserving diffeomorphisms of the annulus and apply it to the analysis of the dispersionless Toda lattice PDE and the solvable rigid body PDE.

Keywords: Toda lattice, Flaschka transformation, symplectic reduction.

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1 Introduction

The Toda lattice system and its many generalizations are fundamental integrable systems that have inspired an extraordinarily rich literature on its geometry and dynamics. A particularly useful form of the classical Toda lattice is the finite non-compact lattice which goes back to the work of Moser [1975]. It models n particles moving on the line under an exponential nearest neighbor potential. A key advance in the study of this system was the introduction of the Flaschka map (Flaschka [1974a,b] and Manakov [1975]). This is a map from the original phase space variables to variables which (in the non-compact case) give the dynamics in Lax pair form on the space of tridiagonal symmetric (Jacobi) matrices. This observation was key to the Lie algebraic generalizations of the Toda system in the work of Bogayavlensky [1976] and Kostant [1979]. The flow of the generalized Toda lattice lies on a particular coadjoint orbit of a Borel group. Also of great interest is the full Toda flow (see Deift et al. [1992]) which is the generalization of the classical finite Toda lattice system to flows on full symmetric matrices, in the simplest setting, and periodic versions of the generalized Toda flow, which involve Kac-Moody Lie algebras (Goodman and Wallach [1982]).

Our goal in this paper is to understand the (symplectic) geometry of the Flaschka map in as much generality as possible and to show, in particular, that it is the inverse of a suitable momentum map.

One can rephrase the idea of the Flaschka map by asking when a given coadjoint orbit is symplectomorphic to a magnetic cotangent bundle. For example, when the given Lie group is an exponential solvable group, such as the group of lower triangular matrices, its simply connected coadjoint orbits are symplectomorphic to the canonical cotangent bundle of $\mathbb{R}^n$. (See Arnal, Currey, and Dali [2009], Pedersen [1988], Pukanszky [1990].) A procedure for constructing such coordinates is described in Symes [1980] and is due to Vergne [1972] (see also Kirillov [1974, pp. 264-267]). Another class of examples is certain coadjoint orbits of semidirect products of Lie groups with representation spaces, in which case there is a magnetic term, generically. For example, the generic orbits of the special Euclidean group in 3-space are magnetic cotangent bundles of 2-spheres.

We remark that there is an alternative approach to proving the existence of global Darboux coordinates for simply connected coadjoint orbits of solvable, connected, simply connected (not necessarily exponential) Lie groups, which was first considered in Pukanszky [1992]. In this setting one makes use of so-called admissible ideals. We will not follow this approach here but refer to Bloch, Gay-Balmaz, and Ratiu [2017] for details of this approach.

The essential ingredient of the approach in this paper is a Lie subalgebra which is a real polarization and, in addition, satisfies the so-called *Pukanszky condition* (see Duval, Elhadad, and Tuynman [1992]). This condition, used for the construction of representations from coadjoint orbits, plays a key role in Duval, Elhadad, and Tuynman [1992] and was used in Symes [1980] to construct the Flaschka transformation (Flaschka [1974a,b], Manakov [1975]) for the Toda lattice (Toda [1970]). In Duval, Elhadad, and Tuynman [1992], a relation was found between the Pukanszky condition and the diffeomorphic character of a momentum map obtained through symplectic reduction. After revisiting this result via the cotangent bundle reduction theorem, we show that there is a remarkable equivalence relation on coadjoint orbits, related to the Pukanszky condition. The associated quotient space turns out to be the base space of a cotangent bundle diffeomorphic to the coadjoint orbit. Such a realization is possible for exponential solvable Lie algebras, since, in this case, there is an explicit construction of a real polarization verifying Pukanszky's condition, via Vergne's algorithm.

We then define the abstract Flaschka map by following the construction of Symes [1980] and Kirillov [1974] and prove that it is the inverse of a very natural momentum map, namely, the momentum map of a Lie group action induced on a symplectically reduced space relative to a specific subgroup naturally associated to the real polarization. From this fact and the cotangent bundle reduction theorem, we obtain that the abstract Flaschka map is a symplectic diffeomorphism from the coadjoint orbit to a magnetic cotangent bundle.

We then show how this situation occurs for the generalized Toda lattice flows associated to semisimple Lie algebras, which generalize the Toda lattice flow on Jacobi matrices. We analyze the situation for both the normal split and compact real forms. The latter form is important for investigating the symplectic geometry of the orbit and, in particular, its relationship to the convexity of the momentum map (see Bloch, Flaschka, and Ratiu [1990]). We also extend the analysis to the generalized rigid body system on Toda orbits.

There is a natural generalization of the Toda lattice system to a flow on the Lie algebra of the Fréchet Lie group of area preserving diffeomorphisms on the annulus $S^1 \times [0,1]$. This Lie algebra consists of divergence free vector fields on the annulus tangent to the boundary and the flow in question is the dispersionless Toda lattice flow which arises, physically, from letting the lattice space tend to zero in a suitable fashion (see Brockett and Bloch [1990], Bloch, Flaschka, and Ratiu [1993], Bloch, Flaschka, and Ratiu [1996], Deift and McLaughlin [1992], Bloch, Golse, Paul, and Uribe [2003]). We derive the Flaschka map in this setting and describe the associated solvable rigid body PDE.

Finally, we introduce the Flaschka transformation for certain coadjoint orbits of semidirect product groups by identifying classes of coadjoint orbits that are symplectomorphic to magnetic cotangent bundles. We illustrate this general result with the example of the Euclidean group in 3-space.

The outline of the paper is as follows. In §2, we provide some background material on the cotangent bundle reduction theorem. In §3, we discuss the Pukanszky condition and its relationship to momentum maps. In §4, we derive the momentum map interpretation of the Flaschka transformation. In §5 we present the Flaschka transformation for the real split and compact real forms of the Toda lattice as well as the solvable rigid body on the Toda orbit. In §6, we generalize the results to the infinite dimensional case by considering the group of area preserving diffeomorphisms on the annulus and we derive the dispersionless Toda lattice equations and the solvable rigid body PDE. In §7, we introduce the Flaschka transformation for certain coadjoint orbits of semidirect products of a Lie group with a vector space and isolate a class of codadjoint orbits that are symplectomorphic to magnetic cotangent bundles.

2 Preliminaries on cotangent bundle reduction

Let G be a Lie group and $H \subset G$ a Lie subgroup (i.e., a subgroup, an initial submanifold of G, and a Lie group relative to this manifold structure). Denote by L_g, respectively R_g, the left and right translations by $g \in G$ on G. Consider the canonical symplectic manifold (T^*G, Ω_{can}) and let H, respectively G, act on T^*G by the cotangent lift of left, respectively right multiplication, i.e.,

$$H \times T^*_g G \ni (h, \alpha_g) \mapsto T^*_{hg} L_{h^{-1}} \alpha_g =: h\alpha_g \in T^*_{hg} G, \tag{2.1}$$

$$T^*G \times G \ni (\alpha_g, f) \mapsto T^*_{gf} R_{f^{-1}} \alpha_g =: \alpha_g f \in T^*_{gf} G. \tag{2.2}$$

The associated momentum mappings are given (see, e.g., Marsden and Ratiu [1999]), respectively, by

$$\mathbf{J}_L : T^*G \to \mathfrak{h}^*, \ \mathbf{J}_L(\alpha_g) = i^*_{\mathfrak{h}} \left(T^*_e R_g \alpha_g\right) = i^*_{\mathfrak{h}} \left(\alpha_g g^{-1}\right) = \left(\alpha_g g^{-1}\right)|_{\mathfrak{h}}, \tag{2.3}$$

$$\mathbf{J}_R : T^*G \to \mathfrak{g}^*, \ \mathbf{J}_R(\alpha_g) = T^*_e L_g \alpha_g = g^{-1}\alpha_g, \tag{2.4}$$

where $i_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{g}$ is the inclusion and $i^*_{\mathfrak{h}} : \mathfrak{g}^* \to \mathfrak{h}^*$ its dual map.

It is useful to have the expressions of the actions and the corresponding momentum maps in the left trivialization $T^*G \ni \alpha_g \mapsto (g, g^{-1}\alpha_g = T^*_e L_g \alpha_g) \in G \times \mathfrak{g}^*$ of T^*G. Formulas (2.1)-(2.4) become

$$h \cdot (g, \mu) = (hg, \mu), \tag{2.5}$$

$$(g, \mu) \cdot g' = \left(gg', \operatorname{Ad}^*_{g'} \mu\right), \tag{2.6}$$

$$\mathbf{J}_L(g, \mu) = \left(\operatorname{Ad}^*_{g^{-1}} \mu\right)|_{\mathfrak{h}}, \tag{2.7}$$

$$\mathbf{J}_R(g, \mu) = \mu, \tag{2.8}$$

where $g, g' \in G$, $h \in H$, $\mu \in \mathfrak{g}^*$, and we denote by the same letter the momentum maps $\mathbf{J}_L : G \times \mathfrak{g}^* \to \mathfrak{h}^*$ and $\mathbf{J}_R : G \times \mathfrak{g}^* \to \mathfrak{g}^*$.

2.1 Symplectic cotangent bundle reduction

Fix $\nu_0 \in \mathfrak{h}^*$ and suppose that ν_0 is H-invariant under the H-coadjoint action, that is, $H_{\nu_0} = H$. Symplectic reduction yields the symplectic manifold

$$\left(\mathbf{J}_L^{-1}(\nu_0)/H, \omega_{\nu_0}\right).$$

We now recall how this reduced space is isomorphic to a cotangent bundle endowed with the canonical symplectic form modified by a magnetic term (see Abraham and Marsden [1978, Theorem 4.3.3], Marsden et al [2007, §2.2]).

First, consider the map

$$\overline{\varphi}_0 : \mathbf{J}_L^{-1}(0) \to T^*(G/H), \;\; \langle \overline{\varphi}_0(\alpha_g), T_g\pi_{G,H}(v_g)\rangle = \langle \alpha_g, v_g\rangle \,,$$

where $\pi_{G,H} : G \to G/H$, $\pi(g) = [g]_H = Hg$ is the quotient map. This map is well-defined, H-invariant, and induces a symplectic diffeomorphism

$$\varphi_0 : \left(\mathbf{J}_L^{-1}(0)/H, \omega_0\right) \to (T^*(G/H), \omega_{can}) \,,$$

where ω_0 is the reduced symplectic form on $\mathbf{J}_L^{-1}(0)/H$ and ω_{can} is the canonical symplectic form on the cotangent bundle $T^*(G/H)$ (Satzer's theorem, Satzer [1977]).

Second, fix $\nu_0 \in \mathfrak{h}^*$. In order to give a realization of the reduced symplectic space $\mathbf{J}_L^{-1}(\nu_0)/H$, we have to introduce one more geometric object. Let $\alpha_{\nu_0} \in \Omega^1(G)$ be such that

$$\alpha_{\nu_0} \text{ is left } H_{\nu_0} = H\text{-invariant} \qquad \text{and} \qquad \alpha_{\nu_0}(g) \in \mathbf{J}_L^{-1}(\nu_0), \;\; \text{for all } g \in G; \tag{2.9}$$

see Abraham and Marsden [1978, Thm 4.3.3], Marsden et al [2007, Thm 2.2.1].

Such a one-form can be constructed with the help of a principal connection one-form $\mathcal{A} \in \Omega^1(G, \mathfrak{h})$ on the left H-principal bundle $\pi_{G,H} : G \to G/H$ as follows (Kummer [1981]):

$$\alpha_{\nu_0}(g) := \mathcal{A}(g)^*\nu_0. \tag{2.10}$$

The equivariance property $\mathcal{A}(hg) \circ T_gL_h = \mathrm{Ad}_h \circ \mathcal{A}(g)$, for all $h \in H$ and $g \in G$, of the left connection one-form $\mathcal{A}$ and $H = H_{\nu_0}$ imply

$$\alpha_{\nu_0}(hg) = h\alpha_{\nu_0}(g), \;\; \forall\, h \in H, \tag{2.11}$$

which immediately implies that $\alpha_{\nu_0} \in \Omega^1(G)$ is invariant under the lift to T^*G of left translation on G by elements of H. Since $\mathcal{A}(g)\,(T_eR_g\eta) = \eta$, for all $\eta \in \mathfrak{h}$, by the defining property of the connection one-form $\mathcal{A}$, it follows that $\mathbf{J}_L(\alpha_{\nu_0}(g)) = \nu_0$ for all $g \in G$. Thus α_{ν_0} so constructed via a principal connection one-form $\mathcal{A} \in \Omega^1(G, \mathfrak{h})$ satisfies properties (2.9).

As we shall see later on, for the Toda system example, we will need $\alpha_{\nu_0} \in \Omega^1(G)$ satisfying (2.9) that is not associated to a connection.

From now on, in this subsection and the next, we shall assume that a one-form $\alpha_{\nu_0} \in \Omega^1(G)$ satisfying (2.9) is given.

The map

$$\mathrm{Shift}_{\nu_0} : (T^*G, \Omega_{can}) \to (T^*G, \Omega_{can} - \pi_G^*\mathbf{d}\alpha_{\nu_0}) \,, \quad \mathrm{Shift}_{\nu_0}(\alpha_g) := \alpha_g - \alpha_{\nu_0}(g),$$

where $\pi_G : T^*G \to G$ is the cotangent bundle projection, is a symplectic diffeomorphism. This map restricts to a H-equivariant diffeomorphism $\mathrm{shift}_{\nu_0} : \mathbf{J}_L^{-1}(\nu_0) \to \mathbf{J}_L^{-1}(0)$, which induces a diffeomorphism $\widetilde{\mathrm{shift}}_{\nu_0} : \mathbf{J}_L^{-1}(\nu_0)/H \to \mathbf{J}_L^{-1}(0)/H$. We conclude that the map

$$\varphi_{\nu_0} := \varphi_0 \circ \widetilde{\mathrm{shift}}_{\nu_0} : \left(\mathbf{J}_L^{-1}(\nu_0)/H, \omega_{\nu_0}\right) \to (T^*(G/H), \omega_{can} - B_{\nu_0}) \tag{2.12}$$

is a symplectic diffeomorphism, where ω_{ν_0} is the reduced symplectic form, ω_{can} is the canonical symplectic form on $T^*(G/H)$, and $B_{\nu_0} \in \Omega^2\left(T^*(G/H)\right)$ is defined by

$$B_{\nu_0} := \pi_{G/H}^*\beta_{\nu_0},$$

where $\pi_{G/H} : T^*(G/H) \to G/H$ is the cotangent bundle projection, and $\beta_{\nu_0} \in \Omega^2(G/H)$ is the unique 2-form such that $\pi_{G,H}^*\beta_{\nu_0} = \mathbf{d}\alpha_{\nu_0}$.

2.2 Description of the reduced G-action and momentum map

Since the actions of H and G commute, G acts symplectically on the right on the reduced space $\left(\mathbf{J}_L^{-1}(\nu_0)/H, \omega_{\nu_0}\right)$ and admits the equivariant momentum map

$$\mathbf{J}_R^{\nu_0} : \mathbf{J}_L^{-1}(\nu_0)/H \to \mathfrak{g}^*, \quad \mathbf{J}_R^{\nu_0}\left([\alpha_g]_H\right) = g^{-1}\alpha_g, \tag{2.13}$$

where $\alpha_g \in T_g^*G$ is an arbitrary element in the equivalence class $[\alpha_g]_H \in \mathbf{J}^{-1}(\nu_0)/H$.

Theorem 2.1. *The right action of G on the cotangent bundle $T^*(G/H)$ induced by the natural right action of G on $\mathbf{J}_L^{-1}(\nu_0)/H$ is given by*

$$\beta_{[g]_H}\cdot f=\Phi_f^{T^*}\left(\beta_{[g]_H}\right)+C\left([g]_H,f\right), \tag{2.14}$$

where Φ^{T^*} denotes the cotangent lift of the right action $\Phi: G/H\times G\to G/H$ given by $\Phi_f\left([g]_H\right):=[gf]_H$ and $C: G/H\times G\to T^*(G/H)$ is defined by

$$C\left([g]_H,f\right):=\overline{\varphi}_0\left(\alpha_{\nu_0}(g)f-\alpha_{\nu_0}(gf)\right)\in T^*_{[gf]_H}(G/H).$$

The momentum map of the right G-action (2.14) is G-equivariant and has the expression

$$\mathbf{J}_R^{\nu_0}:(T^*(G/H),\omega_{can}-B_{\nu_0})\to\mathfrak{g}^*,\quad \mathbf{J}_R^{\nu_0}\left(\beta_{[g]}\right)=\mathbf{J}_{can}(\beta_{[g]})+\overline{\alpha}_{\nu_0}\left([g]_H\right), \tag{2.15}$$

where $\overline{\alpha}_{\nu_0}: G/H\to\mathfrak{g}^*$ is defined by $\overline{\alpha}_{\nu_0}\left([g]_H\right):=g^{-1}\alpha_{\nu_0}(g)$ for $g\in[g]_H$. (We denote by the same symbol $\mathbf{J}_R^{\nu_0}$ the momentum maps (2.13) and (2.15) since they are related by the symplectic diffeomorphism (2.12); see also the diagram below.)

Proof. Given $\alpha_g\in T^*G$, define $\beta_g:=\operatorname{Shift}_{\nu_0}(\alpha_g)=\alpha_g-\alpha_{\nu_0}(g)$. It is readily seen that the right action of $f\in G$ on β_g induced by the cotangent lift of right translation on α_g is

$$\beta_g\cdot f=\beta_g f+c(g,f),\quad\text{where}\quad c(g,f):=\alpha_{\nu_0}(g)f-\alpha_{\nu_0}(gf)\in T^*_{gf}G. \tag{2.16}$$

Note that this action consistently preserves $\mathbf{J}_L^{-1}(0)$, since $\mathbf{J}_L(c(g,f))=0$. The H-invariance (2.11) of α_{ν_0} implies the equality $c(hg,f)=hc(g,f)$, for all $h\in H$.

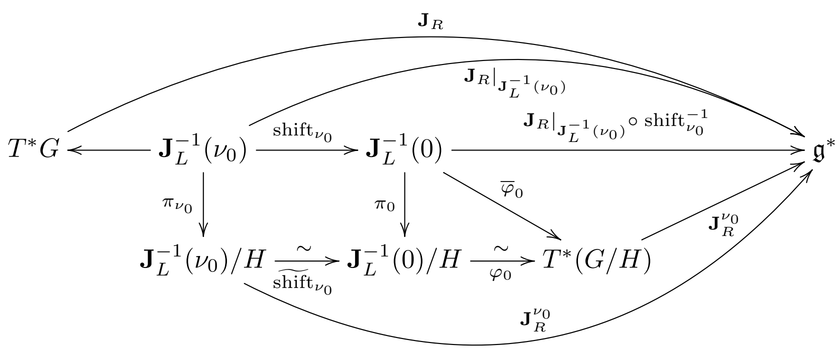
The definitions of the maps φ_0 and $\overline{\varphi}_0$ imply that the G-action induced on $T^*(G/H)$ by (2.16) is given by (2.14).

Using the shift map $\operatorname{Shift}_{\nu_0}$, we conclude that the momentum map associated to the right G-action (2.16) on β_g is given by $\beta_g\mapsto g^{-1}\beta_g+g^{-1}\alpha_{\nu_0}(g)$, and thus the induced momentum map on $T^*(G/H)$ is $\mathbf{J}_R^{\nu_0}\left(\beta_{[g]_H}\right)=g^{-1}\beta_g+g^{-1}\alpha_{\nu_0}(g)$, where $\beta_g\in\mathbf{J}_L^{-1}(0)$ is such that $\overline{\varphi}_0(\beta_g)=\beta_{[g]_H}$. We can rewrite this expression as

$$\mathbf{J}_R^{\nu_0}\left(\beta_{[g]_H}\right)=\mathbf{J}_{can}\left(\beta_{[g]_H}\right)+\overline{\alpha}_{\nu_0}\left([g]_H\right),$$

where $\overline{\alpha}_{\nu_0}: G/H\to\mathfrak{g}^*$ is defined by $\alpha\left([g]_H\right):=g^{-1}\alpha_{\nu_0}(g)$ for $g\in[g]_H$. This map is well-defined thanks to the H-invariance property (2.11). ■

The diagram below illustrates the various maps involved in Theorem 2.1. The group G acts on the right on all the spaces in this diagram. Its associated momentum map is denoted $\mathbf{J}_R$. Note the same notation $\mathbf{J}_R^{\nu_0}$ for the momentum maps $\mathbf{J}_L^{-1}(\nu_0)/H\to\mathfrak{g}^*$ and $T^*(G/H)\to\mathfrak{g}^*$ (because of the symplectic diffeomorphism (2.12) in the last line)



For later use, it is worth recording some formulas appearing in the theorem above, if T^*G is left trivialized as $G\times\mathfrak{g}^*$. Given $\nu_0\in\mathfrak{h}^*$, let $\widetilde{\nu}_0\in\mathfrak{g}^*$ be an arbitrary extension. Let $\mathfrak{h}^\circ:=\{\mu\in\mathfrak{g}^*\mid\langle\mu,\eta\rangle=0,\ \forall\eta\in\mathfrak{h}\}$ be the annihilator of $\mathfrak{h}$ in $\mathfrak{g}^*$. Then

$$\begin{aligned}\mathbf{J}_L^{-1}(\nu_0)&\overset{(2.7)}{=}\left\{(g,\mu)\in G\times\mathfrak{g}^*\mid\left(\operatorname{Ad}^*_{g^{-1}}\mu\right)|_{\mathfrak{h}}=\nu_0\right\}=\left\{(g,\mu)\in G\times\mathfrak{g}^*\mid\operatorname{Ad}^*_{g^{-1}}\mu-\widetilde{\nu}_0\in\mathfrak{h}^\circ\right\}\\&=\left\{\left(g,\operatorname{Ad}^*_g\left(\widetilde{\nu}_0+\mathfrak{h}^\circ\right)\right)\mid g\in G\right\}\end{aligned} \tag{2.17}$$

which does not depend on the choice of the extension $\widetilde{\nu}_0 \in \mathfrak{g}^*$ of ν_0. Thus, the reduced space

$$\mathbf{J}_L^{-1}(\nu_0)/H \quad \text{is diffeomorphic to} \quad \left\{ \left(g, \operatorname{Ad}_g^* \left(\widetilde{\nu}_0 + \mathfrak{h}^\circ\right)\right) \mid g \in G \right\} / H. \tag{2.18}$$

Formulas (2.13) and (2.8) imply $\mathbf{J}_R^{\nu_0}([g,\mu]_H) = \mu$ and hence (2.18) yields

$$\operatorname{range} \mathbf{J}_R^{\nu_0} = \left\{ \operatorname{Ad}_g^* \left(\widetilde{\nu}_0 + \mathfrak{h}^\circ\right) \mid g \in G \right\}. \tag{2.19}$$

3 Pukanszky's condition and reduction

In this section, we recall, in our setting, the notion of polarization, Pukanszky's condition, and their relationship to reduction. The results on polarizations and Pukanszky's condition can be found, in the more general setting of the complexification of a Lie algebra, in Kostant [1970], Bernat et al. [1972, Chapter 4], and Duval, Elhadad, and Tuynman [1992]. The reduction result in the second subsection is due to Duval, Elhadad, and Tuynman [1992]. Due to the importance of these results later on (see e.g., §7.2) and the fact that the proofs simplify since we are working only on real Lie algebras, we provide below short full proofs of all statements.

3.1 Real polarizations and Pukanszky's condition

Definition 3.1. *Let $\mathfrak{g}$ be a Lie algebra and $\mu_0 \in \mathfrak{g}^*$. Given a linear subspace $\mathfrak{a} \subset \mathfrak{g}$, define*

$$\mathfrak{a}^{\perp_{\mu_0}} := \{\xi \in \mathfrak{g} \mid \langle \mu_0, [\xi, \eta] \rangle = 0, \ \forall\, \eta \in \mathfrak{a}\}.$$

Note that $\mathfrak{a} \subseteq \left(\mathfrak{a}^{\perp_{\mu_0}}\right)^{\perp_{\mu_0}}$, $\mathfrak{g} = \left(\mathfrak{g}^{\perp_{\mu_0}}\right)^{\perp_{\mu_0}}$, and that if $\mathfrak{a} \subseteq \mathfrak{b}$ for some other linear subspace $\mathfrak{b}$, then $\mathfrak{b}^{\perp_{\mu_0}} \subseteq \mathfrak{a}^{\perp_{\mu_0}}$. In particular, $\mathfrak{g}^{\perp_{\mu_0}} = \mathfrak{g}_{\mu_0} = \{\xi \in \mathfrak{g} \mid \operatorname{ad}_\xi^* \mu_0 = 0\} \subset \mathfrak{a}^{\perp_{\mu_0}}$ for any subspace $\mathfrak{a} \subset \mathfrak{g}$.

We also have

$$\mathfrak{a}^{\perp_{\mu_0}} = \left(\operatorname{ad}_{\mathfrak{a}}^* \mu_0\right)^\circ. \tag{3.1}$$

Indeed, $\xi \in \left(\operatorname{ad}_{\mathfrak{a}}^* \mu_0\right)^\circ$ if and only if for any $\eta \in \mathfrak{a}$ we have $0 = \left\langle \operatorname{ad}_\eta^* \mu_0, \xi \right\rangle = -\langle \mu_0, [\xi, \eta] \rangle$ which is equivalent to $\xi \in \mathfrak{a}^{\perp_{\mu_0}}$.

By standard linear algebra, we have

$$\dim(\mathfrak{a}) + \dim(\mathfrak{a}^{\perp_{\mu_0}}) = \dim(\mathfrak{g}) + \dim(\mathfrak{a} \cap \mathfrak{g}^{\perp_{\mu_0}}). \tag{3.2}$$

Suppose that $\mathfrak{g}_{\mu_0} \subseteq \mathfrak{a}$. Then (3.2) and $\mathfrak{g}^{\perp_{\mu_0}} = \mathfrak{g}_{\mu_0}$ imply that

$$\dim(\mathfrak{a}) + \dim(\mathfrak{a}^{\perp_{\mu_0}}) = \dim(\mathfrak{g}) + \dim(\mathfrak{g}_{\mu_0}). \tag{3.3}$$

Applying formula (3.3) to $\mathfrak{a}^{\perp_{\mu_0}}$, we get $\dim(\mathfrak{a}) = \dim\left(\left(\mathfrak{a}^{\perp_{\mu_0}}\right)^{\perp_{\mu_0}}\right)$ which proves that

$$\left(\mathfrak{a}^{\perp_{\mu_0}}\right)^{\perp_{\mu_0}} = \mathfrak{a} \quad \text{if} \quad \mathfrak{g}_{\mu_0} \subseteq \mathfrak{a}. \tag{3.4}$$

Finally, note that if G is a Lie group with Lie algebra $\mathfrak{g}$, then $\operatorname{Ad}_g(\mathfrak{a}^{\perp_{\mu_0}}) = \left(\operatorname{Ad}_g \mathfrak{a}\right)^{\perp_{\operatorname{Ad}^*_{g^{-1}} \mu_0}}$, for all $g \in G$.

Definition 3.2. *Let G be a Lie group and $\mu_0 \in \mathfrak{g}^*$. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called* ***real polarization associated to*** *μ_0 if*

(i) $\{\operatorname{Ad}_g \mathfrak{h} \mid g \in G_{\mu_0}\} = \mathfrak{h}$*;*

(ii) $\mathfrak{h}^{\perp_{\mu_0}} = \mathfrak{h}$.

Note that if $\mathfrak{h} \subset \mathfrak{g}$ is a real polarization associated to $\mu_0 \in \mathfrak{g}^*$, then

$$\mathfrak{g}_{\mu_0} \subseteq \mathfrak{h}. \tag{3.5}$$

Indeed, if $\xi \in \mathfrak{g}_{\mu_0}$ then for any $\eta \in \mathfrak{h}$ we have $0 = \left\langle \operatorname{ad}_\xi^* \mu_0, \eta \right\rangle = \langle \mu_0, [\xi, \eta] \rangle$ which shows that $\xi \in \mathfrak{h}^{\perp_{\mu_0}} \overset{\text{(ii)}}{=} \mathfrak{h}$.

Remark 3.3. For a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and $\mu_0 \in \mathfrak{g}^*$, it is easy to see that the following conditions are equivalent:

(i) $\mathfrak{h}^{\perp_{\mu_0}} = \mathfrak{h}$,

(ii) $(\ \langle \mu_0, [\xi, \eta]\rangle = 0,\ \forall \eta \in \mathfrak{h}\ ) \quad \Longleftrightarrow \quad \xi \in \mathfrak{h}$,

(iii) $\langle \mu_0, [\mathfrak{h}, \mathfrak{h}]\rangle = 0 \quad \text{and} \quad 2\dim(\mathfrak{h}) = \dim(\mathfrak{g}) + \dim(\mathfrak{g}_{\mu_0})$.

Note that the second equation in (**iii**) can be written as $\dim(\mathfrak{h}^\circ) = \frac{1}{2}\dim(\mathcal{O}_{\mu_0})$, where $\mathfrak{h}^\circ \subset \mathfrak{g}^*$ is the annihilator of $\mathfrak{h}$.

Definition 3.4. *Let G be a Lie group, $\mu_0 \in \mathfrak{g}^*$, and $\mathfrak{h} \subset \mathfrak{g}$ a real polarization associated to μ_0.*

(i) *Define $H_\circ$ as the connected Lie subgroup of G whose Lie algebra is $\mathfrak{h}$.*

(ii) *Define the subset $H := H_\circ G_{\mu_0} \subset G$.*

Remark 3.5. From (**i**) in Definition 3.2, it follows that H is a subgroup of G. To see this, it suffices to note that $\mathrm{Ad}_{G_{\mu_0}} \mathfrak{h} = \mathfrak{h}$ implies that $gH_\circ g^{-1} = H_\circ$ for all $g \in G_{\mu_0}$. This latter identity also immediately implies that $H_\circ G_{\mu_0} = G_{\mu_0} H_\circ$.

We need the following result of Kostant (see Kostant [1970]; see also Vergne's article in Bernat et al. [1972, Chapter 4]). This result is also cited in Duval, Elhadad, and Tuynman [1992, Lemma 3.6] since it is crucial for their developments. Due to its importance in Subsection 7.2, we include the proof.

Lemma 3.6. *Let G be a connected Lie group, $\mu_0 \in \mathfrak{g}^*$, and $\mathfrak{h} \subset \mathfrak{g}$ a real polarization associated to μ_0. Then,*

(i) *$H_\circ$ and H are a closed subgroups of G having the same Lie algebra $\mathfrak{h}$; $H_\circ$ is the connected component of the identity in H;*

(ii) *$\nu_0 := i_{\mathfrak{h}}^* \mu_0 \in \mathfrak{h}^*$ is Ad_H^*-invariant, i.e., $H_{\nu_0} = H$ (Ad_H^* denotes the H-coadjoint action on $\mathfrak{h}^*$);*

(iii) *$\mathfrak{h}^\circ \subset \mathfrak{g}^*$ and $\mu_0 + \mathfrak{h}^\circ \subset \mathfrak{g}^*$ are Ad_h^*-invariant for any $h \in H$.*

Note that if G_{μ_0} is connected, then H is also connected and hence $H = H_\circ$.

Proof. (i) We reproduce the proof in Kostant [1970]. Note that

$$\left\langle \mathrm{ad}_\xi^* \mu_0, \eta \right\rangle = 0,\ \ \forall \eta \in \mathfrak{h} \quad \text{is equivalent to} \quad \xi \in \mathfrak{h}. \tag{3.6}$$

Indeed, $0 = \left\langle \mathrm{ad}_\xi^* \mu_0, \eta \right\rangle = \langle \mu_0, [\xi, \eta]\rangle$ for all $\eta \in \mathfrak{h}$ means that $\xi \in \mathfrak{h}^{\perp_{\mu_0}} = \mathfrak{h}$ by Definition 3.2(ii).

For $\eta, \zeta \in \mathfrak{h}$, $t \in \mathbb{R}$, we have

$$\begin{aligned}\left\langle \mathrm{Ad}_{\exp(t\zeta)}^* \mu_0 - \mu_0, \eta \right\rangle &= \int_0^t \frac{d}{ds}\left\langle \mathrm{Ad}_{\exp(s\zeta)}^* \mu_0, \eta \right\rangle ds = \int_0^t \frac{d}{ds}\left\langle \mu_0, \sum_{n=0}^\infty \frac{s^n}{n!}\mathrm{ad}_\zeta^n \eta \right\rangle ds \\ &= \sum_{n=1}^\infty \frac{1}{n!} t^n \left\langle \mathrm{ad}_\zeta^* \mu_0, \mathrm{ad}_\zeta^{n-1}\eta \right\rangle \overset{(3.6)}{=} 0\end{aligned}$$

since $\mathrm{ad}_\zeta^{n-1}\eta \in \mathfrak{h}$ for $n \geq 1$.

Because $H_\circ$ is generated by elements $\exp\zeta$ for $\zeta \in \mathfrak{h}$, we have

$$\langle \mathrm{Ad}_h^* \mu_0 - \mu_0, \eta\rangle = 0, \qquad \forall \eta \in \mathfrak{h}, \quad h \in H_\circ. \tag{3.7}$$

It is clear that this identity also holds if $h \in \overline{H_\circ}$. If λ is in the Lie algebra of $\overline{H_\circ}$, replace above h by $\exp(t\lambda)$ and take $\frac{d}{dt}\big|_{t=0}$ of the resulting relation to get $\langle \mathrm{ad}_\lambda^* \mu_0, \eta\rangle = 0$, for all $\eta \in \mathfrak{h}$. By (3.6), it follows that $\lambda \in \mathfrak{h}$. This shows that $H_\circ$ and $\overline{H_\circ}$ have the same Lie algebra $\mathfrak{h}$. Since $H_\circ$ is a Lie subgroup of $\overline{H_\circ}$, this shows that $H_\circ$ is open, hence closed in $\overline{H_\circ}$. But $\overline{H_\circ}$ is connected which implies that $H_\circ = \overline{H_\circ}$.

Next, we show that H is closed in G. Note first that for any $h \in H_\circ$, $k \in G_{\mu_0}$, and $\eta \in \mathfrak{h}$, we have

$$\langle \mathrm{Ad}_{kh}^* \mu_0 - \mu_0, \eta\rangle = \langle \mathrm{Ad}_h^* \mathrm{Ad}_k^* \mu_0 - \mu_0, \eta\rangle = \langle \mathrm{Ad}_h^* \mu_0 - \mu_0, \eta\rangle \overset{(3.7)}{=} 0.$$

Thus, for any $g \in H := H_\circ G_{\mu_0} = G_{\mu_0} H_\circ$ and $\eta \in \mathfrak{h}$, we have $\left\langle \mathrm{Ad}_g^* \mu_0 - \mu_0, \eta \right\rangle = 0$ and hence

$$\left\langle \mathrm{Ad}_{g'}^* \mu_0 - \mu_0, \eta \right\rangle = 0, \qquad \forall \eta \in \mathfrak{h}, \quad g' \in \overline{H}. \tag{3.8}$$

Let H_c be the connected component of the identity in $\overline{H}$, $\mathfrak{h}_c$ its Lie algebra, and $\zeta \in \mathfrak{h}_c$ arbitrary. Put $g' = \exp(t\zeta) \in H_c$ in (3.8) and then take $\frac{d}{dt}\big|_{t=0}$ of the resulting relation. This yields $\left\langle \operatorname{ad}^*_\zeta \mu_0, \eta \right\rangle = 0$ for any $\eta \in \mathfrak{h}$, which implies that $\zeta \in \mathfrak{h}$ by (3.6). Hence $\mathfrak{h}_c \subseteq \mathfrak{h}$. However, since $H_\circ \subseteq H \subseteq \overline{H}$ and $H_\circ$ is a connected closed subgroup of G, it is necessarily a subgroup of the connected component of the identity H_c in $\overline{H}$. In particular, $\mathfrak{h} \subseteq \mathfrak{h}_c$, which shows that $\mathfrak{h}_c = \mathfrak{h}$. Thus $H_\circ \subseteq H_c$ both have the same Lie algebra, which implies that $H_\circ$ is open, hence closed, in H_c. Connectedness of H_c implies then that $H_\circ = H_c$, i.e., $H_\circ$ is the connected component of the identity in $\overline{H}$.

Summarizing, we have the following inclusions of topological groups $H_\circ \subseteq H \subseteq \overline{H} \subseteq G$, where $H_\circ$ is the connected component of the identity in $\overline{H}$. However, $H_\circ$ is also the connected component of the identity in the topological group H. Indeed, if H' is the connected component in the identity of the topological group H, then $\overline{H'}$ is a closed connected subgroup of $\overline{H}$ and thus $\overline{H'} \subseteq H_\circ$. On the other hand, $H_\circ \subseteq H'$, since $H_\circ$ is a connected topological subgroup of H, and hence $H_\circ = \overline{H_\circ} \subseteq \overline{H'}$, which shows that $\overline{H'} = H_\circ \subseteq H'$. Consequently, we have $H_\circ = H'$, i.e., $H_\circ$ is the connected component of the identity in H.

Since $H_\circ$ is the connected component of the identity in both H and $\overline{H}$, it follows that $\overline{H}$ is the union of H and some connected components of $\overline{H}$ that are disjoint from H. Assume that $C \neq \varnothing$ is a connected component of $\overline{H}$ satisfying $C \cap H = \varnothing$. Since C is necessarily open in $\overline{H}$, there is some open subset U of G such that $C = U \cap \overline{H}$. But then $\varnothing = C \cap H = U \cap \overline{H} \cap H = U \cap H$, which is impossible since U contains points in $\overline{H}$. Thus H and $\overline{H}$ have the same connected components and are hence equal.

This proves that H is closed in G and that $H_\circ$ is its connected component of the identity.

(ii) We prove that $(\operatorname{Ad}_H)^*_{(h_\circ g)^{-1}} \nu_0 = \nu_0$ for all $h_\circ \in H_\circ$ and $g \in G_{\mu_0}$. For all $\xi \in \mathfrak{h}$, we have

$$\begin{aligned}\left\langle (\operatorname{Ad}_H)^*_{(h_\circ g)^{-1}} \nu_0, \xi \right\rangle &= \left\langle i^*_{\mathfrak{h}} \mu_0, (\operatorname{Ad}_H)_{g^{-1}h_\circ^{-1}} \xi \right\rangle = \left\langle \operatorname{Ad}^*_{g^{-1}} \mu_0, \operatorname{Ad}_{h_\circ^{-1}} \xi \right\rangle = \left\langle \mu_0, \operatorname{Ad}_{h_\circ^{-1}} \xi \right\rangle \\ &= \left\langle \nu_0, \operatorname{Ad}_{h_\circ^{-1}} \xi \right\rangle\end{aligned}$$

since $g \in G_{\mu_0}$ and $\operatorname{Ad}_{h_\circ^{-1}} \xi \in \mathfrak{h}$. In order to show that the right hand side equals $\langle \nu_0, \xi \rangle$, which is the statement in (ii), it suffices to prove that

$$\frac{d}{dt} \langle \nu_0, \operatorname{Ad}_{\exp t\eta} \xi \rangle = 0, \quad \text{for all} \quad t \in \mathbb{R}, \quad \xi, \eta \in \mathfrak{h}$$

because $H_\circ$ is generated by a neighborhood of the identity. We have

$$\frac{d}{dt} \langle \nu_0, \operatorname{Ad}_{\exp t\eta} \xi \rangle = \left\langle \nu_0, \frac{d}{dt} \operatorname{Ad}_{\exp t\eta} \xi \right\rangle = \langle \nu_0, \operatorname{Ad}_{\exp t\eta} [\eta, \xi] \rangle = \langle \nu_0, [\eta, \operatorname{Ad}_{\exp t\eta} \xi] \rangle = 0$$

because $\langle \mu_0, [\mathfrak{h}, \mathfrak{h}] \rangle = 0$ by the polarization condition in Remark 3.3(iii).

(iii) It is easy to verify that $\operatorname{Ad}^*_h \mathfrak{h}^\circ \subset \mathfrak{h}^\circ$ for all $h \in H$.

To prove the second assertion, we show below that $\operatorname{Ad}^*_h(\mu_0 + \nu) - \mu_0 \in \mathfrak{h}^\circ$ for any $h \in H$ and $\nu \in \mathfrak{h}^\circ$. Indeed, if $\eta \in \mathfrak{h}$, we have

$$\langle \operatorname{Ad}^*_h(\mu_0 + \nu) - \mu_0, \eta \rangle = \langle \mu_0 + \nu, \operatorname{Ad}_h \eta \rangle - \langle \mu_0, \eta \rangle = \langle \mu_0, \operatorname{Ad}_h \eta \rangle - \langle \mu_0, \eta \rangle$$

because $\operatorname{Ad}_h \eta \in \mathfrak{h}$ and $\nu \in \mathfrak{h}^\circ$. Since all elements of H are generated by products of the form $gh_\circ$, where $g \in G_{\mu_0}$, $h_\circ \in H_\circ$, it suffices to assume that $h = gh_\circ$. But then the right hand side of the previous identity equals $\langle \mu_0, \operatorname{Ad}_{h_\circ} \eta \rangle - \langle \mu_0, \eta \rangle$. However, $H_\circ$ is generated by elements of the form $\exp \zeta$ for $\zeta \in \mathfrak{h}$. We have

$$\begin{aligned}\frac{d}{dt} \langle \mu_0, \operatorname{Ad}_{\exp(t\zeta)} \eta \rangle &= \left\langle \mu_0, \frac{d}{dt} \operatorname{Ad}_{\exp(t\zeta)} \eta \right\rangle = \left\langle \mu_0, \operatorname{Ad}_{\exp(t\zeta)} [\zeta, \eta] \right\rangle \\ &= \left\langle \mu_0, \left[\zeta, \operatorname{Ad}_{\exp(t\zeta)} \eta\right] \right\rangle = 0\end{aligned}$$

by Remark 3.3(iii), since $\zeta, \operatorname{Ad}_{\exp(t\zeta)} \eta \in \mathfrak{h}$. This shows that $\left\langle \mu_0, \operatorname{Ad}_{\exp(t\zeta)} \eta \right\rangle = \langle \mu_0, \eta \rangle$, for any $\eta, \zeta \in \mathfrak{h}$, which finishes the proof. ■

Lemma 3.7 (Pukanszky's condition)**.** (Duval, Elhadad, and Tuynman [1992]) *Let G be a Lie group and $\mu_0 \in \mathfrak{g}^*$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a real polarization associated to μ_0 and define H as above. Then the following are equivalent:*

(i) $\mu_0 + \mathfrak{h}^\circ \subseteq \mathcal{O}_{\mu_0}$*;*

(ii) $\{\mathrm{Ad}_h^* \mu_0 \mid h \in H\} = \mu_0 + \mathfrak{h}^\circ$, *for all* $h \in H$*;*

(iii) $\{\mathrm{Ad}_h^* \mu_0 \mid h \in H\}$ *is closed in* $\mathfrak{g}^*$*.*

If any of these equivalent conditions hold, we say that the real polarization $\mathfrak{h}$ *associated to* $\mu_0 \in \mathfrak{g}^*$ *satisfies* ***Pukanszky's condition****.*

Proof. The implications (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are obvious.

We next prove that (iii) $\Rightarrow$ (ii). By Lemma 3.6(iii) we know that $\mathrm{Ad}_h^* \mathfrak{h}^\circ \subset \mathfrak{h}^\circ$ and $\mathrm{Ad}_h^*(\mu_0 + \mathfrak{h}^\circ) \subseteq \mu_0 + \mathfrak{h}^\circ$ for any $h \in H$, which implies $\mathrm{Ad}_h^* \mu_0 \subseteq \mu_0 + \mathfrak{h}^\circ$ for any $h \in H$. Therefore $\{\mathrm{Ad}_h^* \mu_0 \mid h \in H\} \subseteq \mu_0 + \mathfrak{h}^\circ$ which, by hypothesis, is closed. We show below that this is also an open set which proves the desired implication.

The closed Lie subgroup $H \subset G$ acts on $\mathfrak{g}^*$ by the restriction of the G-coadjoint action. Therefore, $\{\mathrm{Ad}_h^* \mu_0 \mid h \in H\} =: H \cdot \mu_0$ is diffeomorphic to H/H_{μ_0}, where $H_{\mu_0} = H \cap G_{\mu_0}$. Since the Lie algebra of H_{μ_0} equals $\left\{\xi \in \mathfrak{h} \mid \mathrm{ad}_\xi^* \mu_0 = 0\right\} = \mathfrak{h} \cap \mathfrak{g}_{\mu_0} = \mathfrak{g}_{\mu_0}$, we have $\dim(H \cdot \mu_0) = \dim \mathfrak{h} - \dim \mathfrak{g}_{\mu_0} = \dim \mathfrak{g} - \dim \mathfrak{h} = \dim(\mu_0 + \mathfrak{h}^\circ)$ by Remark 3.3(iii). This shows that $H \cdot \mu_0$ is open in $\mu_0 + \mathfrak{h}^\circ$, as required.

Finally, we show that (i) $\Rightarrow$ (iii). As in the proof of (iii) $\Rightarrow$ (ii), from Lemma 3.6(iii) we conclude that $\mathrm{Ad}_h^* \mu_0 \subset \mu_0 + \mathfrak{h}^\circ$ for all $h \in H$, i.e., $H \cdot \mu_0 \subset \mu_0 + \mathfrak{h}^\circ$, which in turn implies that $\mathrm{Ad}_h^* \mu \subset \mu_0 + \mathfrak{h}^\circ$ for any $\mu \in H \cdot \mu_0$ and any $h \in H$. Now let $\rho \in \overline{H \cdot \mu_0}$. Since $\mu_0 + \mathfrak{h}^\circ$ is obviously closed, it follows that $\mathrm{Ad}_h^* \rho \in \mu_0 + \mathfrak{h}^\circ \subset \mathcal{O}_{\mu_0}$, for any $h \in H$ (the second inclusion is the working hypothesis). In particular, $\rho \in \mathcal{O}_{\mu_0}$ so the last part of the proof in (iii) $\Rightarrow$ (ii) applies and we get $\dim(H \cdot \rho) = \dim(\mu_0 + \mathfrak{h}^\circ)$. Together with $H \cdot \rho \subset \mu_0 + \mathfrak{h}^\circ$, this shows that $H \cdot \rho$ is open in $\mu_0 + \mathfrak{h}^\circ$ and hence $H \cdot \rho \cap H \cdot \mu_0 \neq \varnothing$, which in turn implies the existence of some $h \in H$ such that $\rho = \mathrm{Ad}_h^* \mu_0 \subset H \cdot \mu_0$. Thus, $H \cdot \mu_0$ is closed, so (iii) holds. ■

Remark 3.8. In Symes [1980], Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ satisfying $\mathfrak{h}^{\perp_{\mu_0}} = \mathfrak{h}$ and $\mu_0 + \mathfrak{h}^\circ \subset \mathcal{O}_{\mu_0}$ are called admissible subordinate subalgebras. For exponentially solvable Lie groups, in Symes [1980] there is a concrete construction of a real polarization satisfying Pukanszky's condition; we shall verify this in §5.3.

3.2 Pukanszky's conditions and momentum maps

Below is a reformulation of the results of Duval, Elhadad, and Tuynman [1992] in terms of the setting recalled in §2.1.

Theorem 3.9 (Pukanszky's conditions and momentum maps)**.** (Duval, Elhadad, and Tuynman [1992]) *Let* G *be a Lie group,* $\mu_0 \in \mathfrak{g}^*$*, and denote by* $\mathcal{O}_{\mu_0}$ *the coadjoint orbit of* μ_0*. Let* $\mathfrak{h} \subset \mathfrak{g}$ *be a real polarization associated to* μ_0 *and define* H *as above. Let* $\nu_0 := i_{\mathfrak{h}}^* \mu_0$*. Then the following are equivalent:*

(i) $\mathfrak{h}$ *verifies Pukanszky's conditions;*

(ii) *The reduced momentum map* $\mathbf{J}_R^{\nu_0} : (T^*(G/H), \omega_{can} - B_{\nu_0}) \to \mathfrak{g}^*$ *is onto* $\mathcal{O}_{\mu_0}$*;*

(iii) *The symplectic action of* G *on* $T^*(G/H)$ *is transitive;*

(iv) $\mathbf{J}_R^{\nu_0} : (T^*(G/H), \omega_{can} - B_{\nu_0}) \to \left(\mathcal{O}_{\mu_0}, \omega_{\mathcal{O}_{\mu_0}}\right)$ *is a symplectic diffeomorphism, where* $\omega_{\mathcal{O}_{\mu_0}}$ *is the minus orbit symplectic form, i.e.,*

$$\omega_{\mathcal{O}_{\mu_0}}(\mu)(\mathrm{ad}_\xi^* \mu, \mathrm{ad}_\eta^* \mu) = -\langle \mu, [\xi, \eta] \rangle, \quad \mu \in \mathcal{O}_{\mu_0}, \quad \xi, \eta \in \mathfrak{g}.$$

Proof. The proof simplifies considerably if one works with the abstract reduced symplectic manifold $\mathbf{J}_L^{-1}(\nu_0)/H$ realized as $\left\{\left(g, \mathrm{Ad}_g^*(\widetilde{\nu}_0 + \mathfrak{h}^\circ)\right) \mid g \in G\right\}/H$ ($\mu_0 \in \mathfrak{g}^*$ is an extension of $\nu_0 \in \mathfrak{h}^*$ and use (2.18)) instead of the magnetic cotangent bundle in the statement to which it is symplectomorphic.

(iii) $\Rightarrow$ **(ii)** is obvious since $\mathbf{J}_R^{\nu_0}([e, \mu_0]_H) = \mu_0$.

(iv) $\Rightarrow$ **(iii)** is obvious.

(i) $\Leftrightarrow$ **(ii)**. By (2.19) (with $\widetilde{\nu}_0 = \mu_0$) it follows that $\mu_0 \in \mathrm{range}\, \mathbf{J}_R^{\nu_0}$ and hence, again by (2.19), we conclude that $\mathcal{O}_{\mu_0} \subseteq \mathrm{range}\, \mathbf{J}_R^{\nu_0}$.

If **(i)** holds, i.e., $\mathfrak{h}$ satisfies Pukanszky's condition, we have $\mu_0 + \mathfrak{h}^\circ \subseteq \mathcal{O}_{\mu_0}$ (see Lemma 3.7(i)) and hence for any $g \in G$ we have $\mathrm{Ad}_g^*(\mu_0 + \mathfrak{h}^\circ) \subseteq \mathrm{Ad}_g^* \mathcal{O}_{\mu_0} = \mathcal{O}_{\mu_0}$, which shows that $\mathrm{range}\, \mathbf{J}_R^{\nu_0} \subseteq \mathcal{O}_{\mu_0}$.

Conversely, assume **(ii)** holds, which implies, by (2.19), that $\mu_0 + \mathfrak{h}^\circ \subseteq \mathcal{O}_{\mu_0}$, hence **(i)**.

(i) $\Rightarrow$ **(iv)**. Assume that $\mathfrak{h}$ is a real polarization associated to μ_0 satisfying Pukanszky's condition. By the equivalence **(i)** $\Leftrightarrow$ **(ii)**, it follows that range $\mathbf{J}_R^{\nu_0} = \mathcal{O}_{\mu_0}$. We will prove first that $\mathbf{J}_R^{\nu_0}$ is injective. Let $[g_1, \mathrm{Ad}^*_{g_1}(\mu_0 + \nu_1)]_H, [g_2, \mathrm{Ad}^*_{g_1}(\mu_0+\nu_2)]_H \in \left\{\left(g, \mathrm{Ad}^*_g(\mu_0+\mathfrak{h}^\circ)\right) \mid g \in G\right\}/H$, $\nu_1, \nu_2 \in \mathfrak{h}^\circ$, be such that

$$\mathrm{Ad}^*_{g_1}(\mu_0+\nu_1) = \mathbf{J}_R^{\nu_0}\left(\left[g_1, \mathrm{Ad}^*_{g_1}(\mu_0+\nu_1)\right]_H\right) = \mathbf{J}_R^{\nu_0}\left(\left[g_2, \mathrm{Ad}^*_{g_2}(\mu_0+\nu_2)\right]_H\right) = \mathrm{Ad}^*_{g_2}(\mu_0+\nu_2).$$

By Lemma 3.7(ii), there exist $h_1, h_2 \in H$ such that $\mu_0 + \nu_1 = \mathrm{Ad}^*_{h_1}\mu_0$ and $\mu_0+\nu_2 = \mathrm{Ad}^*_{h_2}\mu_0$, so that from the relation above we get $\mu_0 = \mathrm{Ad}^*_{h_2 g_2 g_1^{-1} h_1^{-1}}\mu_0$, i.e., $h_2 g_2 g_1^{-1} h_1^{-1} \in G_{\mu_0} \subset H$ (see Definition 3.4(ii)). Therefore $g_2 g_1^{-1} \in H$ and the relation above becomes $\mu_0 + \nu_1 = \mathrm{Ad}^*_{g_2 g_1^{-1}}(\mu_0+\nu_2)$. We conclude

$$\begin{aligned}\left[g_2, \mathrm{Ad}^*_{g_2}(\mu_0+\nu_2)\right]_H &= \left[g_2, \mathrm{Ad}^*_{g_1}\mathrm{Ad}^*_{g_2 g_1^{-1}}(\mu_0+\nu_2)\right]_H \\ &= \left[(g_2 g_1^{-1})g_1, \mathrm{Ad}^*_{g_1}(\mu_0+\nu_1)\right]_H = \left[g_1, \mathrm{Ad}^*_{g_1}(\mu_0+\nu_1)\right]_H\end{aligned}$$

by (2.5) since $g_2 g_1^{-1} \in H$, which concludes the proof that $\mathbf{J}_R^{\nu_0}$ is injective.

Thus, $\mathbf{J}_R^{\nu_0}$ is a smooth bijective map. Since it is an equivariant momentum map, it is Poisson and since both its domain of definition $(T^*(G/H), \omega_{can} - B_{\nu_0})$ and $\left(\mathcal{O}_{\mu_0}, \omega_{\mathcal{O}_{\mu_0}}\right)$ are symplectic, it follows that $\mathbf{J}_R^{\nu_0}$ is a symplectic map. However, any symplectic map is an immersion and since $\dim\mathcal{O}_{\mu_0} = \dim\mathfrak{g} - \dim\mathfrak{g}_{\mu_0} = 2\dim\mathfrak{g} - 2\dim\mathfrak{h} = \dim T^*(G/H)$ by Remark 3.3(iii), we conclude that $\mathbf{J}_R^{\nu_0}$ is a local diffeomorphism, hence a symplectic diffeomorphism. ■

4 The Flaschka transformation is a momentum map

As we shall see in this section, there is a remarkable equivalence relation on coadjoint orbits verifying Pukanszky's condition. The associated quotient space turns out to be the base space G/H of the cotangent bundle diffeomorphic to the coadjoint orbit, see Theorem 3.9. Such a realization is possible for exponential solvable Lie algebras, since in this case, there is an explicit construction of real polarization verifying Pukanszky's condition, via Vergne's algorithm, as explained in Symes [1980]. We then define the abstract Flaschka map by following the construction of Symes [1980] and Kirillov [1974], and show that its definition requires the choice of a smooth section s_{μ_0} of the submersion associated to the equivalence relation on the coadjoint orbit. Finally, we prove the main result of this section, namely that the abstract Flaschka map is the inverse of the reduced momentum map $\mathbf{J}_R^{\nu_0}$. The choice of the section s_{μ_0} is equivalent to the choice of the one-form α_{ν_0} verifying (2.9) and is needed to identify the symplectic reduced space with a magnetic cotangent bundle. From this and Theorem 3.9(iv), we obtain the result that the abstract Flaschka map is a symplectic diffeomorphism from the coadjoint orbit to a magnetic cotangent bundle.

4.1 The Pukanszky homogeneous space G/H

In this paragraph we first review, following Symes [1980], an equivalence relation on coadjoint orbit. This relation is associated to a real polarization satisfying Pukanszky condition. Then we show that the quotient space is diffeomorphic to a homogeneous space G/H.

Let G be a Lie group, $\mu_0 \in \mathfrak{g}^*$, and $\mathfrak{h}(\mu_0) := \mathfrak{h} \subset \mathfrak{g}$ a real polarization associated to μ_0 and verifying Pukanszky's condition. Recall that $H_{\nu_0} = H$ by Lemma 3.6(ii). Let $\mathcal{O}_{\mu_0} = \{\mathrm{Ad}^*_g\mu_0 \mid g \in G\}$ be the coadjoint orbit containing μ_0.

For an arbitrary $\mu = \mathrm{Ad}^*_g\mu_0 \in \mathcal{O}_{\mu_0}$, define the Lie subalgebra

$$\mathfrak{h}(\mu) = \mathfrak{h}\left(\mathrm{Ad}^*_g\mu_0\right) := \mathrm{Ad}_{g^{-1}}\left(\mathfrak{h}(\mu_0)\right). \tag{4.1}$$

It easy to check that $\mathfrak{h}(\mu)$ is a real polarization associated to μ verifying Pukanszky's condition $\mu + \mathfrak{h}(\mu)^\circ \subset \mathcal{O}_\mu$ in Lemma 3.7(i). It is worth noting that the statement in Lemma 3.7(ii) reads

$$\mathrm{Ad}^*_{H^\mu}\mu = \mu + \mathfrak{h}(\mu)^\circ, \quad \text{where} \quad H^\mu := g^{-1}Hg,$$

Note that the Lie algebra of H^μ is $\mathfrak{h}(\mu) + \mathfrak{g}_\mu = \mathfrak{h}(\mu)$.

Consider the relation $\sim$ on the coadjoint orbit $\mathcal{O}_{\mu_0}$ defined, for $\nu, \gamma \in \mathcal{O}_{\mu_0}$, by:

$$\nu \sim \gamma \quad \text{if and only if} \quad \nu \in \gamma + \mathfrak{h}(\gamma)^\circ. \tag{4.2}$$

Note that, for $\nu, \gamma \in \mathcal{O}_{\mu_0}$, Lemma 3.7(ii) implies that

$$\nu \sim \gamma \;\Leftrightarrow\; \nu \in \operatorname{Ad}^*_{H^\gamma} \gamma, \;\text{ where } H^\gamma := g^{-1}Hg, \text{ for } g \in G, \text{ satisfying } \gamma = \operatorname{Ad}^*_g \mu_0. \tag{4.3}$$

This is an equivalence relation. The associated quotient space is denoted $N_{\mu_0} := \mathcal{O}_{\mu_0}/\sim$, with quotient map

$$\pi_{\mu_0} : \mathcal{O}_{\mu_0} \to N_{\mu_0}, \quad \mu \mapsto \pi_{\mu_0}(\mu) =: [\mu]_\sim.$$

In particular, as a subset in $\mathcal{O}_{\mu_0}$, we have $[\mu_0]_\sim = \operatorname{Ad}^*_H \mu_0$, by (4.3). Note that, for $g \in G$ and $\mu \in \mathcal{O}_{\mu_0}$, we have

$$\left[\operatorname{Ad}^*_g \mu\right]_\sim = \operatorname{Ad}^*_g[\mu]_\sim := \left\{\operatorname{Ad}^*_g \nu \mid \nu \in [\mu]_\sim\right\}. \tag{4.4}$$

Indeed, using (4.3), and defining $\mu := \operatorname{Ad}^*_g \mu_0$, we can write

$$\left[\operatorname{Ad}^*_g \mu_0\right]_\sim = \operatorname{Ad}^*_{H^\mu} \mu = \operatorname{Ad}^*_{g^{-1}Hg} \mu = \operatorname{Ad}^*_g \operatorname{Ad}^*_H \mu_0 = \operatorname{Ad}^*_g[\mu_0]_\sim$$

and hence, if $\nu = \operatorname{Ad}^*_f \mu_0$, we have

$$\left[\operatorname{Ad}^*_g \nu\right]_\sim = \left[\operatorname{Ad}^*_{fg} \mu_0\right]_\sim = \operatorname{Ad}^*_{fg}[\mu_0]_\sim = \operatorname{Ad}^*_g \operatorname{Ad}^*_f [\mu_0]_\sim = \operatorname{Ad}^*_g \left[\operatorname{Ad}^*_f \mu_0\right]_\sim = \operatorname{Ad}^*_g [\nu]_\sim .$$

Theorem 4.1. *Let G be a connected Lie group, $\mu_0 \in \mathfrak{g}^*$, and $\mathfrak{h} \subset \mathfrak{g}$ a real polarization associated to μ_0 verifying the Pukanszky condition. Then the map given by*

$$\Sigma : G/H \to N_{\mu_0}, \quad Hg = [g]_H \mapsto [\nu]_\sim := \left[\operatorname{Ad}^*_g \mu_0\right]_\sim \tag{4.5}$$

is a well-defined bijection. The quotient space N_{μ_0} carries a unique smooth manifold structure relative to which the quotient map $\pi_{\mu_0} : \mathcal{O}_{\mu_0} \to N_{\mu_0}$ is a smooth submersion. Moreover, relative to this differentiable structure on N_{μ_0}, the bijection Σ is a diffeomorphism.

Proof. Since for any $h \in H$ and $g \in G$, we have

$$\left[\operatorname{Ad}^*_{hg} \mu_0\right]_\sim \overset{(4.4)}{=} \operatorname{Ad}^*_g \operatorname{Ad}^*_h [\mu_0]_\sim \overset{(4.3)}{=} \operatorname{Ad}^*_g \operatorname{Ad}^*_h \operatorname{Ad}^*_H \mu_0 = \operatorname{Ad}^*_g \operatorname{Ad}^*_H \mu_0 \overset{(4.3)}{=} \left[\operatorname{Ad}^*_g \mu_0\right]_\sim,$$

the map Σ is well-defined. It is clearly surjective. To show that it is injective, let $g_1, g_2 \in G$, be such that $\left[\operatorname{Ad}^*_{g_1} \mu_0\right]_\sim = \left[\operatorname{Ad}^*_{g_2} \mu_0\right]_\sim$. By (4.4), this is equivalent to $\operatorname{Ad}^*_{g_1 g_2^{-1}}[\mu_0]_\sim = [\mu_0]_\sim$, that is, $\operatorname{Ad}^*_{g_1 g_2^{-1}} \operatorname{Ad}^*_H \mu_0 = \operatorname{Ad}^*_H \mu_0$, because of (4.3). This proves that there exist $h_1, h_2 \in H$, such that $h_1 g_1 g_2^{-1} h_2^{-1} \in G_{\mu_0} \subset H$, and hence $[g_1]_H = [g_2]_H$.

Let us consider the quotient maps $\pi_{G,H} : G \to G/H$ and $\pi_{G,G_{\mu_0}} : G \to G/G_{\mu_0}$ which are smooth surjective submersions. Since, for all $g_1, g_2 \in G$, we have $\pi_{G,G_{\mu_0}}(g_1) = \pi_{G,G_{\mu_0}}(g_2) \Rightarrow \pi_{G,H}(g_1) = \pi_{G,H}(g_2)$, it follows that the quotient map $\rho_{\mu_0} : G/G_{\mu_0} \ni \pi_{G,G_{\mu_0}}(g) \mapsto \pi_{G,H}(g) \in G/H$ is smooth (see, e.g., Abraham, Marsden and Ratiu [1988, Prop. 3.5.21]). Moreover, since $\rho_{\mu_0} \circ \pi_{G,G_{\mu_0}} = \pi_{G,H}$ is a submersion and $\pi_{G,G_{\mu_0}}$ is onto, it follows that ρ_{μ_0} is a smooth surjective submersion (see, e.g., Abraham, Marsden and Ratiu [1988, Ex 3.5-5(iv)]). This implies that $\pi_{\mu_0} = \Sigma \circ \rho_{\mu_0} : G/G_{\mu_0} \simeq \mathcal{O}_{\mu_0} \to N_{\mu_0}$ is a smooth surjective submersion which shows that the manifold structure of N_{μ_0} is the quotient manifold structure induced by π_{μ_0} (see, e.g., Abraham, Marsden and Ratiu [1988, Prop. 3.5.21]). ■

Remark 4.2. (**1**) From this theorem, and Theorem 2.1, it follows that the action of G on $T^*N_{\mu_0}$ has the form

$$\gamma_{[\nu]_\sim} \cdot f = \Phi^{T^*}_f\left(\gamma_{[\nu]_\sim}\right) + C\left([\nu]_\sim, f\right),$$

where Φ^{T^*} denotes the cotangent lifted action of the right action of G on N_{μ_0} given by

$$\Phi_f\left([\nu]_\sim\right) = \left[\operatorname{Ad}^*_f \nu\right]_\sim .$$

The momentum map defined by this action is

$$\mathbf{J}^{\nu_0}_R : T^*N_{\mu_0} \to \mathfrak{g}^*, \quad \mathbf{J}^{\nu_0}_R\left(\gamma_{[\nu]_\sim}\right) = \mathbf{J}_{can}\left(\gamma_{[\nu]_\sim}\right) + \overline{\alpha}_{\nu_0}\left([\nu]_\sim\right). \tag{4.6}$$

Here, $C\left([\nu]_\sim, f\right) := C\left([g]_H, f\right)$ and $\overline{\alpha}_{\nu_0}\left([\nu]_\sim\right) := \overline{\alpha}_{\nu_0}\left([g]_H\right)$, where $[\nu]_\sim$ and $[g]_H$ are related by the diffeomorphism (4.5).

(**2**) By the results of Duval, Elhadad, and Tuynman [1992] recalled in §3, we know that

$$\mathbf{J}^{\nu_0}_R : \left(T^*N_{\mu_0}, \Omega_{can} - B_{\nu_0}\right) \to \left(\mathcal{O}_{\mu_0}, \omega_{\mathcal{O}_{\mu_0}}\right) \tag{4.7}$$

is a symplectic diffeomorphism; recall that $\omega_{\mathcal{O}_{\mu_0}}$ is the $(-)$-orbit symplectic form.

4.2 Momentum map interpretation of the Flaschka transformation

In this section, we recall the abstract formulation of the Flaschka transformation given by Symes [1980]. To make this map well defined in general, we need to introduce a smooth section $s_{\mu_0} : N_{\mu_0} \to \mathcal{O}_{\mu_0} \subset \mathfrak{g}^*$ of the surjective submersion $\pi_{\mu_0} : \mathcal{O}_{\mu_0} \to N_{\mu_0}$. Then we show that the momentum map $\mathbf{J}_R^{\nu_0} : T^*(G/H) \to \mathcal{O}_{\mu_0} \subset \mathfrak{g}^*$ obtained in §2.1 and §3 is the inverse of the abstract Flaschka map. In the process, we show that there is a a bijective correspondence between such sections s_{μ_0} and one-forms $\alpha_{i_{\mathfrak{h}}^*\mu_0}$ verifying (2.9).

Remark 4.3. Historically, Flaschka (Flaschka [1974a,b]) and Manakov (Manakov [1975]) independently introduced this transformation starting with a cotangent bundle in order to obtain a Lax equation for the Toda system (Toda [1970]). At the time, the link between Lax equations and Hamiltonian systems on Lie-Poisson spaces was not known. We have adopted here the definition given in Symes [1980] which defines the Flaschka transformation as a map from the coadjoint orbit to a cotangent bundle, i.e., it is the inverse of the original map. The motivation for this definition is based on one of the major challenges in the study of Lie-Poisson Hamiltonian systems (integrable or not): determine which coadjoint orbits are (magnetic) cotangent bundles and identify classes of Lie algebras for which the generic coadjoint orbits are (magnetic) cotangent bundles. We shall give examples of both situations in subsequent sections.

The abstract Flaschka map. The ***abstract Flaschka map*** $F : \mathcal{O}_{\mu_0} \to T^*N_{\mu_0}$ is defined by its restrictions $F|_{[\mu]_\sim}$ to the equivalence classes $[\mu]_\sim \subset \mathcal{O}_{\mu_0}$, that is, by the collection of maps

$$F|_{[\mu]_\sim} : [\mu]_\sim \to T^*_{[\mu]_\sim} N_{\mu_0}. \tag{4.8}$$

Given a section $s_{\mu_0} : N_{\mu_0} \to \mathcal{O}_{\mu_0}$, the map $F|_{[\mu]_\sim}$ is, in turn, defined by

$$\left\langle F|_{[\mu]_\sim}(s_{\mu_0}([\mu]_\sim) + \sigma), v_{[\mu]_\sim} \right\rangle := \langle \sigma, \xi \rangle \,, \tag{4.9}$$

where $\xi \in \mathfrak{g}$ is such that

$$v_{[\mu]_\sim} = T_{\bar{\mu}}\pi_{\mu_0}\left(\mathrm{ad}^*_\xi \bar{\mu}\right), \quad \bar{\mu} := s_{\mu_0}([\mu]_\sim). \tag{4.10}$$

Note that the section s_{μ_0} is used to choose a particular element in the equivalence class $[\mu]_\sim$, so that any element $\tau \in [\mu]_\sim$ can be uniquely written as $\tau = s_{\mu_0}([\mu]_\sim) + \sigma$, where $\sigma \in \mathfrak{h}(\bar{\mu})^\circ$.

Let us show that the map F is well defined, that is, it does not depend on the choice of ξ such that (4.10) holds.

Suppose that $v_{[\mu]_\sim} = T_{\bar{\mu}}\pi_{\mu_0}\left(\mathrm{ad}^*_\xi \bar{\mu}\right) = T_{\bar{\mu}}\pi_{\mu_0}\left(\mathrm{ad}^*_{\xi'} \bar{\mu}\right)$. Since $\pi_{\mu_0} \circ \psi = \Sigma \circ \rho_{\mu_0}$, where $\psi : G/G_{\mu_0} \to \mathcal{O}_{\mu_0}$ is the diffeomorphism defined by $\psi(G_{\mu_0}g) := \mathrm{Ad}^*_g \mu_0$, we have

$$0 = T_{\bar{\mu}}\pi_{\mu_0}\left(\mathrm{ad}^*_{\xi-\xi'} \bar{\mu}\right) = T_{Hg}\Sigma\left(T_{G_{\mu_0}g}\rho_{\mu_0}\left(T_{\bar{\mu}}\psi^{-1}(\mathrm{ad}^*_{\xi-\xi'}\bar{\mu})\right)\right),$$

which implies that $T_{G_{\mu_0}g}\rho_{\mu_0}\left(T_{\bar{\mu}}\psi^{-1}(\mathrm{ad}^*_{\xi-\xi'}\bar{\mu})\right) = 0$. Let us compute the kernel of $T_{G_{\mu_0}g}\rho_{\mu_0}$. We have

$$\begin{aligned}
\ker\left(T_{G_{\mu_0}g}\rho_{\mu_0}\right) &= \{T_g\pi_{G,G_{\mu_0}}(v_g) \mid T_{G_{\mu_0}g}\rho_{\mu_0}(T_g\pi_{G,G_{\mu_0}}(v_g)) = 0\} \\
&= \{T_g\pi_{G,G_{\mu_0}}(v_g) \mid T_g\pi_{G,H}(v_g) = 0\} \\
&= \{T_g\pi_{G,G_{\mu_0}}(\eta g) \mid \eta \in \mathfrak{h}(\mu_0)\}.
\end{aligned}$$

Therefore, we have

$$\mathrm{ad}^*_{\xi-\xi'}\bar{\mu} = T_{G_{\mu_0}g}\psi\left(T_g\pi_{G,G_{\mu_0}}(\eta g)\right) = T_g(\psi \circ \pi_{G,G_{\mu_0}})(\eta g) = T_g\,\mathrm{Ad}^*_{\_}\,\mu_0(\eta g).$$

However, since

$$\begin{aligned}
T_g\,\mathrm{Ad}^*_{_}\,\mu_0(\eta g) &= T_e(\mathrm{Ad}^*_{_}\,\mu_0 \circ R_g)(\eta) = T_e(\mathrm{Ad}^*_g \circ \mathrm{Ad}^*_{_}\,\mu_0)(\eta) = \mathrm{Ad}^*_g\,\mathrm{ad}^*_\eta\,\mu_0 \\
&= \mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}}\eta}\,\mathrm{Ad}^*_g\,\mu_0 = \mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}}\eta}\,\bar{\mu} \in T_{\bar{\mu}}\mathcal{O}_{\mu_0},
\end{aligned}$$

we conclude $\xi - \xi' - \mathrm{Ad}_{g^{-1}}\eta \in \mathfrak{g}_{\bar{\mu}}$ for some $\eta \in \mathfrak{h}(\mu_0)$ and hence $\xi - \xi' \in \mathfrak{h}(\bar{\mu}) + \mathfrak{g}_{\bar{\mu}} = \mathfrak{h}(\bar{\mu})$. Therefore, $\langle \sigma, \xi \rangle = \langle \sigma, \xi' \rangle$ since $\sigma \in \mathfrak{h}(\bar{\mu})^\circ$. So we have shown that the definition does not depend on the choice of $\xi \in \mathfrak{g}$ verifying (4.10).

The diagram below illustrates the various maps used in the definition of F. The group G acts on the right on each of these spaces and all arrows represent G-equivariant maps.

$$
\begin{array}{ccccc}
G & \xrightarrow{\pi_{G,G_{\mu_0}}} & G/G_{\mu_0} & \xrightarrow{\psi} & \mathcal{O}_{\mu_0} \\
 & \searrow^{\pi_{G,H}} & \downarrow{\scriptstyle \rho_{\mu_0}} & & \downarrow{\scriptstyle \pi_{\mu_0}} \\
 & & G/H & \xrightarrow[\Sigma]{} & N_{\mu_0}
\end{array}
\tag{4.11}
$$

Momentum map interpretation. In order to give the momentum map interpretation of the abstract Flaschka transformation, we show that there is a a bijective correspondence between the sections s_{μ_0} of π_{μ_0} and one-forms $\alpha_{i_{\mathfrak{h}}^*\mu_0}$ verifying (2.9).

Lemma 4.4. *Let $\mu_0 \in \mathfrak{g}^*$ and $\mathfrak{h}$ be a real polarization associated to μ_0 and verifying the Pukanszky condition. Define $\nu_0 := i_{\mathfrak{h}}^*\mu_0 \in \mathfrak{h}^*$, where $i_{\mathfrak{h}} : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is the inclusion. Then there is a bijective correspondence between the sections s_{μ_0} of π_{μ_0} and one-forms α_{ν_0} verifying (2.9), given by*

$$s_{\mu_0}([\mu]_\sim) = g^{-1}\alpha_{\nu_0}(g) = \overline{\alpha}_{\nu_0}([g]_H), \tag{4.12}$$

where $g \in G$ is such that $\mu = \operatorname{Ad}_g^* \mu_0$.

Proof. Let $\alpha_{\nu_0} \in \Omega^1(G)$ satisfy conditions (2.9) and define $s_{\mu_0}([\operatorname{Ad}_g^*\mu_0]_\sim) := g^{-1}\alpha_{\nu_0}(g)$. Using the diffeomorphism (4.5) and the H-invariance of α_{ν_0}, one verifies that the map s_{μ_0} is well-defined. We will show that $s_{\mu_0}([\operatorname{Ad}_g^*\mu_0]_\sim) \in \operatorname{Ad}_g^*\mu_0 + \mathfrak{h}\left(\operatorname{Ad}_g^*\mu_0\right)^\circ = [\operatorname{Ad}_g^*\mu_0]_\sim \subset \mathcal{O}_{\mu_0}$. Indeed, we have the following equivalences:

$$
\begin{aligned}
\mathbf{J}_L(\alpha_{\nu_0}(g)) = \nu_0 &\iff i_{\mathfrak{h}}^*(\alpha_{\nu_0}(g)g^{-1}) = \nu_0 = i_{\mathfrak{h}}^*\mu_0 \\
&\iff \alpha_{\nu_0}(g)g^{-1} \in \mu_0 + \mathfrak{h}(\mu_0)^\circ \\
&\iff g^{-1}\alpha_{\nu_0}(g) \in \operatorname{Ad}_g^*\mu_0 + \operatorname{Ad}_g^*\mathfrak{h}(\mu_0)^\circ \\
&\iff s_{\mu_0}([\operatorname{Ad}_g^*\mu_0]_\sim) \in \operatorname{Ad}_g^*\mu_0 + \mathfrak{h}\left(\operatorname{Ad}_g^*\mu_0\right)^\circ
\end{aligned}
\tag{4.13}
$$

which shows that $\mathbf{J}_L(\alpha_{\nu_0}(g)) = \nu_0$ is equivalent to s_{μ_0} being a section of π_{μ_0}.

Conversely, let s_{μ_0} be a section of π_{μ_0} and define

$$\alpha_{\nu_0}(g)(v_g) := \left\langle g s_{\mu_0}\left([\operatorname{Ad}_g^*\mu_0]_\sim\right), v_g\right\rangle = \left\langle s_{\mu_0}\left([\operatorname{Ad}_g^*\mu_0]_\sim\right), g^{-1}v_g\right\rangle,$$

for any $v_g \in T_gG$. We show that α_{ν_0} is H-invariant. We have

$$
\begin{aligned}
\alpha_{\nu_0}(hg)(hv_g) &= \left\langle s_{\mu_0}\left([\operatorname{Ad}_g^*\operatorname{Ad}_h^*\mu_0]_\sim\right), g^{-1}v_g\right\rangle \\
&= \left\langle s_{\mu_0}\left([\operatorname{Ad}_g^*(\mu_0+\lambda)]_\sim\right), g^{-1}v_g\right\rangle, \quad \lambda \in \mathfrak{h}(\mu_0)^\circ \\
&= \left\langle s_{\mu_0}\left([\operatorname{Ad}_g^*\mu_0]_\sim\right), g^{-1}v_g\right\rangle = \alpha_{\nu_0}(g)(v_g).
\end{aligned}
$$

In the second equality we used Lemma 3.7**(ii)**. In the third equality, we used the fact that $\operatorname{Ad}_g^*\lambda \in \mathfrak{h}\left(\operatorname{Ad}_g\mu_0\right)^\circ$. From the equivalences (4.13) we conclude that $\mathbf{J}_L(\alpha_{\nu_0}(g)) = \nu_0$. ■

We now use Lemma 4.4 and Theorem 4.1 to prove the following result.

Theorem 4.5. *Let $\mu_0 \in \mathfrak{g}^*$ and $\mathfrak{h}$ a real polarization associated to μ_0 verifying the Pukanszky condition. Define $\nu_0 := i_{\mathfrak{h}}^*\mu_0 \in \mathfrak{h}^*$. Fix a one-form $\alpha_{\nu_0} \in \Omega^1(G)$ verifying the conditions (2.9) and consider the abstract Flaschka transformation $F : \mathcal{O}_{\mu_0} \to T^*(G/H)$ associated to the section $s_{\mu_0} := \overline{\alpha}_{\nu_0} \circ \Sigma^{-1}$. Then F is a smooth diffeomorphism whose inverse is the reduced momentum map associated to the symplectic reduction of T^*G by H at ν_0, that is,*

$$F^{-1} = \mathbf{J}_R^{\nu_0} : T^*(G/H) \to \mathcal{O}_{\mu_0}.$$

Therefore, F is a symplectic diffeomorphism relative to the minus coadjoint orbit symplectic form on $\mathcal{O}_{\mu_0}$ and the magnetic form $\omega_{can} - B_{\nu_0}$ on $T^*(G/H)$, as defined in §2.1.

Proof. From Theorem 3.9**(iv)**, we know that $\mathbf{J}_R^{\nu_0}$ is a symplectic diffeomorphism. We shall verify that its inverse is F. It suffices to show that $\mathbf{J}_R^{\nu_0} \circ F = id_{\mathcal{O}_{\mu_0}}$.

The infinitesimal generator of the G-action on $\mathcal{O}_{\mu_0}$ and N_{μ_0} read, respectively,

$$\xi_{\mathcal{O}_{\mu_0}}(\nu) = \operatorname{ad}_\xi^* \nu \quad \text{and} \quad \xi_{N_{\mu_0}}([\nu]_\sim) = T_\nu \pi_{\mu_0}\left(\operatorname{ad}_\xi^* \nu\right)$$

(see diagram (4.11)), so the cotangent lift momentum map is $\left\langle \mathbf{J}_{can}(\gamma_{[\nu]_\sim}), \xi \right\rangle = \left\langle \gamma_{[\nu]_\sim}, T_\nu \pi_{\mu_0}\left(\operatorname{ad}_\xi^* \nu\right)\right\rangle$.

We thus have, on $[\mu]_\sim \subset \mathcal{O}_{\mu_0}$, choosing $s_{\mu_0} := \overline{\alpha}_{\nu_0}$, and denoting $\bar{\mu} := \overline{\alpha}_{\nu_0}([\mu]_\sim)$,

$$\begin{aligned}\left\langle \left(\mathbf{J}_R^{\nu_0} \circ F|_{[\mu]_\sim}\right)(\bar{\mu}+\sigma), \xi \right\rangle &= \left\langle \mathbf{J}_{can}\left(F|_{[\mu]_\sim}(\bar{\mu}+\sigma)\right) + \bar{\mu}, \xi \right\rangle \\ &= \left\langle F|_{[\mu]_\sim}(\bar{\mu}+\sigma), T_{\bar{\mu}}\pi_{\mu_0}\left(\operatorname{ad}_\xi^* \bar{\mu}\right)\right\rangle + \langle \bar{\mu}, \xi\rangle \\ &= \langle \sigma, \xi\rangle + \langle \bar{\mu}, \xi \rangle \\ &= \langle \bar{\mu}+\sigma, \xi\rangle .\end{aligned}$$

Thus $\mathbf{J}_R^{\nu_0} \circ F = id_{\mathcal{O}_{\mu_0}}$. ■

5 The Flaschka transformation for the Toda systems

In this section we discuss the Flaschka transformation for the Toda system (introduced in Toda [1970]) which was the setting for the original mapping defined independently in Flaschka [1974a,b] and Manakov [1975]. We also present the Flaschka map for the full Lie algebraic generalizations of the Toda flow as proposed by Bogayavlensky [1976] and Kostant [1979]. In these cases, the magnetic term in the symplectic structure on the cotangent bundle vanishes. Finally, in order to illustrate the geometric nature of the Flaschka map (in the sense that it provides canonical variables for all Hamiltonian systems on the coadjoint orbit), we consider free rigid body systems on the Toda coadjoint orbit as well as the canonical Flaschka variables for these systems by using the general theory developed earlier.

5.1 Preliminaries

We present the essential background and notation for Lie-Poisson equations on duals of Lie subalgebras and the classical structure theory of semisimple complex and real split Lie algebras.

Lie-Poisson equations on subalgebras. We begin by recalling various formulas used in the theory of integrable systems on duals of Lie subalgebras of real semisimple Lie algebras (see, e.g. Ratiu [1980], Reyman and Semenov-Tian-Shansky [1994], Symes [1980]). Let $\mathfrak{g}$ be a real Lie algebra admitting an invariant symmetric bilinear non-degenerate form $\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. For example, if $\mathfrak{g}^{\mathbb{C}}$ is a complex semisimple Lie algebra, recall that its (complex valued) Killing form is given by

$$\kappa(\xi, \eta) := \operatorname{Tr}(\operatorname{ad}_\xi \circ \operatorname{ad}_\eta), \quad \xi, \eta \in \mathfrak{g}^{\mathbb{C}}.$$

If $\mathfrak{g}$ is the normal real form of $\mathfrak{g}^{\mathbb{C}}$ then we choose $\gamma := \kappa|_{\mathfrak{g}\times\mathfrak{g}}$. If $\mathfrak{l}$ is the compact real form of $\mathfrak{g}^{\mathbb{C}}$ then we choose $\gamma := \operatorname{Im}\kappa|_{\mathfrak{l}\times\mathfrak{l}}$.

Identify the dual space $\mathfrak{g}^*$ with $\mathfrak{g}$ using γ. Relative to this pairing, the infinitesimal coadjoint operator is given by

$$\operatorname{ad}_\xi^\dagger \zeta = -[\xi, \zeta], \quad \xi, \zeta \in \mathfrak{g}.$$

The Lie-Poisson bracket of $F, H \in C^\infty(\mathfrak{g})$ is given by

$$\{F, H\}(\xi) = \pm\gamma\left(\xi, [\nabla F(\xi), \nabla H(\xi)]\right)$$

and the Hamiltonian vector field has the expression

$$X_H(\xi) = \mp[\xi, \nabla H(\xi)] = \mp \operatorname{ad}_{\nabla H(\xi)}^\dagger \xi,$$

where $\nabla H(\xi)$ denotes the gradient of H relative to γ. Thus, a function $C \in C^\infty(\mathfrak{g})$ is is in the center of the Lie-Poisson algebra of $\mathfrak{g}^* = \mathfrak{g}$, i.e., is a Casimir function on $\mathfrak{g}^* = \mathfrak{g}$, if and only if $[\nabla C(\zeta), \zeta] = 0$ for all $\zeta \in \mathfrak{g}$.

Let $\mathfrak{s}$ and $\mathfrak{k}$ be two Lie subalgebras of $\mathfrak{g}$ and assume that we have the vector space direct sum decomposition

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{k}. \tag{5.1}$$

Denote by $\pi_{\mathfrak{s}} : \mathfrak{g} \to \mathfrak{s}$ and $\pi_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ the associated projections. We get the natural identifications

$$\mathfrak{s}^* = \mathfrak{k}^\perp := \{\xi \in \mathfrak{g} \mid \gamma(\xi, \eta) = 0, \ \forall \eta \in \mathfrak{k}\}, \qquad \mathfrak{k}^* = \mathfrak{s}^\perp := \{\zeta \in \mathfrak{g} \mid \gamma(\zeta, \sigma) = 0, \ \forall \sigma \in \mathfrak{s}\}.$$

Denote by $\pi_{\mathfrak{s}^\perp} : \mathfrak{g} \to \mathfrak{s}^\perp$ and $\pi_{\mathfrak{k}^\perp} : \mathfrak{g} \to \mathfrak{k}^\perp$ the projections associated to the direct sum $\mathfrak{g} = \mathfrak{s}^\perp \oplus \mathfrak{k}^\perp$.

We now write the Lie-Poisson equations on $\mathfrak{s}^* = \mathfrak{k}^\perp$. First, we compute the coadjoint representation $(\mathrm{ad}_{\mathfrak{s}})^\dagger : \mathfrak{s} \times \mathfrak{k}^\perp \to \mathfrak{k}^\perp$ of the Lie algebra $\mathfrak{s}$ on its dual $\mathfrak{k}^\perp$ in terms of the bracket on $\mathfrak{g}$, i.e., the operators $(\mathrm{ad}_{\mathfrak{s}})^\dagger_\xi : \mathfrak{s}^* = \mathfrak{k}^\perp \to \mathfrak{s}^* = \mathfrak{k}^\perp$ for $\xi \in \mathfrak{s}$. Given $\xi, \zeta \in \mathfrak{s}$ and $\eta \in \mathfrak{s}^* = \mathfrak{k}^\perp$, we have

$$\gamma\left((\mathrm{ad}_{\mathfrak{s}})^\dagger_\xi \eta, \zeta\right) = \gamma\left(\eta, [\xi, \zeta]\right) = -\gamma\left([\xi, \eta], \zeta\right) = -\gamma\left(\pi_{\mathfrak{k}^\perp}([\xi, \eta]), \zeta\right),$$

and hence

$$(\mathrm{ad}_{\mathfrak{s}})^\dagger_\xi \eta = -\pi_{\mathfrak{k}^\perp}([\xi, \eta]). \tag{5.2}$$

Second, if $h : \mathfrak{s}^* = \mathfrak{k}^\perp \to \mathbb{R}$, the functional derivative of h at $\mu \in \mathfrak{s}^* = \mathfrak{k}^\perp$ relative to the duality pairing between $\mathfrak{s}$ and $\mathfrak{k}^\perp$ given by γ, has the expression

$$\frac{\delta h}{\delta \mu} = \pi_{\mathfrak{s}}(\nabla h(\mu)), \tag{5.3}$$

where $\nabla h(\mu) \in \mathfrak{g}$ is the gradient relative to γ. Third, from the general theory of Lie-Poisson systems (see, e.g. Marsden and Ratiu [1999, §10.7]) and (5.2), (5.3), the ($\mp$) Lie-Poisson equations on $\mathfrak{k}^\perp$ are

$$\partial_t \mu = \pm (\mathrm{ad}_{\mathfrak{s}})^\dagger_{\frac{\delta h}{\delta \mu}} \mu \iff \partial_t \mu = \mp \pi_{\mathfrak{k}^\perp}\left([\pi_{\mathfrak{s}}(\nabla h(\mu)), \mu]\right) \tag{5.4}$$

where $\mu(t) \in \mathfrak{k}^\perp$.

Let us compute the Lie-Poisson equations on $\mathfrak{s}^* = \mathfrak{k}^\perp$ in the particular case when the Hamiltonian h is the restriction to $\mathfrak{k}^\perp$ of a Casimir function on $\mathfrak{g}^* = \mathfrak{g}$, i.e., $[\nabla h(\zeta), \zeta] = 0$ for all $\zeta \in \mathfrak{g}$. Therefore, $[\pi_{\mathfrak{s}}(\nabla h(\zeta)), \zeta] = [\nabla h(\zeta) - \pi_{\mathfrak{k}}(\nabla h(\zeta)), \zeta] = -[\pi_{\mathfrak{k}}(\nabla h(\zeta)), \zeta]$. Note that $[\xi, \mu] \in \mathfrak{k}^\perp$ for all $\xi \in \mathfrak{k}$ and $\mu \in \mathfrak{k}^\perp$. Thus the Lie-Poisson equations (5.4) on $\mathfrak{k}^\perp$ become

$$\partial_t \mu = \mp \pi_{\mathfrak{k}^\perp}\left([\pi_{\mathfrak{s}}(\nabla h(\mu)), \mu]\right) = \pm \left[\pi_{\mathfrak{k}}(\nabla h(\mu)), \mu\right].$$

For example, consider the Hamiltonian $h : \mathfrak{g}^* = \mathfrak{g} \to \mathbb{R}$ defined by $h(\zeta) = \frac{1}{2}\gamma(\zeta, \zeta)$. We have $\nabla h(\zeta) = \zeta$ and therefore h is a Casimir function on $\mathfrak{g}^*$. Its restriction to $\mathfrak{s}^* = \mathfrak{k}^\perp$ gives hence rise to the ($\mp$) Lie-Poisson equations

$$\partial_t \mu = \pm \left[\pi_{\mathfrak{k}}(\mu), \mu\right], \quad \mu \in \mathfrak{s}^* = \mathfrak{k}^\perp, \tag{5.5}$$

also knowns as the *full Toda equations.*

Root space decomposition and real forms. We summarize the relevant facts about semisimple Lie algebras that will be used later on. We fix terminology, conventions, and notation, since these are not uniform in the literature. Our sources are Cahn [1984], Humphreys [1980], Knapp [2002], and Samelson [1989] (even though they do not follow the same conventions; for example the Cartan matrix in Knapp [2002] is the transpose of the one in Humphreys [1980] while Cahn [1984] tends to prefer the compact real form associated to a Dynkin diagram). We follow the conventions in Humphreys [1980] for the Dynkin diagrams and the Cartan matrices. Our goal here is to bring all this background information, which is necessary for the remaining developments in this paper, together in a uniform fashion.

Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semisimple Lie algebra and select a Cartan subalgebra $\mathfrak{c}^{\mathbb{C}}$. Let $r := \dim_{\mathbb{C}} \mathfrak{c}^{\mathbb{C}}$ be the *rank* of the Lie algebra. The Killing form κ is nondegenerate on $\mathfrak{c}^{\mathbb{C}}$. For every $\lambda \in \left(\mathfrak{c}^{\mathbb{C}}\right)^*$ (the complex dual of $\mathfrak{c}^{\mathbb{C}}$) there exists a unique element $t_\lambda \in \mathfrak{c}^{\mathbb{C}}$ such that $\langle \lambda, \eta \rangle = \kappa(t_\lambda, \eta)$, for all $\eta \in \mathfrak{c}^{\mathbb{C}}$, where $\langle \, , \rangle : \left(\mathfrak{c}^{\mathbb{C}}\right)^* \times \mathfrak{c}^{\mathbb{C}} \to \mathbb{C}$ is the natural duality pairing. Thus we get a symmetric bilinear form on $\left(\mathfrak{c}^{\mathbb{C}}\right)^*$, also denoted by κ, namely $\kappa(\lambda, \gamma) := \kappa(t_\lambda, t_\gamma)$.

For $\alpha \in \left(\mathfrak{c}^{\mathbb{C}}\right)^*$, write $\mathfrak{g}^{\mathbb{C}}_\alpha := \{\xi \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}_\eta \xi = \langle \alpha, \eta \rangle \xi, \ \forall \eta \in \mathfrak{c}\}$. We let $\Delta := \left\{\alpha \in \left(\mathfrak{c}^{\mathbb{C}}\right)^* \mid \alpha \neq 0, \ \mathfrak{g}^{\mathbb{C}}_\alpha \neq \{0\}\right\}$ be the *space of roots.* Then Δ is a finite subset of $\left(\mathfrak{c}^{\mathbb{C}}\right)^*$, $0 \notin \Delta$, $\mathrm{span}_{\mathbb{C}} \Delta = \left(\mathfrak{c}^{\mathbb{C}}\right)^*$, and if both $\alpha, z\alpha \in \Delta$ for some

$z \in \mathbb{C}$ then $z = \pm 1$ (the only multiples of a root α which are also roots are $\pm\alpha$). In addition, if $\alpha, \beta \in \Delta$, then $\beta - \frac{2\kappa(\beta,\alpha)}{\kappa(\alpha,\alpha)}\alpha \in \Delta$ where $\frac{2\kappa(\beta,\alpha)}{\kappa(\alpha,\alpha)} \in \mathbb{Z}$ are the *Cartan integers* and $\kappa(\beta,\alpha)$, $\kappa(\alpha,\alpha) \in \mathbb{Q}$. For all $\alpha \in \Delta$, we have $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}}_\alpha = 1$; $\mathfrak{g}^{\mathbb{C}}_\alpha$ is the *α-root space*. The $\mathbb{Q}$-vector space $\operatorname{span}_{\mathbb{Q}} \Delta$ is a $\mathbb{Q}$-subspace of $\left(\mathfrak{c}^{\mathbb{C}}\right)^*$ (viewed as a $\mathbb{Q}$-vector space) and $\dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \Delta = r$. In addition, $\kappa : \operatorname{span}_{\mathbb{Q}} \Delta \times \operatorname{span}_{\mathbb{Q}} \Delta \to \mathbb{Q}$ is positive definite. This implies that $\left(\operatorname{span}_{\mathbb{R}} \Delta = \mathbb{R} \otimes_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} \Delta, \kappa\right)$ is a real inner product space of real dimension r whose complexification is $\left(\mathfrak{c}^{\mathbb{C}}\right)^*$.

Choose a *fundamental set of roots*, or a *base*, $\Pi = \{\alpha_1, ..., \alpha_r\}$ of Δ, i.e., Π is a basis of $\operatorname{span}_{\mathbb{R}} \Delta$ (and hence also of the complex vector space $\left(\mathfrak{c}^{\mathbb{C}}\right)^*$) such that any $\alpha \in \Delta$ can be written as $\alpha = \sum_{i=1}^r m_i \alpha_i$, where $m_i \in \mathbb{Z}$ and either all $m_i \geq 0$, or all $m_i \leq 0$. Since the only roots having only one $m_{i_0} = 1$ and all other $m_i = 0$, $i \neq i_0$, are the elements of Π, the elements of the base Π are also called *simple roots*. If $\alpha = \sum_{i=1}^r m_i \alpha_i$, the number $\sum_{i=1}^r m_i \in \mathbb{Z}$ is the *height* of the root $\alpha \in \Delta$. Thus the only roots of height one are the simple roots. If $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, then $\kappa(\alpha, \beta) \leq 0$ and $\alpha - \beta \notin \Delta$.

Denote by $\Delta_\pm := \left\{\alpha = \sum_{i=1}^r m_i \alpha_i \in \Delta \mid m_i \gtreqless 0, \ \forall i = 1, \ldots, r\right\}$ the subsets of *positive*, resp. *negative roots.* We have $\Delta_- = -\Delta_+$ and the corresponding *root space decomposition* of $\mathfrak{g}^{\mathbb{C}}$ (a vector space direct sum) is

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{c}^{\mathbb{C}} \oplus \coprod_{\alpha \in \Delta_+} \left(\mathfrak{g}^{\mathbb{C}}_\alpha \oplus \mathfrak{g}^{\mathbb{C}}_{-\alpha}\right). \tag{5.6}$$

The spaces $\mathfrak{g}^{\mathbb{C}}_\alpha$ and $\mathfrak{g}^{\mathbb{C}}_\beta$ are κ-orthogonal unless $\beta = -\alpha$; $\mathfrak{c}^{\mathbb{C}}$ is κ-orthogonal to all $\mathfrak{g}^{\mathbb{C}}_\alpha$ and $\kappa|_{\mathfrak{g}^{\mathbb{C}}_\alpha \times \mathfrak{g}^{\mathbb{C}}_\alpha} \equiv 0$. In addition,

$$\kappa(\eta_1, \eta_2) = \sum_{\alpha \in \Delta} \langle \alpha, \eta_1 \rangle \langle \alpha, \eta_2 \rangle, \quad \text{for all} \quad \eta_1, \eta_2 \in \mathfrak{c}^{\mathbb{C}}.$$

If $\alpha \in \Delta$, the only multiples of α which are roots are $\pm\alpha$. If $\alpha, \beta, \alpha + \beta \in \Delta$, then $\left[\mathfrak{g}^{\mathbb{C}}_\alpha, \mathfrak{g}^{\mathbb{C}}_\beta\right] = \mathfrak{g}^{\mathbb{C}}_{\alpha+\beta}$ and $\mathfrak{g}^{\mathbb{C}}$ is generated as a Lie algebra by the root spaces $\left\{\mathfrak{g}^{\mathbb{C}}_\alpha \mid \alpha \in \Delta\right\}$; $\dim_{\mathbb{C}} \left[\mathfrak{g}^{\mathbb{C}}_\alpha, \mathfrak{g}^{\mathbb{C}}_{-\alpha}\right] = 1$ and $\alpha|_{\left[\mathfrak{g}^{\mathbb{C}}_\alpha, \mathfrak{g}^{\mathbb{C}}_{-\alpha}\right]}$ does not vanish identically.

Given $\alpha \in \Delta$, define $h_\alpha := \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \in \mathfrak{c}^{\mathbb{C}}$ and let $h_i := h_{\alpha_i} \in \mathfrak{c}^{\mathbb{C}}$. For every $\xi_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha$, $\xi_\alpha \neq 0$, there exists a unique $\xi_{-\alpha} \in \mathfrak{g}^{\mathbb{C}}_{-\alpha}$ such that $[\xi_\alpha, \xi_{-\alpha}] = h_\alpha$. If $\alpha, \beta \in \Delta$, the Cartan integers have the alternate expressions

$$\frac{2\kappa(\beta, \alpha)}{\kappa(\alpha, \alpha)} = \langle \beta, h_\alpha \rangle = \frac{\kappa(\beta, \beta)}{2} \kappa(h_\alpha, h_\beta) \in \mathbb{Z}$$

and $\beta - \langle \beta, h_\alpha \rangle \alpha \in \Delta$.

If $\alpha, \beta \in \Delta$, $\beta \neq \pm\alpha$, let $p, q \in \mathbb{N}$, $p, q \geq 0$, be the largest integers for which $\beta - p\alpha$, $\beta + q\alpha \in \Delta$. Then $\{\beta + j\alpha \in \Delta \mid j = -p, \ldots, q\}$ is the *α-string through β*; it is uninterrupted,

$$p - q = \langle \beta, h_\alpha \rangle = 0, \pm 1, \pm 2, \pm 3, \tag{5.7}$$

and contains at most four roots. In addition, at most two root lengths occur in this string and if $q \geq 1$, then

$$p + 1 = q \frac{\kappa(\alpha + \beta, \alpha + \beta)}{\kappa(\beta, \beta)}.$$

Fix a *Chevalley basis* of $\mathfrak{g}^{\mathbb{C}}$ associated to Π, i.e., a basis $\left\{h_i, e_\alpha \mid i = 1, ..., r, \ \alpha \in \Delta, \ e_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha\right\}$ of $\mathfrak{g}^{\mathbb{C}}$ satisfying the additional conditions

- $[e_\alpha, e_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta$;
- if $\alpha, \beta, \alpha + \beta \in \Delta$, then the structure constants $N_{\alpha,\beta}$ defined by $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$.

The structure constants relative to the Chevalley basis necessarily lie in $\mathbb{Z}$ and h_α is a $\mathbb{Z}$-linear combination of $h_1, \ldots, h_r$. Moreover, we have

$$[h_\alpha, e_\beta] = \frac{2\kappa(\alpha, \beta)}{\kappa(\alpha, \alpha)} e_\beta, \quad \kappa(e_\alpha, e_{-\alpha}) = \frac{2}{\kappa(\alpha, \alpha)}, \quad N^2_{\alpha,\beta} = q(p+1) \frac{\kappa(\alpha + \beta, \alpha + \beta)}{\kappa(\beta, \beta)}, \tag{5.8}$$

where $\beta - p\alpha, \ldots, \beta + q\alpha$ is the α-string through β. In addition,

$$q = 0 \Longrightarrow [e_\alpha, e_\beta] = 0 \quad \text{and} \quad q \geq 1 \Longrightarrow [e_\alpha, e_\beta] = \pm(p+1) e_{\alpha+\beta}, \text{ i.e., } N^2_{\alpha,\beta} = (p+1)^2. \tag{5.9}$$

The Chevalley basis is not unique but there is little room to change it. Once the fundamental set of roots Π of Δ is chosen, the basis $\{h_1, \ldots, h_r\}$ of $\mathfrak{c}^{\mathbb{C}}$ is completely determined. The e_α can be multiplied with arbitrary non-zero constants $c_\alpha \in \mathbb{C}$, as long as they satisfy the conditions $c_\alpha c_{-\alpha} = 1$ and $c_\alpha c_\beta = \pm c_{\alpha+\beta}$ for any $\alpha, \beta, \alpha+\beta \in \Delta$. The signs $\pm$ in (5.9) are also determined up to multiplication of $N_{\alpha,\beta} = \pm(p+1)$ by the factor $\varepsilon_\alpha \varepsilon_\beta \varepsilon_{\alpha+\beta}$, where $\varepsilon_\alpha = \pm 1$ is an arbitrary coefficient of e_α subject to the condition $\varepsilon_{-\alpha} = \varepsilon_\alpha$ for all $\alpha, \beta, \alpha+\beta \in \Delta$.

Note that for any basis $\left\{h_i, e_\alpha \mid i = 1, ..., r,\ \alpha \in \Delta,\ e_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}\right\}$ of $\mathfrak{g}^{\mathbb{C}}$ (in particular, a Chevalley basis), the root space decomposition (5.6) can be written as

$$\mathfrak{g}^{\mathbb{C}} = \operatorname{span}_{\mathbb{C}}\{h_1, \ldots, h_r\} \oplus \coprod_{\alpha \in \Delta_+} \mathbb{C}\left(e_\alpha + e_{-\alpha}\right). \tag{5.10}$$

The *Cartan matrix* associated to $\mathfrak{g}^{\mathbb{C}}$ is, by definition, the matrix in $\mathrm{GL}(r, \mathbb{Z})$ with entries

$$C_{ij} = \langle \alpha_i, h_j \rangle = \frac{2\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)} = \frac{\kappa(\alpha_i, \alpha_i)}{2} \kappa(h_i, h_j). \tag{5.11}$$

The Cartan matrix depends only on the ordering of the simple roots $\{\alpha_1, \ldots, \alpha_r\} = \Pi$; it does not depend on the base Π. Note that $C_{ii} = 2$, $C_{ij} \leq 0$, $C_{ij} = 0$ if and only if $C_{ji} = 0$, for all $i, j = 1, \ldots, r$, $i \neq j$, and if $D := \operatorname{diag}\left(\sqrt{\kappa(\alpha_1, \alpha_1)}, \ldots, \sqrt{\kappa(\alpha_r, \alpha_r)}\right)$, then

$$D^{-1} C D = \left[2\kappa \left(\frac{\alpha_i}{\sqrt{\kappa(\alpha_i, \alpha_i)}}, \frac{\alpha_j}{\sqrt{\kappa(\alpha_j, \alpha_j)}} \right) \right]$$

is a symmetric positive definite $r \times r$ matrix. Fix the standard basis given by the column vectors $\{(1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, 1)^T\} \subset \mathbb{C}^r$ in which the Cartan matrix C has the entries (5.11). The dual basis is then formed by the row vectors $\{(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\} \subset (\mathbb{C}^r)^*$. The simple roots of $\mathfrak{g}^{\mathbb{C}}$, expanded in this dual basis of $(\mathbb{C}^r)^* \cong \left(\mathfrak{c}^{\mathbb{C}}\right)^*$, are the row vectors of C.

There is a simple algorithm that determines all the roots of $\mathfrak{g}^{\mathbb{C}}$ from the Cartan matrix and the identity (5.7). Of course, it suffices to construct the positive roots. Start with a height one root α_i, i.e., a simple root. Then necessarily $p = 0$ in the α_j-string through α_i, where α_j is another simple root, since $\alpha_i - \alpha_j$ is never a root. Thus, by (5.7), we get

$$q = -\frac{2\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)} = -C_{ij}.$$

So, in order that this string exist, which means that at least $\alpha_i + \alpha_j \in \Delta$, we must have $C_{ij} \neq 0$. If this happens, then $\alpha_i + \alpha_j$ equals the sum of the i^{th} and j^{th} row of C. If not, then $\alpha_i + \alpha_j \notin \Delta$. This determines a height two root. One proceeds now in the same way for all i to obtain all height two roots. Assume, recursively, that $\beta = \sum_{i=1}^r m_i \alpha_i$, $m_i \in \mathbb{Z}$, is a height $k := \sum_{i=1}^r m_i > 0$ positive root. But then, for α_j a simple root, the fragment of the string $\beta, \beta - \alpha_j, \ldots \beta - p\alpha_j$ is already known, i.e., $p \geq 0$ is determined. By (5.7),

$$p - q = \frac{2\kappa(\beta, \alpha_j)}{\kappa(\alpha_j, \alpha_j)} = \sum_{i=1}^r m_i \frac{2\kappa(\alpha_i, \alpha_j)}{\kappa(\alpha_j, \alpha_j)} = \sum_{i=1}^r m_i C_{ij}$$

which shows that if $\beta + \alpha_j \in \Delta$, then $q = p - \sum_{i=1}^r m_i C_{ij} > 0$. If this happens, then one adds the j^{th} row of C to the root β expressed in the dual basis associated to the matrix representation (5.11) to obtain the root $\beta + \alpha_j$. If not, then $\beta + \alpha_j \notin \Delta$. One proceeds to do this for all roots of height k and obtains the roots of height $k+1$.

Fix a Chevalley basis $\{h_i, e_\alpha \mid i = 1, ..., r,\ \alpha \in \Delta\}$ of $\mathfrak{g}^{\mathbb{C}}$. The *normal real form* of $\mathfrak{g}^{\mathbb{C}}$ is the real Lie algebra $\mathfrak{g} := \operatorname{span}_{\mathbb{R}}\{h_i, e_\alpha \mid i = 1, ..., r,\ \alpha \in \Delta\}$. Then $\mathfrak{c} := \operatorname{span}_{\mathbb{R}}\{h_i \mid i = 1, ..., r\}$ is a real Cartan subalgebra of $\mathfrak{g}$ and we have the root space decomposition

$$\mathfrak{g} = \mathfrak{c} \oplus \coprod_{\alpha \in \Delta_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) = \operatorname{span}_{\mathbb{R}}\{h_1, \ldots, h_r\} \oplus \coprod_{\alpha \in \Delta_+} \mathbb{R}\left(e_\alpha + e_{-\alpha}\right), \tag{5.12}$$

where $\mathfrak{g}_\alpha = \mathbb{R} e_\alpha$, for all $\alpha \in \Delta$, i.e., the real Lie algebra $\mathfrak{g}$ is *split*. In addition, the complexification of $\mathfrak{g}$ is $\mathfrak{g}^{\mathbb{C}}$, i.e., $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ and, similarly, $\mathfrak{c}^{\mathbb{C}} = \mathfrak{c} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{c} \oplus i\mathfrak{c}$.

It is useful to think of $\mathfrak{g}$ as the fixed point set of an anticomplex involution on $\mathfrak{g}^{\mathbb{C}}$. A map $\varkappa : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is a complex (respectively anticomplex) Lie algebra involution if $\varkappa \circ \varkappa = id$, $\varkappa(\xi+\eta) = \varkappa(\xi) + \varkappa(\eta)$, $\varkappa([\xi,\eta]) = [\varkappa(\xi), \varkappa(\eta)]$, and $\varkappa(z\xi) = z\varkappa(\xi)$ (respectively $\varkappa(z\xi) = \bar{z}\varkappa(\xi)$) for all $\xi, \eta \in \mathfrak{g}^{\mathbb{C}}$ and $z \in \mathbb{C}$. Define the anticomplex Lie algebra involution $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ by specifying its values on the basis $\{h_i, e_{\pm\alpha} \mid i = 1, \ldots, r,\ \alpha \in \Delta_+\}$ of $\mathfrak{g}^{\mathbb{C}}$, namely, $\tau(h_i) = h_i$, $\tau(e_{\pm\alpha}) = e_{\pm\alpha}$, for all $i = 1, \ldots, r$, $\alpha \in \Delta_+$, and then extend it to $\mathfrak{g}^{\mathbb{C}}$ by anticomplex linearity. Then $\mathfrak{g} = \{\xi \in \mathfrak{g}^{\mathbb{C}} \mid \tau(\xi) = \xi\}$ and $\mathfrak{c} = \{\eta \in \mathfrak{c}^{\mathbb{C}} \mid \tau(\eta) = \eta\}$. For example, if $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(r+1, \mathbb{C})$, then the anticomplex involution τ defined above is given by $\tau(\xi) = \bar{\xi}$ for all $\xi \in \mathfrak{sl}(r+1,\mathbb{C})$ and hence $\mathfrak{g} = \mathfrak{sl}(r+1,\mathbb{R})$, as expected.

There are two important real vector space decompositions of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{g}$, respectively. As usual, fix a Chevalley basis $\{h_i, e_\alpha \mid i = 1, ..., r,\ \alpha \in \Delta\}$ of $\mathfrak{g}^{\mathbb{C}}$. Define the complex nilpotent Lie subalgebras $\mathfrak{n}_{\pm}^{\mathbb{C}} := \coprod_{\alpha \in \Delta_\pm} \mathbb{C} e_\alpha$ of $\mathfrak{g}^{\mathbb{C}}$. Then $\mathfrak{b}_{\pm}^{\mathbb{C}} := \mathfrak{n}_{\pm}^{\mathbb{C}} \oplus \mathfrak{c}$ are *Borel subalgebras*, i.e., maximal solvable Lie subalgebras of $\mathfrak{g}^{\mathbb{C}}$. Define the *compact real form* of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{l} := \left\{ \mathrm{i}\sum_{j=1}^{r} a_j h_j + \sum_{\alpha\in\Delta_+} x_\alpha (e_\alpha - e_{-\alpha}) + \mathrm{i}\sum_{\alpha\in\Delta_+} y_\alpha (e_\alpha + e_{-\alpha}) \;\middle|\; a_j, x_\alpha, y_\alpha \in \mathbb{R} \right\}; \tag{5.13}$$

$\mathfrak{l}$ is a real compact Lie algebra (i.e., the Killing form κ is negative definite on $\mathfrak{l}$), $\mathrm{i}\mathfrak{c}$ is its Cartan subalgebra, $\mathfrak{l} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}^{\mathbb{C}}$, and $\mathfrak{l}$ is the fixed point set of the anticomplex involution $\sigma : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ given on the Chevalley basis by $\sigma(h_j) = -h_j$ and $\sigma(e_\alpha) = -e_{-\alpha}$ for all $j = 1, \ldots, r$, $\alpha \in \Delta$. For example, if $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(r+1,\mathbb{C})$, then $\sigma(\xi) = -(\bar{\xi})^{\mathsf{T}}$ for all $\xi \in \mathfrak{sl}(r+1,\mathbb{C})$, and $\mathfrak{l} = \mathfrak{su}(r+1)$.

For an arbitrary complex Lie algebra $\mathfrak{a}$ denote by $\mathfrak{a}_{\mathbb{R}}$ the same Lie algebra but thought of as a real Lie algebra; thus, $\dim \mathfrak{a}_{\mathbb{R}} = 2\dim_{\mathbb{C}} \mathfrak{a}$. Thus, we have the vector space direct sum decomposition (the Iwasawa decomposition of $\mathfrak{g}^{\mathbb{C}}$ viewed as a real Lie algebra)

$$\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}} = \mathfrak{l} \oplus \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}. \tag{5.14}$$

The projections associated to this direct sum are:

$$\begin{aligned} &\pi_{\mathfrak{l}}\left((a_k + \mathrm{i}b_k)h_k + (x_{-\alpha} + \mathrm{i}y_{-\alpha})e_{-\alpha} + (x_\alpha + \mathrm{i}y_\alpha)e_\alpha\right) \\ &\qquad = \mathrm{i}b_k h_k + x_\alpha(e_\alpha - e_{-\alpha}) + \mathrm{i}y_\alpha(e_\alpha + e_{-\alpha}) && (5.15) \\ &\pi_{\mathfrak{c}}\left((a_k + \mathrm{i}b_k)h_k + (x_{-\alpha} + \mathrm{i}y_{-\alpha})e_{-\alpha} + (x_\alpha + \mathrm{i}y_\alpha)e_\alpha\right) = a_k h_k && (5.16) \\ &\pi_{\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}}\left((a_k + \mathrm{i}b_k)h_k + (x_{-\alpha} + \mathrm{i}y_{-\alpha})e_{-\alpha} + (x_\alpha + \mathrm{i}y_\alpha)e_\alpha\right) \\ &\qquad = \left((x_\alpha + x_{-\alpha}) - \mathrm{i}(y_\alpha - y_{-\alpha})\right)e_{-\alpha} && (5.17) \end{aligned}$$

where $k = 1, \ldots, r$, $\alpha \in \Delta_+$, $a_k, b_k, x_\alpha, y_\alpha \in \mathbb{R}$.

The imaginary part of the Killing form of $\mathfrak{g}^{\mathbb{C}}$, i.e., $\operatorname{Im}\kappa : (\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}} \times (\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}} \to \mathbb{R}$, is a nondegenerate pairing by which we identify the dual $\left((\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}}\right)^*$ with $(\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}}$. With this identification, the dual space to $\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ is identified with $\mathfrak{l}^\circ = \mathfrak{l}^\perp = \mathfrak{l}$ and the dual space to $\mathfrak{l}$ with $\left(\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right)^\circ = \left(\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right)^\perp = \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$. Thus we have

$$\left((\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}}\right)^* = (\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}} = \mathfrak{l} \oplus \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}.$$

In addition, the imaginary part $\operatorname{Im}\kappa$ of the Killing form is a real symmetric nondegenerate invariant bilinear form on $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$, $\operatorname{Im}\kappa(\mathfrak{l},\mathfrak{l}) = \operatorname{Im}\kappa\left(\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}, \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right) = 0$ and hence $\mathfrak{l}$ and $\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ are dual to each other relative to $\operatorname{Im}\kappa$, i.e., $\mathfrak{l}^* \cong \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$, an isomorphism of real vector spaces.

The anticomplex Lie algebra involutions σ and τ commute and hence $\sigma \circ \tau$ is a complex Lie algebra involution. Its fixed point-set coincides with the complexification of the *compact normal Lie algebra* $\mathfrak{k} := \mathfrak{l} \cap \mathfrak{g}$. For example, if $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(r+1,\mathbb{C})$, then $(\sigma \circ \tau)(\xi) = -\xi^{\mathsf{T}}$ so that its fixed point set is $\mathfrak{so}(r+1,\mathbb{C})$ which is the complexification of $\mathfrak{so}(r+1,\mathbb{R}) = \mathfrak{sl}(r+1,\mathbb{R}) \cap \mathfrak{su}(r+1)$.

Next, we turn to the Iwasawa decomposition of $\mathfrak{g}$. Define the real nilpotent Lie subalgebras $\mathfrak{n}_\pm := \coprod_{\alpha\in\Delta_\pm} \mathbb{R} e_\alpha$ of $\mathfrak{g}$. Then $\mathfrak{b}_\pm := \mathfrak{n}_\pm \oplus \mathfrak{c}$ are *Borel subalgebras*, i.e., maximal solvable Lie subalgebras of $\mathfrak{g}$. We have the direct sum (Iwasawa decomposition of the real normal form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$)

$$\mathfrak{g} = \mathfrak{b}_- \oplus \mathfrak{k} = \mathfrak{n}_- \oplus \mathfrak{c} \oplus \mathfrak{k}. \tag{5.18}$$

This direct sum induces a decomposition of the dual

$$\mathfrak{g}^* = \mathfrak{g} = \mathfrak{b}_-^\perp \oplus \mathfrak{k}^\perp,$$

where $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ via the real valued symmetric invariant nondegenerate bilinear form $\kappa|_{\mathfrak{g}\times\mathfrak{g}}$, the orthogonal spaces are also relative to $\kappa|_{\mathfrak{g}\times\mathfrak{g}}$, and

$$\begin{aligned} \mathfrak{b}_- &= \coprod_{\alpha\in\Delta_+} \mathbb{R}e_{-\alpha} \oplus \coprod_{i=1}^{r} \mathbb{R}h_i = \mathfrak{b}_-^{\mathbb{C}} \cap \mathfrak{g} \qquad & \mathfrak{k} &= \coprod_{\alpha\in\Delta_+} \mathbb{R}(e_\alpha - e_{-\alpha}) = \mathfrak{l}\cap\mathfrak{g} \\ \mathfrak{k}^* &\cong \mathfrak{b}_-^\perp = \coprod_{\alpha\in\Delta_+} \mathbb{R}e_{-\alpha} & \mathfrak{b}_-^* &\cong \mathfrak{k}^\perp = \coprod_{i=1}^{r} \mathbb{R}h_i \oplus \coprod_{\alpha\in\Delta_+} \mathbb{R}(e_\alpha + e_{-\alpha}). \end{aligned} \tag{5.19}$$

The projections associated to these direct sum decompositions are

$$\pi_{\mathfrak{b}_-}(c_k h_k + a_\alpha e_\alpha + a_{-\alpha}e_{-\alpha}) = c_k h_k + (a_\alpha + a_{-\alpha})e_{-\alpha} \tag{5.20}$$
$$\pi_{\mathfrak{k}}(c_k h_k + a_\alpha e_\alpha + a_{-\alpha}e_{-\alpha}) = a_\alpha(e_\alpha - e_{-\alpha}) \tag{5.21}$$
$$\pi_{\mathfrak{k}^\perp}(c_k h_k + a_\alpha e_\alpha + a_{-\alpha}e_{-\alpha}) = c_k h_k + a_\alpha(e_\alpha + e_{-\alpha}) \tag{5.22}$$
$$\pi_{\mathfrak{b}_-^\perp}(c_k h_k + a_\alpha e_\alpha + a_{-\alpha}e_{-\alpha}) = (a_{-\alpha} - a_\alpha)e_{-\alpha}, \tag{5.23}$$

where $k = 1,\ldots,r$, $\alpha\in\Delta_+$, and $c_k, a_\alpha, a_{-\alpha}\in\mathbb{R}$. Note that while $\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ is a nilpotent Lie sub algebra of $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$, $\mathfrak{c}\oplus\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}} \neq (\mathfrak{b}_-^{\mathbb{C}})_{\mathbb{R}}$ since $(\mathfrak{b}_-^{\mathbb{C}})_{\mathbb{R}}$ contains the imaginary part of the Cartan algebra $\mathfrak{c}^{\mathbb{C}}$.

Since the Lie algebra is semisimple, we have the isomorphisms $\mathfrak{g}\simeq \mathrm{ad}_{\mathfrak{g}} = \mathrm{Der}\,\mathfrak{g}$, that is, the ideal $\mathrm{ad}_{\mathfrak{g}} := \{\mathrm{ad}_\xi \mid \xi\in\mathfrak{g}\}$ of *inner derivations* on $\mathfrak{g}$ coincides with the Lie algebra $\mathrm{Der}\,\mathfrak{g}$ of all *derivations* on $\mathfrak{g}$. Therefore, the connected Lie group with Lie algebra $\mathfrak{g}$ is $G := I(\mathfrak{g}) := \{e^{\mathrm{ad}_\xi} \mid \xi\in\mathfrak{g}\}$, the connected component of the Lie group of all Lie algebra automorphisms $A(\mathfrak{g})$ of $\mathfrak{g}$. An *automorphism of the root space* Δ is, by definition, a linear automorphism $\phi : \mathfrak{c}^* \to \mathfrak{c}^*$ such that $\phi(\Delta) = \Delta$. This implies that $\frac{\kappa(\phi(\beta),\phi(\alpha))}{\kappa(\phi(\alpha),\phi(\alpha))} = \frac{\kappa(\beta,\alpha)}{\kappa(\alpha,\alpha)}$ for any $\alpha,\beta\in\Delta$. Let $A(\Pi) := \{\phi \text{ automorphism of } \Delta \mid \phi(\Pi) = \Pi\}$; this is a subgroup of the automorphism group of Δ isomorphic to the automorphism group of the Dynkin diagram. The exact sequence of group homomorphisms

$$1 \longrightarrow I(\mathfrak{g}) \longrightarrow A(\mathfrak{g}) \longrightarrow A(\Pi) \longrightarrow 1$$

splits, i.e., there is a group homomorphism section $A(\Pi)\longrightarrow A(\mathfrak{g})$. For the Dynkin diagrams A_1, B_r, C_r, G_2, F_4, E_7, E_8, we have $A(\Pi) = \{1\}$. For the Dynkin diagrams A_r with $r\geq 2$, D_r with $r\geq 5$, and E_6, we have $A(\Pi) = \mathbb{Z}/2\mathbb{Z}$. Finally, the Dynkin diagram D_4 has $A(\Pi) = \mathfrak{S}_3$, the group of permutations of three elements.

The exponential map $\exp : \mathfrak{g}\to G = I(\mathfrak{g})$ is given by $\exp\xi = e^{\mathrm{ad}_\xi}$. Let $B_- := \{e^{\mathrm{ad}_\zeta} \mid \zeta\in\mathfrak{b}_-\}\subset G$ be the connected Lie subgroup with Lie algebra $\mathfrak{b}_-$. The coadjoint action of the Lie group B_- on the dual $\mathfrak{b}_-^* = \mathfrak{k}^\perp$ of its Lie algebra $\mathfrak{b}_-$, expressed in terms of the Lie group structure of G via the pairing defined by the Killing form κ, is given by

$$(\mathrm{Ad}_{B_-})_b^\dagger\mu = \pi_{\mathfrak{k}^\perp}(\mathrm{Ad}_{b^{-1}}\mu), \quad b\in B_-, \quad \mu\in\mathfrak{b}_-^* = \mathfrak{k}^\perp. \tag{5.24}$$

Indeed, for any $\eta\in\mathfrak{b}_-$ we have

$$\kappa\left((\mathrm{Ad}_{B_-})_b^\dagger\mu,\eta\right) = \kappa\left(\mu,\mathrm{Ad}_b\,\eta\right) = \kappa\left(\mathrm{Ad}_{b^{-1}}\mu,\eta\right) = \kappa\left(\pi_{\mathfrak{k}^\perp}\left(\mathrm{Ad}_{b^{-1}}\mu\right),\eta\right).$$

5.2 The real split normal form Toda equations on a coadjoint orbit

Fix a Chevalley basis of $\mathfrak{g}^{\mathbb{C}}$ and work with the Iwasawa decomposition (5.18) of the normal real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$, i.e., in the decomposition (5.1) we take $\mathfrak{s} = \mathfrak{b}_-$ and $\gamma = \kappa$, the Killing form of $\mathfrak{g}$. Using the same notations as before, we consider the element

$$\mu_0 = \sum_{\alpha_i\in\Pi}(e_{\alpha_i} + e_{-\alpha_i})\in\mathfrak{b}_-^* = \mathfrak{k}^\perp. \tag{5.25}$$

We have the following result due to Kostant [1979] for a different representation of the Jacobi orbit.

Theorem 5.1. *The B_--coadjoint orbit of μ_0 in $\mathfrak{b}_-^*$ is the $2r$-dimensional manifold*

$$\mathcal{O}_{\mu_0} = \left\{ \sum_{i=1}^{r} c_i h_i + \sum_{\alpha_i\in\Pi} a_i(e_{\alpha_i} + e_{-\alpha_i}) \;\middle|\; c_i\in\mathbb{R},\ a_i > 0,\ \forall\, i = 1,\ldots,r \right\}.$$

Proof. Let $\mu \in \mathfrak{b}_-^*$ be an element of the form $\mu = \sum_{i=1}^r c_i h_i + \sum_{\alpha_i \in \Pi} a_i(e_{\alpha_i} + e_{-\alpha_i})$, with $a_i > 0$, $i = 1, ..., r$. We shall show that $\mu \in \mathcal{O}_{\mu_0}$ by constructing an element $b \in B_-$ such that $(\mathrm{Ad}_{B_-})_b^\dagger \mu = \mu_0$. Let $A := \{e^{\mathrm{ad}_\xi} \mid \xi \in \mathfrak{c}\} \subset B_-$ be the Lie group associated to the Cartan subalgebra $\mathfrak{c}$.

Consider the linear system

$$\sum_{j=1}^r \lambda_j \langle \alpha_i, h_j \rangle = \log a_i, \quad \text{for } i = 1, ..., r. \tag{5.26}$$

Since the matrix of the system is the Cartan matrix of $\mathfrak{g}$, it is invertible and hence this system has a unique solution $\lambda_1, ..., \lambda_r$. We define $\xi := \sum_{j=1}^r \lambda_j h_j \in \mathfrak{c}$ and $h := \exp \xi \in A$.

We now choose $\eta \in \mathfrak{n}_-$ such that $\eta = -\sum_{i=1}^r c_i e_{-\alpha_i} + \zeta$, where $\zeta \in [\mathfrak{n}_-, \mathfrak{n}_-]$ is arbitrary, and define

$$b := h \exp \eta \in B_-$$

which implies

$$(\mathrm{Ad}_{B_-})_{b^{-1}}^\dagger \mu_0 \overset{(5.24)}{=} \pi_{\mathfrak{k}^\perp}\left(\mathrm{Ad}_{h \exp \eta}\, \mu_0\right) = \pi_{\mathfrak{k}^\perp}(\mathrm{Ad}_h \,\mathrm{Ad}_{\exp \eta}\, \mu_0) = \pi_{\mathfrak{k}^\perp}\left(e^{\mathrm{ad}_\xi} e^{\mathrm{ad}_\eta} \mu_0\right).$$

We observe that

$$e^{\mathrm{ad}_\eta} \mu_0 = \sum_{k=0}^\infty \frac{1}{k!} (\mathrm{ad}_\eta)^k \mu_0 = \mu_0 + [\eta, \mu_0] + \rho_1 = \mu_0 + \sum_{i=1}^r c_i h_i + \rho_1,$$

where $\rho_1 \in \mathfrak{n}_-$ and in the last equality we used

$$[\eta, \mu_0] = \left[-\sum_{i=1}^r c_i e_{-\alpha_i} + \zeta, \sum_{j=1}^r \left(e_{\alpha_j} + e_{-\alpha_j} \right) \right] = -\sum_{i,j=1}^r c_i \left[e_{-\alpha_i}, e_{\alpha_j} \right] + \rho_2 = \sum_{i=1}^r c_i h_i + \rho_2,$$

where $\rho_2 \in \mathfrak{n}_-$. We now compute $e^{\mathrm{ad}_\xi} e^{\mathrm{ad}_\eta} \mu_0 = e^{\mathrm{ad}_\xi}(\mu_0 + \sum_{i=1}^r c_i h_i + \rho_1)$. Since $\xi = \sum_{j=1}^r \lambda_j h_j \in \mathfrak{c}$, we have $e^{\mathrm{ad}_\xi} e_{-\alpha_i} = \sum_{k=1}^\infty \frac{1}{k!} \mathrm{ad}_\xi^k e_{-\alpha_i} =: \rho_4 \in \mathfrak{n}_-$, $\sum_{k=1}^\infty \frac{1}{k!} \mathrm{ad}_\xi^k \rho_1 =: \rho_5 \in \mathfrak{n}_-$, so, using the identity $[\xi, e_{\alpha_i}] = \sum_{j=1}^r \lambda_j [h_j, e_{\alpha_i}] = \sum_{j=1}^r \lambda_j \langle \alpha_i, h_j \rangle e_{\alpha_i} \overset{(5.26)}{=} (\log a_i) e_{\alpha_i}$, we get

$$\begin{aligned}
e^{\mathrm{ad}_\xi} \sum_{i=1}^r (e_{\alpha_i} + e_{-\alpha_i} + c_i h_i + \rho_1) &= \sum_{i=1}^r \left(\sum_{k=1}^\infty \frac{1}{k!} \mathrm{ad}_\xi^k e_{\alpha_i} + \sum_{k=1}^\infty \frac{1}{k!} \mathrm{ad}_\xi^k e_{-\alpha_i} + c_i \sum_{k=1}^\infty \frac{1}{k!} \mathrm{ad}_\xi^k h_i + \sum_{k=1}^\infty \frac{1}{k!} \mathrm{ad}_\xi^k \rho_1 \right) \\
&= \sum_{i=1}^r \sum_{k=1}^\infty \frac{1}{k!} \mathrm{ad}^k_{\sum_{j=1}^r \lambda_j h_j} e_{\alpha_i} + \rho_4 + \sum_{i=1}^r c_i \sum_{j=1}^r \sum_{k=1}^\infty \frac{1}{k!} \mathrm{ad}^k_{\lambda_j h_j} h_i + \rho_5 \\
&= \sum_{i=1}^r \left(\left(e_{\alpha_i} + \sum_{j=1}^r \lambda_j \langle e_{\alpha_i}, h_j \rangle e_{\alpha_i} + \frac{1}{2!} \sum_{j=1}^r \sum_{l=1}^r \lambda_l \lambda_j \langle e_{\alpha_i}, h_j \rangle \langle e_{\alpha_i}, h_l \rangle e_{\alpha_i} + \cdots \right) + c_i h_i + \rho_4 + \rho_5 \right) \\
&= \sum_{i=1}^r \left(e^{\sum_{j=1}^r \lambda_j \langle e_{\alpha_i}, h_j \rangle} e_{\alpha_i} + c_i h_i + \rho_4 + \rho_5 \right) \\
&= \sum_{i=1}^r \left(c_i h_i + e^{\log a_i} e_{\alpha_i} \right) + \rho_3 \\
&= \sum_{i=1}^r (c_i h_i + a_i e_{\alpha_i}) + \rho_3,
\end{aligned}$$

where $\rho_3 := \rho_4 + \rho_5 \in \mathfrak{n}_-$. Therefore,

$$(\mathrm{Ad}_{B_-})_{b^{-1}}^\dagger \mu_0 = \sum_{i=1}^r \pi_{\mathfrak{k}^\perp} \left(c_i h_i + a_i e_{\alpha_i} + \rho_3 \right) \overset{(5.22)}{=} \sum_{i=1}^r \left(c_i h_i + a_i \left(e_{\alpha_i} + e_{-\alpha_i} \right) \right) = \mu. \tag{5.27}$$

We have thus shown that any element of the form $\mu = \sum_{i=1}^r c_i h_i + \sum_{\alpha_i \in \Pi} a_i(e_{\alpha_i} + e_{-\alpha_i})$, with $a_i > 0$, $i = 1, ..., r$ is in the coadjoint orbit of μ_0.

Conversely, the same computation shows that any element in $\mathcal{O}_{\mu_0}$ has this form. ∎

For $\mu \in \mathcal{O}_{\mu_0}$ we have

$$\pi_{\mathfrak{k}}(\mu) \overset{(5.21)}{=} \sum_{\alpha_i \in \Pi} a_i(e_{\alpha_i} - e_{-\alpha_i}),$$

so that the $(-)$ Lie-Poisson equations (5.5) on $\mathcal{O}_{\mu_0} \subset \mathfrak{k}^{\perp}$ recover the Toda lattice equations on semisimple Lie algebras in standard notation ($L = \mu$ and $P = \pi_{\mathfrak{k}}(\mu)$):

$$\partial_t L = [P, L], \quad L = \sum_{i=1}^{r} c_i h_i + \sum_{\alpha_i \in \Pi} a_i(e_{\alpha_i} + e_{-\alpha_i}), \quad P = \sum_{\alpha_i \in \Pi} a_i(e_{\alpha_i} - e_{-\alpha_i}). \tag{5.28}$$

The Hamiltonian, evaluated on $\mathcal{O}_{\mu_0}$, is

$$h(\mu) = \frac{1}{2}\kappa(\mu, \mu) = \sum_{i,j=1}^{r} \frac{c_i C_{ij} c_j}{|\alpha_i|^2} + \sum_{i=1}^{r} a_i^2 \frac{2}{|\alpha_i|^2}. \tag{5.29}$$

We close this section by explicitly computing the Toda equations (5.28) in the coordinates given by the choice of the base Π of Δ. We have

$$\begin{aligned}
[P, L] &= \left[\sum_{\alpha_i \in \Pi} a_i(e_{\alpha_i} - e_{-\alpha_i}), \sum_{j=1}^{r} c_j h_j + \sum_{\alpha_j \in \Pi} a_j(e_{\alpha_j} + e_{-\alpha_j}) \right] \\
&= \sum_{\alpha_i \in \Pi} \sum_{j=1}^{r} a_i c_j \left[e_{\alpha_i} - e_{-\alpha_i}, h_j \right] + \sum_{\alpha_i \in \Pi} \sum_{\alpha_j \in \Pi} a_i a_j \left[e_{\alpha_i} - e_{-\alpha_i}, e_{\alpha_j} + e_{-\alpha_j} \right]
\end{aligned}$$

Since

$$[e_{\alpha_i}, h_j] - [e_{-\alpha_i}, h_j] \overset{(5.8)}{=} -\frac{2\kappa(\alpha_j, \alpha_i)}{\kappa(\alpha_j, \alpha_j)} e_{\alpha_i} + \frac{2\kappa(\alpha_j, -\alpha_i)}{\kappa(\alpha_j, \alpha_j)} e_{-\alpha_i} \overset{(5.11)}{=} -C_{ij}\left(e_{\alpha_i} + e_{-\alpha_i}\right)$$

and, using the properties of the Chevalley basis, we get

$$\begin{aligned}
&\sum_{\alpha_i \in \Pi} \sum_{\alpha_j \in \Pi} a_i a_j \left[e_{\alpha_i} - e_{-\alpha_i}, e_{\alpha_j} + e_{-\alpha_j} \right] \\
&\quad = \sum_{\alpha_i \in \Pi} a_i^2 \left[e_{\alpha_i} - e_{-\alpha_i}, e_{\alpha_i} + e_{-\alpha_i} \right] + \sum_{\alpha_i \neq \alpha_j \in \Pi} a_i a_j \left[e_{\alpha_i} - e_{-\alpha_i}, e_{\alpha_j} + e_{-\alpha_j} \right] \\
&\quad = \sum_{i=1}^{r} 2a_i^2 h_i.
\end{aligned}$$

Indeed, the second term vanishes, because it equals

$$\sum_{\alpha_i \neq \alpha_j \in \Pi} a_i a_j \left(\left[e_{\alpha_i}, e_{\alpha_j}\right] + \left[e_{\alpha_i}, e_{-\alpha_j}\right] - \left[e_{-\alpha_i}, e_{\alpha_j}\right] - \left[e_{-\alpha_i}, e_{-\alpha_j}\right] \right) = 0;$$

the second and third summands vanish since $\alpha_i - \alpha_j$ is not a root and the sum of the first and last summands is $\left[e_{\alpha_i}, e_{\alpha_j}\right] - \left[e_{-\alpha_i}, e_{-\alpha_j}\right] = N_{\alpha_i,\alpha_j} e_{\alpha_i+\alpha_j} - N_{-\alpha_i,-\alpha_j} e_{-\alpha_i-\alpha_j} = N_{\alpha_i,\alpha_j}\left(e_{\alpha_i+\alpha_j} + e_{-\alpha_i-\alpha_j}\right)$ and hence

$$\begin{aligned}
&\sum_{\alpha_i \neq \alpha_j \in \Pi} a_i a_j N_{\alpha_i,\alpha_j}\left(e_{\alpha_i+\alpha_j} + e_{-\alpha_i-\alpha_j}\right) \\
&\quad = \sum_{i<j} a_i a_j N_{\alpha_i,\alpha_j}\left(e_{\alpha_i+\alpha_j} + e_{-\alpha_i-\alpha_j}\right) + \sum_{i>j} a_i a_j N_{\alpha_i,\alpha_j}\left(e_{\alpha_i+\alpha_j} + e_{-\alpha_i-\alpha_j}\right) = 0
\end{aligned}$$

because $N_{\alpha_i,\alpha_j} = -N_{\alpha_j,\alpha_i}$. Thus, the Toda equations on $\mathcal{O}_{\mu_0}$ are

$$\dot{c}_i = 2a_i^2, \qquad \dot{a}_i = -a_i \sum_{j=1}^{r} C_{ij} c_j, \qquad i = 1, \ldots, r. \tag{5.30}$$

5.3 Real polarization and Flaschka map

We continue to work with the chosen Chevalley basis of $\mathfrak{g}^{\mathbb{C}}$. Let us show that $(B_-)_{\mu_0} = \exp[\mathfrak{n}_-, \mathfrak{n}_-]$ and therefore $(\mathfrak{b}_-)_{\mu_0} = [\mathfrak{n}_-, \mathfrak{n}_-]$. Take $b \in B_-$ such that $(\mathrm{Ad}_B)^\dagger_b \mu_0 = \mu_0$. We can write $b = h \exp \eta$, where $\eta \in \mathfrak{n}_-$ and $h \in A$. Using the same notations and computations as in Lemma 5.1, we conclude from (5.27) that

$$\sum_{i=1}^r (c_i h_i + a_i (e_{\alpha_i} + e_{-\alpha_i})) = \sum_{\alpha \in \Delta_+} (e_\alpha + e_{-\alpha}).$$

Therefore, $c_i = 0$ and $a_i = 1$ for all $i = 1, ..., r$. This shows that $b \in \exp[\mathfrak{n}_-, \mathfrak{n}_-]$.

The nilpotent Lie subalgebra $\mathfrak{n}_- = \sum_{\alpha \in \Delta_+} \mathbb{R} e_{-\alpha}$ is a real polarization associated to μ_0 (so we will work from now on with $\mathfrak{h}(\mu_0) = \mathfrak{h} = \mathfrak{n}_-$ in Definition 3.2 and §4.1). Indeed, the conditions in Remark 3.3**(iii)** are satisfied and $(\mathrm{Ad}_B)^\dagger_{(B_-)_{\mu_0}} \mathfrak{n}_- = e^{\mathrm{ad}_{[\mathfrak{n}_-, \mathfrak{n}_-]}} \mathfrak{n}_- = \mathfrak{n}_-$. We have $\mathfrak{n}_-^\circ = \mathfrak{c}$ (remember that the annihilator is taken in $\mathfrak{b}_-$). The connected subgroup $H_\circ$ in Definition 3.4**(i)** with Lie algebra $\mathfrak{n}_-$ is $H_\circ = N_- = \exp \mathfrak{n}_-$, the lower unipotent group. Therefore, the group H in Definition 3.4**(ii)** is, in this case, $H = H_\circ (B_-)_{\mu_0} = N_-$. The real polarization $\mathfrak{n}_-$ satisfies the Pukanszky condition in Lemma 3.7**(i)** since $\mu_0 + \mathfrak{c} \subset \mathcal{O}_{\mu_0}$, from Lemma 5.1.

Since $\mathrm{Ad}_{B_-} \mathfrak{n}_- = e^{\mathrm{ad}_{\mathfrak{b}_-}} \mathfrak{n}_- = \mathfrak{n}_-$ it follows that $\mathfrak{h}(\mu) = \mathfrak{h}(\mu_0) = \mathfrak{n}_-$, for all $\mu \in \mathcal{O}_{\mu_0}$ (see (4.1) for the notation). Therefore, the equivalence relation (4.2) is in this case: $\mu \sim \nu$ if and only if $\mu - \nu \in \mathfrak{n}_-^\circ = \mathfrak{c}$ if and only if the coefficients of e_{α_i} of μ and ν coincide. Therefore, we have the diffeomorphism

$$\mathcal{O}_{\mu_0} / \sim \longrightarrow \mathbb{R}^r_+, \quad [\mu]_\sim = \left[\sum_{i=1}^r c_i h_i + \sum_{i=1}^r a_i (e_{\alpha_i} + e_{-\alpha_i}) \right]_\sim \mapsto (a_1, ..., a_r).$$

Recall that $\pi_{\mu_0} : \mathcal{O}_{\mu_0} \ni \mu \mapsto \pi_{\mu_0}(\mu) =: [\mu]_\sim \in N_{\mu_0}$ denotes the quotient projection.

In order to define the Flaschka map $F : \mathcal{O}_{\mu_0} \to T^* \mathbb{R}^r_+$, we need to fix a section $s_{\mu_0} : \mathbb{R}^r_+ \to \mathcal{O}_{\mu_0}$ (see (4.9)). We will choose $s_{\mu_0}(a_1, ..., a_r) := \sum_{i=1}^r a_i (e_{\alpha_i} + e_{-\alpha_i})$. From (4.10), given $(v_1, ..., v_r) \in \mathbb{R}^r$, we need to find $\xi \in \mathfrak{b}_-$ such that $T\pi_{\mu_0}((\mathrm{ad}_{\mathfrak{b}_-})^\dagger_\xi \bar{\mu}) = (v_1, ..., v_r)$. For $\xi = \sum_{i=1}^r \xi_i h_i$ and $\bar{\mu} = \sum_{i=1}^r a_i (e_{\alpha_i} + e_{-\alpha_i})$, we have

$$(\mathrm{ad}_{\mathfrak{b}_-})^\dagger_\xi \bar{\mu} \overset{(5.2)}{=} -\pi_{\mathfrak{k}^\perp} [\xi, \bar{\mu}] = -\sum_{i,j=1}^r \xi_j a_i C_{ij} (e_{\alpha_i} + e_{-\alpha_i}),$$

where $[C_{ij}]$ is the Cartan matrix of $\mathfrak{g}^{\mathbb{C}}$, and hence we have

$$T\pi_{\mu_0} \left((\mathrm{ad}_{\mathfrak{b}_-})^\dagger_\xi \bar{\mu} \right) = \left(a_1, ..., a_r, -a_1 \sum_{j=1}^r \xi_j C_{1j}, ..., -a_r \sum_{j=1}^r \xi_j C_{rj} \right).$$

We choose $\xi_1, ..., \xi_r$ as the unique solution of the linear system $\sum_{j=1}^r \xi_j C_{ij} = -\frac{v_i}{a_i}$. For $\sigma = \sum_{i=1}^r c_i h_i \in \mathfrak{c}$ and using (4.10), we have

$$\begin{aligned} \left\langle F|_{[\mu]_\sim} (\bar{\mu} + \sigma), (v_1, ..., v_r) \right\rangle &= \kappa(\sigma, \xi) = \sum_{i,j=1}^r c_i \xi_j \kappa(h_i, h_j) = \sum_{i,j=1}^r c_i \frac{2}{\kappa(\alpha_i, \alpha_i)} \xi_j C_{ij} \\ &= -\sum_{i=1}^r c_i \frac{2}{\kappa(\alpha_i, \alpha_i)} \frac{v_i}{a_i}, \end{aligned}$$

so we proved the following statement.

Proposition 5.2. *The Flaschka map $F : \mathcal{O}_{\mu_0} \to T^* \mathbb{R}^r_+$ is given by*

$$F \left(\sum_{i=1}^r c_i h_i + \sum_{i=1}^r a_i (e_{\alpha_i} + e_{-\alpha_i}) \right) = \left(a_1, ..., a_r, -\frac{2}{|\alpha_1|^2} \frac{c_1}{a_1}, ..., -\frac{2}{|\alpha_r|^2} \frac{c_r}{a_r} \right), \tag{5.31}$$

where $|\alpha_i|^2 := \kappa(\alpha_i, \alpha_i)$. The inverse is

$$F^{-1}(u_1, ..., u_r, v_1, ..., v_r) = -\sum_{i=1}^r \frac{|\alpha_i|^2 u_i v_i}{2} h_i + \sum_{i=1}^r u_i (e_{\alpha_i} + e_{-\alpha_i}). \tag{5.32}$$

5.4 Momentum maps and symplectic structures

The goal of this section is to prove that the B_-- coadjoint orbit $\mathcal{O}_{\mu_0}$ given in Theorem 5.1 is symplectically diffeomorphic to the standard (i.e., zero magnetic term) cotangent bundle $T^*\mathbb{R}^r_+ = T^*(B_-/N_-)$.

Lemma 5.3. *Consider the section $s_{\mu_0} = \overline{\alpha}_{\nu_0} : \mathbb{R}^r_+ \to \mathcal{O}_{\mu_0}$ used above in the definition of the Flaschka map F, that is,*

$$\overline{\alpha}_{\nu_0}(a_1, ..., a_r) = \sum_{i=1}^r a_i \left(e_{\alpha_i} + e_{-\alpha_i}\right).$$

Let $\alpha_{\nu_0} \in \Omega^1(B_-)$ be the associated one-form given by $\alpha_{\nu_0}(b) := b\overline{\alpha}_{\nu_0}\left(\left[\left(\mathrm{Ad}_{B_-}\right)^\dagger_b \mu_0\right]_\sim\right)$. Then $\mathbf{d}\alpha_{\nu_0} = 0$ and hence $B_{\nu_0} = 0$ (in Theorem 3.9).

Proof. Let $\xi, \eta \in \mathfrak{b}_-$ and denote by ξ^L, η^L the associated left invariant vector fields on B_-. We have

$$\begin{aligned}
\mathbf{d}\left(\alpha_{\nu_0}\left(\xi^L\right)\right)(\eta^L)(b) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\alpha_{\nu_0}\left(\xi^L\right)(b\exp(\eta\varepsilon)\right) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \kappa\left(\overline{\alpha}_{\nu_0}(\pi_{\mu_0}((\mathrm{Ad}_{B_-})^\dagger_{b\exp(\varepsilon\eta)}\mu_0)), \xi\right) \\
&= \kappa\left(T(\overline{\alpha}_{\nu_0}\circ\pi_{\mu_0})((\mathrm{ad}_{\mathfrak{b}_-})^\dagger_\eta(\mathrm{Ad}_{B_-})^\dagger_b\mu_0), \xi\right) \\
&\overset{(5.2)}{=} -\kappa\left((\overline{\alpha}_{\nu_0}\circ\pi_{\mu_0})(\pi_{\mathfrak{k}^\perp}([\eta,\mu])), \xi\right),
\end{aligned}$$

where we denoted $\mu := (\mathrm{Ad}_{B_-})^\dagger_b\mu_0$ and used the linearity of the map

$$(\overline{\alpha}_{\nu_0}\circ\pi_{\mu_0})\left(\sum_{i=1}^r c_i h_i + a_i(e_{\alpha_i} + e_{-\alpha_i})\right) = \sum_{i=1}^r a_i(e_{\alpha_i} + e_{-\alpha_i}) \tag{5.33}$$

when regarded as a map on $\mathfrak{b}^*_- = \mathfrak{k}^\perp$. Denoting $\xi = \sum_{i=1}^r x_i h_i + \sum_{i=1}^r \xi_i e_{-\alpha_i} + \bar{\xi} \in \mathfrak{b}_-$ and $\eta = \sum_{i=1}^r y_i h_i + \sum_{i=1}^r \eta_i e_{-\alpha_i} + \bar{\eta} \in \mathfrak{b}_-$, where $\bar{\xi}, \bar{\eta} \in [\mathfrak{n}_-, \mathfrak{n}_-]$, and using the properties of the Chevalley basis, we have

$$\pi_{\mathfrak{k}^\perp}([\eta,\mu]) = -\sum_{i=1}^r \eta_i a_i h_i + \sum_{i,j=1}^r y_i a_j \left\langle \alpha_j, h_i\right\rangle (e_{\alpha_j} + e_{-\alpha_j}), \tag{5.34}$$

so that

$$\begin{aligned}
\mathbf{d}\left(\alpha_{\nu_0}\left(\xi^L\right)\right)\eta^L(b) &= -\kappa\left((\overline{\alpha}_{\nu_0}\circ\pi_{\mu_0})(\pi_{\mathfrak{k}^\perp}([\eta,\mu])), \xi\right) = -\sum_{i,j}^r \kappa\left(y_i a_j \left\langle\alpha_j, h_i\right\rangle (e_{\alpha_j} + e_{-\alpha_j}), \xi\right) \\
&= -\sum_{i,j}^r y_i a_j \xi_j \left\langle\alpha_j, h_i\right\rangle \kappa\left(e_{\alpha_j}, e_{-\alpha_j}\right).
\end{aligned}$$

We also compute

$$\begin{aligned}
[\xi,\eta] &= -\sum_{i,j=1}^r x_i\eta_j \left\langle\alpha_j, h_i\right\rangle e_{-\alpha_j} + \sum_{i,j=1}^r y_i\xi_j \left\langle\alpha_j, h_i\right\rangle e_{-\alpha_j} + \zeta, \quad \zeta \in [\mathfrak{n}_-, \mathfrak{n}_-] \\
&= \sum_{i,j=1}^r (y_i\xi_j - x_i\eta_j)\left\langle\alpha_j, h_i\right\rangle e_{-\alpha_j} + \zeta,
\end{aligned}$$

so that

$$\alpha_{\nu_0}(b)\left(\left[\xi^L, \eta^L\right]\right)(b) = \left\langle(\overline{\alpha}_{\nu_0}\circ\pi_{\mu_0})(\mu), [\xi,\eta]\right\rangle \overset{(5.33)}{=} \sum_{i,j=1}^r a_j(y_i\xi_j - x_i\eta_j)\left\langle\alpha_j, h_i\right\rangle \kappa(e_{\alpha_j}, e_{-\alpha_j})$$

and we get

$$\mathbf{d}\alpha_{\nu_0}(\xi^L, \eta^L) = \mathbf{d}\left(\alpha_{\nu_0}\left(\eta^L\right)\right)\xi^L - \mathbf{d}\left(\alpha_{\nu_0}\left(\xi^L\right)\right)\eta^L - \alpha_{\nu_0}\left(\left[\xi^L, \eta^L\right]\right) = 0,$$

as stated. ∎

Using the same notations as above, the $(-)$-orbit symplectic form on $\mathcal{O}_{\mu_0}$ is

$$\begin{aligned}\omega_{\mathcal{O}_{\mu_0}}(\mu)\left(\left(\mathrm{ad}_{\mathfrak{b}_-}\right)^\dagger_\xi\mu, \left(\mathrm{ad}_{\mathfrak{b}_-}\right)^\dagger_\eta\mu\right) &= -\kappa(\mu,[\xi,\eta]) \\ &= -\sum_{i,j=1}^r a_j(y_i\xi_j - x_i\eta_j)\left\langle\alpha_j,h_i\right\rangle\frac{2}{|\alpha_j|^2}\,,\end{aligned} \tag{5.35}$$

where we used (5.8).

Using formula (5.32), we compute the push-forward of $\omega_{\mathcal{O}_{\mu_0}}$ to $T^*\mathbb{R}^r_+$ as follows.

$$\begin{aligned}&(F_*\omega_{\mathcal{O}_{\mu_0}})(u_1,...,u_r,v_1,...,v_r)\left((\delta u_1,...,\delta v_r),(\bar\delta u_1,...,\bar\delta v_r)\right)\\ &=\omega_{\mathcal{O}_{\mu_0}}(F^{-1}(u_1,...,v_r))\left(T_{(u_1,...,v_r)}F^{-1}(\delta u_1,...,\delta v_r),T_{(u_1,...,v_r)}F^{-1}(\bar\delta u_1,...,\bar\delta v_r)\right)\\ &=\omega_{\mathcal{O}_{\mu_0}}\left(\sum_{i=1}^r\left(-\frac{|\alpha_i|^2u_iv_i}{2}h_i+u_i(e_{\alpha_i}+e_{-\alpha_i})\right)\right)\\ &\qquad\left(\sum_{i=1}^r\left(-\frac{|\alpha_i|^2}{2}(\delta u_iv_i+u_i\delta v_i)h_i+\delta u_i(e_{\alpha_i}+e_{-\alpha_i})\right),\right.\\ &\qquad\quad\left.\sum_{i=1}^r\left(-\frac{|\alpha_i|^2}{2}(\bar\delta u_iv_i+u_i\bar\delta v_i)h_i+\bar\delta u_i(e_{\alpha_i}+e_{-\alpha_i})\right)\right).\end{aligned}$$

From formula (5.34), we deduce that the vectors above are of the form $(\mathrm{ad}_{\mathfrak{b}_-})^\dagger_\xi\mu$, $(\mathrm{ad}_{\mathfrak{b}_-})^\dagger_\eta\mu$, respectively, if $\xi=\sum_{i=1}^r x_ih_i+\xi_ie_{-\alpha_i}+\bar\xi$, $\eta=\sum_{i=1}^r y_ih_i+\eta_ie_{-\alpha_i}+\bar\eta$, where $\bar\xi,\bar\eta\in[\mathfrak{n}_-,\mathfrak{n}_-]$ and

$$\begin{aligned}\xi_j &= -\frac{|\alpha_j|^2}{2u_j}(\delta u_jv_j+u_j\delta v_j) & \eta_j &= -\frac{|\alpha_j|^2}{2u_j}(\bar\delta u_jv_j+u_j\bar\delta v_j)\\ \sum_{i=1}^r C_{ji}x_i &= -\frac{\delta u_j}{u_j} & \sum_{i=1}^r C_{ji}y_i &= -\frac{\bar\delta u_j}{u_j}.\end{aligned} \tag{5.36}$$

With this choice, the expression above becomes

$$-\left\langle\mu,[\xi,\eta]\right\rangle = -\sum_{i=1}^r\frac{|\alpha_j|^2}{2}\kappa(e_{\alpha_i},e_{-\alpha_i})(\bar\delta u_i\delta v_i-\bar\delta v_i\delta u_i)\overset{(5.8)}{=}-\sum_{i=1}^r(\bar\delta u_i\delta v_i-\bar\delta v_i\delta u_i),$$

proving that

$$F_*\omega_{\mathcal{O}_{\mu_0}}=\sum_{i=1}^r du_i\wedge dv_i,$$

which is the canonical symplectic form on $T^*\mathbb{R}^r_+$, written in the Darboux coordinates u_i,v_i. If one wants to identify the orbit $\mathcal{O}_{\mu_0}$ with $T^*\mathbb{R}^r$, the following symplectic diffeomorphism

$$\begin{aligned}&\Xi : T^*\mathbb{R}^r_+\ni(u_1,\dots,u_r,v_1,\dots,v_r)\longmapsto\\ &\qquad(q_1:=\log u_1,\dots,q_r=\log u_r,p_1=u_1v_1,\dots,p_r=u_rv_r)\in T^*\mathbb{R}^r\end{aligned} \tag{5.37}$$

produces Darboux coordinates, i.e., $(\Xi\circ F)_*\omega_{\mathcal{O}_{\mu_0}}=\sum_{i=1}^r dq_i\wedge dp_i$. The inverse of (5.37) has the expression $u_i=e^{q_i},\quad v_i=p_ie^{-q_i}$, $i=1,\dots,r$.

Theorem 5.4. *The map $\Xi\circ F:\mathcal{O}_{\mu_0}\to T^*\mathbb{R}^r$ given by (5.31) and (5.37) is a symplectic diffeomorphism between the coadjoint orbit $\mathcal{O}_{\mu_0}$ endowed with the $(-)$-orbit symplectic form (5.35) and the canonical phase space $(T^*\mathbb{R}^r,\sum_{i=1}^r dq_i\wedge dp_i)$.*

Remark 5.5. When $\mathfrak{g}=\mathfrak{sl}(r+1,\mathbb{R})$, the map $(\Xi\circ F)^{-1}:T^*\mathbb{R}^r\to\mathcal{O}_{\mu_0}$ is the original Flaschka transformation if the total linear momentum of the Toda system is set equal to zero.

In terms of the variables (q_i,p_i), the Hamiltonian (5.29) reads

$$h(q_i,p_i)=\frac{1}{4}\sum_{i,j=1}^r p_iC_{ij}p_j|\alpha_j|^2+\sum_{i=1}^r e^{2q_i}\frac{2}{|\alpha_i|^2}=\frac{1}{2}\sum_{i,j=1}^r p_ip_j\kappa(\alpha_i,\alpha_j)+\sum_{i=1}^r e^{2q_i}\frac{2}{|\alpha_i|^2}\,.$$

We now change variables to rewrite the first term in the classical form $\frac{1}{2}\sum_{i=1}^{2} p_i^2$.

The $r \times r$ matrix $K_{ij} = \kappa(\alpha_i, \alpha_j) = \frac{1}{2} C_{ij} |\alpha_j|^2$, where $C := [C_{ij}]$ is the Cartan matrix, is positive definite and therefore we can form its square root $\sqrt{K^{-1}}$. (The list of the matrices K for all simple Lie algebra is given in the appendix.) We define the symplectic change of variables

$$(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{R}^r \longmapsto (\mathbf{Q}, \mathbf{P}) := (\sqrt{K^{-1}}\mathbf{q}, \sqrt{K}\mathbf{p}) \in T^*\mathbb{R}^r$$

obtained by cotangent-lift of the linear map $\mathbf{q} \mapsto \sqrt{K^{-1}}\mathbf{q}$. In the new variables, the Toda Hamiltonian reads

$$h(Q_i, P_i) = \frac{1}{2}\sum_{i=1}^{r} P_i^2 + \sum_{i=1}^{r} e^{2(\sqrt{K}\mathbf{Q})_i} \frac{2}{|\alpha_i|^2}.$$

This is one possible form of the generalized Toda system associated to a semisimple Lie algebra, as presented in Kostant [1979, formula (0.1.7)].

5.5 The Toda equations on compact real forms

The Toda system has a formulation using the Iwasawa decomposition (5.14) of $\mathfrak{g}^{\mathbb{C}}$, i.e., $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}} = \mathfrak{l} \oplus \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$, where the phase space is a coadjoint orbit of the real Lie subgroup of $G^{\mathbb{C}}$ whose real Lie algebra is $\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$. As we shall see, the equations of motion are identical. This formulation of the Toda system has been used before in Bloch, Flaschka, and Ratiu [1990] and it admits an infinite dimensional analogue in the study of the dispersionless Toda PDE, as will be shown later.

The Toda orbit in the compact real form. In the general theory presented at the beginning of Section 5.1, we take the decomposition (5.14), i.e., $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}} = \mathfrak{l} \oplus \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$, and for γ the real non-degenerate symmetric bilinear form $\operatorname{Im}\kappa : (\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}} \times (\mathfrak{g}^{\mathbb{C}})_{\mathbb{R}} \to \mathbb{R}$ that identifies $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$ with its dual. Let

$$\mathscr{B}_- := \left\{ e^{\operatorname{ad}_\xi} \mid \xi \in \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}} \right\}$$

be the connected Lie group with Lie algebra $\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$. We note that $\mathfrak{b}_- = \mathfrak{c} \oplus \mathfrak{n}_- \subsetneq \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}} \subsetneq (\mathfrak{b}_-^{\mathbb{C}})_{\mathbb{R}}$ and that $B_- \subsetneq \mathscr{B}_-$.

The dual $I^* : (\mathfrak{c} \oplus (\mathfrak{n}_-^{\mathbb{C}})_{\mathbb{R}})^* = \mathfrak{l} \to (\mathfrak{b}_-)^* = \mathfrak{k}^\perp$ of the Lie algebra inclusion $I : \mathfrak{b}_- = \mathfrak{c} \oplus \mathfrak{n}_- \to \mathfrak{c} \oplus (\mathfrak{n}_-^{\mathbb{C}})_{\mathbb{R}}$ given by

$$\begin{aligned} I^* &\left(\sum_{k=1}^{r} \mathrm{i} s_k h_k + \sum_{\alpha \in \Delta_+} \left(x_\alpha (e_\alpha - e_{-\alpha}) + \mathrm{i} y_\alpha (e_\alpha + e_{-\alpha}) \right) \right) \\ &= \sum_{k=1}^{r} s_k h_k + \sum_{\alpha \in \Delta_+} y_\alpha (e_\alpha + e_{-\alpha}). \end{aligned} \tag{5.38}$$

is a surjective linear Poisson map $I^* : \mathfrak{l} \to \mathfrak{k}^\perp$.

Given any $\nu_0 \in \left(\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right)^* = \mathfrak{l}$, the $\mathscr{B}_-$-coadjoint orbit is

$$\mathcal{O}_{\nu_0} = \left\{ \left(\operatorname{Ad}_{\mathscr{B}_-}\right)_b^* \nu_0 \mid b \in \mathscr{B}_- \right\} = \left\{ \pi_{\mathfrak{l}}(\operatorname{Ad}_{b^{-1}} \nu_0) \mid b \in \mathscr{B}_- \right\}.$$

In particular, the phase space of the Toda system is the orbit through

$$\nu_0 := \sum_{\alpha_k \in \Pi} \mathrm{i}(e_{\alpha_k} + e_{-\alpha_k}) \in (\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}})^* = \mathfrak{l}. \tag{5.39}$$

Note that $I^*(\nu_0) \overset{(5.25)}{=} \mu_0 \in \mathfrak{k}^\perp$.

In the following theorem, we endow the coadjoint orbits $\mathcal{O}_{\nu_0}, \mathcal{O}_{\mu_0}$ with their respective minus orbit symplectic forms.

Theorem 5.6. *The $\mathscr{B}_-$-coadjoint orbit of ν_0 in $\mathfrak{l}$ is*

$$\mathcal{O}_{\nu_0} = \left\{ \sum_{k=1}^{r} \mathrm{i} s_k h_k + \sum_{\alpha_k \in \Pi} R_k \mathrm{i}(e_{\alpha_k} + e_{-\alpha_k}) \,\middle|\, s_k \in \mathbb{R},\ R_k > 0,\ \forall\, k = 1, ..., r \right\}. \tag{5.40}$$

In addition, $I^*|_{\mathcal{O}_{\nu_0}} : \mathcal{O}_{\nu_0} \to \mathcal{O}_{\mu_0}$ is a symplectic diffeomorphism.

Remark 5.7. More generally, Bloch, Flaschka, and Ratiu [1996] consider coadjoint orbits of elements of the form $\nu_0 = \sum_{\alpha_k \in \Pi} \delta_k (e^{\mathrm{i}\theta_k} e_{\alpha_k} - e^{-\mathrm{i}\theta_k} e_{-\alpha_k})$, where $\delta_k \in \{0, 1\}$. In this case

$$\mathcal{O}_{\nu_0} = \left\{ \sum_{i=k}^{r} \mathrm{i} s_k h_k + \sum_{\alpha_k \in \Pi} \delta_k R_k \mathrm{i}(e^{\mathrm{i}\theta_k} e_{\alpha_k} - e^{-\mathrm{i}\theta_k} e_{-\alpha_k}) \,\middle|\, s_k \in \mathbb{R},\ R_k > 0,\ \forall\, k = 1, ..., r \right\}.$$

Proof. The proof is done in several steps.

Step 1. The following formula holds

$$\mathcal{O}_{\nu_0} = \{ \pi_{\mathfrak{l}} \operatorname{Ad}_b \nu_0 \mid b \in B_- \}. \tag{5.41}$$

The key point of this formula is that it is enough to consider the coadjoint action of B_- on ν_0 instead of the larger subgroup $\mathscr{B}_-$.

Since any element in $\mathscr{B}_-$ is of the form $e^{\operatorname{ad}_{\xi+\eta}}$, where $\xi \in \mathfrak{b}_- := \mathfrak{c} \oplus \mathfrak{n}_-$ and $\eta \in \mathrm{i}\mathfrak{n}_-$, it suffices to show that

$$\pi_{\mathfrak{l}} \left(e^{\operatorname{ad}_{\xi+\eta}} \nu_0 \right) = \pi_{\mathfrak{l}} \left(e^{\operatorname{ad}_\xi} \nu_0 \right). \tag{5.42}$$

To do this, we express the left hand side using the exponential series and we get

$$\pi_{\mathfrak{l}} \left(e^{\operatorname{ad}_{\xi+\eta}} \nu_0 \right) = \pi_{\mathfrak{l}} \left(\nu_0 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\operatorname{ad}_\xi + \operatorname{ad}_\eta \right)^k \nu_0 \right) = \pi_{\mathfrak{l}} \left(e^{\operatorname{ad}_\xi} \nu_0 \right) + \pi_{\mathfrak{l}} f(\xi, \eta),$$

where

$$f(\xi, \eta) = \operatorname{ad}_{\zeta_1} \operatorname{ad}_{\zeta_2} \cdots \operatorname{ad}_{\zeta_p} \nu_0, \qquad p = 1, 2, \ldots,$$

and each ζ_i is either ξ or η but there is at least one η. Since $\xi \in \mathfrak{c} \oplus \mathfrak{n}_-$, it follows that

$$[\xi, \nu_0] \in \mathrm{i} \left(\coprod_{\alpha_m \in \Pi} \mathfrak{g}_{\alpha_m} \oplus \mathfrak{c} \oplus \mathfrak{n}_- \right)$$

and hence

$$\operatorname{ad}_\xi^a \nu_0 \in \mathrm{i} \left(\coprod_{\alpha_m \in \Pi} \mathfrak{g}_{\alpha_m} \oplus \mathfrak{c} \oplus \mathfrak{n}_- \right), \qquad \forall\, a = 1, 2, \ldots. \tag{5.43}$$

Since $\eta \in \mathrm{i}\mathfrak{n}_-$, we have $[\eta, \nu_0] \in \mathfrak{c} \oplus \mathfrak{n}_-$ we get

$$\operatorname{ad}_\eta^a \nu_0 \in \mathfrak{c} \oplus \mathfrak{n}_-^{\mathbb{C}}, \qquad \forall\, a = 1, 2, \ldots. \tag{5.44}$$

By (5.43), it follows $\operatorname{ad}_\eta \operatorname{ad}_\xi^a \nu_0 \in \mathfrak{c} \oplus \mathfrak{n}_-$ for all $a = 1, 2, \ldots$, and hence, from (5.43), we conclude that

$$\operatorname{ad}_{\zeta_1} \cdots \operatorname{ad}_{\zeta_p} \operatorname{ad}_\eta \operatorname{ad}_\xi^a \nu_0 \in \mathfrak{c} \oplus \mathfrak{n}_-^{\mathbb{C}} \tag{5.45}$$

for all $a, p = 1, 2, \ldots$, and ζ_i equal to ξ or η. Finally, since $\operatorname{ad}_\xi \operatorname{ad}_\eta^a \nu_0 \in \mathfrak{c} \oplus \mathfrak{n}_-^{\mathbb{C}}$, relations (5.43), (5.44), and (5.45), imply that $f(\xi, \eta) \in \mathfrak{c} \oplus \mathfrak{n}_-^{\mathbb{C}}$. However, by (5.15), we have $\pi_{\mathfrak{l}} f(\xi, \eta) \in \pi_{\mathfrak{l}} \left(\mathfrak{c} \oplus \mathfrak{n}_-^{\mathbb{C}} \right) = 0$ which proves (5.42).

Step 2. Since $\left(\operatorname{Ad}_{\mathscr{B}_-}\right)_b \circ I = I \circ \left(\operatorname{Ad}_{B_-}\right)_b$, for all $b \in B_-$, using the expressions of the coadjoint action, and $I^*(\nu_0) = \mu_0$, dualizing the previous identity, we get

$$I^* \left(\pi_{\mathfrak{l}} \operatorname{Ad}_{b^{-1}} \nu_0 \right) = I^* \left(\left(\operatorname{Ad}_{\mathscr{B}_-}\right)_b^* \nu_0 \right) = \left(\operatorname{Ad}_{B_-}\right)_b^* \mu_0 = \pi_{\mathfrak{k}^\perp} \operatorname{Ad}_{b^{-1}} \mu_0, \quad \forall\, b \in B_-. \tag{5.46}$$

This shows that $I^*|_{\mathcal{O}_{\nu_0}} : \mathcal{O}_{\nu_0} \to \mathcal{O}_{\mu_0}$ is surjective since

$$I^*\left(\mathcal{O}_{\nu_0}\right) \stackrel{(5.41)}{=} \{\pi_{\mathfrak{k}^\perp} \operatorname{Ad}_{b^{-1}} \mu_0 \mid b \in B_-\} \stackrel{(5.24)}{=} \mathcal{O}_{\mu_0}.$$

Step 3. We show now that the map $I^*|_{\mathcal{O}_{\nu_0}}$ is injective. By Step 1, an arbitrary element of $\mathcal{O}_{\nu_0}$ is of the form $\pi_{\mathfrak{l}} \operatorname{Ad}_b \nu_0$, where $b \in B_-$. Let $\pi_{\mathfrak{l}} \operatorname{Ad}_{b_1} \nu_0, \pi_{\mathfrak{l}} \operatorname{Ad}_{b_2} \nu_0 \in \mathcal{O}_{\nu_0}$ be such that

$$\pi_{\mathfrak{k}^\perp} \operatorname{Ad}_{b_1} \mu_0 \stackrel{(5.46)}{=} I^*\left(\pi_{\mathfrak{l}} \operatorname{Ad}_{b_1} \nu_0\right) = I^*\left(\pi_{\mathfrak{l}} \operatorname{Ad}_{b_2} \nu_0\right) \stackrel{(5.46)}{=} \pi_{\mathfrak{k}^\perp} \operatorname{Ad}_{b_2} \mu_0,$$

which means $\operatorname{Ad}_{b_2} \mu_0 - \operatorname{Ad}_{b_2} \mu_0 \in \ker \pi_{\mathfrak{k}^\perp} = \mathfrak{n}_-$ by (5.22). Since $b_j = e^{\operatorname{ad}_{\xi^j}}$ with $\xi^j = h^j + \sum_{k=1}^r \xi_k^j e_{-\alpha_k} + \zeta^j \in \mathfrak{b}_- := \mathfrak{c} \oplus \mathfrak{n}_-$, $h^j \in \mathfrak{c}$, and $\zeta^j \in [\mathfrak{n}_-, \mathfrak{n}_-]$, this is equivalent to

$$\sum_{j=1}^r \left(e^{\operatorname{ad}_{\xi^1}} e_{\alpha_j} - e^{\operatorname{ad}_{\xi^2}} e_{\alpha_j}\right) \in \mathfrak{n}_-. \tag{5.47}$$

A direct computation shows that if $\xi = h + \sum_{j=1}^r \xi_k e_{-\alpha_k} + \zeta \in \mathfrak{b}_-$, where $h \in \mathfrak{c}$, $\zeta \in [\mathfrak{n}_-, \mathfrak{n}_-]$, and $\xi_k \in \mathbb{R}$, then for each fixed $j = 1, \ldots, r$, we have

$$e^{\operatorname{ad}_\xi} e_{\alpha_j} = e^{\langle \alpha_j, h\rangle} e_{\alpha_j} - \frac{1}{\langle \alpha_j, h\rangle}\left(e^{\langle \alpha_j, h\rangle} - 1\right) \xi_j h_j + \lambda$$

where $\lambda \in \mathfrak{n}_-$, $h_j := \left[e_{\alpha_j}, e_{-\alpha_j}\right]$. Therefore, (5.47) becomes

$$\sum_{j=1}^r \left(\left(e^{\langle \alpha_j, h^1\rangle} - e^{\langle \alpha_j, h^2\rangle}\right) e_{\alpha_j} - \left(\frac{e^{\langle \alpha_j, h^1\rangle} - 1}{\langle \alpha_j, h^1\rangle} \xi_j^1 - \frac{e^{\langle \alpha_j, h^2\rangle} - 1}{\langle \alpha_j, h^2\rangle} \xi_j^2\right) h_j\right) = 0.$$

This implies $\left\langle \alpha_j, h^1 - h^2\right\rangle = 0$ for all $j = 1, \ldots, r$, and hence $h^1 = h^2$. Since the function $x \mapsto (e^x - 1)/x$ is strictly positive for $x \in \mathbb{R}$, from the second summand in the expression above we conclude that $\xi_j^1 = \xi_j^2$, for all $j = 1, \ldots, r$, which shows that $\xi^1 - \xi^2 \in [\mathfrak{n}_-, \mathfrak{n}_-]$. This implies that $\pi_{\mathfrak{l}} \operatorname{Ad}_{b_1} \nu_0 = \pi_{\mathfrak{l}} \operatorname{Ad}_{b_2} \nu_0$ because for any $\lambda \in \mathfrak{c} \oplus \coprod_{j=1}^r \mathfrak{g}_{-\alpha_j}$, $\rho \in [\mathfrak{n}_-, \mathfrak{n}_-]$, we have

$$\pi_{\mathfrak{l}}\left(e^{\operatorname{ad}_{\lambda+\rho}} \nu_0\right) = \pi_{\mathfrak{l}}\left(e^{\operatorname{ad}_\lambda} \nu_0\right)$$

which is easily seen by counting the heights of the roots whose corresponding root spaces contain the elements λ and ρ.

Step 4. At this point we have the necessary information to show that $I^*|_{\mathcal{O}_{\nu_0}}$ is a diffeomorphism. By Steps 2 and 3, the map $I^*|_{\mathcal{O}_{\nu_0}}$ is bijective. The map $I^* : \mathfrak{l} \to \mathfrak{k}^\perp$ is clearly smooth as the dual of a linear map. Endow the coadjoint orbits $\mathcal{O}_{\nu_0}$ and $\mathcal{O}_{\mu_0}$ with their natural manifold structures (that makes them diffeomorphic to the quotient of the group giving the orbit by the stabilizer subgroup). These orbits are initial manifolds (see Ortega and Ratiu [2004, §1.1.8 and Proposition 2.3.12] or Michor [2008, §2]), and hence necessarily injectively immersed. Therefore, the composition $I^*|_{\mathcal{O}_{\nu_0}} : \mathcal{O}_{\nu_0} \hookrightarrow \mathfrak{l} \to \mathfrak{k}^\perp$ is smooth. However, the range of this map is the initial submanifold $\mathcal{O}_{\mu_0}$ and hence, by the definition of an initial submanifold, the map $I^*|_{\mathcal{O}_{\nu_0}} : \mathcal{O}_{\nu_0} \to \mathcal{O}_{\mu_0}$ is smooth. Since the (connected components of) coadjoint orbits are the symplectic leaves of Lie-Poisson spaces and I^* is Poisson, it follows that $I^*|_{\mathcal{O}_{\nu_0}}$ is symplectic, in particular an immersion. We shall prove below that $\dim \mathcal{O}_{\nu_0} = 2r$. Since $\dim \mathcal{O}_{\mu_0} = 2r$ by Theorem 5.1, this implies that $I^*|_{\mathcal{O}_{\nu_0}} : \mathcal{O}_{\nu_0} \to \mathcal{O}_{\mu_0}$ is a symplectic diffeomorphism.

To compute the dimension of $\mathcal{O}_{\nu_0}$ we check by a direct computation that the isotropy subalgebra of ν_0 in $\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ is $\left\{\xi \in \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}} \mid \pi_{\mathfrak{l}}\left(\operatorname{ad}_\xi \nu_0\right) = 0\right\} = [\mathfrak{n}_-, \mathfrak{n}_-] \oplus i\mathfrak{n}_-$ which has dimension $\dim\left(\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right) - 2r$. Thus $\dim \mathcal{O}_{\nu_0} = 2r$.

Step 5. The formula of $\mathcal{O}_{\nu_0}$ follows directly from the expression of I^*. ■

The Flaschka map on $\mathcal{O}_{\nu_0}$. With these preliminaries, we compute the Flaschka map for the $\mathscr{B}_-$-coadjoint orbit $\mathcal{O}_{\nu_0}$. We begin by noting that $\mathfrak{h}(\nu_0) := \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ is a real polarization associated to $\nu_0 \in \mathfrak{l}$. Indeed, condition (i) in Definition 3.2 holds with $\left(\mathscr{B}_-\right)_{\nu_0} = \left\{e^{\operatorname{ad}_\xi} \mid \xi \in [\mathfrak{n}_-, \mathfrak{n}_-] \oplus i\mathfrak{n}_-\right\}$, as an easy direct verification shows. Instead of checking condition (ii) in Definition 3.2, we show (iii) in Remark 3.3. If $\sum_{\alpha\in\Delta_+}(x_\alpha + \mathrm{i}y_\alpha)e_{-\alpha} \in \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$, we find that

$$\operatorname{Im}\kappa\left(\nu_0, \left[\sum_{\alpha\in\Delta_+}(x_\alpha + \mathrm{i}y_\alpha)e_{-\alpha}, \sum_{\beta\in\Delta_+}(x_\beta + \mathrm{i}y_\beta)e_{-\beta}\right]\right) = 0.$$

Next, let $|\Delta_+|$ denote the number of positive roots of $\mathfrak{g}^{\mathbb{C}}$. Then

$$2\dim\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}} = 4|\Delta_+| = (r + 2|\Delta_+|) + (2|\Delta_+| - r) = \dim\left(\mathfrak{c}\oplus\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right) + \dim\left([\mathfrak{n}_-, \mathfrak{n}_-]\oplus \mathrm{i}\mathfrak{n}_-\right),$$

which proves both conditions in Remark 3.3(iii).

Using the explicit expression (5.40) of $\mathcal{O}_{\nu_0}$ and the fact that the annihilator of $\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ in $\mathfrak{l}$ is $\mathrm{i}\mathfrak{c}$, we conclude that $\nu_0 + \left(\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right)^\circ \subset \mathcal{O}_{\nu_0}$ which verifies condition (i) in Lemma 3.7, i.e., the polarization $\mathfrak{h}(\nu_0) = \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ satisfies the Pukanszky condition.

An easy verification shows that $\mathrm{Ad}_{\mathscr{B}_-}\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}} = \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ so that $\mathfrak{h}(\nu) = \mathfrak{h}(\nu_0) = \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ for all $\nu \in \mathcal{O}_{\nu_0}$. Consequently, $\mu, \nu \in \mathcal{O}_{\nu_0}$ satisfy $\mu \sim \nu$, i.e., $\mu - \nu \in \mathrm{i}\mathfrak{c}$, if and only if they have the same coefficients of $\mathrm{i}(e_{\alpha_k} + e_{-\alpha_k})$. Therefore the map

$$\mathcal{O}_{\nu_0}/\!\sim\ \ni [\nu]_\sim \longmapsto (R_1, \ldots, R_k) \in \mathbb{R}_+^r,$$

where $\nu := \sum_{k=1}^r \mathrm{i}s_k h_k + \sum_{\alpha_k\in\Pi} R_k \mathrm{i}(e_{\alpha_k} + e_{-\alpha_k})$, $R_k > 0$, is a diffeomorphism. Recall that $\pi_{\nu_0} : \mathcal{O}_{\nu_0} \ni \nu \mapsto \pi_{\nu_0}(\nu) =: [\nu]_\sim \in N_{\nu_0}$ denotes the quotient projection.

To define the Flaschka map $F : \mathcal{O}_{\nu_0} \to T^*\mathbb{R}_+^r$, we choose the section $s_{\nu_0} : \mathbb{R}_+^r \to \mathcal{O}_{\nu_0}$ (see (4.9)), given by $s_{\nu_0}(R_1, ..., R_r) := \sum_{k=1}^r \mathrm{i}R_k(e_{\alpha_k} + e_{-\alpha_k})$. From (4.10), given $(v_1, ..., v_r) \in \mathbb{R}^r$, we need to find $\xi \in \mathfrak{c}\oplus\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ such that $-T\pi_{\nu_0}(\pi_{\mathfrak{l}}\,\mathrm{ad}_\xi\,\bar{\nu}) = (v_1, ..., v_r)$. For $\xi = \sum_{k=1}^r \xi_k h_k$ and $\bar{\nu} = \sum_{k=1}^r \mathrm{i}R_k(e_{\alpha_k} + e_{-\alpha_k})$, we have

$$\pi_{\mathfrak{l}}[\xi, \bar{\nu}] = \mathrm{i}\sum_{j,k=1}^r \xi_j R_k C_{kj}(e_{\alpha_k} + e_{-\alpha_k}),$$

where $[C_{kj}]$ is the Cartan matrix of $\mathfrak{g}^{\mathbb{C}}$, and hence we have

$$-T\pi_{\nu_0}\left(\pi_{\mathfrak{l}}[\xi, \bar{\nu}]\right) = \left(R_1, ..., R_r, -R_1\sum_{j=1}^r \xi_j C_{1j}, ..., -R_r\sum_{j=1}^r \xi_j C_{rj}\right).$$

We choose $\xi_1, ..., \xi_r$ as the unique solution of the linear system $\sum_{j=1}^r \xi_j C_{kj} = -\frac{v_k}{R_k}$. For $\sigma = \mathrm{i}\sum_{k=1}^r s_k h_k \in \mathrm{i}\mathfrak{c}$ and using (4.10), we have

$$\begin{aligned}\left\langle F|_{[\nu]_\sim}(\bar{\nu} + \sigma), (v_1, ..., v_r)\right\rangle &= \operatorname{Im}\kappa(\sigma, \xi) = \sum_{k,j=1}^r s_k\xi_j\kappa(h_k, h_j) = \sum_{k,j=1}^r s_k\frac{2}{\kappa(\alpha_k, \alpha_k)}\xi_j C_{kj}\\ &= -\sum_{k=1}^r s_k\frac{2}{\kappa(\alpha_k, \alpha_k)}\frac{v_k}{R_k},\end{aligned}$$

which shows that the Flaschka map $F : \mathcal{O}_{\nu_0} \to T^*(\mathcal{B}_-/H) = T^*\mathbb{R}_+^r$ is

$$F\left(\mathrm{i}\sum_{k=1}^r s_k h_k + \mathrm{i}\sum_{k=1}^r R_k(e_{\alpha_k} + e_{-\alpha_k})\right) = \left(R_1, ..., R_r, -\frac{2}{|\alpha_1|^2}\frac{s_1}{R_1}, ..., -\frac{2}{|\alpha_r|^2}\frac{s_r}{R_r}\right), \tag{5.48}$$

where $|\alpha_k|^2 := \kappa(\alpha_k, \alpha_k)$. The inverse $F^{-1} : T^*(\mathscr{B}_-/H) \to \mathcal{O}_{\nu_0} \subset \mathfrak{l}$ is

$$F^{-1}(u_1, ..., u_r, v_1, ..., v_r) = -\sum_{k=1}^r \frac{|\alpha_k|^2 u_k v_k}{2}h_k + \sum_{k=1}^r u_k(e_{\alpha_k} + e_{-\alpha_k}). \tag{5.49}$$

Note that the Flaschka map (5.48) is obtained by composing (5.31) with I^* on the right. The considerations above prove the following result.

Corollary 5.8. *Let $\nu_0 \in \mathfrak{l}$ be given by (5.39) and $\mathcal{O}_{\nu_0}$ its $\mathscr{B}_-$-coadjoint orbit (see (5.40)) endowed with the $(-)$ orbit symplectic form. The Flaschka symplectic diffeomorphism $F : \mathcal{O}_{\nu_0} \to T^*\mathbb{R}_+^r$ onto the canonical cotangent bundle is given by (5.48) with inverse (5.49).*

One can pass to the canonical cotangent bundle $T^*\mathbb{R}^r$ with global coordinates (q_i, p_i), as before, by defining $q_i := \log R_i$ and $p_i := -2s_i/|\alpha_i|^2$.

The Toda equations on $\mathcal{O}_{\nu_0}$. Define $k : \mathcal{O}_{\nu_0} \to \mathbb{R}$ by

$$k(\nu) := h(I^*(\nu)) = \frac{1}{2}\kappa(I^*(\nu), I^*(\nu)) = -\frac{1}{2}\kappa(\nu,\nu) \in \mathbb{R}. \tag{5.50}$$

Note that k extends to the function $\nu \mapsto -\frac{1}{2}\operatorname{Re}\kappa(\nu,\nu)$, where $\nu \in \left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$, which we shall also call $k : \left(\mathfrak{g}^{\mathbb{C}}\right)^*_{\mathbb{R}} = \left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}} \to \mathbb{R}$; k is a Casimir function on $\left(\mathfrak{g}^{\mathbb{C}}\right)^*_{\mathbb{R}}$ since it is bi-invariant. We have $\nabla k(\nu) = -\mathrm{i}\nu$, where the gradient is taken relative to $\operatorname{Im}\kappa$ and hence the Lie-Poisson equations (5.5) on $\left(\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right)^* \cong \mathfrak{l}$ become in this case

$$\dot{\nu} = -\left[\pi_{\mathfrak{l}}(\mathrm{i}\nu), \nu\right]. \tag{5.51}$$

On the $\mathscr{B}_-$-coadjoint orbit $\mathcal{O}_{\nu_0} \subset \mathfrak{l}$ these equations are identical to (5.30) which is seen by setting $s_k = c_k$ and $R_k = a_k$. Equations (5.51) are the *full Toda equations on the compact real form* $\mathfrak{l}$ of $\mathfrak{g}^{\mathbb{C}}$. Note that (5.51) is not obtained by simply pulling back by $I^* : \mathfrak{l} \to \mathfrak{k}^{\perp}$ the full Toda equations (5.5) associated to the normal real form of $\mathfrak{g}^{\mathbb{C}}$; however, due to Theorem 5.6, equations (5.51) on $\mathcal{O}_{\nu_0}$ are the pull back by I^* of the Toda equations (5.30) on $\mathcal{O}_{\mu_0}$.

5.6 The free rigid body system on the Toda orbit $\mathcal{O}_{\mu_0}$

Work on integrable systems on Borel subalgebras of semisimple Lie algebras can be found in Arhangel'skiĭ [1980], Trofimov [1980]; for an excellent survey see Trofimov and Fomenko [1987]. However, we want to point out that the system considered in this section is not the one studied in the papers just cited. We present a family of Hamiltonian systems generated by the restriction of the kinetic energy function defined in Mishchenko and Fomenko [1976, 1982] for any complex semisimple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ to the real Borel subalgebra $\mathfrak{b}_-$ of the normal real normal form $\mathfrak{g}$. Our goal is to find Flaschka variables for these systems using the general theory developed earlier.

Consider the decomposition of the dual of the lower Borel subalgebra $\mathfrak{b}_-^* = \mathfrak{k}^{\perp}$ into its Cartan subalgebra component and the rest, i.e., $\mu \in \mathfrak{k}^{\perp}$ is decomposed as $\mu = \mu_1 + \mu_2$, where $\mu_1 \in \coprod_{\alpha\in\Delta_+}\mathbb{R}(e_\alpha + e_{-\alpha})$ and $\mu_2 \in \mathfrak{c}$. We take the Hamiltonian $h : \mathfrak{b}_-^* \to \mathbb{R}$ given by

$$h(\mu) = \frac{1}{2}\kappa(\mu, \varphi_{a,b,D}(\mu)), \tag{5.52}$$

where $\varphi_{a,b,D} : \mathfrak{b}_-^* \to \mathfrak{b}_-^*$ has the form

$$\varphi_{a,b,D}(\mu) = \operatorname{ad}_a^{-1}\operatorname{ad}_b(\mu_1) + D(\mu_2).$$

Here $a, b \in \mathfrak{c}$, and a is a semisimple regular element, that is, $\langle\alpha, a\rangle \neq 0$ for all $\alpha \in \Delta$, and $D : \mathfrak{c} \to \mathfrak{c}$ is an arbitrary real linear operator symmetric relative to κ. In coordinates, this means that $\sum_{k=1}^r D_i^k \kappa(h_j, h_k) = \sum_{k=1}^r D_j^k \kappa(h_i, h_k)$, for all $i, j = 1, ..., r$, where the matrix $\left[D_i^j\right]$ of D in the basis $\{h_1, \ldots, h_k\}$ is defined, as customary, by $D(h_i) =: D_i^j h_j$.

We now consider the Jacobi orbit through μ_0 and we write any $\mu \in \mathcal{O}_{\mu_0}$ as $\mu = \sum_{i=1}^r c_i h_i + a_i(e_{\alpha_i} + e_{-\alpha_i})$. Using the expression (5.11) of the entries of the Cartan matrix, the Hamiltonian (5.52) becomes

$$\begin{aligned} h(\mu) &= \frac{1}{2}\kappa(\mu_2, D(\mu_2)) + \sum_{\alpha_i\in\Pi} a_i^2 \frac{\langle\alpha_i, b\rangle}{\langle\alpha_i, a\rangle}\kappa(e_{\alpha_i}, e_{-\alpha_i}) \\ &= \sum_{i,j,k=1}^r \frac{c_i C_{ik} D_j^k c_j}{|\alpha_i|^2} + \sum_{\alpha_i\in\Pi} 2\frac{a_i^2}{|\alpha_i|^2}\frac{\langle\alpha_i, b\rangle}{\langle\alpha_i, a\rangle}. \end{aligned} \tag{5.53}$$

Note that $\nabla h(\mu) = \varphi_{a,b,D}(\mu)$. If $a = b$ and $D_i^j = \delta_i^j$, we recover the Toda Hamiltonian (5.29) on $\mathcal{O}_{\mu_0}$.

We now compute the $(-)$ Lie-Poisson equations

$$\partial_t \mu = -\pi_{\mathfrak{k}^{\perp}}\left[\pi_{\mathfrak{b}_-}(\varphi_{a,b,D}(\mu)), \mu\right] \tag{5.54}$$

on the orbit $\mathcal{O}_{\mu_0}$ (see (5.4); note that this is not a Lax equation). This is just the rigid body system on the Toda orbit. We have

$$\varphi_{a,b,D}(\mu) = \operatorname{ad}_a^{-1}\operatorname{ad}_b(\mu_1) + D(\mu_2) = \sum_{i=1}^r a_i \frac{\langle\alpha_i, b\rangle}{\langle\alpha_i, a\rangle}(e_{\alpha_i} + e_{-\alpha_i}) + D(\mu_2),$$

so that

$$\pi_{\mathfrak{b}_-}(\varphi_{a,b,D}(\mu)) \overset{(5.20)}{=} \sum_{i=1}^{r} 2a_i \frac{\langle \alpha_i, b\rangle}{\langle \alpha_i, a\rangle} e_{-\alpha_i} + D(\mu_2)$$

and hence

$$\pi_{\mathfrak{k}^\perp}\left([\pi_{\mathfrak{b}_-}(\varphi_{a,b,D}(\mu)), \mu]\right) \overset{(5.22)}{=} \sum_{i=1}^{r}\left(-2a_i^2 \frac{\langle \alpha_i, b\rangle}{\langle \alpha_i, a\rangle} h_i + a_i \langle \alpha_i, D(\mu_2)\rangle (e_{\alpha_i} + e_{-\alpha_i})\right).$$

Thus Hamilton's equations are

$$\partial_t c_i = -2a_i^2 \frac{\langle \alpha_i, b\rangle}{\langle \alpha_i, a\rangle}, \qquad \partial_t a_i = a_i \langle \alpha_i, D(\mu_2)\rangle = a_i \sum_{j,k=1}^{r} C_{ik} D_j^k c_j. \tag{5.55}$$

In terms of the Flaschka variables $(q_i, p_i) \in T^*\mathbb{R}^r$, the Hamiltonian (5.53) becomes

$$h(q_i, p_i) = \frac{1}{4} \sum_{i,j,k=1}^{r} p_i C_{ik} D_j^k |\alpha_j|^2 p_j + 2 \sum_{i=1}^{r} e^{2q_i} \frac{\langle \alpha_i, b\rangle}{|\alpha_i|^2 \langle \alpha_i, a\rangle}.$$

Note that the matrix with entries $S_{ij} := \sum_{k=1}^{r} C_{ik} D_j^k |\alpha_j|^2$ is symmetric.

Example. For $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$, the most general operator $D : \mathfrak{c} \to \mathfrak{c}$, symmetric relative to κ, has the form

$$\begin{bmatrix} 2D_1^2 - d & D_2^1 \\ D_1^2 & 2D_2^1 - d \end{bmatrix}$$

for any $D_1^2, D_2^2, d \in \mathbb{R}$. The Hamiltonian (5.53) has the form

$$h(c_1, c_2, a_1, a_2) = 3\Big((3D_1^2 - 2\lambda)c_1^2 + 2dc_1c_2 + (3D_2^1 - 2d)c_2^2\Big) + 6\left(a_1^2 \frac{2B_1 - B_2}{2A_1 - A_2} + a_2^2 \frac{2B_2 - B_1}{2A_2 - A_1}\right)$$

and the associated equations of motion are

$$\begin{aligned} \dot{c}_1 &= -2a_1^2 \frac{2B_1 - B_2}{2A_1 - A_2}, & \dot{a}_1 &= a_1\left((3D_1^2 - 2d)c_1 + dc_2\right), \\ \dot{c}_2 &= -2a_2^2 \frac{2B_2 - B_1}{2A_2 - A_1}, & \dot{a}_2 &= a_2\left((3D_2^1 - 2d)c_2 + dc_1\right). \end{aligned}$$

In canonical Flaschka variables (q_i, p_i), the Hamiltonian has the form

$$\begin{aligned} h(q_1, q_2, p_1, p_2) =& \frac{1}{12}\Big((3D_1^2 - 2d)p_1^2 + 2dp_1p_2 + (3D_2^1 - 2d)p_2^2\Big) \\ &+ 6\left(e^{2q_1} \frac{2B_1 - B_2}{2A_1 - A_2} + e^{2q_2} \frac{2B_2 - B_1}{2A_2 - A_1}\right). \quad \blacklozenge \end{aligned}$$

It is known that the Toda system on $\mathcal{O}_{\nu_0}$ coincides with the gradient system on the (co)adjoint orbit in $\mathfrak{l}$ containing ν_0 relative to the normal metric and the height function given by $f(\nu) = \kappa(\nu, \mathrm{i}\delta)$, where $\delta := \sum_{j=1}^{r} \lambda_j$ and λ_i are defined by $\kappa(\lambda_i, h_j) = \delta_{ij}$ for all $i, j = 1, \ldots r$; see Bloch [1990]; Bloch, Brockett, and Ratiu [1990, 1992]. A direct lengthy computation leads to the following result.

Proposition 5.9. *The rigid body equation (5.54) on the Toda orbit $\mathcal{O}_{\nu_0}$ is gradient relative to the normal metric on the coadjoint orbit of $\mathfrak{l}$ containing ν_0 (for any function) if and only if $\varphi_{a,b,D}$ is a multiple of the identity, i.e., the rigid body equations are the Toda system up to a reparametrization of time.*

As in the case of the Toda system, there is a formulation of the free rigid body on the coadjoint orbit $\mathcal{O}_{\nu_0}$ given in (5.40). The Hamiltonian on $\mathcal{O}_{\nu_0}$ reads

$$k(\nu) = \frac{1}{2}\kappa\left(\varphi_{a,b,D}(I^*\nu), I^*\nu\right) = -\frac{1}{2}\kappa\left(\varphi_{a,b,D}(\nu), \nu\right). \tag{5.56}$$

We extend this function to $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$ and call it also k by defining $k(\nu) := -\frac{1}{2}\operatorname{Re}\kappa\left(\varphi_{a,b,D}(\nu),\nu\right)$ for all $\nu \in \left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}}$.

Hamilton's equations (5.4), where $\mathfrak{s} = \mathfrak{c} \oplus \left(\mathfrak{n}_{-}^{\mathbb{C}}\right)_{\mathbb{R}}$ and $\mathfrak{k}^{\perp} = \mathfrak{l}$, become hence

$$\dot{\nu} = -\pi_{\mathfrak{l}}\left[\pi_{\mathfrak{c}\oplus\left(\mathfrak{n}_{-}^{\mathbb{C}}\right)_{\mathbb{R}}}\left(\mathrm{i}\varphi_{a,b,D}(\nu)\right), \nu\right], \tag{5.57}$$

where the projections $\pi_{\mathfrak{l}}$ and $\pi_{\mathfrak{c}\oplus\left(\mathfrak{n}_{-}^{\mathbb{C}}\right)_{\mathbb{R}}}$ are given by (5.15)–(5.17). These are the *full rigid body equations* on $\mathfrak{l}$, the generalization of the full Toda system in the symmetric form associated to $\mathfrak{sl}(r+1,\mathbb{R})$ studied in Deift et al. [1992].

6 The Flaschka map and the diffeomorphism group of the annulus

There is a natural generalization of the Toda flow to a Hamiltonian system, the *dispersionless Toda PDE*, on the Lie algebra of the Fréchet Lie group $\mathrm{SDiff}(\mathcal{A})$ of measure preserving diffeomorphisms of the annulus $\mathcal{A} = S^1 \times [0,1]$. The flow of this evolutionary PDE arises physically by letting the lattice space in the finite Toda system tend to zero in a suitable fashion (see Brockett and Bloch [1990], Bloch, Flaschka, and Ratiu [1993], Bloch, Flaschka, and Ratiu [1996], Deift and McLaughlin [1992], Bloch, Golse, Paul, and Uribe [2003]). In this section we derive the Flaschka map in this infinite dimensional setting and carry out the same program for the corresponding solvable rigid body PDE.

We remark that while $\mathrm{SDiff}(\mathcal{A})$ behaves in many ways like a compact group (in fact like the special unitary group) it has no natural complexification – see e.g. Lempert [1997], Neretin [1996], Pressley and Segal [1986], Bloch, Flaschka, and Ratiu [1993], Bloch, Flaschka, and Ratiu [1996], Bloch, El Hadrami, Flaschka, and Ratiu [1997]. In this paper we work exclusively with the Lie algebra, which has a complexification.

6.1 The Flaschka map for the dispersionless Toda PDE

The Lie algebra of the Fréchet Lie group $\mathrm{SDiff}(\mathcal{A})$ of the annulus $\mathcal{A} = S^1 \times [0,1]$ consists of divergence free vector fields X on $\mathcal{A}$ tangent to the boundary. The line integral of X on each component of the boundary vanishes because X is tangent to the boundary. Thus, using Stokes' Theorem, the integral on any closed loop also vanishes because the divergence of X is zero. Consequently, X admits a stream function x, i.e., X is a Hamiltonian vector field relative to the area form on $\mathcal{A}$ and its stream function x is the Hamiltonian.

Since X is tangent to the boundary, we have

$$\frac{\partial x}{\partial \theta}(z_0,\theta) = 0, \quad \text{for} \quad z_0 = 0, 1. \tag{6.1}$$

Recall that the operation that produces the Hamiltonian vector field from a function is a Lie algebra (anti)homomorphism. However, this linear operation has a kernel, namely the constants. Thus we can identify the set of functions modulo $\mathbb{R}$ with the Lie algebra of $\mathrm{SDiff}(\mathcal{A})$. This isomorphism is best realized by working only with Hamiltonian functions x with zero average on $\mathcal{A}$. Since the average of the Poisson bracket of any two functions vanishes, the linear map that associates to a function with zero average its Hamiltonian vector field is a Lie algebra (anti)homomorphism.

Thus, we will work from now on with

$$\mathcal{F}(\mathcal{A},\mathbb{R}) := \left\{ x \in C^\infty(\mathcal{A},\mathbb{R}) \;\middle|\; \frac{\partial x}{\partial \theta}(z_0,\theta) = 0, \text{ for } z_0 = 0,1 \text{ and } \int_{\mathcal{A}} x(z,\theta)dz\,d\theta = 0 \right\} \tag{6.2}$$

endowed with the Lie bracket

$$\{x,y\}(z,\theta) := \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial z} - \frac{\partial y}{\partial \theta}\frac{\partial x}{\partial z}, \tag{6.3}$$

for all $x, y \in \mathcal{F}(\mathcal{A},\mathbb{R})$.

The complexification of this Lie algebra is

$$\mathcal{F}(\mathcal{A},\mathbb{C}) := \left\{ x \in C^\infty(\mathcal{A},\mathbb{C}) \;\middle|\; \frac{\partial x}{\partial \theta}(z_0,\theta) = 0, \text{ for } z_0 = 0,1 \text{ and } \int_{\mathcal{A}} x(z,\theta)dz\,d\theta = 0 \right\} \tag{6.4}$$

endowed with the same bracket (6.3). We shall work with Fourier series expansion of elements of $\mathcal{F}(\mathcal{A},\mathbb{C})$ and various subalgebras, i.e., we write

$$x(z,\theta) = \sum_{n=-\infty}^{\infty} x_n(z) e^{2\pi \mathrm{i} n\theta},$$

where, in view of (6.2), we have $x_n(0) = x_n(1) = 0$ for all $n \in \mathbb{Z}$ and $\int_0^1 x_0(z)dz = 0$.

Note that we have the formula

$$\left\{x_n(z)e^{2\pi \mathrm{i} n\theta}, x_m(z)e^{2\pi \mathrm{i} m\theta}\right\} = 2\pi \mathrm{i} \left(n x_n(z)x_m'(z) - m x_n'(z)x_m(z)\right) e^{2\pi \mathrm{i}(n+m)\theta} \tag{6.5}$$

We think of a complex valued function on the annulus $x(z,\theta)$ as an infinite matrix indexed by two indices: a discrete one $n \in \mathbb{Z}$ and a continuous one $z \in [0,1]$. The index n encodes the "diagonal" on which the element lies; it is the analogue of the height of a root in the root space decomposition of a complex semisimple Lie algebra. The index z "counts" how many root vectors one has for a given height n. At the "index (n,z)" we write the "matrix element" $\mathrm{i}x_n(z)$.

The Cartan subalgebra of $\mathcal{F}(\mathcal{A},\mathbb{C})$ is $\mathcal{F}([0,1],\mathbb{C})$. Hence the roots of the complex Lie algebra $\mathcal{F}(\mathcal{A},\mathbb{C})$ are the linear maps $\alpha_{n,z} : \mathcal{F}([0,1],\mathbb{C}) \to \mathbb{C}$, $\alpha_{n,z} = 2\pi i n \partial_z$, with corresponding root space $\operatorname{span}_{\mathbb{C}} e^{2\pi i n\theta}$, where the $\alpha_{n,z}$ are thought of as linear functionals on $\mathcal{F}([0,1],\mathbb{C})$. Viewed as distributions, the roots are $\alpha_{n,z}(s) = 2\pi i n \delta'(s-z)$, where $z, s \in [0,1]$ and $n \in \mathbb{Z}$. (See Saveliev and Vershik [1989], Bloch, Flaschka, and Ratiu [1996], Bloch, El Hadrami, Flaschka, and Ratiu [1997].)

Regarded as a real Lie algebra, $\mathcal{F}(\mathcal{A},\mathbb{C})$ is denoted by $\mathcal{F}(\mathcal{A},\mathbb{C})_{\mathbb{R}}$. Define the Lie subalgebra

$$\mathcal{F}_-(\mathcal{A},\mathbb{C}) := \left\{ x(z,\theta) = \sum_{n=-\infty}^{-1} x_n(z)e^{2\pi i n\theta} \in \mathcal{F}(\mathcal{A},\mathbb{C}) \;\middle|\; x_n(z) \in \mathbb{C} \right\}.$$

In Bloch, Flaschka, and Ratiu [1996] it was argued that the infinite dimensional generalization of the decomposition into real Lie subalgebras

$$\left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}} = \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}} \oplus \mathfrak{l}$$

is

$$\mathcal{F}(\mathcal{A},\mathbb{C})_{\mathbb{R}} = \mathcal{F}([0,1],\mathrm{i}\mathbb{R}) \oplus \mathcal{F}_-(\mathcal{A},\mathbb{C})_{\mathbb{R}} \oplus \mathcal{F}(\mathcal{A},\mathbb{R}), \tag{6.6}$$

where we impose the boundary conditions (6.1) and the averages of the functions in each space above is zero.

A weakly nondegenerate invariant bilinear symmetric form on $\mathcal{F}(\mathcal{A},\mathbb{C})$ is

$$\kappa(x_1,x_2) := -\int_{\mathcal{A}} x_1(z,\theta)x_2(z,\theta)dzd\theta,$$

i.e., we have

$$\kappa\left(\{x_1,x_2\},x_3\right) = \kappa\left(x_1,\{x_2,x_3\}\right)$$

an identity which is obtained by integration by parts whose boundary terms vanish because of (6.1) and the periodicity of the functions involved. The sign on the right hand side of κ is chosen such that κ is negative on $\mathfrak{l} = \mathcal{F}(\mathcal{A},\mathbb{R})$, in complete analogy with the fact that the Killing form is negative definite on the compact real form of a finite dimensional complex Lie algebra. Note that, as expected, κ is positive on the "Cartan subalgebra" $\mathfrak{c} = \mathcal{F}([0,1],\mathrm{i}\mathbb{R})$.

We use $\operatorname{Im}\kappa : \mathcal{F}(\mathcal{A},\mathbb{C})_{\mathbb{R}} \times \mathcal{F}(\mathcal{A},\mathbb{C})_{\mathbb{R}} \to \mathbb{R}$ to identify the dual $\left(\mathcal{F}(\mathcal{A},\mathbb{C})_{\mathbb{R}}\right)^*$ with $\mathcal{F}(\mathcal{A},\mathbb{C})_{\mathbb{R}}$. With this identification, the dual of the real Lie algebra $\mathcal{F}(\mathcal{A},\mathrm{i}\mathbb{R}) \oplus \mathcal{F}_-(\mathcal{A},\mathbb{C})$ is $\mathfrak{l} := \mathcal{F}(\mathcal{A},\mathbb{R})$. The projections associated to the direct sum decomposition (6.6) are

$$\pi_{\mathfrak{l}}\left(\sum_{n=-\infty}^{\infty} x_n(z)e^{2\pi \mathrm{i} n\theta}\right) = \frac{x_0(z)+\overline{x}_0(z)}{2} + \sum_{n=1}^{\infty}\left(x_n(z)e^{2\pi \mathrm{i} n\theta} + \overline{x}_n(z)e^{-2\pi \mathrm{i} n\theta}\right) \tag{6.7}$$

$$\pi_{\mathfrak{c}}\left(\sum_{n=-\infty}^{\infty} x_n(z)e^{2\pi \mathrm{i} n\theta}\right) = \frac{x_0(z)-\overline{x}_0(z)}{2} \tag{6.8}$$

$$\pi_{\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}}\left(\sum_{n=-\infty}^{\infty} x_n(z)e^{2\pi \mathrm{i} n\theta}\right) = \sum_{n=-\infty}^{-1}\left(x_n(z) - \overline{x}_{-n}(z)\right)e^{2\pi \mathrm{i} n\theta}. \tag{6.9}$$

The $(-)$ Lie-Poisson bracket on $\mathfrak{l} = \mathcal{F}(\mathcal{A},\mathbb{R})$ is given by

$$\{F,G\}_{LP}(x) = \operatorname{Im}\int_{\mathcal{A}} x(z,\theta)\left\{\frac{\delta F}{\delta x},\frac{\delta G}{\delta x}\right\}(z,\theta)dz\,d\theta, \tag{6.10}$$

where $F, G : \mathcal{F}([0,1], \mathbb{R}) \to \mathbb{R}$ and the functional derivatives are computed using the paring $\operatorname{Im}\kappa$.

We use the analogies from the finite dimensional case and take $\nu_0(z,\theta) := 2\cos(2\pi\theta)$ and

$$\mathcal{O}_{\nu_0} := \left\{ u(z) + 2v(z)\cos(2\pi\theta) \;\middle|\; u(z), v(z) \in \mathbb{R},\ \int_0^1 u(z)dz = 0, \right.$$
$$\left. v(z) > 0 \text{ for } z \in]0,1[\text{ and } v(0) = v(1) = 0 \right\}. \tag{6.11}$$

It was shown in Bloch, Flaschka, and Ratiu [1996, Theorem 4.3] that $\mathcal{O}_{\nu_0}$ is a Poisson submanifold of the Lie-Poisson space $\mathfrak{l}$. Define

$$\mathfrak{V} := \left\{ u, v : [0,1] \to \mathbb{R} \;\middle|\; \int_0^1 u(z)dz = 0,\ v(z) > 0 \text{ for } v \in]0,1[\text{ and } v(0) = v(1) = 0 \right\}$$

and endow it with the Poisson bracket

$$\{f, g\}_{\mathfrak{V}}(u, v) = 2\pi \int_0^1 v \left(\frac{\delta f}{\delta v} \partial_z \left(\frac{\delta g}{\delta u} \right) - \frac{\delta g}{\delta v} \partial_z \left(\frac{\delta f}{\delta u} \right) \right) dz, \tag{6.12}$$

where the functional derivatives are computed using the L^2-pairing on $\mathfrak{V}$. In Bloch, Flaschka, and Ratiu [1996, Theorem 4.3] it was shown (with different sign conventions and up to a factor of 2π) that the map $\Phi : \mathcal{O}_{\nu_0} \ni u(z) + 2v(z)\cos(2\pi\theta) \longmapsto (u(z), v(z)) \in \mathfrak{V}$ is a Poisson diffeomorphism, i.e.,

$$\{f, g\}_{\mathfrak{V}} \circ \Phi = \{f \circ \Phi, g \circ \Phi\}_{LP} \qquad f, g : \mathfrak{V} \to \mathbb{R}.$$

Define

$$\mathfrak{W} := \{ w, v : [0,1] \to \mathbb{R} \mid w(0) = w(1) = 0,\ v(z) > 0 \text{ for } v \in]0,1[\text{ and } v(0) = v(1) = 0 \}$$

and $\Psi : \mathfrak{V} \to \mathfrak{W}$ by $\Psi(u, v) := \left(\int_0^z u(s)ds, v \right)$. Since for $f : \mathfrak{V} \to \mathbb{R}$, we have

$$\frac{\delta(f \circ \Psi)}{\delta v} = \frac{\delta f}{\delta v} \quad \text{and} \quad \frac{\delta(f \circ \Psi)}{\delta u} = \int_z^1 \frac{\delta f}{\delta w}(s)ds\,,$$

it follows that the push forward of (6.12) by Ψ is

$$\{\mathscr{f}, \mathscr{g}\}_{\mathfrak{W}}(w, v) = -2\pi \int_0^1 v \left(\frac{\delta \mathscr{g}}{\delta v} \frac{\delta \mathscr{f}}{\delta w} - \frac{\delta \mathscr{f}}{\delta v} \frac{\delta \mathscr{g}}{\delta w} \right) dz \tag{6.13}$$

where $\mathscr{f}, \mathscr{g} : \mathfrak{W} \to \mathbb{R}$. For a given function $\mathscr{h} : \mathfrak{W} \to \mathbb{R}$, the Hamiltonian vector field relative to the Poisson bracket (6.13) has the expression

$$X_{\mathscr{h}}(w, v) = 2\pi \left(v \frac{\delta \mathscr{h}}{\delta v}, -v \frac{\delta \mathscr{h}}{\delta w} \right). \tag{6.14}$$

Note that, formally, the Poisson bracket (6.13) is associated to the weak symplectic form $\Omega_{\mathfrak{W}}$ given by

$$\Omega_{\mathfrak{W}}(w, v)\left((\delta w_1, \delta v_1), (\delta w_2, \delta v_2) \right) = \frac{1}{2\pi} \int_0^1 \frac{1}{v(z)} \left(\delta w_1(z) \delta v_2(z) - \delta v_1(z) \delta w_2(z) \right) dz. \tag{6.15}$$

The pull back of $\Omega_{\mathfrak{W}}$ to $\mathcal{O}_{\nu_0}$ is

$$\begin{aligned} &\left((\Phi \circ \Psi)^* \Omega_{\mathfrak{W}} \right) (u + 2v\cos(2\pi\theta)) \left(\delta u_1 + 2\delta v_1 \cos(2\pi\theta), \delta u_2 + 2\delta v_2 \cos(2\pi\theta) \right) \\ &= \frac{1}{2\pi} \int_0^1 \frac{1}{v(z)} \left(\left(\int_0^z \delta u_1(s)ds \right) \delta v_2(z) - \left(\int_0^z \delta u_2(s)ds \right) \delta v_1(z) \right) dz. \end{aligned} \tag{6.16}$$

We shall prove now that this symplectic form coincides with the orbit symplectic form

$$\begin{aligned} &\omega_{\mathcal{O}_{\nu_0}}(u + 2v\cos(2\pi\theta)) \left(\pi_{\mathfrak{l}}\{x, u + 2v\cos(2\pi\theta)\}, \pi_{\mathfrak{l}}\{y, u + 2v\cos(2\pi\theta)\} \right) \\ &:= \operatorname{Im} \int_{\mathcal{A}} (u(z) + 2v(z)\cos(2\pi\theta)) \{x, y\}(z, \theta) dz\, d\theta\,, \end{aligned} \tag{6.17}$$

induced by the Lie-Poisson bracket on $\mathcal{O}_{\nu_0}$, where $x(z,\theta) = \mathrm{i}x_0(z) + \sum_{n<0} x_n(z)e^{2\pi \mathrm{i} n\theta}$, $y(z,\theta) = \mathrm{i}y_0(z) + \sum_{n<0} y_n(z)e^{2\pi \mathrm{i} n\theta} \in \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$, i.e., $x_0(z), y_0(z) \in \mathbb{R}$.

A direct computation, using (6.7), yields (we denote, for convenience, sometimes $' := \partial_z$)

$$\left(\pi_{\mathfrak{l}}\{x, u + 2v\cos(2\pi\theta)\}\right)(z,\theta) = 2\pi\left((v\,\mathrm{Im}(x_{-1})' + 2vx_0'\cos(2\pi\theta)\right)$$

$$\int_{\mathcal{A}} (u(z) + 2v(z)\cos(2\pi\theta))\,\{x,y\}(z,\theta)dz\,d\theta = -2\pi\int_0^1 v(z)\left(x_0'(z)y_{-1}(z) - y_0'(z)x_{-1}(z)\right)dz$$

which shows that

$$\begin{aligned}
&\omega_{\mathcal{O}_{\nu_0}}(u + 2v\cos(2\pi\theta))\left(\pi_{\mathfrak{l}}\{x, u+2v\cos(2\pi\theta)\}, \pi_{\mathfrak{l}}\{y, u+2v\cos(2\pi\theta)\}\right)\\
&= \omega_{\mathcal{O}_{\nu_0}}(u + 2v\cos(2\pi\theta))\left(2\pi\left((v\,\mathrm{Im}(x_{-1})' + 2vx_0'\cos(2\pi\theta)\right), 2\pi\left((v\,\mathrm{Im}(y_{-1})' + 2vy_0'\cos(2\pi\theta)\right)\right)\\
&= -2\pi\int_0^1 v(z)\left(x_0'(z)\,\mathrm{Im}\,y_{-1}(z) - y_0'(z)\,\mathrm{Im}\,x_{-1}(z)\right)dz.
\end{aligned}$$

Defining

$$\begin{cases}
\delta u_1 := 2\pi\left(v\,\mathrm{Im}\,x_{-1}\right)' \implies \mathrm{Im}\,x_{-1} = \dfrac{1}{2\pi v}\displaystyle\int_0^z \delta u_1(s)ds\\
\delta u_2 := 2\pi\left(v\,\mathrm{Im}\,y_{-1}\right)' \implies \mathrm{Im}\,y_{-1} = \dfrac{1}{2\pi v}\displaystyle\int_0^z \delta u_2(s)ds\\
\delta v_1 := 2\pi v x_0' \implies x_0' = \dfrac{1}{2\pi v}\delta v_1\\
\delta v_2 := 2\pi v y_0' \implies y_0' = \dfrac{1}{2\pi v}\delta v_2
\end{cases}$$

the previous formula becomes

$$\frac{1}{2\pi}\int_0^1 \frac{1}{v(z)}\left(\left(\int_0^z \delta u_1(s)ds\right)\delta v_2(z) - \left(\int_0^z \delta u_2(s)ds\right)\delta v_1(z)\right)dz$$

which proves that $(\Phi\circ\Psi)^*\,\Omega_{\mathfrak{W}} = \omega_{\mathcal{O}_{\nu_0}}$. We summarize the considerations above in the following theorem.

Theorem 6.1. *The submanifold $\mathcal{O}_{\nu_0} = \{u + 2v\cos(2\pi\theta) \mid u, v : [0,1] \to \mathbb{R}, v(z) > 0$ for $z \neq 0, 1,\ v(0) = v(1) = 0\}$ is a Poisson submanifold of the $(-)$-Lie-Poisson space $\mathcal{F}(\mathcal{A}, \mathbb{R})$ (see (6.10)). The Poisson structure arises formally from the weak $(-)$-orbit symplectic form (6.17). On the space $\mathfrak{W} = \{(w,v) \mid w, v : [0,1] \to \mathbb{R}, v(z) > 0$ for $v \in]0,1[$ and $v(0) = v(1) = 0\}$, the $(-)$-Lie-Poisson bracket takes the more convenient form (6.13), the expression of the Hamiltonian vector field is (6.14), and the associated weak symplectic form is (6.15).*

At this point we have all the ingredients to compute the Flaschka map. We begin with the computation of the isotropy Lie algebra of ν_0. If $x(z,\theta) = \mathrm{i}x_0(z) + \sum_{n<0} x_n(z)e^{2\pi \mathrm{i} n\theta} \in \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$, where $x_0(z) \in \mathbb{R}$ and $x_n(z) \in \mathbb{C}$, using the formula for the coadjoint action

$$\mathrm{ad}_x^*\,\nu_0 = -\pi_{\mathfrak{l}}\{x, \nu_0\},$$

of $\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ on $\mathfrak{l}$, we get

$$\left(\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right)_{\nu_0} = \left\{x_{-1}(z)e^{-2\pi \mathrm{i}\theta} + \sum_{n<-1} x_n(z)e^{2\pi \mathrm{i} n\theta}\ \middle|\ x_{-1}(z) \in \mathbb{R},\ x_n(z) \in \mathbb{C}\right\}$$

which is the analogue of $[\mathfrak{n}_-, \mathfrak{n}_-] \oplus \mathrm{i}\mathfrak{n}_-$ in finite dimensions.

Let $\mathscr{B}_-$ be the formal adjoint group with Lie algebra $\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$. Its adjoint action on $\mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ is hence

$$\sum_{k=0}^{\infty} \frac{1}{k!}\,\mathrm{ad}_x^k\,y$$

where $x, y \in \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ and $\mathrm{ad}_x\,y := \{x, y\}$.

Now we verify that $\mathfrak{h}(\nu_0) := \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ is a polarization associated to $\nu_0 = 2\cos(2\pi\theta)$. Using (6.5) and the formal exponential series, one can see immediately that the conditions in Definition 3.2 hold. Since the annihilator

$\left(\left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}\right)^\circ$ in $\mathfrak{l}$ equals $\mathcal{F}([0,1],\mathbb{R})$, it is clear that $\nu_0 + \mathcal{F}([0,1],\mathbb{R}) \subset \mathcal{O}_{\nu_0}$, i.e., this polarization satisfies the Pukanszky condition (see Lemma 3.7). The associated equivalence relation is hence

$$u_1(z) + 2v_1(z)\cos(2\pi\theta) \sim u_2(z) + 2v_2(z)\cos(2\pi\theta) \iff v_1(z) = v_2(z).$$

Thus, $N_{\nu_0} = \mathcal{O}_{\nu_0}/\!\sim\, = \{v(z) \mid v(z) > 0\}$.

To define the Flaschka map $F : \mathcal{O}_{\nu_0} \to T^*\mathcal{F}([0,1],\mathbb{R}_+) = \mathcal{F}([0,1],\mathbb{R}_+) \times \mathcal{F}([0,1],\mathbb{R})$, we choose the section $s_{\nu_0} : \mathcal{F}([0,1],\mathbb{R}_+) \to \mathcal{O}_{\nu_0}$, given by $(s_{\nu_0}(v))(z) := 2v(z)\cos(2\pi\theta)$. Given $f \in \mathcal{F}([0,1],\mathbb{R})$, we need to find $x \in \mathfrak{c} \oplus \left(\mathfrak{n}_-^{\mathbb{C}}\right)_{\mathbb{R}}$ such that $-T\pi_{\nu_0}\left(\pi_{\mathfrak{l}}\left(\{x, 2v\cos(2\pi\theta)\}\right)\right) = f$. For $x(z,\theta) = \mathrm{i}x_0(z)$, $x_0(z) \in \mathbb{R}$, we have

$$\pi_{\mathfrak{l}}\left(\{x, 2v\cos(2\pi\theta)\}\right) = \pi_{\mathfrak{l}}\left(2\pi x_0'(z)v(z)\left(e^{2\pi\mathrm{i}\theta} - e^{-2\pi\mathrm{i}\theta}\right)\right) \overset{(6.7)}{=} 4\pi x_0'(z)v(z)\cos(2\pi\theta).$$

Thus we get

$$T\pi_{\nu_0}\left(\pi_{\mathfrak{l}}\left(\{x, 2v\cos(2\pi\theta)\}\right)\right) = 2\pi x_0'(z)v(z)$$

and hence the desired x is

$$x(z,\theta) = \mathrm{i}x_0(z) = -\frac{\mathrm{i}}{2\pi}\int_0^z \frac{f(s)}{v(s)}ds.$$

Recall that $w'(z) = u(z)$. Thus, following the general theory in finite dimensions, the Flaschka map is given by

$$\begin{aligned}\langle F_v\left(2v\cos(2\pi\theta) + u\right), f\rangle &= -\operatorname{Im}\int_{\mathcal{A}} u(z)x(z,\theta)dz\,d\theta = \frac{1}{2\pi}\int_0^1 u(z)\left(\int_0^z \frac{f(s)}{v(s)}ds\right)dz \\ &= -\frac{1}{2\pi}\int_0^1 w(z)\frac{f(z)}{v(z)}dz + \left[\frac{1}{2\pi}w(z)\int_0^z \frac{f(s)}{v(s)}ds\right]_{z=0}^{z=1} \\ &= -\frac{1}{2\pi}\int_0^1 w(z)\frac{f(z)}{v(z)}dz\end{aligned}$$

since $w(1) = 0$, which yields

$$F_v(2v\cos(2\pi\theta) + u) = \left(v(z), -\frac{1}{2\pi v(z)}\int_0^z u(s)ds\right).$$

This map F is, formally, a symplectic diffeomorphism between $\left(\mathcal{O}_{\nu_0}, \omega_{\mathcal{O}_{\nu_0}}\right)$ and the weak symplectic vector space $(T^*\mathcal{F}([0,1],\mathbb{R}_+) = \mathcal{F}([0,1],\mathbb{R}_+) \times \mathcal{F}([0,1],\mathbb{R}),\ \Omega_{\mathrm{can}})$, as can also be shown by a direct verification. This formula is the analogue of (5.31) for the finite dimensional normal real form and (5.48) for the compact real form.

Proceeding as in the finite dimensional case, we define

$$q(z) := \log v(z), \qquad p(z) := -\frac{1}{2\pi}w(z) \tag{6.18}$$

Note that $q(0) = q(1) = -\infty$ and $p(0) = p(1) = 0$. We require $(q,p) \in T^*\mathcal{Q}$, where

$$\mathcal{Q} := \{q : [0,1] \to \mathbb{R} \mid q(0) = q(1) = -\infty\}$$

and hence

$$\mathcal{P} := \mathcal{Q}^* := \left\{p : [0,1] \to \mathbb{R} \;\middle|\; \int_0^1 q(z)p(z)dz < \infty,\ \forall q \in \mathcal{Q}\right\}, \qquad T^*\mathcal{Q} = \mathcal{Q} \times \mathcal{P}.$$

Note that all elements of $\mathcal{P}$ necessarily satisfy $p(0) = p(1) = 0$. It is readily verified (like in the finite dimensional case at the end of Section 5.4) that the diffeomorphism (6.18) is symplectic relative to the canonical forms on $T^*\mathcal{F}([0,1],\mathbb{R}_+)$ and $T^*\mathcal{Q}$. Taking into account Theorem 6.1, this proves the following result, extending the Flaschka map from the finite Toda system to the dispersionless Toda PDE.

Theorem 6.2. *The diffeomorphism* $\mathcal{O}_{\nu_0} \ni u + 2v\cos(2\pi\theta) \mapsto (q,p) := \left(\log v, -\frac{1}{2\pi}\int_0^z u(s)ds\right) \in T^*\mathcal{Q}$ *is symplectic relative to the minus orbit symplectic form on* $\mathcal{O}_{\nu_0}$ *and the canonical symplectic form on* $T^*\mathcal{Q}$.

Therefore, this map transforms the Lie-Poisson system

$$\partial_t w = 2\pi v \frac{\delta \hbar}{\delta v}, \qquad \partial_t v = -2\pi v \frac{\delta \hbar}{\delta w} \tag{6.19}$$

with Hamiltonian $\hbar$ on the coadjoint orbit $\mathcal{O}_{\nu_0}$ to a canonical Hamiltonian system for the transformed Hamiltonian $h(q,p) = \hbar(w,v)$. The Hamiltonian for the Toda PDE (the analogue of (5.50)) is

$$\begin{aligned} H(u + 2v\cos(2\pi\theta)) &= \frac{1}{2}\int_{\mathcal{A}} (u + 2v\cos(2\pi\theta))^2 dz\, d\theta = \frac{1}{2}\int_0^1 \left(u^2 + 2v^2\right) dz \\ &= \frac{1}{2}\int_0^1 \left((w')^2 + 2v^2\right) dz \end{aligned} \tag{6.20}$$

and the associated equations of motion are

$$\partial_t w = 4\pi v^2, \qquad \partial_t v = 2\pi v \partial_z^2 w.$$

Taking into account that $w' = u$, the above equations become the standard Toda PDE, up to a factor of 2π (Takasaki and Takebe [1991], Bloch, Flaschka, and Ratiu [1996])

$$\partial_t u = 4\pi \partial_z (v^2), \qquad \partial_t v = 2\pi v \partial_z u.$$

In terms of the canonical variables $(q,p) \in T^*\mathcal{Q}$, the Hamiltonian is

$$h(q,p) = \frac{1}{2}\int_0^1 \left((2\pi p')^2 + 2e^{2q}\right) dz$$

and hence Hamilton's equations are

$$\partial_t q = -(2\pi)^2 \partial_z^2 p, \qquad \partial_t p = -2e^{2q}.$$

Eliminating p, we get

$$\partial_t^2 q = 8\pi^2 \partial_z^2 \left(e^{2q}\right).$$

6.2 The Flaschka map for the solvable rigid body PDE

We shall study a Hamiltonian PDE which is the continuum analogue of the rigid body equation on the Toda orbit presented in Subsection 5.6.

We begin by computing the sectional operator. If $a \in \mathfrak{c} = \mathcal{F}([0,1], \mathrm{i}\mathbb{R})$ and $x \in \left(\mathfrak{g}^{\mathbb{C}}\right)_{\mathbb{R}} = \mathcal{F}(\mathcal{A}, \mathbb{C})_{\mathbb{R}}$, then

$$\operatorname{ad}_a x = \{a, x\} = -\partial_z a(z) \partial_\theta x(z,\theta).$$

As expected, the θ-Fourier expansion of $\operatorname{ad}_a x$ does not have a constant term (i.e., it has no "Cartan component"). To invert the adjoint operator, i.e., solve the equation $\operatorname{ad}_a x = y$ for x, where both x, y do not have a constant term in the Fourier expansion we need $\partial_z a(z) \neq 0$ for all $z \in [0,1]$ and then

$$\left(\operatorname{ad}_a^{-1} y\right)(z,\theta) = -\frac{1}{\partial_z a}\int_0^\theta y(z,\psi) d\psi + \frac{1}{\partial_z a}\int_0^1 \left(\int_0^\theta y(z,\psi) d\psi\right) d\theta.$$

Note that the constant appearing in the second summand is chosen such that

$$\int_0^1 \left(\operatorname{ad}_a^{-1} y\right)(z,\theta) d\theta = 0,$$

i.e., $\left(\operatorname{ad}_a^{-1} y\right)(z,\theta)$ does not have a constant term in the θ-Fourier expansion, as required. Thus, if $a, b \in \mathcal{F}([0,1], \mathrm{i}\mathbb{R})$ and $\partial_z a(z) \neq 0$ for all $z \in [0,1]$, we have

$$\left(\operatorname{ad}_a^{-1} \operatorname{ad}_b x\right)(z,\theta) = \frac{\partial_z b}{\partial_z a}\left(x(z,\theta) - \int_0^1 x(z,\theta) d\theta\right)$$

and hence the sectional operator associated to $a, b \in \mathcal{F}([0,1], \mathrm{i}\mathbb{R})$ and a L^2-symmetric linear operator $D : \mathfrak{c}^{\mathbb{C}} := \mathcal{F}([0,1], \mathbb{C}) \to \mathcal{F}([0,1], \mathbb{C})$ has the expression

$$\varphi_{a,b,D}(x)(z,\theta) := \frac{\partial_z b}{\partial_z a}\left(x(z,\theta) - \int_0^1 x(z,\theta)d\theta\right) + D\left(\int_0^1 x(\cdot,\theta)d\theta\right)(z). \tag{6.21}$$

Note that on the coadjoint orbit $\mathcal{O}_{\nu_0}$ (see (6.11)), the sectional operator simplifies to

$$\varphi_{a,b,D}\left(u(z) + 2v(z)\cos 2\pi\theta\right) = Du(z) + \frac{\partial_z b}{\partial_z a} 2v(z)\cos 2\pi\theta.$$

Therefore the rigid body PDE Hamiltonian (the analogue of (5.56)) is

$$\begin{aligned} h(u + 2v\cos 2\pi\theta) &= \frac{1}{2}\int_{\mathcal{A}} \left(u(z) + 2v(z)\cos 2\pi\theta\right)\varphi_{a,b,D}\left(u(z) + 2v(z)\cos 2\pi\theta\right) dz d\theta \\ &= \int_0^1 \left(\frac{1}{2}uDu + \frac{\partial_z b}{\partial_z a}v^2\right) dz = \int_0^1 \left(\frac{1}{2}w' D(w') + \frac{\partial_z b}{\partial_z a}v^2\right) dz = \hbar(w,v). \end{aligned}$$

As expected, if D is the identity operator and $a = b$, this Hamiltonian coincides with the Toda PDE Hamiltonian (6.20). Thus, Hamilton's equations (6.19) become in this case

$$\partial_t u = 4\pi\partial_z\left(v^2\frac{\partial_z b}{\partial_z a}\right), \qquad \partial_t v = 2\pi v\partial_z(Du).$$

In terms of the Flaschka canonical variables (6.18),

$$h(q,p) = \frac{1}{2}\int_0^1 \left(2\pi p' D(2\pi p') + 2\frac{\partial_z b}{\partial_z a}e^{2q}\right) dz,$$

the associated Hamilton's equations are

$$\partial_t q = -(2\pi)^2\partial_z\left(D\partial_z p\right), \qquad \partial_t p = -2\frac{\partial_z b}{\partial_z a}e^{2q}.$$

Eliminating p, we get

$$\partial_t^2 q = 8\pi^2\partial_z\left(D\left(\partial_z\left(\frac{\partial_z b}{\partial_z a}e^{2q}\right)\right)\right).$$

7 The Flaschka map for semidirect products

So far, we have found Flaschka maps for coadjoint orbits of various Lie subalgebras of semisimple Lie algebras. We present below the Flaschka map for a very different class of coadjoint orbits, namely those in semidirect products that are topologically cotangent bundles. This time around, the target of the Flaschka map is a magnetic cotangent bundle.

Recall from Sections 2 and 4 some crucial results. Let G be a Lie group, $\mu_0 \in \mathfrak{g}^*$, G_{μ_0} the coadjoint isotropy subgroup of μ_0, and $\mathfrak{h}$ a real polarization associated to μ_0 (Definition 3.2). Let $H_\circ$ be the connected Lie subgroup whose Lie algebra is $\mathfrak{h}$. Let $H := H_\circ G_{\mu_0}$ and recall (Lemma 3.6) that both $H_\circ$ and H are closed Lie subgroups of G and that $H_{\nu_0} = H$, where $\nu_0 = i_{\mathfrak{h}}^*\mu_0 \in \mathfrak{h}^*$ and $i_{\mathfrak{h}} : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is the inclusion. Assume that $\mathfrak{h}$ satisfies Pukanszky's conditions (Lemma 3.7). Then we have the following symplectic diffeomorphisms

$$\left(\mathbf{J}_L^{-1}(\nu_0)/H, \omega_{\nu_0}\right) \xrightarrow{\varphi_{\nu_0}} (T^*(G/H), \omega_{can} - B_{\nu_0}) \underset{F}{\overset{\mathbf{J}_R^{\nu_0}}{\rightleftarrows}} \left(\mathcal{O}_{\mu_0}, \omega^-_{\mathcal{O}_{\mu_0}}\right), \tag{7.1}$$

where $\omega^-_{\mathcal{O}_{\mu_0}}$ is the negative orbit symplectic form. In this section, we apply these results for $G = M \circledS V$, the semidirect product of a Lie group M with a left representation space V.

7.1 Semidirect product reduction

Let M be a Lie group and V a left representation space of M, where the action of M on V is denoted by concatenation. Form the semidirect product $M \circledS V$ with multiplication given by

$$(m_1, u_1)(m_2, u_2) := (m_1 m_2, u_1 + m_1 u_2),$$

where $m_1, m_2 \in M$ and $u_1, u_2 \in V$. Its Lie algebra is the semidirect product Lie algebra $\mathfrak{m} \circledS V$ whose Lie bracket is

$$[(\xi_1, v_1), (\xi_2, v_2)] := ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1),$$

where $\xi_1, \xi_2 \in \mathfrak{m}$, $v_1, v_2 \in V$, and $\eta w := \frac{d}{dt}\big|_{t=0} (\exp t\eta) w$ is the induced $\mathfrak{m}$-representation on V for $\eta \in \mathfrak{m}$, $w \in V$. The adjoint and coadjoint actions of $M \circledS V$ are

$$\begin{aligned} \mathrm{Ad}_{(m,u)}(\xi, v) &= (\mathrm{Ad}_m \xi, mv - (\mathrm{Ad}_m \xi) u) \\ \mathrm{Ad}^*_{(m,u)^{-1}}(\mu, a) &= (\mathrm{Ad}^*_{m^{-1}} \mu + u \diamond (ma), ma), \end{aligned}$$

where $m \in M$, $\xi \in \mathfrak{m}$, $v \in V$, $\mu \in \mathfrak{m}^*$, $a \in V^*$; the operation $\diamond : V \times V^* \to \mathfrak{m}^*$ is defined by $\langle w \diamond b, \eta \rangle := \langle b, \eta w \rangle = -\langle \eta b, w \rangle$ for any $w \in V$, $b \in V^*$, $\eta \in \mathfrak{m}$. Note that the coadjoint orbit $\mathcal{O}_{(\mu,0)} = \mathcal{O}_\mu \times \{0\}$, where $\mathcal{O}_\mu$ is the M-coadjoint orbit containing $\mu \in \mathfrak{m}^*$.

The coadjoint action of $\mathfrak{m} \circledS V$ on $(\mathfrak{m} \circledS V)^*$ has the expression

$$\mathrm{ad}^*_{(\xi,v)}(\mu, a) = \left(\mathrm{ad}^*_\xi \mu - v \diamond a, \xi a\right).$$

Let $\mathcal{J}_L : T^*(M \circledS V) \to (\mathfrak{m} \circledS V)^*_+$,

$$\mathcal{J}_L(\alpha_m, u, a) = \left(\alpha_m m^{-1} + u \diamond a, a\right)$$

be the momentum map of the cotangent lifted left translation; the sign $+$ indicates that $\mathcal{J}_L$ is a Poisson map if one chooses the plus Lie-Poisson bracket on $(\mathfrak{m} \circledS V)^*$. Then, by standard reduction theory (Marsden and Weinstein [1974], Abraham and Marsden [1978, §4.3], Ortega and Ratiu [2004, §6.2]), the reduced symplectic manifold $\mathcal{J}_L^{-1}(\mu, a)/(M \circledS V)_{(\mu,a)}$ is symplectically diffeomorphic to $\mathcal{O}_{(\mu,a)}$ endowed with the minus orbit symplectic form $\omega^-_{\mathcal{O}_{(\mu,a)}}$; this symplectic diffeomorphism is induced on the quotient by the momentum map $\mathcal{J}_R : T^*(M \circledS V) \to (\mathfrak{m} \circledS V)^*_-$ of the lifted right translation,

$$\mathcal{J}_R(\alpha_m, u, a) = \left(m^{-1}\alpha_m, m^{-1}a\right).$$

Let $\mathcal{J}^a_L : T^*M \to \mathfrak{m}^*_a$ be the momentum map of the cotangent lift of left translation on M of the isotropy subgroup $M_a = \{m \in M \mid ma = m\}$, $\mathfrak{m}_a = \{\xi \in \mathfrak{m} \mid \xi a = 0\}$ the Lie algebra of M_a, and $\mu_a = \mu|_{\mathfrak{m}_a}$. Its expression is

$$\mathcal{J}^a_L(\alpha_m) = \left(\alpha_m m^{-1}\right)|_{\mathfrak{m}_a}$$

The semidirect product reduction theorem (see Guillemin and Sternberg [1984], Marsden, Ratiu and Weinstein [1984a,b], Ratiu [1981, 1982], Marsden et al [2007, §4.3]) states that the reduced symplectic manifold $\left(\mathcal{J}^a_L\right)^{-1}(\mu_a)/(M_a)_{\mu_a}$ is symplectically diffeomorphic to $\left(\mathcal{O}_{(\mu,a)}, \omega^-_{\mathcal{O}_{(\mu,a)}}\right)$; here $(M_a)_{\mu_a}$ is the coadjoint isotropy subgroup of M_a at $\mu_a \in \mathfrak{m}^*_a$.

We conclude this subsection by showing that

$$\left(\mathcal{J}^a_L\right)^{-1}(\mu_a)/(M_a)_{\mu_a} = (\mathbf{J}^a_L)^{-1}(\mu_a, a)/(M_a \circledS V)_{(\mu_a,a)}$$

as reduced symplectic manifolds, where $\mathbf{J}^a_L : T^*(M \circledS V) \to (\mathfrak{m}_a \circledS V)^*$ is the momentum map of the cotangent lifted action of $M_a \circledS V$ by left translations on $M \circledS V$, i.e.,

$$\mathbf{J}^a_L(\alpha_m, u, a) = \left(\alpha_m m^{-1} + u \diamond a, a\right)|_{\mathfrak{m}_a \circledS V}.$$

To see this, we first note that $(M_a \circledS V)_{(\mu_a,a)} = (M_a)_{\mu_a} \circledS V$ since $i^*_a(v \diamond a) = 0$, where $i_a : \mathfrak{m}_a \hookrightarrow \mathfrak{m}$ is the inclusion. Using again $i^*_a(v \diamond a) = 0$, it is easily seen that $\left(\mathcal{J}^a_L\right)^{-1}(\mu_a) \times V \times \{a\} = (\mathbf{J}^a_L)^{-1}(\mu_a, a)$ which proves the claim.

We summarize the considerations above in the following proposition.

Proposition 7.1. *The reduced symplectic manifold* $(\mathbf{J}_L^a)^{-1}(\mu_a, a)/(M_a \circledS V)_{(\mu_a,a)}$ *is symplectically diffeomorphic to* $\left(\mathcal{O}_{(\mu,a)}, \omega^-_{\mathcal{O}_{(\mu,a)}}\right)$.

The semidirect product reduction theorem gives additional information (see Ratiu [1981, 1982], Marsden et al [2007, Theorem 4.3.2] on the structure of the coadjoint orbits. There is a symplectic embedding of the reduced manifold

$$(\mathcal{J}_L^a)^{-1}(\mu_a)/(M_a)_{\mu_a} \quad \text{into} \quad \left(T^*\left(M/(M_a)_{\mu_a}\right), \omega_{can} - B_{\mu_a}\right),$$

where B_{μ_a} is the closed two-form on $T^*\left(M/(M_a)_{\mu_a}\right)$ obtained as the pull-back of a closed two-form on the base, induced by $\mathbf{d}\alpha_{\mu_a} \in \Omega^2(M)$; $\alpha_{\mu_a} \in \Omega^1(M)$ is chosen such that it is left $(M_a)_{\mu_a}$-invariant and has values in $(\mathcal{J}_L^a)^{-1}(\mu_a)$. This embedding is a symplectic diffeomorphism if and only if $\mathfrak{m}_a = (\mathfrak{m}_a)_{\mu_a}$.

7.2 The Flaschka map

We investigate the sequence of symplectic diffeomorphisms (7.1) for the case of a semidirect product. We take $G := M \circledS V$, $\mu_0 := (\mu, a) \in (\mathfrak{m} \circledS V)^*$. The coadjoint isotropy subgroup is $(M \circledS V)_{(\mu,a)} = \{(m, u) \in M \circledS V \mid m \in M_a,\ u \diamond a = \mu - \mathrm{Ad}^*_{m^{-1}}\mu\}$. If $(\mathfrak{m}_a)_{\mu_a} = \mathfrak{m}_a$, the Lie subalgebra $\mathfrak{h} := \mathfrak{m}_a \circledS V$ is a polarization associated to (μ, a). Indeed, an easy direct verification shows that the condition in Definition 3.2(i) holds. Next we check the condition in Remark 3.3(ii). Let $(\xi, v), (\eta, w) \in \mathfrak{m}_a \circledS V$. Then, denoting by $i_{\mathfrak{m}_a} : \mathfrak{m}_a \hookrightarrow \mathfrak{m}$ the inclusion, we get

$$\begin{aligned}\langle(\mu, a), [(\xi, v), (\eta, w)]\rangle &= \langle\mu, [\xi, \eta]\rangle + \langle a, \xi w - \eta v\rangle = \langle\mu, i_{\mathfrak{m}_a}[\xi, \eta]\rangle - \langle\xi a, w\rangle + \langle\eta a, v\rangle \\ &= \left\langle (\mathrm{ad}_{\mathfrak{m}_a})^*_\xi\, \mu_a, \eta\right\rangle = 0\end{aligned}$$

where in the third equality we used $\xi, \eta \in \mathfrak{m}_a$ and in the fourth $(\mathfrak{m}_a)_{\mu_a} = \mathfrak{m}_a$. We showed that $(\xi, v) \in \mathfrak{m}_a \circledS V$ implies $\langle(\mu, a), [(\xi, v), (\eta, w)]\rangle = 0$ for all $(\eta, w) \in \mathfrak{m}_a \circledS V$. Conversely, suppose that this identity holds, that is,

$$\left\langle \mathrm{ad}^*_\xi\, \mu, \eta\right\rangle + \langle\eta a, v\rangle - \langle\xi a, w\rangle = 0$$

for all $\eta \in \mathfrak{m}_a$ and $w \in V$. The second term vanishes because $\eta \in \mathfrak{m}_a$. If $\eta = 0$ this implies $\xi \in \mathfrak{m}_a$ and hence $(\xi, v) \in \mathfrak{m}_a \circledS V$. Knowing that $\xi \in \mathfrak{m}_a = (\mathfrak{m}_a)_{\mu_a}$ we conclude that the first term also vanishes.

Let $H_\circ$ be the connected Lie subgroup with Lie algebra $\mathfrak{m}_a \circledS V$, that is, $H_\circ = (M_a)_\circ \circledS V$, where $(M_a)_\circ$ is the connected component of the identity of M_a. Thus, the group $H = H_\circ G_{\mu_0}$ in the general theory (see §3) equals

$$\begin{aligned}&((M_a)_\circ \circledS V)(M \circledS V)_{(\mu,a)} = \\ &\left\{(m_1 m_2, u_1 + m_1 u_2) \mid m_1 \in (M_a)_\circ,\ u_1 \in V,\ m_2 \in M_a,\ u_2 \diamond a = \mu - \mathrm{Ad}^*_{m_2^{-1}}\mu\right\} = M_a \circledS V.\end{aligned}$$

Let $i_{\mathfrak{h}} : \mathfrak{m}_a \circledS V \hookrightarrow \mathfrak{m} \circledS V$ be the inclusion, so $\nu_0 = i^*_{\mathfrak{h}}\mu_0 = (\mu_a, a)$, where $\mu_a := \mu|_{\mathfrak{m}_a}$. By Lemma 3.6(ii), we have $(M_a \circledS V)_{(\mu_a,a)} = M_a \circledS V$. Since the coadjoint action of $H = M_a \circledS V$ is

$$(\mathrm{Ad}^*_H)_{(m,u)^{-1}}(\nu, b) = \left(i^*_{\mathfrak{m}_a}\left(\mathrm{Ad}^*_{m^{-1}}\nu + u \diamond mb\right), mb\right),$$

where $m \in M_a$, $u \in V$, $\nu \in \mathfrak{m}^*_a$, $b \in V^*$, we have $(M_a \circledS V)_{(\mu_a,a)} = (M_a)_{\mu_a} \circledS V$ since $i^*_{\mathfrak{m}_a}(u \diamond a) = 0$. This shows that $(M_a)_{\mu_a} = M_a$.

The polarization $\mathfrak{m}_a \circledS V$ associated to (μ, a) satisfies Pukanszky's condition in Lemma 3.7(i). Indeed, since $(\mathfrak{m}_a \circledS V)^\circ = \mathfrak{m}^\circ_a \times \{0\}$, for any $\nu \in \mathfrak{m}^\circ_a$, we have

$$(\mu, a) + (\nu, 0) \in \mathcal{O}_{(\mu,a)} = \{(\mathrm{Ad}^*_{m^{-1}}\mu + u \diamond ma, ma) \mid (m, u) \in M \circledS V\}.$$

This can be seen by choosing $m = e$ and invoking the identity $\mathfrak{m}^\circ_a = \{u \diamond a \mid u \in V\}$ (see, e.g. Marsden et al [2007, Lemma 4.2.7]).

Note that if $a = 0$ then $\mu_a = \mu$, so that we have $M_\mu = M$. All such coadjoint orbits are points. So, without loss of generality, we shall assume below that $a \neq 0$, even though, formally the results hold for $a = 0$.

Since $(M \circledS V)/(M_a \circledS V) = M/M_a = Ma$, Proposition 7.1 and (7.1) imply that we have the following symplectic diffeomorphisms

$$(\mathbf{J}_L^a)^{-1}(\mu_a, a)/(M_a \circledS V)_{(\mu_a,a)} \xrightarrow{\ \sim\ } \left(T^*(Ma), \omega_{can} - B_{(\mu_a,a)}\right) \underset{F}{\overset{\mathbf{J}_R^{(\mu_a,a)}}{\rightleftarrows}} \left(\mathcal{O}_{(\mu,a)}, \omega^-_{\mathcal{O}_{(\mu,a)}}\right)$$

in agreement with Marsden et al [2007, Theorem 4.3.2]. We note that the map Σ in Theorem 4.1 has, in this case, the expression $\mathcal{O}_{(\mu,a)}/\sim\ni[\nu,b]_\sim\mapsto b\in Ma$. We show now that its inverse is given by

$$Ma\ni b\mapsto[\mathrm{Ad}^*_{m^{-1}}\mu,b]_\sim\in\mathcal{O}_{(\mu,a)}/\sim,\quad\text{where}\quad m\in M\quad\text{is such that}\quad b=ma.$$

Indeed, given $(\nu,b)\in\mathcal{O}_{(\mu,a)}$ there is $(m,u)\in M\circledS V$ such that $(\nu,b)=\mathrm{Ad}^*_{(m,u)^{-1}}(\mu,a)=(\mathrm{Ad}^*_{m^{-1}}\mu+u\diamond(ma),ma)$, which is equivalent to $b=ma$ and $\nu=\mathrm{Ad}^*_{m^{-1}}\mu+u\diamond b$. Since $\mathfrak{m}_b^\circ=\{u\diamond b\mid u\in V\}$ (see, e.g. Marsden et al [2007, Lemma 4.2.7]), we conclude that $(\nu,b)\in(\mathrm{Ad}^*_{m^{-1}}\mu,b)+(\mathfrak{m}_b^\circ\times\{0\})=(\mathrm{Ad}^*_{m^{-1}}\mu,b)+(\mathfrak{m}_b\circledS V)\,(\nu,b)^\circ$ which means that $[\nu,b]_\sim=[\mathrm{Ad}^*_{m^{-1}}\mu,b]_\sim$.

It remains to compute the magnetic term for each $(\mu,a)\in(M\circledS V)^*$. Recall from Section 4.2 that the magnetic term $B_{(\mu_a,a)}$ and the Flaschka map F are constructed from a one-form $\alpha_{(\mu_a,a)}\in\Omega^1(M\circledS V)$ which is $M_a\circledS V$-left invariant and takes values in $(\mathbf{J}_L^a)^{-1}(\mu_a,a)$. Imposing these conditions it follows that $\alpha_{(\mu_a,a)}$ has the expression

$$\alpha_{(\mu_a,a)}(m,u)=(\alpha_{\mu_a}(m),u,a)\,, \tag{7.2}$$

where $\alpha_{\mu_a}\in\Omega^1(M)$ is M_a-left invariant and takes values in $\left(\mathcal{J}_L^a\right)^{-1}(\mu_a)$. Since $(M_a)_{\mu_a}=M_a$, these are exactly the hypotheses of the semidirect product reduction theorem that guarantee that the coadjoint orbit $\mathcal{O}_{(\mu,a)}\subset(\mathfrak{m}\circledS V)^*$ is symplectically diffeomorphic to a magnetic cotangent bundle; see the comments at the end of §7.1.

We know from Lemma 4.4 that there is a bijective correspondence between sections $s_{(\mu,a)}:Ma\cong M/M_a\cong(M\circledS V)/(M_a\circledS V)_{(\mu_a,a)}\cong\mathcal{O}_{(\mu,a)}/\sim\to\mathcal{O}_{(\mu,a)}$ (recall that $(M_a\circledS V)_{(\mu_a,a)}=(M_a)_{\mu_a}\circledS V$ and $(M_a)_{\mu_a}=M_a$ by §7.1) and one-form $\alpha_{(\mu_a,a)}\in\Omega^1(M\circledS V)$ are are left $M_a\circledS V$-invariant and take values in $\left(\mathbf{J}_L^a\right)^{-1}(\mu_a,a)$. Using (7.2) and (4.12) we obtain a bijective correspondence between the sections $s_{(\mu,a)}$ and the one-forms $\alpha_{\mu_a}\in\Omega^1(M)$ that are left M_a-invariant and take values in $(\mathcal{J}_L^a)^{-1}(\mu_a)$ given by

$$s_{(\mu,a)}(b)=\left(m^{-1}\alpha_{\mu_a}(m),b\right),\quad\text{where}\quad m\in M\quad\text{is such that}\quad m^{-1}a=b.$$

One-forms $\alpha_{\mu_a}\in\Omega^1(M)$ with the properties indicated above are obtained as μ_a-components of principal connection one-forms on $M\to M/M_a$. For example, if one chooses an arbitrary inner product on $\mathfrak{m}$, such connections are constructed explicitly in Marsden et al [2007, Theorem 4.3.3] (mechanical connections). If $\mu=0$, there is no magnetic term in the cotangent bundle.

On the other hand, if there is a vector space direct sum $\mathfrak{m}=\mathfrak{m}_a\oplus\mathfrak{b}$ with $\mathrm{Ad}_m\mathfrak{b}=\mathfrak{b}$ for all $m\in M_a$, we can define the section $s_{(\mu,a)}:Ma\to\mathcal{O}_{(\mu,a)}$ by $s_{(\mu,a)}(b):=(\mathrm{Ad}^*_{m^{-1}}\widetilde{\mu},b)$, where $ma=b$ and $\langle\widetilde{\mu},\xi+\eta\rangle=\langle\mu_a,\xi\rangle$ for all $\xi\in\mathfrak{m}_a$ and $\eta\in\mathfrak{b}$.

To explicitly write the Flaschka map $F:\mathcal{O}_{(\mu,a)}\to T^*(Ma)$ we use formulas (4.9) and (4.10). Since in this case, b is identified with $[\nu,b]_\sim$, formula (4.9) reads

$$\left\langle F|_b\left(s_{(\mu,a)}(b)+(\sigma,0)\right),v_b\right\rangle=\langle(\sigma,0),(\xi,v)\rangle\,,$$

where $\sigma\in\mathfrak{m}_b^\circ$ and $(\xi,v)\in\mathfrak{m}\circledS V$ is such that $v_b=T_{(m^{-1}\alpha_{\mu_a}(m),b)}\pi\left(\mathrm{ad}_{(\xi,v)}\left(m^{-1}\alpha_{\mu_a}(m),b\right)\right)=-\xi b$, since $\pi:\mathcal{O}_{(\mu,a)}\to Ma$ is given by $\pi(\nu,b)=b$. Thus

$$\langle F(\nu,b),\xi b\rangle=\left\langle m^{-1}\alpha_{\mu_a}(m)-\nu,\xi\right\rangle,\quad\text{where}\quad m^{-1}a=b. \tag{7.3}$$

We summarize the discussion in this subsection in the following statement.

Theorem 7.2. *Given the left representation of a Lie group M on the vector space V, let $a\in V^*$, $\mathfrak{m}_a=\{\xi\in\mathfrak{m}\mid\xi a=0\}$, $\mu\in\mathfrak{m}^*$, $\mu_a:=\mu|_{\mathfrak{m}_a}$, and $(\mathfrak{m}_a)_{\mu_a}:=\left\{\xi\in\mathfrak{m}_a\mid\mathrm{ad}^*_\xi\mu_a=0\right\}$. Assume that $(\mathfrak{m}_a)_{\mu_a}=\mathfrak{m}_a$. Then the Lie subalgebra $\mathfrak{m}_a\circledS V$ of $\mathfrak{m}\circledS V$ is a polarization associated to $(\mu,a)\in(\mathfrak{m}\circledS V)^*$ satisfying Pukanszky's condition. The Flaschka map $F:\left(\mathcal{O}_{(\mu,a)},\omega^-_{\mathcal{O}_{(\mu,a)}}\right)\to\left(T^*(Ma),\omega_{can}-B_{(\mu_a,a)}\right)$ is given by (7.3) and it is a symplectic diffeomorphism. The magnetic term $B_{(\mu_a,a)}\in\Omega^2(T^*(Ma))$ is the pull back by the cotangent bundle projection $T^*(Ma)\to Ma$ of $\beta_{(\mu_a,a)}\in\Omega^2(Ma)$, which in turn is determined by the identity $\rho^*\beta_{(\mu_a,a)}=\mathbf{d}\alpha_{(\mu_a,a)}$, where $\rho:M\to Ma$ is the orbit map. The set of one-forms $\alpha_{(\mu_a,a)}\in\Omega^1(M\circledS V)$ is in bijective correspondence with the the set of forms $\alpha_{\mu_a}\in\Omega^1(M)$, both with their corresponding invariance properties explained above, and the set of sections $s_{(\mu,a)}:Ma\to\mathcal{O}_{(\mu,a)}$. If $\mathbf{d}\alpha_{(\mu_a,a)}=0$, then $B_{(\mu_a,a)}=0$ (in particular, if $\mu=0$).*

7.3 Coadjoint orbits of the Euclidean group

Recall that the special Euclidean group $SE(3) = SO(3) \circledS \mathbb{R}^3 \ni (A, \mathbf{a})$ has multiplication

$$(\mathbf{A}_1, \mathbf{a}_1)(\mathbf{A}_2, \mathbf{a}_2) = (\mathbf{A}_1\mathbf{A}_2, \mathbf{A}_1\mathbf{a}_2 + \mathbf{a}_1).$$

Its Lie algebra $\mathfrak{se}(3)$ is identified with $\mathbb{R}^3 \circledS \mathbb{R}^3$ with Lie bracket

$$[(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)] = (\mathbf{x}_1 \times \mathbf{x}_2, \mathbf{x}_1 \times \mathbf{y}_2 - \mathbf{x}_2 \times \mathbf{y}_1), \qquad \mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2 \in \mathbb{R}^3.$$

It's dual is also identified with $\mathbb{R}^3 \times \mathbb{R}^3$ relative to the dot product taken component-wise:

$$\langle(\mathbf{u}, \mathbf{v}), (\mathbf{x}, \mathbf{y})\rangle := \mathbf{u} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{y}, \qquad \mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

With this identification, the coadjoint action of $SE(3)$ on $\mathfrak{se}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$ has the expression

$$\mathrm{Ad}^*_{(\mathbf{A},\mathbf{a})^{-1}}(\mathbf{u}, \mathbf{v}) = (\mathbf{A}\mathbf{u} + \mathbf{a} \times \mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{v})$$

and hence the induced coadjoint action of $\mathfrak{se}(3)$ is

$$\mathrm{ad}^*_{(\mathbf{x},\mathbf{y})}(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \times \mathbf{x} + \mathbf{v} \times \mathbf{y}, \mathbf{v} \times \mathbf{x}).$$

The only zero dimensional orbit is the origin. The two-dimensional orbits are of the form

$$\mathcal{O}_{(\mathbf{e},\mathbf{0})} = \{(\mathbf{A}\mathbf{e}, \mathbf{0}) \mid \mathbf{A} \in SO(3)\} = S^2_{\|\mathbf{e}\|}, \quad \mathbf{e} \neq \mathbf{0},$$

with symplectic form given by

$$\omega(\mathbf{u}, \mathbf{0})(\mathrm{ad}^*_{(\mathbf{x}_1,\mathbf{y}_1)}(\mathbf{u}, \mathbf{0}), \mathrm{ad}^*_{(\mathbf{x}_2,\mathbf{y}_2)}(\mathbf{u}, \mathbf{0})) = -\mathbf{u} \cdot (\mathbf{x}_1 \times \mathbf{x}_2)$$

which is $-1/\|\mathbf{e}\|$ times the area element of the sphere $S^2_{\|\mathbf{e}\|}$ of radius $\|\mathbf{e}\|$. All the other orbits are four dimensional, they are topologically equal to cotangent bundles of spheres and, generically, the symplectic form is magnetic. For an elementary direct proof of these statements see Marsden and Ratiu [1999, §14.7]. We shall recover these results below using the Flaschka map.

In what follows we shall use the Lie algebra isomorphism $\widehat{\ } : (\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\,,])$ given by $\widehat{\mathbf{u}}(\mathbf{v}) := \mathbf{u} \times \mathbf{v}$, for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Its inverse is denoted by $^\vee : (\mathfrak{so}(3), [\,,]) \to (\mathbb{R}^3, \times)$.

Given $\mu_0 = (\mathbf{e}, \mathbf{f}) \in \mathfrak{se}(3)^* = \mathbb{R}^3 \times \mathbb{R}^3$, with $\mathbf{f} \neq 0$, we know that $\mathfrak{h} := \mathfrak{so}(3)_{\mathbf{f}} \circledS \mathbb{R}^3 = \mathbb{R}\mathbf{f} \circledS \mathbb{R}^3$ is a real polarization associated to $(\mathbf{e}, \mathbf{f})$ verifying Pukanszky's condition, because $\mathfrak{so}(3)_{\mathbf{f}}$ is an Abelian Lie algebra. Note that $\mathbf{e}_{\mathbf{f}} = \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{f} \in \mathbb{R}\mathbf{f}$ ($\mathbf{e}_{\mathbf{f}}$ is the analogue of μ_a in the general theory).

Thus, by Theorem 7.2, the Flaschka map $F : \left(\mathcal{O}_{(\mathbf{e},\mathbf{f})}, \omega^-_{\mathcal{O}_{(\mathbf{e},\mathbf{f})}}\right) \to \left(T^*S^2_{\|\mathbf{f}\|}, \omega_{can} - B_{(\mathbf{e}_{\mathbf{f}},\mathbf{f})}\right)$ is a symplectic diffeomorphism. As we have seen in Theorem 7.2, we still need to choose the one-form on $SO(3)$ with certain invariance properties in order to compute the magnetic term $B_{(\mathbf{e}_{\mathbf{f}},\mathbf{f})}$. Define the one-form $\alpha_{\mathbf{e}_{\mathbf{f}}} \in \Omega^1(SO(3))$ by $\left\langle \alpha_{\mathbf{e}_{\mathbf{f}}}(\mathbf{A}), \dot{\mathbf{A}} \right\rangle = \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{f}\cdot(\dot{\mathbf{A}}\mathbf{A}^{-1})^\vee$, where $\mathbf{A} \in SO(3)$ and $\dot{\mathbf{A}} \in T_{\mathbf{A}}SO(3)$, or, equivalently, $\alpha_{\mathbf{e}_{\mathbf{f}}}(\mathbf{A}) = \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{A}\widehat{(\mathbf{A}^{-1}\mathbf{f})}$.

It is easily verified that $\alpha_{\mathbf{e}_{\mathbf{f}}}$ is left $SO(3)_{\mathbf{f}} = S^1_{\mathbf{f}} := \{e^{\theta\hat{\mathbf{f}}} \mid \theta \in \mathbb{R}\}$-invariant and that it takes values in $(\mathcal{J}^{\mathbf{f}}_L)^{-1}(\mathbf{e}_{\mathbf{f}})$, where $\mathcal{J}^{\mathbf{f}}_L(\alpha_{\mathbf{A}}) = \left(\alpha_{\mathbf{A}}\mathbf{A}^{-1}\right)|_{\mathfrak{so}(3)_{\mathbf{f}}}$, $\alpha_{\mathbf{A}} \in T^*_{\mathbf{A}}SO(3)$. In addition, we have

$$\mathbf{d}\alpha_{\mathbf{e}_{\mathbf{f}}}(\mathbf{A})\left(\dot{\mathbf{A}}_1, \dot{\mathbf{A}}_2\right) = \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{f}\cdot\left((\dot{\mathbf{A}}_1\mathbf{A}^{-1})^\vee \times (\dot{\mathbf{A}}_2\mathbf{A}^{-1})^\vee\right).$$

This equality follows from the general formula $\mathbf{d}\alpha(g)(u_g, v_g) = \left\langle \mu_0, [u_g g^{-1}, v_g g^{-1}]\right\rangle$, when $\alpha(g)(u_g) = \left\langle \mu_0, u_g g^{-1}\right\rangle$ on any Lie group G. Therefore, $\beta_{(\mathbf{e}_{\mathbf{f}},\mathbf{f})} \in \Omega^2(S^2_{\|\mathbf{f}\|})$ has the expression

$$\beta_{(\mathbf{e}_{\mathbf{f}},\mathbf{f})}(\mathbf{v})(\mathbf{x}_1 \times \mathbf{v}, \mathbf{x}_2 \times \mathbf{v}) := \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{v}\cdot(\mathbf{x}_1 \times \mathbf{x}_2), \quad \mathbf{v} \in S^2_{\|\mathbf{f}\|}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3.$$

The magnetic term $B_{(\mathbf{e}_{\mathbf{f}},\mathbf{f})}$ is the pull back of $\beta_{(\mathbf{e}_{\mathbf{f}},\mathbf{f})}$ by the cotangent bundle projection $T^*S^2_{\|\mathbf{f}\|} \to S^2_{\|\mathbf{f}\|}$.

Finally, the Flaschka map $F : \mathcal{O}_{(\mathbf{e},\mathbf{f})} \to T^*S^2_{\|\mathbf{f}\|}$ is given by

$$F(\mathbf{u},\mathbf{v}) = \left(\mathbf{v}, \frac{\mathbf{v}\times\mathbf{u}}{\|\mathbf{f}\|^2}\right)$$

Indeed, since $(\mathbf{u},\mathbf{v}) \in \mathcal{O}_{(\mathbf{e},\mathbf{f})}$, there exist $\mathbf{A} \in SO(3)$ and $\mathbf{a} \in \mathbb{R}^3$ such that $\mathbf{u} = \mathbf{A}\mathbf{e} + \mathbf{a} \times \mathbf{A}\mathbf{f}$ and $\mathbf{v} = \mathbf{A}\mathbf{f}$, which shows that $\mathbf{e}\cdot\mathbf{f} = \mathbf{u}\cdot\mathbf{v}$. By (7.3), we have

$$F(\mathbf{u},\mathbf{v}) \cdot (\mathbf{x}\times\mathbf{v}) = \left(\frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{v} - \mathbf{u}\right)\cdot\mathbf{x} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{f}\|^2}\mathbf{v} - \mathbf{u}\right)\cdot\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^3$, that is, $\mathbf{v} \times F(\mathbf{u},\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{f}\|^2}\mathbf{v} - \mathbf{u}$. Knowing that $F(\mathbf{u},\mathbf{v}) \in T^*_{\mathbf{v}}S^2_{\|\mathbf{f}\|}$, which means $F(\mathbf{u},\mathbf{v})\cdot\mathbf{v} = 0$, taking the cross product with $\mathbf{v}$ on the left, we get the desired formula.

Appendix

For all simple Lie algebras, the matrix K is the following

$$\mathsf{A_r}: \qquad K = \frac{1}{2(r+1)} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

$$\mathsf{B_r}: \qquad K = \frac{1}{2(2r-1)} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

$$\mathsf{C_r}: \qquad K = \frac{1}{4(r+1)} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 4 \end{bmatrix}$$

$$\mathsf{D_r}: \qquad K = \frac{1}{4(r-1)} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{bmatrix}$$

$$\mathsf{E_6}: \qquad K = \frac{1}{24} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\mathsf{E_7}: \qquad K = \frac{1}{36} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\mathsf{E_8}: \qquad K = \frac{1}{60} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$\mathsf{F_4}: \qquad K = \frac{1}{18} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix} \qquad\qquad \mathsf{G_2}: \qquad K = \frac{1}{24} \begin{bmatrix} 2 & -3 \\ -3 & 6 \end{bmatrix}$$

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