DISPERSIONLESS TODA AND TOEPLITZ OPERATORS

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ABSTRACT. In this paper we present some results on the dispersionless limit of the Toda lattice equations viewed as the semi-classical limit of an equation involving certain Toeplitz operators. For the non-periodic case the phase space is the Riemann sphere, while in the periodic case it is the torus, $\mathbb{C}/\mathbb{Z}^2$. This implies estimates on the dispersionless limit.

CONTENTS

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1. Introduction

The aim of this paper is to clarify, in a specific example, the link between large systems of ordinary differential equations and certain limiting non-linear partial differential equations. The are two ways of arriving at such a problem. One can start with a dynamical system (system of ODEs) and take a limit where the number of particles tends to infinity (a thermodynamical point of view). On the other hand, one can start with a given PDE and investigate the convergence of numerical schemes obtained by discretization. Both points of view have gained interest recently, the first one in the theory of infinite-dimensional integrable systems and the second in the numerical studies of hyperbolic partial differential equations. We will be concerned in this paper with the example of the dispersionless limit for Toda lattice, but we believe that our methods should be valid in other situations, as we will only use the Lax pair structure of the lattice and not its complete integrability. This work is an extension of that in [7]; however, we also discuss here the case of the periodic Toda lattice, in addition to the non-periodic case.

The Toda system is one of the most studied non-trivial integrable Hamiltonian systems. It consists of N one-dimensional particles interacting through a nearest-neighbor Hamiltonian of the form (see Toda [24]):

$$\mathcal{H}(q_1, ..., q_N; p_1, ..., p_N) \;:=\; \frac{1}{2}\sum_{j=1}^{N} p_i^2 + \sum_{j=1}^{N-1} e^{q_j - q_{j+1}}. \tag{1.1}$$

Flaschka introduced the new coordinates:

$$\left\{ \begin{array}{rcl} a_j &=& \frac{1}{2}e^{\frac{q_j - q_{j+1}}{2}} \\ b_j &=& -\frac{p_j}{2} \end{array} \right. \tag{1.2}$$

in which the Hamiltonian equations associated to (1.1) become:

$$\left\{ \begin{array}{rcl} \dot{a_j} &=& a_j(b_{j+1} - b_j) \\ \dot{b_j} &=& 2(a_j^2 - a_{j-1}^2). \end{array} \right. \tag{1.3}$$

The Flaschka coordinates present of course a slight indetermination for a_0 and a_N, as q_{N+1} and q_0 don't exist. Two choices, both preserved by the flow, are usually considered:

$$\left\{ \begin{array}{lcl} \text{either} & a_N \;=\; a_0 \;=\; 0, & \textit{non periodic case} \\ \text{or} & a_{j+N} = a_j, \;\; b_{j+N} = b_j, & \textit{periodic case.} \end{array} \right. \tag{1.4}$$

The non periodic case is sometimes referred to as the finite nonperiodic Toda lattice (as opposed to the lattice on the line) and was first analyzed in [22].

The key idea behind transforming the $2N$-systems of ODE (1.3) into a system of two PDEs, in the limit where N diverges, consists in formally imposing for the numbers, $a_1, ..., a_N, b_1, ..., b_N$, to be the values of $\frac{1}{N}$-step discretizations of two "nice" functions, a and b, defined on the unit interval. Indeed if we suppose that

$$\left\{ \begin{array}{lcll} a_j^t & = & a^t(\frac{j}{N}) & j = 1, \dots N-1 \\ b_j^t & = & b^t(\frac{j}{N}) & j = 1, \dots N \end{array} \right. \tag{1.5}$$

with $a^t(.)$ and $b^t(.)$, say, in $C^\infty([0,1])$, $a^t(1) = 0 = a^t(1)$ in the non-periodic case and in $C^\infty(S_1)$ in the periodic case, then (1.3) becomes:

$$\left\{ \begin{array}{lcl} N\partial_t a^t(x) & = & a\partial_x b^t(x) \\ N\partial_t b^t(x) & = & 2\partial_x\left(a^t(x)^2\right) \end{array} \right\} + O(\frac{1}{N}) \tag{1.6}$$

where $x = j/N$, which, after a rescaling of time

$$s := \frac{t}{N}, \tag{1.7}$$

gives, in the limit as $N \to \infty$, the system:

$$\left\{ \begin{array}{lcl} \partial_s a & = & a\partial_x b \\ \partial_s b & = & 2\partial_x a^2. \end{array} \right. \tag{1.8}$$

(See [13], [5], [6], [14], [7].)

The question addressed in this paper (see theorems below) is the following: *Suppose that we know a smooth solution of (1.8) defined for $s \le s_c$. Is it possible to compute the solution of the family of systems (1.3), with initial conditions $a_j^{t=0}$ and $b_j^{t=0}$ given by (1.5), for $t < t_c := Ns_c$, modulo an error of order $\frac{1}{N^p}$, for some (possibly any) $p > 0$?*

Before we state the precise results and the strategy of their proofs we would like to comment a bit on the equation (1.8). The limit equation (1.8) is nonlinear and therefore might (and does) develop shocks. For example, if we suppose that the initial condition $(a^{t=0}, b^{t=0})$ satisfies $b^{t=0} = 2a^{t=0}$, which is easily shown to be a condition preserved by the equation, then (1.8) reduces to the famous Burger's equation

$$\partial_t a = \partial_x a^2 \tag{1.9}$$

which is known to have smooth solution only for finite time. Therefore, in general there will be a critical time, s_c, beyond which a solution of (1.8) cannot be extended smoothly. This is *not* to be seen as a contradiction with the fact that the initial system, (1.3), being Hamiltonian, has a solution for all times. Recall that to obtain (1.8) we have performed a rescaling of time; the natural time for the original system in N times bigger than the natural time of the equation which develops shocks. What we learn from these remarks is that the evolution of the N-particle Toda system fails to be "uniformly nice" for times of order N, as $N \to \infty$.

The first result is the following, for the periodic case:

Theorem 1.1. *Let $a,b \in C^\infty(\mathbb{R})$ be 1-periodic functions, and let a_j^t and b_j^t, i=1,...,N be the solution of the periodic Toda flow*

$$\left\{\begin{array}{lcl} \dot{a}_j &=& a_j(b_{j+1}-b_j) \\ \dot{b}_j &=& 2(a_j^2 - a_{j-1}^2) \end{array}\right. \tag{1.10}$$

with initial condition:

$$\left\{\begin{array}{lcl} a_j^{t=0} &=& a(\frac{j}{N}) \\ b_j^{t=0} &=& b(\frac{j}{N}). \end{array}\right. \tag{1.11}$$

Let us moreover suppose that there exists $s_c > 0$ such that the system:

$$\left\{\begin{array}{lcl} \partial_s a &=& a\partial_x b \\ \partial_s b &=& 2\partial_x a^2 \end{array}\right. \tag{1.12}$$

with initial conditions:

$$\left\{\begin{array}{lcl} a^{s=0}(x) &=& a(x) \\ b_j^{s=0}(x) &=& b(x) \end{array}\right. \tag{1.13}$$

has a smooth periodic solution for $s < s_c$. Then there exists two sequences of smooth functions (determined by $a^s(x)$ and $b^s(x)$), $a_k^s(x)$ and $b_k^s(x)$, $k = 1, 2, \dots,$ defined on $[0, s_c) \times \mathbb{R}$ and periodic in x, such that for all integers $K > 0$ and for each $\epsilon > 0$, there exist $C_K > 0$ such that for $t \leq N(s_c - \epsilon)$,

$$\forall j = 1, ..., N, \quad \left| a_j^t - \left(a^{\frac{t}{N}}(\frac{j}{N}) + \sum_{k=1}^{K-1} N^{-k} a_k^{\frac{t}{N}}(\frac{j}{N}) \right) \right| \leq C_K N^{-K} \tag{1.14}$$

and

$$\forall j = 1, ..., N, \quad \left| b_j^t - \left(b^{\frac{t}{N}}(\frac{j}{N}) + \sum_{k=1}^{K-1} N^{-k} b_k^{\frac{t}{N}}(\frac{j}{N}) \right) \right| \leq C_K N^{-K}. \tag{1.15}$$

In particular, as $N \to \infty$, $\frac{j}{N} \to x$, and $\frac{t}{N} \to s < s_c$,

$$a_j^t \to a^s(x) \quad \textit{and} \quad b_j^t \to b^s(x) \tag{1.16}$$

where a^s and b^s are solutions of (1.12).

The result in the non-periodic case is slightly more complicated to state, due to having to impose boundary conditions on the function a. We will consider functions, a, in the following space: Denote by $C^\infty([0,1])$ the space of functions having a smooth extension to an open neighborhood of $[0,1]$, and let us introduce the space

$$\mathcal{A} := \{ \, a \in C^0([0,1]) \, ; \, \frac{a(x)}{\sqrt{x(1-x)}} \in C^\infty([0,1]) \, \}. \tag{1.17}$$

Functions in $\mathcal{A}$ are smooth in $(0,1)$ and must satisfy $a(0) = 0 = a(1)$ (the choice of this space for the function a corresponds to working with Toeplitz operators on the Riemann sphere, as will be explained below). The function, b, will be chosen in $C^\infty[0,1]$.

Theorem 1.2. *Let $a \in \mathcal{A}$ and $b \in C^\infty(\mathbb{R})$, and let a_j^t, j=1,...,N-1 and b_j^t, j=1,...,N be the solution of the non-periodic Toda flow*

$$\left\{\begin{array}{lcl} \dot{a}_j &=& a_j(b_{j+1}-b_j) \\ \dot{b}_j &=& 2(a_j^2 - a_{j-1}^2) \end{array}\right. \tag{1.18}$$

with initial condition:

$$\left\{\begin{array}{lcl} a_j^{t=0} &=& a(\frac{j}{N}) \\ b_j^{t=0} &=& b(\frac{j}{N}). \end{array}\right. \tag{1.19}$$

Let us moreover suppose that there exists $s_c > 0$ such that the system:

$$\left\{\begin{array}{lcl} \partial_s a &=& a\partial_x b \\ \partial_s b &=& 2\partial_x a^2 \end{array}\right. \tag{1.20}$$

with initial conditions:

$$\left\{\begin{array}{lcl} a^{s=0}(x) &=& a(x) \\ b^{s=0}(x) &=& b(x) \end{array}\right. \tag{1.21}$$

has a solution with $a^s \in \mathcal{A}$ and $b^s \in C^\infty([0,1])$, for $s < s_c$.

Then there exists two sequences of smooth functions (determined by $a^s(x)$ and $b^s(x)$), $a_k^s(x)$ and $b_k^s(x)$, $k = 1, 2, \dots$, defined on $[0, s_c) \times (0,1)$ with $a_k^s(\cdot) \in \mathcal{A}$ for each $s \in [0, s_c)$, such that for all integers $K > 0$ and for each $\epsilon > 0$, there exist $C_K > 0$ such that for $t \leq N(s_c - \epsilon)$,

$$\forall j = 1, ..., N-1, \quad \left| a_j^t - \left(a^{\frac{t}{N}}(\frac{j}{N}) + \sum_{k=1}^{K-1} N^{-k} a_k^{\frac{t}{N}}(\frac{j}{N}) \right) \right| \leq C_K N^{-K} \tag{1.22}$$

and

$$\forall j = 1, ..., N, \quad \left| b_j^t - \left(b^{\frac{t}{N}}(\frac{j}{N}) + \sum_{k=1}^{K-1} N^{-k} b_k^{\frac{t}{N}}(\frac{j}{N}) \right) \right| \leq C_K N^{-K}. \tag{1.23}$$

In particular, as $N \to \infty$, $\frac{i}{N} \to x$, and $\frac{t}{N} \to s < s_c$,

$$a_j^t \to a^s(x) \quad \text{and} \quad b_j^t \to b^s(x). \tag{1.24}$$

In addition to these two Theorems, we have also proved spectral estimates for the matrices with initial conditions (1.19), (1.21). More precisely, we have proved that the asymptotic density of states is given by the formulae of [14]. For the precise statements see section 6.

The proofs of Theorems 1.1 and 1.2 will use extensively the well-known "Lax pair" formulation (see [15], [22]): the system (1.3) can be written in the form

$$\frac{dL}{dt} = [L(t)\, ,\, B(L(t))]. \tag{1.25}$$

In the non-periodic case ($a_N(t) = a_0(t) = 0$) L is the matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & \cdots & & a_{N-1} & b_N \end{pmatrix} \tag{1.26}$$

and

$$B(L) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ 0 & -a_2 & 0 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & \cdots & & -a_{N-1} & 0 \end{pmatrix}. \tag{1.27}$$

Notice that in this case one has

$$B(L) = [L\,, \mathcal{N}] \tag{1.28}$$

where $\mathcal{N}$ is the $N \times N$ diagonal matrix

$$\mathcal{N} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & \cdots \\ \cdots & \cdots & & \cdots & 0 \\ 0 & \cdots & & 0 & N \end{pmatrix} \tag{1.29}$$

(see [2], [3], [4]).

In the periodic case ($a^t_{i+N} = a^t_i$ and $b^t_{i+N} = b^t_i$)

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ a_N & \cdots & & a_{N-1} & b_N \end{pmatrix}, \tag{1.30}$$

and the matrix $B = B(L)$ is

$$B(L) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & -a_N \\ -a_1 & 0 & a_2 & \cdots & 0 \\ 0 & -a_2 & 0 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ a_N & \cdots & & -a_{N-1} & 0 \end{pmatrix}. \tag{1.31}$$

In this case $B(L)$ is not of the form (1.28), but $L \mapsto B(L)$ is still an (exterior) derivation. Let us recall that the Lax pair formulation implies, in the case of the Toda system, its complete integrability, as (1.25) is easily shown to preserve the spectrum of L, giving enough conserved quantities.

The main idea of this paper consists in regarding the equation

$$\frac{dL}{dt} = [L(t)\,,\,B(L(t))] \tag{1.32}$$

as an equation on operators of semiclassical type, that is, to handle (1.32) with a symbol calculus. Indeed if L and $B(L)$ in (1.32) were semiclassical pseudodifferential operators, $L(x,\hbar D_x)$ and $B(x,\hbar D_x)$, acting on $L^2(\mathbb{R}^2)$, then the (principal) symbol of the right-hand side of (1.32) could be evaluated thanks to the rule (correspondence principle):

$$symbol\left(\frac{1}{i\hbar}[L,B]\right) = \{symbol(L), symbol(B)\}$$

where $\{\ ,\ \}$ is the Poisson bracket on $\mathbb{R}^{2n}$. Of course pseudodifferential operators structurally act on infinite-dimensional Hilbert spaces, and therefore it remains to find a "symbolic calculus" for $N\times N$-matrices. It turns out that such a calculus exists, yielding precise asymptotics as the rank, N, of the matrices tends to infinity (Planck's constant gets identified with $\hbar := \frac{1}{N}$): this is the Toeplitz calculus associated to the quantization of compact Kählerian manifolds.

Physically the fact that the quantization of a compact (and therefore finite-volume) symplectic manifold, M, of dimension n should give rise to a finite-dimensional Hilbert space $\mathcal{H}$ is justified by the *Heisenberg uncertainty principle.* It asserts that any vector (quantum state) must occupy a minimal phase space volume of $(2\pi\hbar)^n$, leading to the estimate that the dimension of $\mathcal{H}$ should be bounded by the volume of M divided by $(2\pi\hbar)^n$.

To quantize a compact symplectic manifold, M, one has to choose a compatible almost-complex structure and the symplectic form of M must satisfy a certain cohomological condition. If the almost-complex structure is integrable, one has a Kähler manifold. Then the Hilbert space which is the quantization of M depends on Planck's constant, $\hbar = 1/N$, and is realized as the space of holomorphic sections of the N-th tensor power of a holomorphic Hermitian line bundle over M whose curvature form is the symplectic form on M. *In many cases this space of sections can be identified with a space of functions with certain analytic properties.* This is the case for the two manifolds which happen to be significant for the purposes of this paper: the Riemann sphere, associated to non-periodic Toda, and the torus $T^2 = \mathbb{C}/\mathbb{Z}^2$, associated to the periodic case.

Although precise constructions will be given in the next sections, we will finish this introduction by summarizing the basic facts about Toeplitz operators and state a crucial Theorem from which Theorem 1.1 follows.

For each N the quantization $\mathcal{H}^S_N$ of the Riemann sphere, S^2, is the space of homogeneous polynomials of degree N in two complex variables. Through stereographic projection, we can identify $\mathcal{H}^S_N$ with the following space:

$$\mathcal{H}^S_N := \{f:\mathbb{C}\to\mathbb{C}\ ;\ f \text{ is entire and } \frac{i}{2}\int_{\mathbb{C}} |f(z)|^2\,\frac{dz\wedge d\overline{z}}{(1+|z|^2)^{N+1}} < +\infty\}. \tag{1.33}$$

The quantization of the torus T^2 is more ambiguous since there exist many flat bundles over T^2. Passing to the universal cover, $\mathbb{C}$, for our purposes we will pick for the quantization of T^2 the following spaces:

(1.34)

$$\mathcal{H}_N^T := \{f : \mathbb{C} \to \mathbb{C} \,;\, f \text{ is entire and } \forall\, m,\, n \in \mathbb{Z},\ f(z+m+in) = e^{N\pi(n^2-2inz)} f(z)\}.$$

The space $\mathcal{H}_N^S$ (resp. $\mathcal{H}_N^T$) is a closed subspace of $L^2(\mathbb{C}, \frac{i}{2}\frac{dz\wedge d\overline{z}}{(1+|z|^2)^{N+1}})$ (resp. $L^2([0,1]\times[0,1])$). Therefore, there exists an orthogonal projection, Π_N^S (resp. Π_N^T), from $L^2(\mathbb{C}, \frac{i}{2}\frac{dz\wedge d\overline{z}}{i(1+|z|^2)^{N+1}})$ to $\mathcal{H}_N^S$ (resp. from $L^2([0,1]\times[0,1])$ to $\mathcal{H}_N^T$).

Since $\mathcal{H}_N^{S,T}$ are spaces of functions on phase space one could think of defining quantization by letting a classical Hamiltonian act by simple multiplication (in analogy with the usual quantization of potentials). This procedure doesn't work as it breaks the analyticity condition in the definition of $\mathcal{H}_N^{S,T}$. But by analogy with the standard Toeplitz matrices one can, after multiplication, project back on the Hilbert space. Although the precise definition somewhat technical (see [9] for the general theory, and, for Toeplitz operators in the present setting, [11]), the following is true:

Given a Toeplitz operator on the sphere, T^S (resp. T^T on the torus), there exists a sequence of C^∞ functions, $\{H_j\}_{j=0,1,\dots}$, on the plane (seen as the sphere) (resp. on the torus) such that:

$$T^{S,T} \sim \sum_{j=0}^{\infty} \Pi_N^{S,T} \tilde{H}_j \Pi_N^{S,T} \tag{1.35}$$

where $\tilde{H}$ stands for the operator of multiplication by H and $\sim$ is meant in the operator norm. Conversely, for any sequence $\{H_j\}_{j=0,1,\dots}$ there is a Toeplitz operator such that the above holds.

H_0 is called the principal symbol of $T^{S,T}$. Since the spaces $\mathcal{H}_N^{S,T}$ are finite dimensional, being a Toeplitz operator is visible only in the limit $N \to \infty$. Moreover the underlying structure of the corresponding phase space is visible through the following result, making the Toeplitz calculus similar to the pseudo-differential one:

Proposition 1.3. *For both the sphere and the torus:*
1. The composition of two Toeplitz operators is a Toeplitz operator whose principal symbol is the product of the principal symbol of the two factors.
2. $\frac{N}{i}[H_T, H_{T'}]$ is a Toeplitz operator whose principal symbol is $\{T_0, T_0'\}$ where $\{\ ,\ \}$ is the Poisson bracket on the sphere and on the torus respectively.

We finally give the Toeplitz formulation of the main result of the paper (see next sections for more details).

The two boundary conditions (1.4) for the Toda systems are associated to the two different geometrical situations described before, as shown by the following:

Lemma 1.4. *1.- Let a, b be two smooth periodic functions, and let L^N be the sequence of matrices*

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ a_N & \cdots & & a_{N-1} & b_N \end{pmatrix} \tag{1.36}$$

where $a_j = a(\frac{j}{N})$, $b_j = b(\frac{j}{N})$. Then L is the matrix of a Toeplitz operator on the torus (with respect to a natural basis), with principal symbol $H_0(\varphi, \theta) = b(\varphi) + 2a(\varphi) cos(\theta)$ (where φ and θ are the natural coordinates on the torus).

2.- Let $b \in C^\infty[0,1]$ and let $a \in \mathcal{A}$, and let L^N be a sequence of matrices

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & \cdots & & a_{N-1} & b_N \end{pmatrix} \tag{1.37}$$

where the entries are defined by: $b_N = b(1)$ and $\forall 1 \leq j \leq N-1$

$$a_j = a(\frac{j}{N}), \quad b_j = b(\frac{j}{N}).$$

Then L^N is the sequence of matrices of a Toeplitz operator on the sphere (with respect to a natural basis), with principal symbol $H_0(h, \theta) = b(h) + 2a(h)\cos(\theta)$ (where h and θ are the natural height and polar angle on the sphere).

The next result, from which Theorem 1.1 follows, is a kind of "Egorov" theorem which states that the Toda equations, although they are nonlinear, propagate a Toeplitz operator into an operator arbitrarily close to a Toeplitz operator, as long as the Toda PDE admits a smooth solution. (Notice, however, the change of time scale.)

Theorem 1.5. *let $L(t)$ satisfy*

$$\frac{dL}{dt} = [L(t)\,,\,B(L(t))]. \tag{1.38}$$

with $B(L)$ defined as before and $L(t=0)$ as in the Lemma. Then, for $s < s_c$, with s_c as before, there exists a Toeplitz operator T_s such that

$$||L(Ns) - T_s||_{HS} \;=\; O(N^{-\infty}) \tag{1.39}$$

where $|| \quad ||_{HS}$ is the Hilbert-Schmidt norm on $\mathcal{H}_N^{S,T}$. For each $\epsilon > 0$, the estimates are uniform on $[0, s_c - \epsilon]$.

It remains to explain how the leading order term of the function $H(s)$ can be computed and the link with the same computation in Theorem 1.1. The situation in the *periodic/non-periodic* case is a little different, depending on whether the matrix $B(L)$ is or is not a commutator.

In the *non-periodic* case $B(L) = [L, \mathcal{N}] = N[L, \frac{\mathcal{N}}{N}]$. It happens that $\frac{\mathcal{N}}{N}$ is a Toeplitz operator whose principal symbol is precisely the function "h" defined in Lemma 1.4. Therefore rewriting (1.38) as $N\frac{dL}{dt} = -\frac{N}{i}[L(t)\,,\,\frac{N}{i}[L(t), \frac{\mathcal{N}}{N}]]$ and using Proposition 1.3 we find that if $H(s) \sim \sum_{k=0}^{\infty} H_k(s)$, H_0 should satisfy

$$\partial_s H_0 \;=\; -\{H_0, \{H_0, h\}\}. \tag{1.40}$$

Identifying $H_0 = a(h) + b(h)2cos(\theta)$ on sees immediately that (1.40) is equivalent to (1.8).

The periodic case is similar if we notice that if $B(L)$ is defined by (1.31), then $\frac{B(L)}{N}$ is a Toeplitz operator of principal symbol $\partial_\varphi a(\theta)2cos(\theta)$.

Let us conclude this long introduction by a final remark: although in the case of Toda the integrability is equivalent to the existence of a Lax pair we didn't use explicitly this integrability in this paper. This makes our treatment of the problem rather different than other approaches, for example the one by Deift and McLaughlin ([14]). On the other hand the rather mysterious link between the different boundary conditions and the different symplectic structures of the underlying dispersionless limit suggests that other systems of ODEs, put in Lax pair form, could be treated the same way and give rise to other "classical" limits.

Finally let us mention that this present paper has been mostly inspired by the papers [2] to [7] and [13]. In particular the idea of relating the dispersionless limit of Toda to semiclassical analysis is due to Flaschka.

The paper is organized as follows: the first three next sections are devoted to a brief description of the Toeplitz quantization of a general Kählerian manifold, and then to the particular cases of the sphere and the torus. In Section 5 we prove the Theorems, and we conclude the paper with some additional results in Section 6.

2. Toeplitz operators

In this section [1] we review some aspects of the semi-classical theory of Toeplitz operators that we will need later on. We begin with a review of the quantization theory of integral Kähler manifolds.

Let X denote an integral Kähler manifold with symplectic form ω. By integral symplectic manifold we mean that the cohomology class defined by the 2-form Ω is integral i.e that the integral of Ω over an arbitrary 2-cycle in X is an integer. In what follows we'll only consider the two cases,

$$X \;=\; \mathbb{P}^1, \quad X \;=\; \mathbb{C}/\mathbb{Z}^2$$

[1] This section is written in order to describe a general theory of Toeplitz quantization, that might be helpful for other situations. IT CAN BE OMITTED BY READERS INTERESTED ONLY IN THE TODA SYSTEM, as the two next sections will present the same results in a concrete way.

(with their standard symplectic forms), so X can simply stand for one of these two manifolds. Let $\mathcal{L} \to X$ be a holomorphic hermitian line bundle provided with a connection ∇ whose curvature form is the symplectic form Ω of X (the condition for X to be integral implies the existence of such a bundle), and for each positive integer N let

$$\mathcal{H}_N = H^0(X, \mathcal{L}^{\otimes N})$$

denote the space of holomorphic sections of the N-th tensor power of $\mathcal{L}$. Another way of saying this is to define $\mathcal{H}_N$ as the space of holomorphic sections of an hermitian line bundle whose curvature is $N\Omega$.

Specifically: in case $X = \mathbb{P}^1$, $\mathcal{H}_N$ is the space of homogeneous polynomials of degree $N-1$ in two complex variables, while if $X = \mathbb{C}/\mathbb{Z}^2$, $\mathcal{H}_N$ can be identified with a space of classical theta functions of degree N (and fixed characteristics) (see Section 4 for details).

The spaces $\mathcal{H}_N$ have a natural Hermitian inner product structure, namely the one inherited from the space of L^2 sections of $\mathcal{L}^{\otimes N}$:

$$< \psi\,,\, \phi >= \int_X < \psi(x)\,,\, \phi(x) > \, d\lambda_x$$

where $d\lambda$ is the Liouville measure of X. We will let

$$d_N := \dim \mathcal{H}_N \tag{2.1}$$

be the dimension of $\mathcal{H}_N$. Then d_N is given by the Riemmann-Roch theorem and is, for N large, a polynomial in N of degree $\dim(X)/2$ and leading coefficient the volume of X.

We next need to discuss the anti-Wick quantization of observables, $H : X \to \mathbb{R}$. This yields a sequence $(T_H^{(N)})$ of operators, where $T_H^{(N)}$ acts on $\mathcal{H}_N$ (which are the simplest Toeplitz operators on X). The definition is as follows. Let

$$\Pi_N : L^2(X, \mathcal{L}^{\otimes N}) \to \mathcal{H}_N$$

be the L^2 orthogonal projector to $\mathcal{H}_N$. Then, given $H \in C^\infty(X, \mathbb{R})$ we can form the operator

$$\begin{array}{rccl} T_H^{(N)} : & \mathcal{H}_N & \to & \mathcal{H}_N \\ & f & \mapsto & \Pi_N(fH), \end{array} \tag{2.2}$$

that is $T_H^{(N)} = \Pi_N \circ \tilde{f}$ where $\tilde{f}$ stands for the operator of multiplication by f.

A general Toeplitz operator is a sequence of operators $T = (T^{(N)})$ where $T^{(N)}$ acts on $\mathcal{H}_N$, such that there is an asymptotic expansion

$$T^{(N)} \sim \sum_{j=0}^{\infty} N^{-j}\, T_{H_j}^{(N)}. \tag{2.3}$$

We will sometimes call the "order" of $T^{(N)}$ the opposite of the first j for which $H_j \neq 0$, but most of the time (and if not specified) we will be dealing with zeroth order Toeplitz operators. H_0 will be called the principal symbol. Since the dimension of $\mathcal{H}_N$ is finite, being a Toeplitz operator is a property only visible in the asymptotics $N \to \infty$. The principal symbol is the basic invariant of a Toeplitz operator and, as we will see, it controls to a large extent the asymptotic behaviour of the $T^{(N)}$. Furthermore, one has the following "symbol calculus":

Theorem 2.1. [8] *Let $S = (S^{(N)})$ and $T = (T^{(N)})$ be two Toeplitz operators of orders m_1 and m_2. Then:*

1. *The composition, $S \circ T := (S^{(N)} \circ T^{(N)})$ is a Toeplitz operator of order $m_1 + m_2$ and its principal symbol is the product $H^S_{m_1} H^T_{m_2}$ of the principal symbols of S and T.*
2. *The commutator, $[S\,,T] := ([S^{(N)}\,, T^{(N)}])$ is a Toeplitz operator of order $m_1 + m_2 - 1$ and its principal symbol is*
$$\frac{1}{i}\,\{H^S_{m_1}\,,\,H^T_{m_2}\}$$
where $\{\,,\,\}$ is the Poisson bracket on X.

The next result evaluates the norm of a Toeplitz operator.

Proposition 2.2. *If $T = (T^{(N)})$ is a Toeplitz operator of order $-m$, then*
$$\|\,T^{(N)}\,\|_{HS} \;\sim\; d_N\,N^{-m}\,\|H_m\|_2.$$
where $\|\quad\|_{HS}$ is the Hilbert Schmidt norm and, once again, $d_N := \dim \mathcal{H}_N$. More properties of Toeplitz calculus can be found in [9], [8] and [10].

3. Quantization of the sphere

The quantization of the Riemann sphere arises in quantum mechanics as spin. The basic quantum angular momentum observables can be identified with the Lie algebra of SU(2). Recall that the irreducible representations of this Lie group can be realized in the spaces
$$\mathcal{G}_N := \{\, f(w_1, w_2)\ ;\ f \text{ a homogeneous polynomial of degree } N-1\,\}.$$
Specifically, if $f \in \mathcal{G}_N$ and $g \in \mathrm{SU}(2)$, then
$$(g\cdot f)(w_1, w_2) = f(g^{-1}\cdot(w_1, w_2))$$
where the action on the right-hand side is the natural action of SU(2) on $\mathbb{C}^2$. These representations, with $N = 1, 2, \cdots$, exhaust the irreducible representations of SU(2). Obviously, these representations are unitary if we put on $\mathcal{G}_N$ the following Hermitian inner product:
$$\langle f_1, f_2\rangle = \int_{S^3} f_1\,\overline{f_2}\,d_{S^3}$$
where $S^3 \subset \mathbb{C}^2$ is the unit sphere and d_{S^3} is its volume form.

Consider the natural action of the circle group, $S^1 \subset \mathbb{C}$, on S^3, given by complex multiplication. By homogeneity, the functions on $\mathcal{G}_N$ transform very simply along the orbits of S^1, and the inner product above is invariant under S^1. We can therefore regard the elements of $\mathcal{G}_N$ as some sort of objects on the abstract quotient, $M := S^3/S^1$, which will be identified below with a sphere of radius $1/2$. The right "objects" are sections of a line bundle over this sphere.

To realize M as a two-dimensional sphere, begin by noticing that M is naturally $\mathbb{CP}^1$, the manifold of complex lines in $\mathbb{C}^2$. Then, define a map:

$$\Phi : \mathbb{CP}^1 \to \mathrm{su}(2) \tag{3.1}$$

by the following rule: *For each $\ell \in \mathbb{CP}^1$, $\Phi(l)$ is the matrix having ℓ as an eigenspace, with associated eigenvalue $i/2$, and $\ell^\perp$ as another eigenspace, with associated eigenvalue $-i/2$.*[2] For example, Φ maps the first axis, $\{(w_1, 0)\}$, to the matrix σ_3 below: the matrices

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.2}$$

form a standard basis of su(2) (so that $[\sigma_1, \sigma_2] = \sigma_3$, etc). Give su(2) the inner product such that the σ_j are orthogonal and have length $1/2$. An intrinsic characterization of this Euclidean inner product on su(2) is:

$$\forall A, \ B \in \mathrm{su}(2) \qquad \langle A, B \rangle = -\frac{1}{2} \operatorname{Tr} AB,$$

which shows that it is SU(2)-invariant. Clearly Φ is an equivariant diffeomorphism onto its image, which is easily seen to be the sphere of radius $1/2$. The description of M as a sphere means that we can speak of (restrictions of) linear functions on $\mathbb{CP}^1$. For the record, we define the following functions on $\mathbb{CP}^1$:

$$\forall j = 1, 2, 3, \quad \forall \ell \in M \qquad x_j(\ell) = 2\,\langle \Phi(\ell), \sigma_j \rangle. \tag{3.3}$$

It is not difficult to get formulae for the previously introduced functions. Let us write $\ell = [w_1 : w_2] \in M$, where $(w_1, w_2) \in S^3$ is a representative. Then one can show that

$$\Phi([w_1 : w_2]) = \frac{i}{2}\begin{pmatrix} |w_1|^2 - |w_2|^2 & 2w_1\overline{w_2} \\ 2w_2\overline{w_1} & |w_2|^2 - |w_1|^2 \end{pmatrix}, \tag{3.4}$$

and more calculations show that:

$$x_1([w_1 : w_2]) = \Re\,(w_1\overline{w_2}), \quad x_2([w_1 : w_2]) = \Im\,(w_1\overline{w_2}), \tag{3.5}$$

$$x_3([w_1 : w_2]) = \frac{1}{2}\left(|w_1|^2 - |w_2|^2\right).$$

[2]The choice of spectrum is dictated by the normalization that the area of S^3/S^2 agrees with the one induced by the Killing form.

To get the description of the quantization of $M = S^3/S^1$ given in §1, consider the map

$$\mathbb{C} \ni z \mapsto \frac{1}{(1+|z|^2)^{1/2}}\,(z,1) \in S^3. \tag{3.6}$$

This is a cross-section of the S^1 action missing precisely one orbit of S^1, namely $\{(e^{i\theta},0), 0 \le \theta < 2\pi\} \subset S^3$. We can therefore think of the $\mathbb{C}$ in (3.6) as the sphere, $M = S^3/S^1$, with a point deleted. This is how one arrives at the Riemann sphere from angular momentum considerations. Finally, we note that in the z coordinate the functions x_j are given by

$$x_1(z) = \Re\frac{z}{1+|z|^2}, \quad x_2(z) = \Im\frac{z}{1+|z|^2}, \quad x_3(z) = \frac{1}{2}\frac{|z|^2-1}{|z|^2+1}. \tag{3.7}$$

3.1. **The Hilbert spaces.** Pulling back a homogeneous polynomial $f(w_1,w_2) \in \mathcal{G}_N$ by the map (3.6), one obtains the function

$$\frac{1}{(1+|z|^2)^{N/2}}\,f(z,1).$$

Notice that $\psi(z) = f(z,1)$ is a polynomial in z of degree less than or equal to $N-1$. Therefore, by pulling back elements from $\mathcal{G}_N$ by (3.6), we obtain the space, $\mathcal{H}_N$, of complex polynomials in one variable of degree $\le N-1$, as we mentioned in §1.

Let us next work out the Hilbert space structure on $\mathcal{H}_N$ corresponding to that of $\mathcal{G}_N$. One can check that if $\rho : S^3 \to \mathbb{R}$ is S^1 invariant, then

$$\int_{S^3} \rho(w)d_{S^3}w = \pi\, i\int_{\mathbb{C}} \rho\left(\frac{z}{(1+|z|^2)^{1/2}}, \frac{1}{(1+|z|^2)^{1/2}}\right)\frac{dz\wedge d\bar z}{(1+|z|^2)^2}.$$

Putting together these remarks (and dropping a factor of 2π, for simplicity) we arrive at the formula

$$\langle\psi_1,\psi_2\rangle = \frac{i}{2}\int_{\mathbb{C}} \psi_1(z)\,\overline{\psi}_2(z)\,\frac{dz\wedge d\bar z}{(1+|z|^2)^{N+1}} \tag{3.8}$$

for the inner product on $\mathcal{H}_N$. A tedious calculation shows:

Lemma 3.1. *For every $N>0$ the vectors*

$$|j\rangle = \sqrt{\frac{N}{\pi}}\sqrt{C^j_{N-1}}\,z^j, \qquad 0 \le j \le N-1$$

form an orthonormal basis of $\mathcal{H}_N$. (Here $C^j_k = \frac{k!}{j!(k-j)!}$ are the binomial coefficients.)

We will use this basis to identify Toeplitz operators on S^2 with certain sequences of matrices. We'll refer to it as the standard basis of $\mathcal{H}_N$.

3.2. **Action of su**(2). SU(2) and its Lie algebra are represented in $\mathcal{G}_N$, and therefore in $\mathcal{H}_N$. Next we state, without proof, how the standard generators of su(2) are represented in $\mathcal{H}_N$.

Lemma 3.2. *Let $J_k^N := i \times$ the operator induced by σ_k on $\mathcal{H}_N$, $k=1,2,3$. Then:*

$$J_1^N = \frac{1}{2}\left((1-z^2)\frac{d}{dz} + (N-1)z\right), \quad J_2^N = \frac{i}{2}\left((1+z^2)\frac{d}{dz} - (N-1)z\right),$$

and

$$J_3^N = z\frac{d}{dz} - \frac{N-1}{2}.$$

In particular, the lowering and raising operators, $J_\pm^N = J_1^N \pm iJ_2^N$, are:

$$J_+^N = -z^2\frac{d}{dz} + (N-1)z, \quad J_-^N = \frac{d}{dz}.$$

It follows that the vectors $|j>$, $j=0,\dots,N-1$ are eigenvectors of J_3^N, with eigenvalue $j - \frac{N-1}{2}$.

Corollary 3.3. *The matrix of J_-^N in the standard basis has zero entries except along the supra-diagonal, along which the entries are equal to*

$$m_j = \sqrt{j(N-j)}, \qquad j = 1,\dots,N-1.$$

3.3. **Toeplitz operators and proof of Lemma 1.4, non-periodic case.** Toeplitz operators on the sphere are defined as in the general case exposed in the preceding section. Given the orthogonal projector

$$\Pi_N : L^2\left(S^2, \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}\right) \to \mathcal{H}_N$$

and given a sequence of C^∞ functions on the sphere H_j a Toeplitz operator is an operator T such that

$$T \sim \sum_{j=0}^{\infty} N^{-j}\Pi_N H_j \Pi_N.$$

The important (and non trivial) result we need for the presentation of this section is the fact that Toeplitz operators form a ring (that is the composition of two Toeplitz is Toeplitz) (see [9] for a proof). Let us mention that another (tedious) way of getting the results of this section is possible, in the spirit of next section.

The following is well-known and can be easily checked:

Lemma 3.4. *For $k=1,2,3$, the sequence of operators $\frac{1}{N}J_k^N$ is a Toeplitz operator with principal symbol equal to the the coordinate function x_k.*

Introduce the fundamental operator,

$$\mathcal{Z}^N := \frac{1}{N}\, J_3^N + \frac{N-1}{2N} = \frac{1}{N}\, z\frac{d}{dz}. \tag{3.9}$$

Notice that the spectrum of $\mathcal{Z}^N$ is simple and equals $\{\, j/N \;;\; 0 \le j \le N-1\,\}$. By the previous lemma, $\mathcal{Z}^N$ is a Toeplitz operator with principal symbol equal to the height function

$$h := x_3 + \frac{1}{2}.$$

Notice that the range of h is $[0,1]$.

Lemma 3.5. *Let α be a smooth function on $[0,1]$. Then there is a Toeplitz operator on the sphere, $\tilde{\alpha}^N$, of order zero, with principal symbol $\alpha\circ h$, which is diagonal in the standard basis of $\mathcal{H}_N$ and whose j-th diagonal entry is $\alpha((j-1)/N)$, $1 \le j \le N$.*

Proof. Let

$$\tilde{\alpha}^N = \frac{1}{2\pi}\int e^{it\mathcal{Z}^N}\,\hat{\alpha}(t)\,dt,$$

where $\hat{\alpha}$ is the Fourier transform of a compactly-supported smooth extension of α. Since $\mathcal{Z}$ is a self-adjoint Toeplitz operator of order zero, $e^{it\mathcal{Z}^N}$ is a unitary Toeplitz operator of order zero (unitary is obvious and the Toeplitz property can be derived by Taylor expansion of the exponential (convergent for each t as $\mathcal{Z}^N$ is bounded), together with the fact that the set of Toeplitz operators is a ring), and we are done. ☐

Next we see how to obtain tri-diagonal Toeplitz operators with the entries described in Lemma 1.4. By the previous lemma, we only have to create Toeplitz operators with the desired off-diagonal matrix elements. This we will do by composing a diagonal Toeplitz operator with the raising and lowering operators.

Lemma 3.6. *Let $a \in C^\infty([0,1])$ be such that $a(h)\cos(\theta)$ is a smooth function on the sphere. Then*

$$\alpha(h) := \frac{1}{\sqrt{h(1-h)}}\, a(h)$$

is a smooth function, and the Toeplitz operator

$$A^N := \frac{1}{N}\, J_-^N \circ \tilde{\alpha}^N \tag{3.10}$$

(where $\tilde{\alpha}^N$ is as in Lemma 3.5) has zero matrix elements except along the supra-diagonal, where the j-th entry is equal to

$$a\Big(\frac{j}{N}\Big), \quad j = 1, \dots, N-1. \tag{3.11}$$

Proof. By symmetry, it suffices to prove the smoothness of α near $h = 0$. Let us first remark that $a(h)\cos(\theta)$ vanishes at the poles, as its integrals over lines of latitude arbitrarily close to the poles are zero. Using (x_1, x_2) as coordinates on the sphere near the south pole, $h = 0$, we introduce the function on the plane, $f(x_1, x_2) := a(h)\cos(\theta)$. If $\rho = \sqrt{x_1^2 + x_2^2}$, we easily find that $\rho = \sqrt{h - h^2}$ and that $f(x_1, x_2) = \frac{a(h)}{\rho} x_1$. By assumption this is C^∞.

We now consider the restriction of f to a meridian near the south pole, say $x_2 = 0$. Since $f(x_1, x_2)$ is C^∞ and vanishes at the origin we have that $\frac{f(x_1,0)}{x_1}$ is C^∞ in x_1. Therefore $g(\rho) := \frac{a(h)}{\rho}$ is C^∞ in $\rho \geq 0$ near 0. This means that $\frac{a(h)}{\sqrt{h}} = \frac{a(h)}{\rho}\frac{\rho}{\sqrt{h}} = g(\rho)\sqrt{1-h} = g(\sqrt{h}\sqrt{1-h})\sqrt{1-h} = \mu(\sqrt{h})$, with μ C^∞ near 0. By a standard argument (cf. [17], p. 371), this implies that $\nu(h) := \frac{a(h)}{\sqrt{h}}$, and therefore $\alpha(h)$, are C^∞ functions of h near 0.

If α is any smooth function on $[0, 1]$, the operator given by (3.10) is as in the conclusion of the Lemma, and an easy matrix calculation shows that its supra-diagonal elements are equal to

$$\alpha(\frac{j}{N}) \sqrt{\frac{j}{N}\left(1 - \frac{j}{N}\right)} \quad j = 1, \dots, N-1, \tag{3.12}$$

which equals $a(j/N)$. □

The proof of the second part of Lemma 1.4 is an easy consequence of the two preceding Lemmas. Let $\mathcal{B}^N$ be defined by:

$$\mathcal{B}^N = \frac{1}{2\pi} \int e^{it(\mathcal{Z}^N + \frac{1}{N})}\, \hat{b}(t)\, dt.$$

This is a diagonal Toeplitz operator, just as in the proof of Lemma 3.5, since $\mathcal{Z}^N + \frac{1}{N}$ is a Toeplitz operator of order zero with the same principal symbol as $\mathcal{Z}^N$. The diagonal entries of $\mathcal{B}^N$ are: $b(j/N),\ j = 1, \dots N)$. Finally, define the Toeplitz operator

$$L^N := \mathcal{B}^N + A^N + \left(A^N\right)^*, \tag{3.13}$$

A^N is as in Lemma 3.6, and $(.)^*$ denotes the adjoint, has the desired matrix elements. The symbol of J_-^N is $x_1 - ix_2$, and therefore the symbol of L^N (as above) is

$$b(h) + 2x_1\, \alpha(h) = b(h) + 2\sqrt{h(1-h)}\, \alpha(h) = b(h) + 2a(h)\, \cos(\theta).$$

Using commutation relations between $\mathcal{Z}^N$ and $J_\pm^N$ we get the

Proposition 3.7. *let L_1 and L_2 two matrices as in Lemma 1.4-2 , therefore Toeplitz operators of principal symbols $H^{1,2}(\theta, h) := a_{1,2}(h) + 2cos(\theta)a_{1,2}(h)$. Then*

L_1L_2 is a Toeplitz operator with principal symbol H^1H^2 and $\frac{1}{iN}[H^1,H^2]$ is a Toeplitz operator and its principal symbol is

$$\{H_0^1,H_0^2\} := \partial_h H_0^1 \partial_\theta H_0^2 - \partial_\theta H_0^1 \partial_h H_0^2.$$

In particular $\frac{1}{iN}B(L^1) := \frac{1}{iN}[L^1,\mathcal{N}]$ is a Toeplitz operator of principal symbol

$$\{H^1,h\} = -\partial_\theta H^1 = 2\sin\theta a(h).$$

4. Quantization on the torus

In this section we carry out in detail the computations necessary for the study of the dispersionless limit of the Toda system in the periodic case. The realization of the space $\mathcal{H}_N^T$ will involve Θ functions that we will present in a slightly more general formulation than the one needed for the main Theorems of this paper, and we will explain at the end of this section how this generality is useful also for Toda. This section is in some sense equivalent to the preceding one and the main result will be the proof, in the periodic case, of the Lemma 1.4.

Let us mention finally that the following "naive" way of quantizing the torus leads also to the same finite dimensional space: if we consider the torus as obtained from T^*S_1 by periodization on the fiber (impulsion) the usual quantum mechanics with Planck constant $\hbar := \frac{1}{N}$ should provide wave functions which are functions ψ on the circle (1-periodic) and whose $\frac{1}{N}$-Fourier transform, as defined by $\hat{\psi}(\xi) := \int_0^1 e^{-i2\pi N\xi x}\psi(x)dx,\ \xi \in \frac{\mathbb{Z}}{N}$, has to be 1-periodic also. Therefore $\hat{\psi}$ is determined by its values for $\xi = 0, \frac{1}{N}, \dots, \frac{N-1}{N}$ and the N-dimensional resulting Hilbert space can by realized as the linear span of the set $\{\delta(x-\frac{k}{N}),\ x \in [0,1], k = 0,\ 1\ , \dots,\ N-1\}$, where δ is the Dirac mass at zero. $\hat{\psi}$

4.1. Θ functions ($\tau = i$).

There are several approaches to the theory of theta functions. Here we review what we need for our purposes for the case of the standard torus $X = \mathbb{C}/\Lambda$ where

$$\Lambda = \mathbb{Z} \oplus i\,\mathbb{Z}$$

($\tau = i$ in the literature). We begin with the classical definition.

4.1.1. *Definition and first properties.*

Definition 4.1. [1], [23]. A theta function of order $N \in \mathbb{Z}^+$ and characteristics $\mu,\ \nu \in \mathbb{R}$ for the square torus is an entire function $f:\ \mathbb{C} \to \mathbb{C}$ such that: $\forall z \in \mathbb{C},\ m+in \in \Lambda$

$$f(z+m+in) = e^{N\pi(n^2-2inz)-2\pi i(\mu m+\nu n)} f(z). \tag{4.1}$$

The space of all such functions will be denoted: $\Theta_N^{\mu,\nu}$.

The parameters $\mu,\ \nu$ arise because one can always tensor a given quantizing line bundle of the torus with the flat line bundle associated to the character of Λ,

$$(m,n) \mapsto e^{2\pi i(m\mu+n\nu)}.$$

We will see the dimension of $\Theta_N^{\mu,\nu}$ is N. Notice that in particular an element $f \in \Theta_N^{\mu,\nu}$ is quasi-periodic with respect to $\mathbb{Z}$: $f(z+m) = e^{-2\pi m\mu} f(z)$. Let us expand it in Fourier series,

$$f(z) = e(-\mu z) \sum_{m=-\infty}^{\infty} a_m \, e(mz) \tag{4.2}$$

where henceforth we use the notation

$$e(z) := e^{2\pi i z}.$$

Let us write the transformation law (4.1) in terms of the Fourier series. The ansatz (4.2) automatically takes care of the quasi-periodicity with respect to $\mathbb{Z}$, so the transformation law reduces to:

$$e(-\mu(z+in)) \sum_m a_m \, e(m(z+in)) = e^{N\pi(n^2-2inz)} \, e(-\nu n) e(-\mu z) \sum_m a_m \, e(mz).$$

Rearranging a bit one gets that the transformation law (4.1) is equivalent to:

$$\forall \, m, \, n \qquad a_{m+Nn} = e^{-\pi(Nn^2+2mn)} \, e^{2\pi n(\mu+i\nu)} \, a_m. \tag{4.3}$$

This shows that the dimension of the space of theta functions of order N is N, as the values of the coefficients $a_0, \dots a_{N-1}$ determine the Fourier series. More precisely:

Proposition 4.2. *For $j = 0, \dots, N-1$, let $\theta_j^{\mu,\nu}(z)$ be defined by the Fourier series*

$$\theta_j^{\mu,\nu}(z) = \sum_{n=-\infty}^{\infty} e^{-\pi(Nn^2+2jn)} \, e^{2\pi n(\mu+i\nu)} \, e(z(j+Nn)).$$

Then $\theta_j^{\mu,\nu} \in \Theta_N^{\mu,\nu}$, and the set $\{\theta_j^{\mu,\nu}, \ j = 0 \dots, N-1\}$ is a basis of the space $\Theta_N^{\mu,\nu}$.

4.1.2. *Line bundles.* [3]

We now review how theta functions arise as holomorphic sections of tensor powers of a line bundle over the torus $X := \mathbb{C}/\Lambda$. This is the point of view of geometric quantization: we think of X as an integral Kahler manifold, and the first step in its quantization is to consider a hermitian holomorphic line bundle $\mathcal{L} \to X$ whose Chern class is the symplectic form of X. Notice that since X is not simply connected, the Chern class condition does not determine the bundle; one can always twist by a flat line bundle. This is what gives rise to theta functions with characteristics.

Any line bundle is of course trivial when pulled-back to $\mathbb{C}$ (even holomorphically) and therefore its sections can be identified with functions on $\mathbb{C}$ satisfying a transformation law that is slightly different than the classical one. More precisely, we can define a line bundle,

$$\mathcal{L} \to X,$$

[3]this paragraph can be omitted by readers not interested by the geometrical aspects of the paper

as a quotient $\mathcal{L} = (\mathbb{C}\times\mathbb{C})/\sim$. Here the equivalence relation is defined by a cocycle,

$$\chi : \mathbb{C} \times \Lambda \to \mathbb{C} \setminus \{0\}$$

as follows:

$$(z, a) \sim (w, b) \quad \Leftrightarrow \quad \exists\lambda \in \Lambda \text{ such that } (w, b) = (z + \lambda\,,\, \chi(z, \lambda)a).$$

The cocycle condition, ensuring that this is indeed an equivalence relation, is:

$$\chi(z, \lambda)\,\chi(z + \lambda, \mu) \;=\; \chi(z, \lambda + \mu). \tag{4.4}$$

It is easy to show that any quotient space $(\mathbb{C} \times \mathbb{C})/\sim$ so defined is indeed a line bundle over X. We observe three features of this construction:

1. The sections of this line bundle are naturally identified with the functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\forall(z, \lambda) \in \mathbb{C} \times \Lambda \qquad f(z + \lambda) \;=\; \chi(z, \lambda)\, f(z). \tag{4.5}$$

   (Indeed the section associated to one such f is defined by:

$$s_f([z]) \;=\; [(z, f(z))]$$

   where the square brackets denote equivalence classes. It is trivial to check that s_f is well-defined.)
2. For any integer N the N-th power of χ, χ^N is again a cocycle. The line bundle it defines is the N-th tensor power of $\mathcal{L}$, $\mathcal{L}^{\otimes N}$.
3. A Hermitian structure on $\mathcal{L}$ is defined by a function $h : \mathbb{C} \to \mathbb{R}^+$ satisfying:

$$h(z) \;=\; |\chi(z, \lambda)|\, h(z + \lambda). \tag{4.6}$$

   (Specifically, the Hermitian metric is: $\|[(z, a)]\| = |a|h(z)$.)

It turns out that the cocycle giving rise to the line bundle we are interested in is:

$$\chi_{\mu,\nu}(z\,,\, m + in) \;=\; (-1)^{mn}\, e^{\pi[z(m-in)+\frac{1}{2}(m^2+n^2)]} e^{-2\pi i[m\mu+n\nu]}. \tag{4.7}$$

Proposition 4.3. *$\chi_{\mu,\nu}$ as defined by (4.7) is indeed a cocycle, i.e. it satisfies (4.4). In addition, the function*

$$h(z) \;=\; e^{-\pi|z|^2/2} \tag{4.8}$$

defines a Hermitian structure on the associated bundle.

The proof is a simple calculation. From this point of view, the definition of theta functions is the following:

Definition 4.4. The space of θ functions of order N and characteristics μ, ν is the space, $\Omega_N^{\mu,\nu}$, of holomorphic sections of the line bundle

$$\mathcal{E}^N_{\mu,\nu} \;:=\; (\mathcal{L}_{0,0})^{\otimes N} \otimes \mathcal{L}_{\mu,\nu}.$$

Here $\mathcal{L}_{\mu,\nu}$ denotes the line bundle associated with the cocycle $\chi_{\mu,\nu}$. Thus $\Omega_N^{\mu,\nu}$ is the space of entire functions $f : \mathbb{C} \to \mathbb{C}$ satisfying: $\forall z \in \mathbb{C},\ m + in \in \Lambda$

$$f(z+m+in) \;=\; (-1)^{Nmn}\, e^{N\pi[z(m-in)+\frac{1}{2}(m^2+n^2)]}\, e^{-2\pi i[m\mu+n\nu]}\, f(z).$$

Both definitions are reconciled as follows:

Proposition 4.5. *For all N, the linear map*

$$\begin{array}{ccc} \Omega_N^{\mu,\nu} & \to & \Theta_N^{\mu,\nu} \\ f(z) & \mapsto & e^{-N\pi z^2/2} f(z) \end{array} \tag{4.9}$$

is an isomorphism.

We'll henceforth use this isomorphism to identify $\Omega_N^{\mu,\nu}$ and $\Theta_N^{\mu,\nu}$.

We now turn our attention to the Hilbert space structure of the space of theta functions. The motivation is clear if we think of the space $\Omega_N^{\mu,\nu}$. Notice that the bundle $\mathcal{L}^{\otimes N}$ has the Hermitian structure defined by the function

$$h(z)^N \;=\; e^{-N\pi|z|^2/2}.$$

This means that if f is a section of $\mathcal{L}^{\otimes N}$, then with respect to the Hermitian metric

$$\text{length of } f(z) \;=\; e^{-N\pi|z|^2/2}\,|f(z)|.$$

Therefore, the space of smooth sections of $\mathcal{L}^{\otimes N}$ has the pre-Hilbert space structure

$$< f\,,\, g > = \int_{\mathcal{F}} f(z)\,\overline{g(z)}\, e^{-N\pi|z|^2}\, dx\, dy, \tag{4.10}$$

where $\mathcal{F}$ is a fundamental domain (e.g. $\mathcal{F} = [0,1]\times[0,1]$) and $z = x+iy$. It is easy to check that the expression above is independent of the choice of $\mathcal{F}$. We will endow Ω_N with this Hermitian inner product.

Let us now work out the formula for the inner product on $\Theta_N^{\mu,\nu}$, obtained by decreeing that the basic isomorphism (4.9) be unitary. Clearly to obtain the weight for the inner product on $\Theta_N^{\mu,\nu}$ the weight $e^{-N\pi|z|^2}$ in (4.10) ought to be multiplied by $|e^{-N\pi z^2/2}|^{-2} = |e^{N\pi z^2}| = e^{N\pi(x^2-y^2)}$. One obtains:

$$\forall f,\ g \in \Theta_N^{\mu,\nu} \qquad < f\,,\, g > := \int_{\mathcal{F}} f(z)\,\overline{g(z)}\, e^{-2N\pi y^2}\, dx\, dy. \tag{4.11}$$

4.1.3. *Without bundles.* In this paper we are only interested in theta functions with characteristics satisfying

$$\mu = 0.$$

Indeed the x and y directions do not play symmetrical roles in the periodic Toda problem. Henceforth we will take μ to be zero.

The next lemma gives in particular the Hilbert structure of $\Theta^{0,\nu}$

Lemma 4.6. *Let us endow $\Theta^{0,\nu}$ with the following Hilbert structure:*

$$\forall f,\ g \in \Theta^{0,\nu} \qquad < f\,,\,g >:= \int_{\mathcal{F}} f(z)\,\overline{g(z)}\,e^{-2N\pi y^2}\,dx\,dy.$$

where $\mathcal{F} \ = \ [0,1]\times[0,1]$.

Then the basis $\{\theta_0^{0,\nu},\dots,\theta_{N-1}^{0,\nu}\}$ *of* $\Theta^{0,\nu}$ *is orthogonal, and* $\forall j$

$$\|\,\theta_j^{0,\nu}\,\|^2 \ = \ \frac{e^{2\pi j^2/N}}{\sqrt{2N}}.$$

This will be a consequence of a more general calculation that we'll perform below (case $H=1$ in Proposition 4.7).

4.2. **Toeplitz quantization.** Let $T_H^{(N,\nu)}$ be the Toeplitz operator with multiplier H acting on the space $\Theta^{0,\nu}$. Our goal here is to compute the matrix associated to $T_H^{(N,\nu)}$ where H is of the form

$$H(x,y) = u(y) + 2\cos(2\pi x)\,v(y) \tag{4.12}$$

in the orthonormal basis

$$\vartheta_j^{0,\nu} \ := \ (2N)^{1/4}\,e^{-\pi j^2/N}\,\theta_j^{0,\nu}, \qquad j=0,\dots,N-1 \tag{4.13}$$

(c.f.Lemma 4.6).

4.2.1. *Multipliers $H = u(y)$.* We will first consider the case $v=0$, i.e. H of the form

$$H(x,y) \ = \ u(y), \tag{4.14}$$

where u is smooth periodic function of period 1. Let $\hat{u}(n)$ denote the n-th Fourier coefficient of u.

Proposition 4.7. *Let H be given by (4.14). Then the matrix of $T_H^{(N,\nu)}$ in the basis (4.13) is diagonal and the j-th diagonal element is equal to*

$$\lambda_j^{(N)} \ = \ \sum_{m=-\infty}^{\infty} \hat{u}(m)\,e^{-\pi m^2/2N}\,e^{-2\pi i m j/N}. \tag{4.15}$$

Proof. The matrix coefficient $a_{lj}^{(N)} =< u\vartheta_l^{0,\nu},\vartheta_j^{0,\nu} >$ of $T_H^{(N,\nu)}$ is

$$a_{jl}^{(N)} = \sqrt{2N}e^{-\pi[j^2+l^2]/N}\ < H\theta_l^{0,\nu},\theta_j^{0,\nu} > .$$

Writing out the inner-product $< H\theta_l^{0,\nu},\theta_j^{0,\nu} >$ and recalling that H is a function of y, one sees that $a_{jl}^{(N)}$ is zero unless $l=j$. If this is the case, then

$$< u(y)\theta_j^{0,\nu},\theta_j^{0,\nu} >= \sum_m e^{-2\pi[Nm^2+2mj]} \int_0^1 e^{-2N\pi y^2}\,e^{-4\pi y[j+mN]}\,u(y)\,dy \ =$$

$$= \int_0^1 \left(e^{-2\pi[Ny^2+2yj]} \sum_m e^{-2\pi[Nm^2+2m(j+Ny)]} \right) u(y)\, dy = \tag{4.16}$$

$$= \int_0^1 \lambda_j^{(N)}(y)\, u(y)\, dy,$$

where $\lambda_j^{(N)}(y)$ is defined to be the expression in parenthesis in (4.16). By completing the square, the kernel $\lambda_j^{(N)}(y)$ can be transformed into

$$\lambda_j^{(N)}(y) = e^{2\pi j^2/N} \sum_m e^{-2\pi N(m+y+j/N)^2}.$$

Applying the Poisson summation formula one gets:

$$\lambda_j^{(N)}(y) = \frac{e^{2\pi j^2/N}}{\sqrt{2N}} \sum_m e^{-\pi m^2/2N}\, e^{2\pi i(y+j/N)m}.$$

Substituting back in (4.16) and collecting factors (including the normalization of the $\vartheta_j^{0,\nu}$) one gets the result. (If we take $H=1$ we obtain: $\lambda_j^{(N)} = 1$, which proves that the normalization of the $\vartheta_j^{0,\nu}$ is correct.) □

Corollary 4.8. A. *For each N, the basis $\{\theta_j^{0,\nu}\,;\, j = 0,\dots,N-1\}$ is a basis of eigenvectors of the Toeplitz operator $T^{(N,\nu)}_{\cos(2\pi y)}$. Specifically:*

$$T^{(N,\nu)}_{\cos(2\pi y)}(\theta_j^{0,\nu}) = \cos(2\pi j/N)\, e^{-\pi/2N}\, \theta_j^{0,\nu}, \quad 0,\dots,N-1. \tag{4.17}$$

B. *If H_1, H_2 are periodic functions of y, then $\forall N$ $[T^{(N,\nu)}_{H_1}, T^{(N,\nu)}_{H_2}] = 0$.*
C. *If H is a trigonometric polynomial in y, then the eigenvalues $\lambda_j^{(N)}$ of $T_H^{(N,\nu)}$ satisfy:*

$$\lambda_j^{(N)} \sim \sum_m \hat{u}(m)\, e^{2\pi i m j/N} = u(-j/N). \tag{4.18}$$

uniformly in j, as $N \to \infty$.

4.2.2. *Multipliers $H = 2\cos(x)v(y)$.* We now compute

$$< 2\cos(2\pi x)v(y)\theta_j^{0,\nu}\,,\, \theta_l^{0,\nu} >= 2\sum_{n,m} e^{-\pi[N(n^2+m^2)+2(jn+lm)]} \cdot e^{2\pi i\nu(n-m)}.$$

$$\cdot\, \widehat{\cos(2\pi x)}[N(m-n)+l-j] \int_0^1 e^{-2\pi[Ny^2+y(j+l+N(n+m))]}\, v(y)\, dy. \tag{4.19}$$

The only non-zero terms are the following:
(a) $N(m-n)+l-j=1$, that is $Nm+l = Nn+j+1$. If $j \le N-2$, this implies that $l=j+1$ and $m=n$. Otherwise, $j=N-1$ in which case $l=0$ and $m=n+1$.

(b) $N(m-n)+l-j=-1$, that is $Nm+l+1=Nn+j$. Here simply the roles of (m,l) and (n,j) are reversed: In this case either ($j=l+1$ and $m=n$) or ($l=N-1$ and $j=0$ and $n=m+1$).

It follows that the matrix of T_H is tri-diagonal except for possibly non-zero entries at the upper-right hand and lower left-hand corners, exactly as in the periodic Toda lattice. Let us now compute the non-zero matrix entries. Up to a permutation of j, l there are only two possibilities:
(i) $l=j+1$ and $m=n$. In this case (4.19) becomes

$$< 2\cos(2\pi x)v(y)\theta_j\,,\,\theta_{j+1} >= \int_0^1 f_j(y)\,v(y)\,dy \tag{4.20}$$

where

$$f_j(y) := \sum_n e^{-\pi[2Nn^2+2(jn+(j+1)n)]}\, e^{-2\pi[Ny^2+y(2j+1+2nN)]}. \tag{4.21}$$

Once again we complete the square:

$$f_j(y) = e^{\frac{2\pi}{N}(j+\frac{1}{2})^2}\sum_n e^{-2\pi N[y+n+(j+\frac{1}{2})/N]^2}.$$

As before we can apply the Poisson summation formula to obtain that

$$f_j(y) = \frac{1}{\sqrt{2N}}\, e^{\frac{2\pi}{N}(j+\frac{1}{2})^2}\sum_m e^{2\pi i m[y+(j+\frac{1}{2})/N]}\, e^{-\pi m^2/2N},$$

Taking into account the normalization of the ϑ_j, we obtain:

$$\begin{aligned} < 2\cos(2\pi x)v(y)\vartheta_j\,,\,\vartheta_{j+1} > &= e^{-\pi/2N}\sum_m e^{-\pi m^2/2N}\, e^{-2\pi i m(j+\frac{1}{2})/N}\,\hat{v}(m) \\ &\sim v\left(-(j+\frac{1}{2})/N\right). \end{aligned} \tag{4.22}$$

(ii) $j=N-1$, $l=0$ and $m=n+1$. In this case (4.19) becomes

$$< 2\cos(2\pi x)v(y)\theta_{N-1}\,,\,\theta_0 >= \int_0^1 g_j(y)\,v(y)\,dy \tag{4.23}$$

where

$$g_j(y) := e^{-2\pi i\nu}\sum_n e^{-\pi[2Nn^2+4Nn+N-2n]}\, e^{-2\pi N[y^2+2y(1+n-\frac{1}{2N})]}. \tag{4.24}$$

Completing the square:

$$g_j(y) = e^{-2\pi i\nu}\, e^{\pi[N-2-\frac{1}{2N}]}\sum_n e^{-2\pi N(y+n+1-\frac{1}{2N})^2}. \tag{4.25}$$

Applying the Poisson summation formula:

$$g_j(y) = \frac{1}{\sqrt{2N}} e^{-2\pi i\nu}\, e^{\pi[N-2-\frac{1}{2N}]}\sum_m e^{2\pi i m[y+1-\frac{1}{2N}]}\, e^{-\pi m^2/2N}.$$

Finally we obtain:

$$< 2\cos(2\pi x)v(y)\vartheta_{N-1}\,,\,\vartheta_0 >= e^{-2\pi i\nu}\,e^{-3\pi/2N}\sum_m e^{2\pi i m\frac{1}{2N}}\,e^{-\pi m^2/2N}\,\hat{v}(m)$$

$$\sim e^{-2\pi i\nu}\,v\left(\frac{1}{2N}\right). \tag{4.26}$$

Putting all this together we can state:

Theorem 4.9. *As $N\to\infty$, the matrix of $T_H^{(N,\nu)}$ is asymptotic to the periodic Jacobi matrix*

(4.27)

$$\begin{pmatrix} u(1) & v(1-1/2N) & 0 & \cdots & e^{-2\pi i\nu}v(1/2N) \\ v(1-1/2N) & u(1-1/N) & v(1-3/2N) & \cdots & 0 \\ 0 & v(1-3/2N) & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ e^{2\pi i\nu}v(1/2N) & 0 & \cdots & v(3/2N) & u(1/N) \end{pmatrix}$$

More precisely, given a matrix of the form (4.27) one can find two functions

$$u_N\sim u+\sum_{j=1}^{\infty}N^{-j}u_j \quad \text{and} \quad v_N\sim v+\sum_{j=1}^{\infty}N^{-j}v_j$$

such that, $\forall N$ (4.27) is the matrix of $T_{H_N}^{(N,\nu)}$ where

$$H_N(x,y)=u_N(y)+2cos(2\pi x)v_N(y)$$

Finally if u and v are regular enough for the following expressions to exist, we have:

$$u_N=e^{+\frac{\pi\Delta}{2N}}u \quad \text{and} \quad v_N=e^{+\frac{\pi\Delta}{2N}}v.$$

Here Δ stands for the (positive) Laplacian on the torus.

Remarks:

1. The Fourier series that appear in the exact expression for the matrix coefficients are precisely the Fourier series of the functions
$$e^{-\pi\Delta/2N}(u), \qquad e^{-\pi\Delta/2N}(v).$$
Here $\Delta=-\frac{d^2}{dx^2}$ is the Laplacian on the circle, and therefore the functions above are the solutions to the heat equation with initial condition u (resp. v) at time $\pi/2N$.
2. Sometimes Toda systems with quasi-periodic boundary conditions $a_N = e^{-2\pi\nu}a_0$ are considered. The "spectral parameter" $e^{-2\pi\nu}$ is of crucial importance for the connection with algebraic geometry. Although we won't go in this direction, let us mention that the results of this paper extend naturally to this setting.

4.3. **Proof of Lemma 1.4 and other results.** The Lemma 1.4 is an easy consequence of Theorem 4.9 if we take for $u = b$ and $v(.) = a(. - \frac{1}{2N}) = a(.) + \frac{1}{2N}a'(.) + \frac{1}{2!2N^2}a''(.) + \cdots = e^{\frac{1}{2N}\partial_{\cdot}}a(.)$, and consider matrices on the rearranged basis:

$$\{(2N)^{1/4}e^{-\pi(N-1)^2/N}\theta^{0,\nu}_{N-1}, \ldots, (2N)^{1/4}\theta^{0,\nu}_0\}$$

with $\nu = 0$. In particular we have the more precise result

Proposition 4.10. *let*

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ a_N & \cdots & & a_{N-1} & b_N \end{pmatrix} \tag{4.28}$$

with $a_i = a(\frac{i}{N})$, $b_i = b(\frac{i}{N})$, $a, b \in C^\infty(S^1)$.
Then L is, in the basis $\{(2N)^{1/4}e^{-\pi(N-1)^2/N}\theta^{0,0}_{N-1}, \ldots, (2N)^{1/4}\theta^{0,0}_0\}$ of $\Theta^{0,0}_N$, the matrix of the Toeplitz operator T^T_H of (total) symbol

$$H_N(\varphi, \theta) \;=\; e^{+\frac{\pi}{2}\frac{\Delta\varphi}{N}}\{b(\varphi) + 2e^{\frac{i}{2N}D_\varphi}a(\varphi)\cos(\theta)\} \tag{4.29}$$

where $D_\varphi := -i\frac{d}{d\varphi}$ and Eq.(4.29) has to be understood as an asymptotic expansion (possibly convergent if a and b are regular enough).

By easy computations and the same argument we get the following Proposition, also a consequence of the general theory:

Proposition 4.11. *let L_1 and L_2 two matrices as in Proposition 4.10, therefore Toeplitz operators of principal symbols $H^{1,2}(\theta, \varphi) := a_{1,2}(\varphi) + 2cos(\theta)a_{1,2}(\varphi)$. Then L_1L_2 is a Toeplitz operator with principal symbol H^1H^2 and $\frac{1}{iN}[H^1, H^2]$ is a Toeplitz operator whose principal symbol is*

$$\{H^1_0, H^2_0\} := \partial_\varphi H^1_0 \partial_\theta H^2_0 - \partial_\theta H^1_0 \partial_\varphi H^2_0.$$

Finally we will need the

Proposition 4.12. *Let*

$$B(L) = \begin{pmatrix} 0 & a_1 & 0 & \cdots & -a_N \\ -a_1 & 0 & a_2 & \cdots & 0 \\ 0 & -a_2 & 0 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ a_N & \cdots & & -a_{N-1} & 0 \end{pmatrix} \tag{4.30}$$

with $a_i = a(\frac{i}{N})$, $a \in C^\infty(S^1)$.
Then $B(L)$ is, in the basis $\{(2N)^{1/4}e^{-\pi(N-1)^2/N}\theta^{0,0}_{N-1}, \ldots, (2N)^{1/4}\theta^{0,0}_0\}$ of $\Theta^{0,0}_N$, the matrix of the Toeplitz operator of (total) symbol

$$-\frac{\partial}{\partial\theta}H_N(\varphi, \theta) \;=\; ie^{+\frac{\pi}{2}\frac{\Delta\varphi}{N}}\{2e^{\frac{i}{2N}D_\varphi}a(\varphi)\sin(\theta)\} \tag{4.31}$$

where H_N is as in Proposition 4.10.

5. Proofs

The Lemma 1.4 has been already proved in the preceding sections. In essence it means that in the different types of sampling used in this paper (that is: periodic and non-periodic) are Toeplitz operators.

Let us first prove that Theorem 1.5 implies Theorem 1.1.

5.1. From Theorem 1.5 to Theorem 1.1.

Let us consider the Toeplitz operator $T_{H,K} = \sum_0^K N^{-k}\mathcal{T}_k$ obtained from the operator T_N constructed in Theorem 1.5 by truncation at order K. Let us consider for simplicity the non-periodic case, the periodic one being exactly the same. We know that each $\mathcal{T}_k$ has a tridiagonal form and matrix elements which are discretizations of some functions. Let us define the functions a_k and b_k in Theorem 1.1 as

$$\mathcal{T}_k = \begin{pmatrix} b_k(1/N) & a_k(1/N) & 0 & \cdots & 0 \\ a_k(1/N) & b_k(2/N) & a_k(2/N) & \cdots & 0 \\ 0 & a_k(2/N) & b_k(3/N) & a_k\cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & \cdots & & a_k(1-1/N) & b_k(1) \end{pmatrix} \tag{5.1}$$

Since we know that $||L(t) - T_{H,N}||_{HS}$ is of order $N^{-(K+1)}$ it is enough to use the fact that the operator norm is bounded by the Hilbert-Schmidt norm to finish the proof of Theorem 1.1.

The rest of this section is devoted to a proof of Theorem 1.5. It remains to prove the Theorem 1.5. Let us recall that the main idea in the proof that a given operator O is a Toeplitz operator consists in finding a "symbol" H whose corresponding Toeplitz operator T_H is close to O, and then proceed by induction. The first part of the proof of the Theorem 1.5 will follow this way and will provide a Toeplitz operator T solving the equation

$$\dot{T} = [T, B(T)] \tag{5.2}$$

modulo a remainder of arbitrary order in N^{-1}. We will prove then that the true solution of (5.2) is close to T in an adapted norm.

5.2. Construction of a Toeplitz approximate solution.

In order to carry the proofs in the periodic and non-periodic case together we will denote by x both the variables h and φ.

Let us recall that if T_H is a Toeplitz operator of symbol $H(x,\theta) = b(x) + 2a(x)cos(\theta)$ then $\frac{1}{iN}B(T_H)$ is also a Toeplitz operator and its principal symbol is

$$-2\partial_\theta a(x)cos\theta.$$

In what follows X will stand for either the sphere or the torus.

Lemma 5.1. *Let $L_0 = \{L_0^{(N)}\}$ be a self-adjoint Toeplitz operator on X, tri-diagonal in the standard basis. Let $H_0 : X \to \mathbb{R}$ be its principal symbol. Let $J = [0, \tau]$ be a closed one-sided neighborhood of zero in $\mathbb{R}$, and assume that there exists a smooth solution $H : J \times X \to \mathbb{R}$ of the initial-value problem*

$$\left\{ \begin{array}{rcl} \frac{\partial}{\partial s} H & = & \{H \, , \, \partial_x H\} \\ H|_{s=0} & = & H_0 \end{array} \right. \tag{5.3}$$

Then there exists a smooth one-parameter family of self-adjoint Toeplitz operators, $\Lambda(t)$, of order zero, with $\Lambda^{(N)}(t)$ defined for $\frac{t}{N} \in J$, such that

$$\left\{ \begin{array}{rcl} \frac{d}{dt} \Lambda & = & [\Lambda \, , \, B(\Lambda)] + R \\ \Lambda|_{t=0} & = & L_0 + S \end{array} \right. \tag{5.4}$$

where the norms of R and S are of arbitrary order in N^{-1}. Moreover, Λ can be chosen to be tridiagonal.

Proof. The method is standard in pseudodifferential operator theory: one proceeds by successive approximations. Let $\Lambda_0(s)$ be a s. a. time-dependent Toeplitz operator of order zero with principal symbol equal to H. We can choose $\Lambda_o(s)$ to be tri-diagonal. By the principal symbol calculus and equation (5.3),

$$\frac{d}{ds} \Lambda_0 = N \, [\Lambda_0 \, , \, B(\Lambda_0)] + \mathcal{R}_0,$$

where $\mathcal{R}_0$ is a Toeplitz operator of order -1. Let

$$\Lambda_1 = \Lambda_0 + S$$

where S is a Toeplitz operator of order -1. Then a simple calculation shows that

$$\frac{d}{ds} \Lambda_1 = N \, [\Lambda_1 \, , \, B(\Lambda_1)] + \mathcal{R}_1,$$

with $\mathcal{R}_1$ of order -2, provided the symbol σ of S satisfies

$$\frac{d}{ds} \sigma = \{H \, , \, \partial_\theta \sigma\} + \{\sigma \, , \, \partial_\theta H\} - \rho_0 \tag{5.5}$$

where ρ_0 is the principal symbol of $\mathcal{R}_0$. This equation is an inhomogeneous linearization of the non-linear equation, (5.3), around the solution H. Although it appears to be a second-order equation in σ, since Λ_0 was chosen tri-diagonal then σ is also tri-diagonal and (5.5) is easily seen to be equivalent to a hyperbolic first-order 2×2 system. Therefore, it certainly has a smooth solution given any smooth initial condition. The initial condition σ_0 is chosen in such a way that $L_0 - \Lambda_1|_{s=0}$ is of order -2. That is, σ_0 is the principal symbol of $L_0 - \Lambda_0|_{s=0}$. In addition, since tri-diagonality of the symbols is preserved by the equation they must satisfy, the symbols will be tri-diagonal and hence we can choose the Toeplitz operators to be tridiagonal as well.

Proceeding inductively in this fashion one finds a self-adjoint time-dependent Toeplitz operator, Λ_∞, such that

$$\Lambda_\infty|_{s=0} \;=\; L_0 \;+\; O(N^{-\infty})$$

and

$$\frac{d}{ds}\Lambda_\infty - N[\Lambda_\infty\,,\,[\Lambda_\infty\,,\,\mathcal{Z}]] \;=\; O(N^{-\infty}).$$

It now suffices to make the change of time variables, $t := sN$, to conclude the proof.

□

5.3. **Approximating the true solution.** To prove that the operator Λ_∞ is close to the exact solution of the Toda system, L, we need a stability result that we will derive from some a priori estimates.

Therefore let us denote:

$$L(t) \;=\; \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \text{ or } a_N \\ a_1 & b_2 & a_2 & \cdots & 0 \\ 0 & a_2 & b_3 & a_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 \text{ or } a_N & \cdots & & a_{N-1} & b_N \end{pmatrix} \tag{5.6}$$

the solution of the Toda flow with an initial condition $L(0) = L_0$ where L_0 is a Toeplitz operator as in Lemma 5.1. Moreover let

$$\Lambda_\infty(t) \;=\; \begin{pmatrix} B_1 & A_1 & 0 & \cdots & 0 \text{ or } A_N \\ A_1 & B_2 & A_2 & \cdots & 0 \\ 0 & A_2 & B_3 & A_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 \text{ or } A_N & \cdots & & A_{N-1} & B_N \end{pmatrix} \tag{5.7}$$

the Toeplitz operator just constructed in the preceding subsection, and let

$$L(t) - \Lambda_\infty(t) \;=\; \begin{pmatrix} \beta_1 & \alpha_1 & 0 & \cdots & 0 \text{ or } \alpha_N \\ \alpha_1 & \beta_2 & \alpha_2 & \cdots & 0 \\ 0 & \alpha_2 & \beta_3 & \alpha_3 \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 \text{ or } \alpha_N & \cdots & & \alpha_{N-1} & \beta_N \end{pmatrix}. \tag{5.8}$$

We will obtain an estimate on the "energy"

$$E := \sum_{j=1}^{N}\left(2\alpha_j^2 + \frac{1}{2}\beta_j^2\right).$$

By hypothesis $E(t=0) = O(N^{-\infty})$ and by the constructions of $L(t)$ and Λ_∞ we have:

$$\left\{\begin{array}{rcl} \dot{a}_j & = & a_j(b_{j+1} - b_j) \\ \dot{b}_j & = & 2(a_j^2 - a_{j-1}^2) \end{array}\right. \tag{5.9}$$

and

$$\left\{\begin{array}{rcl} \dot{A}_j & = & A_j(B_{j+1} - B_j) + F(N) \\ \dot{B}_j & = & 2(A_j^2 - A_{j-1}^2) + G(N) \end{array}\right. \tag{5.10}$$

where F and G are fastly decreasing. From (5.9) and (5.10) we compute:

$$\begin{cases} \dot{\alpha}_j &= A_j(\beta_{j+1}-\beta_j)+\alpha_j(B_{j+1}-B_j)+\alpha_j(\beta_{j+1}-\beta_j)+F(N) \\ \dot{\beta}_j &= 4A_j\alpha_j-4A_{j-1}\alpha_{j-1}+2(\alpha_j^2-\alpha_{j-1}^2)+G(N) \end{cases} \tag{5.11}$$

Multiplying the first equation by $2\alpha_j$ and the second one by $\frac{1}{2}\beta_j$, we get, after some cancellations (both for the non periodic case with $\alpha_0 = \alpha_N = 0$ and the periodic one with $\alpha_{i+N}-\alpha_i = \beta_{i+N}-\beta_i = 0$), that

(5.12)
$$\dot{E} = 2\sum \alpha_j^2(B_{j+1}-B_j)+\sum \alpha_j^2(\beta_{j+1}-\beta_j)+2F(N)\sum \alpha_j+\frac{1}{2}G(N)\sum \beta_j.$$

Let us now remark that, from the preceding sections, since Λ_∞ is a Toeplitz operator, $B_{j+1}-B_j = O(N^{-1})$ (the O depending actually on the supremum norm of the derivative of the symbol of Λ_∞). In particular $|B_{j+1}-B_j| \leq \frac{C}{N}\sup\{|\partial_x b^t(x)|\}$.

Denoting $K_N := [2N|F(N)|^2+\frac{N}{2}|G(N)|^2]^{1/2}$ and $C_N = \frac{C}{N}\big(\sup\{|\partial_x b^t(x)|\}+1\big)$, where b^t is the solution of the limiting PDE, we claim that E satisfies the inequation:

$$\dot{E} \leq C_N E+\sqrt{2}E^{3/2}+K_N E^{1/2}. \tag{5.13}$$

(Indeed the first sum in (5.12) is less than $C_N E$, the second sum can be seen to be less than $\sqrt{2}E^{3/2}$, and the sum of the remainder terms, by the Cauchy-Schwartz inequality, is less than $K_N E^{1/2}$.)

Let us next compute the solution of the equation

$$\dot{X} = C_N X+\sqrt{2}X^{3/2}+K_N X^{1/2} \tag{5.14}$$

with $X(0) = E(0)$. In terms of $Y := \sqrt{X}$, the equation becomes:

$$\dot{Y} = \frac{1}{2}C_N Y+\frac{1}{\sqrt{2}}Y^2+K_N, \tag{5.15}$$

which can be solved explicitly. The result is:

$$\sqrt{X(t)} \;=\; Y(t) = \frac{s_+ - s_- D e^{\frac{s_+-s_-}{\sqrt{2}}t}}{1-De^{\frac{s_+-s_-}{\sqrt{2}}t}}, \tag{5.16}$$

with

$$s_\pm \;=\; -\frac{C_N}{2\sqrt{2}} \pm \sqrt{\frac{C_N}{8}-\sqrt{2}K_N}, \tag{5.17}$$

and

$$D = \frac{Y(0)-s_+}{Y(0)-s_-} = \frac{\sqrt{E(0)}-s_+}{\sqrt{E(0)}-s_-}. \tag{5.18}$$

Keeping in mind that $K_N = O(N^{-\infty})$ and that $C_N = O(N^{-1})$, we see that, in particular, $-s_+ \sim 2\frac{K_N}{C_N} = O(N^{-\infty})$, $s_- \sim -\frac{C_N}{2} = O(N^{-1})$ (and negative), and $0 \leq D = O(N^{-\infty})$.

Since $Y(0) = \frac{S_+ - s_- D}{1-D} \geq 0$ and $-s_- De^{\frac{s_+ - s_-}{\sqrt{2}}t}$ is an increasing function of $t \geq 0$ (as $s_+ \sim -\frac{C}{N}$, $C > 0$), $Y(t)$ remains positive and $O(N^{-\infty})$ for $\frac{t}{N}$ bounded by any constant. Therefore $X(t) = O(N^{-\infty})$, uniformly for such t's.

The fact that E is bounded by X (for each N) is standard; let us recall briefly the argument. Let X_ϵ be the solution of

$$\dot{X}_\epsilon = (C_N + \epsilon)X_\epsilon + \sqrt{2}X_\epsilon^{3/2} + K_N X_\epsilon^{1/2} \tag{5.19}$$

with $X_\epsilon(0) = E(0)$, and let $t_* \geq 0$ be a time such that $E(t) \leq X_\epsilon(t)$ for all $t \in [0, t_*]$. Then

$$\dot{E}(t_*) \leq C_N E + \sqrt{2}E^{3/2} + K_N E^{1/2} \leq C_N X_\epsilon + \sqrt{2}X_\epsilon^{3/2} + K_N X_\epsilon^{1/2}$$

$$< (C_N + \epsilon)X_\epsilon + \sqrt{2}X_\epsilon^{3/2} + K_N X_\epsilon^{1/2} = \dot{X}_\epsilon(t_*).$$

Therefore $\dot{E}(t_*) < \dot{X}_\epsilon(t_*)$, which implies that there exists $\delta > 0$ such that $E(t) \leq X_\epsilon(t)$ for all $t \in [0, t_* + \delta]$. Therefore, the set $\{t \geq 0\ ;\ E(t) \leq X_\epsilon(t)\}$ is open. Since it is obviously closed, it must equal the entire interval of definition of $E(t)$. Since this is valid for all ϵ, we have that $E(t) \leq X(t)$ on the whole interval of definition of E.

We have just proved that $E = O(N^{-\infty})$, uniformly for $\frac{t}{N}$ bounded. It is then enough to notice that E control the Hilbert-Schmidt norm of $L(t) - \Lambda_\infty$ to conclude the proof of Theorem 1.5.

6. Final remarks and complementary results

1. It is well known that the solution of the Toda flow is unitarily conjugate to the initial condition.

   Let us decompose the operator $e^{L(0)t}$ into its unitary $U(t)$ and upper-triangular parts (QR decomposition or Gramm-Schmidt factorization method). Then the solution of the Toda flow satisfies

   $$L(t) = U(t)L(0)U(t)^{-1}$$

   The equation for $U(t)$ is easy to derive using the Toda equation; we find

   $$\dot{U}U^{-1} = B\left(UL(0)U^{-1}\right)$$

   where $B(L)$ is as before. This equation has a natural limit as $N \to \infty$ using the results of the preceding sections (especially Propositions 3.7 and 4.11). Although we didn't check all the details it seems natural to us that $U(\frac{s}{N})$ should be a Fourier integral operator (in the framework of Toeplitz quantization) associated to an Hamiltonian flow of time-dependent Hamiltonian h_s satisfying the equation

   $$\partial_s h_s = 2\sin\theta a^s(x)\ (= -\partial_\theta\left(b^s(x) + 2\cos\theta a^s(x)\right)),$$

that is

$$h_s(\theta, x) = 2\sin\theta \int_0^s a^u(x)du,$$

where as before x runs for both φ and h, and a^u is the solutions of the limiting equation.

We note also that it is possible to get an explicit solution via factorization to our dispersionless system (1.8) for certain classes of initial data using the natural homomorhism between the Lie algebra $su(2)$ and the Poisson algebra on the sphere. This includes the explicit solution discussed in [13] and in [20]. (A more general approach to obtaining solutions, though not via factorization is also discussed in [20].)

2. Toeplitz operators share with pseudodifferential operators basically all their spectral properties. Let us quote some general theorems on the asymptotics of the spectrum of Toeplitz operators acting on holomorphic sections of quantizing line bundles over compact Riemann surfaces. These are special cases of very general results, see [10] and [11].

Let X be a compact Riemann surface equipped with a Riemannian metric. Let ω denote the area form of X. Assume given a holomorphic Hermitian line bundle $\mathcal{L} \to X$ whose curvature is ω. Let $\{T_H^{(N)}\}$ be a Toeplitz operator of order zero with principal symbol $H : X \to \mathbb{R}$; for each integer N, $T_H^{(N)}$ acts on the space of holomorphic sections of $\mathcal{L}^N$. Denote by

$$E_1(N) \leq E_2(N) \leq \cdots \leq E_{d_N}(N)$$

the eigenvalues, with multiplicities, of T_H^N. Here d_N denotes the dimension of the space of holomorphic sections of $\mathcal{L}^N$. By the Riemann-Roch and Kodaira vanishing theorems, for N large $d_N = N(g-2) - (g-1)$ where g is the genus of X. The question we address here is: what is the asymptotic behaviour of the eigenvalues $E_j(N)$ as $N \to \infty$?

Notice first that since $\{T_H^{(N)}\}$ is taken to be of order zero there exists a constant $C > 0$ such that for all j and N $|E_j(N)| \leq C$. The first result gives the asymptotic density of states:

Theorem 6.1. *For any continuous test function φ, one has:*

$$\lim_{N\to\infty} \frac{1}{d_N} \sum_{j=1}^{d_N} \varphi(E_j(N)) = \int_X \ (\varphi \circ H) \ \omega.$$

If φ is smooth one has a full asymptotic expansion in decreasing integral powers of N.

It follows that in the situation of the dispersionless limit of the Toda lattice, the classical integrals of motion $\mathrm{Tr}(L^k)$ are asymptotic to $d_N \int_X H^k\omega$.

It is perhaps worth pointing out that one can re-state Theorem 6.1 as follows: the sequence of spectral measures $\frac{1}{d_N}\sum_{j=1}^{d_N}\delta(E-E_j(N))$ converges weakly to the measure dm^H defined by:

$$\int_{\mathbb{R}} \varphi(E)\ dm_E^H := \int_X (\varphi\circ H)\ \omega.$$

On the set of regular values of H the measure dm^H is absolutely continuous with respect to Lebesgue measure:

$$dm_H = \chi(E)\ dE$$

where $\chi(E)$ is the derivative of the "distribution function"

$$\chi(E) = \frac{d}{dE}\ \text{Area}\ H^{-1}(-\infty, E].$$

It is possible to analyze the asymptotics of the spectrum at a finer scale; this is done by either the trace formula or the Bohr-Sommerfeld quantization condition. To state how that goes, we introduce the following notation.

For every regular value, E, of the classical Hamiltonian, let γ_j^E denote the connected components of $H^{-1}(E)$:

$$H^{-1}(E) = \coprod_j \gamma_j^E,$$

and for each j let $\mathfrak{h}_j^E \in [0, 2\pi)$ denote the holonomy angle of the oriented curve γ_j^E with respect to the connection of the quantizing line bundle, $\mathcal{L}$. Let T_j^E denote the smallest positive period of γ_j^E.

Then one has:

Theorem 6.2. *For every regular value E of H and any smooth test function φ with compactly supported Fourier transform one has:* [4]

$$\sum_j \varphi\left(N(E_j(N)-E)\right) = \sum_k \varphi\left(\frac{2\pi\left(k+\frac{1}{2}\right) - N\mathfrak{h}_j^E}{T_j^E}\right) + O\left(\frac{1}{N}\right).$$

(In fact one has a full asymptotic expansion in decreasing powers of N.)

This is closely related to the Bohr-Sommerfeld quantization rule: The energy levels E_l away from critical values of the classical Hamiltonian are within $O(1/N^2)$ of solutions to the equation in E

$$\mathfrak{h}_j^E = \frac{2\pi}{N}\left(k+\frac{1}{2}\right)$$

for some connected component γ_j^E of $H^{-1}(E)$.

[4] Assuming the subprincipal symbol is zero

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