Asymptotic Hamiltonian Dynamics: the Toda lattice, the three-wave interaction and the nonholonomic Chaplygin sleigh

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Abstract

In this paper we discuss asymptotic stability in energy-preserving systems which have an almost Poisson structure. In particular we consider a class of Poisson systems which includes the Toda lattice. In standard Hamiltonian systems one of course does not expect asymptotic stability. The key here is the structure of the phase space of the Poisson or almost Poisson systems and the nature of the equilibria. Our systems are in some sense generalizations of the integrable Toda lattice system but are not integrable in general. As a particular example we point out an interesting connection between three mechanical two degree-of-freedom systems that exhibit asymptotic stability. Two of them are classical Hamiltonian systems while the third is a nonholonomic system. Nonholonomic systems are generally energy-preserving but not Hamiltonian, but the system analyzed here turns out to have a phase space which is the union of Hamiltonian ones. We also discuss various higher-dimensional examples.

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Contents

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1 Introduction

A basic observation about Hamiltonian systems is that they satisfy Liouville's theorem, preserving volume in phase space, and thus cannot exhibit asymptotic stability. A reflection of this fact is that the spectrum of their linearization about a fixed point is symmetric about the imaginary axis. However, with a slight change in one's point of view this observation has been shown to be false in the following sense: Hamiltonian systems can exhibit asymptotic stability with respect to some of their variables. Further, if the system is integrable, the flow restricts to a level set of the integrals and the flow on this level set (a Lagrange sub-manifold with dimension equal to half that of the full phase space) can exhibit asymptotic stability on a compactification of the phase space. Moreover, to the casual observer of the system it certainly appears that the entire system is asymptotically stable. This is the case for the non-periodic Toda lattice as discussed for example in the work of Moser [1974] and more recently in Bloch [1990], Bloch, Brockett and Ratiu [1990, 92] and Bloch, Flaschka and Ratiu [1990].

Another class of dynamical systems that exhibits surprising asymptotic stability behavior is the class of nonholonomic systems. Nonholonomic systems are mechanical systems subject to nonintegrable constraints. They are not in general Hamiltonian, but in the absence of external dissipative forces they are always energy preserving. Such systems however do not necessarily preserve volume in the phase space, unlike Hamiltonian systems, and thus can exhibit asymptotic stability – see for example Zenkov, Bloch and Marsden [1998] and references therein

Our main goal here is to describe a large class of almost Poisson systems (i.e. systems with a bracket that fails to satisfy Jacobi) and Poisson (Hamiltonian) systems which exhibit asymptotic stability and which generalize the Toda lattice flows. We also describe some connections between these systems and nonholonomic systems. It is known that nonholonomic systems preserve energy but may exhibit asymptotic stability and in addition have a natural bracket which fails to satisfy the Jacobi identity (see for example Koon and Marsden [1997], Zenkov, Bloch and Marsden [1998], Van der Schaft and Maschke [1994] and Ruina [1998].) We do not explore the nonholonomic connection in much detail here but intend to return to this in a future publication. We do however point out a very interesting connection between the Toda lattice flow and a nonholonomic system (the Chapylgin system) in low dimensions and discuss briefly the higher-dimensional Suslov problem.

In the general setting we describe a large class of flows, both Poisson and not, which are of similar structure to the non-periodic Toda lattice flow and which exhibit asymptotic stability and preserve energy. The key in the Poisson setting to this behavior is the failure of the leaves of the Poisson structure to be compact in the full phase space. In this sense they are similar to the Toda lattice flow, but they are not in general integrable.

In the two-dimensional setting we show here that a basic example of a nonholonomic system that exhibits asymptotic stability, the so-called Chaplygin sleigh, see Neimark and Fufaev [1972], Ruina [1998], is in fact a "union" of two Hamiltonian systems. It is the the 2-dimensional classical Toda lattice flow but where the potential or off-diagonal variables are allowed to change sign as discussed in Tomei [1984], Brockett and Faybusovich [1991], Kodama and Ye [1996], Gekhtman and Shapiro [1997] and Bloch and Gekhtman [1998].

We discuss also the classic explicit solution method for Toda lattice equations and, briefly, that for the Chaplygin sleigh. In the process we of course obtain a Lax pair formulation for the Chaplygin sleigh. We also point out these systems are in turn equivalent to the equations for a certain regime of the semi-resonant three-wave interaction (see Ablowitz, Funk and Newell [1972], and also Alber, Luther, Marsden and Robbins [1998a, b] for example).

Some of the ideas in this paper are briefly described in Bloch [1999].

The structure of the paper is as follows: we first present a general discussion of the Toda lattice flow in arbitrary dimensions. We then discuss a general theory of asymptotic behavior for both Poisson and almost Poisson systems and illustrate this with some four-dimensional examples. We then consider in detail the two-dimensional case and relate it to the Chaplygin sleigh and the three-wave interaction. We compare and contrast the phase portraits of these systems and discuss some of the details of the three wave interaction. Finally we discuss the signed Toda lattice and the higher-dimensional Suslov problem.

2 The Toda Lattice

An important and beautiful mechanical system that describes the interaction of particles on the line (i.e. in one dimension) is the Toda lattice.

We shall describe here the non-periodic finite Toda lattice as analyzed by Moser [1974].

The model consists of n particles moving freely on the x–axis and interacting under an exponential potential. Denote the position of the kth particle by x_k. The Hamiltonian is then given by

$$H(x,y) = \frac{1}{2}\sum_{k=1}^{n} y_k^2 + \sum_{k-1}^{n-1} e^{(x_k - x_{k+1})} .$$

The associated Hamiltonian equations are

$$\begin{aligned} \dot{x}_k &= \frac{\partial H}{\partial y_k} = y_k & (2.1) \\ \dot{y}_k &= -\frac{\partial H}{\partial x_k} = e^{x_{k-1}-x_k} - e^{x_k - x_{k-1}} , & (2.2) \end{aligned}$$

where we use the convention $e^{x_0 - x_1} = e^{x_n - x_{n+1}} = 0$, which corresponds to formally setting $x_0 = -\infty$ and $x_{n+1} = +\infty$.

This system of equations has an extraordinarily rich structure. Part of this is revealed by Flaschka's observation (Flaschka [1974])that a change of variables enables one to write the equation in Lax pair form. To achieve this, set

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2} \quad , b_k = -\frac{1}{2} y_k . \tag{2.3}$$

The equations of motion then become

$$\begin{aligned} \dot{a}_k &= a_k(b_{k+1} - b_k) , \quad k = 1, \cdots, n-1 & (2.4) \\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2) , \quad k = 1, \cdots, n & (2.5) \end{aligned}$$

with the boundary conditions $a_0 = a_n = 0$ and where the $a_i > 0$. Note that for $a_i = 0$, which corresponds to an infinite separation of particles in original configuration space, there is a singularity in the Flaschka transformation.

This system can be written in the matrix form

$$\frac{d}{dt} L = [B, L] = BL - LB, \tag{2.6}$$

where

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix} \qquad B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix} .$$

One can express the Hamiltonian structure in Flaschka's variables through the Poisson matrix

$$J = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \tag{2.7}$$

where A has entries $a_{ii} = a_i$, $a_{ii+1} = -a_i$ and all other entries are zero. L is assumed to be traceless.

The flow is then given by

$$\dot{q} = J\mathrm{grad}H \tag{2.8}$$

where $q = [b_1, \cdots, b_{n-1}, a_1, \cdots, a_{n-1}]$ and $H = 1/2\mathrm{Tr}L^2$ is the Hamiltonian and grad is taken with respect to the standard metric.

If $O(t)$ is the orthogonal matrix solving the equation

$$\frac{d}{dt}O = BO\,, \quad O(0) = \text{ Identity}$$

then from (2.6), we have

$$\frac{d}{dt}(O^{-1}LO) = 0\,.$$

Thus, $O^{-1}LO = L(0)$, i.e. $L(t)$ is related to $L(0)$ by a similarity transformation and thus the eigenvalues of L, which are real and distinct, are preserved along the flow. This is enough to show that this system is explicitly solvable or integrable. Another basis for the integrals is given by the powers of L: $1/2\mathrm{Tr}L^k$, with the case $k = 2$ giving the Hamiltonian. This system may be viewed as a Hamiltonian system on the dual of the upper triangular (upper Borel) matrices.

There is, however, much more structure in this system. For instance, if N is the matrix $\mathrm{diag}[1, 2, \cdots, n]$ the Toda flow (2.6) can be written in the form

$$\dot{L} = [L, [L, N]]\,. \tag{2.9}$$

This shows that the flow is also gradient (on a level set of its integrals). This is exhibited by writing the equation in the so-called double bracket form of Brockett [1988] (see Bloch [1990], Bloch Brockett and Ratiu [1990, 1992]). These equations are in fact the gradient flow of the function $\mathrm{Tr}LN$ with respect to the normal metric. (The original observation of gradient flow behavior goes back to Moser [1974].) From this observation it is easy to show that the flow tends asymptotically to a diagonal matrix with the eigenvalues of $L(0)$ on the diagonal and ordered according to magnitude, recovering the observation of Moser, Symes [1981] and Deift, Nanda and Tomei [1983].

A very important feature of the tridiagonal non-periodic Toda lattice flow is that it can be solved explicitly as follows:

Let the initial data be given by $L(0) = L_0$. Use the Gram-Schmidt process to factorize a symmetric matrix L as $L = k(L)u(L)$ where $k(L)$ is orthogonal and $u(L)$ is upper triangular.

Then the explicit solution of the Toda flow is given by

$$L(t) = (\mathrm{Ad}k(\exp(tL_0))L_0\,. \tag{2.10}$$

2.1 Example 1

Here we simulate the Toda lattice in four dimensions. Below we shall compare this flow with other flows with similar qualitative behavior.

In this case

$$J(a) = \begin{bmatrix} 0 & 0 & a_1 & -a_1 \\ 0 & 0 & 0 & a_2 \\ -a_1 & 0 & 0 & 0 \\ a_1 & -a_2 & 0 & 0 \end{bmatrix} \tag{2.11}$$

and

$$H(a,b) = a_1^2 + a_2^2 + b_1^2 + b_2^2 + b_1 b_2 \,. \tag{2.12}$$

This gives the equations of motion:

$$\dot{a_1} = a_1(b_1 - b_2) \tag{2.13}$$
$$\dot{a_2} = a_2(b_1 + 2b_2) \tag{2.14}$$
$$\dot{b_1} = -2a_1^2 \tag{2.15}$$
$$\dot{b_2} = 2(a_1^2 - a_2^2) \tag{2.16}$$

Note here that using $b_1 + b_2 + b_3 = 0$, these equations are identical to the Toda lattice equations discussed above. In particular $\mathrm{Tr}LN$ is, in this case, $2(a_1^2 + a_2^2)$ and provides a Lyapunov function for the flow.

3 Asymptotically Stable Almost Poisson Systems

We consider in this section a general class of Poisson-like systems that exhibit asymptotic behavior.

We begin by recalling some basic facts about Poisson and almost Poisson systems (see for example Marsden and Ratiu [1994] and da Silva and Weinstein [1999]).

Definition 3.1 *A Poisson manifold is a pair $(M, \{,\})$ where M is a smooth manifold and $\{,\}$ is a Poisson bracket, i.e. a bilinear operation on the C^∞ functions on M, $C^\infty(M)$, such that (i)$(C^\infty(M), \{,\})$ is a Lie algebra and (ii) $\{,\}$ is a derivation in each factor (the Leibniz identity is satisfied).*

Somewhat more generally we have:

Definition 3.2 *An almost Poisson manifold is a pair $(M, \{,\})$ where M is a smooth manifold and (i)$\{,\}$ defines an almost Lie algebra structure on the C^∞ functions on M, i.e. the bracket satisfies all conditions for a Lie algebra except that the Jacobi identity is not satisfied and (ii) $\{,\}$ is a derivation in each factor.*

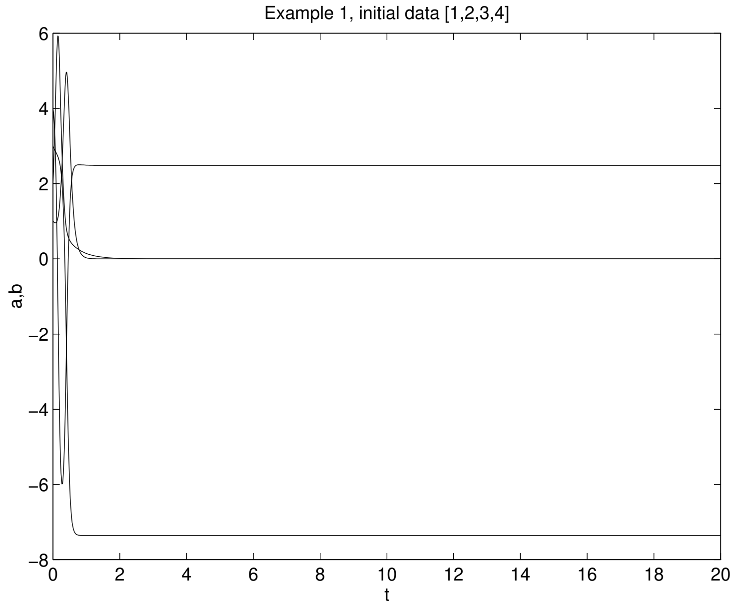

Figure 2.1: Toda.

In either case we can define a field Π of skew-symmetric bilinear forms on T^*M, i.e. $\Pi \in \Gamma(TM \wedge TM)$, by

$$\{f, g\} = \Pi(df, dg)\,. \tag{3.1}$$

The field Π is called a *bivector field.*

Conversely, a bivector field Π defines an almost Poisson structure on M by $\{f, g\}_\Pi = \Pi(df, dg)$.

An almost Poisson structure on M will be Poisson if its *Jacobiator*, defined by

$$J(f, g, h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}g\} \tag{3.2}$$

vanishes.

A Poisson manifold M is stratified by its *symplectic leaves*: $z_1, z_2 \in M$ are in the same symplectic leaf if there exists a smooth curve in M joining z_1, z_2, each segment of which is the trajectory of a locally defined Hamiltonian vector field.

On a Poisson manifold (M, Π) a point $0 \in M$ has a coordinate neighborhood with coordinates $(q_1, \cdots, q_k, p_1, \cdots, p_k, y_1, \cdots, y_l)$ centered at 0 such that

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^{l} \phi_{i,j}(y) \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \tag{3.3}$$

where $\phi_{ij}(0) = 0$ (see Weinstein [1983]).

Given an almost Poisson structure with local coordinate form $\pi_{ij}(z)$ and a function H on M one can define an *almost Poisson* vector field on M by

$$\dot{z}_i = \pi_{ij}(z)\frac{\partial H}{\partial z_j}\,. \tag{3.4}$$

There are various special cases of interest one can consider. Firstly if Π is full rank everywhere it defines a symplectic structure on the manifold, and if it is locally full rank it defines a local symplectic structure.

Another case of interest to us here is a linear (almost) Poisson structure where the (almost) Poisson tensor locally has the form $\pi_{ij}(z) = \sum_k c_{ijk} z_k$.

We are particularly interested here in the case where the (almost) Poisson tensor is of zero rank and hence in the notation above we may write $\pi_{ij}(z) = \phi_{ij}(y)$ where $\phi_{ij}(0) = 0$.

Nevertheless, it is important for the class of systems we consider (which includes the Toda lattice flow discussed above) to preserve aspects of the canonical structure. Thus in the first instance we consider systems that are locally of the form

$$\dot{z} = J(q)\nabla H(z) \tag{3.5}$$

where z is the column vector $z = [q,p]^T = [q_1,\cdots,q_n,p_1,\cdots,p_n]^T$ and $J(q)$ is a skew-symmetric matrix with $J(0) = 0$. It is interesting to consider flows of this type both when J is a Poisson matrix and when it is not. The latter case is important for nonholonomic systems where the matrix in generically not Poisson and where asymptotic behavior is often exhibited (see Zenkov, Bloch and Marsden [1998], Koon and Marsden [1997], van der Schaft and Maschke [1994]). Here however we shall concern ourselves largely with the case where J is Poisson as we are interested in dual Hamiltonian and asymptotically stable systems.

The Poisson nature of J can easily be checked directly in specific cases by checking the Jacobi identity on coordinate functions (see for example Marsden and Ratiu [1994]).

In our setting this reduces to checking

$$J_{li}\frac{\partial J_{jk}}{\partial q_l} + J_{lj}\frac{\partial J_{ki}}{\partial q_l} + J_{lk}\frac{\partial J_{ij}}{\partial q_l} = 0 \tag{3.6}$$

for all variables. We consider here examples where this is satisfied and examples where it fails.

The system 3.5 has a manifold of equilibria given by $J(q) = 0$. The linearization of the flow at any of these equilibria is given by

$$\delta\dot{z}_i = \sum_{k=1}^{n} A_{ik}\delta q_k \tag{3.7}$$

where

$$A_{ik} = \sum_j \frac{\partial J_{ij}}{\partial q_k} \nabla H_j \tag{3.8}$$

and $\nabla H = (\frac{\partial H}{\partial q_1}, \cdots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \cdots, \frac{\partial H}{\partial p_n})$, all evaluated at the given equilibrium $(q,p) = (0, p_0)$.

We are interested in the stability of the equilibria.

We can prove the following result:

Theorem 3.3 *Consider the system 3.5 where J is linear in q and of the form*

$$J(q) = \begin{bmatrix} 0 & \hat{J}(q) \\ -\hat{J}(q)^T & 0 \end{bmatrix} \tag{3.9}$$

and $H = q^T Q q + p^T P p$ where Q and P are constant symmetric matrices. If the matrix A given by 3.8 has all eigenvalues with negative real parts, the given equilibrium is (locally) asymptotically stable.

Proof: By assumption the system has n stable eigenvalue and n zero eigenvalues. Stability in this instance follows from the center manifold theorem (see for example Carr [1981] or Guckenheimer and Holmes [1983]). In this instance we have the equations of motion

$$\begin{aligned} \dot{q} &= \hat{J}(q) P p & (3.10) \\ \dot{p} &= -\hat{J}^T(q) Q q \,. & (3.11) \end{aligned}$$

This has a manifold of equilibria of the form $(q,p) = (0, p_0)$. Set $x = p - p_0$ and $y = q$.

Then the equations of motion become

$$\begin{aligned} \dot{x} &= -\hat{J}^T(y) Q y & (3.12) \\ \dot{y} &= \hat{J}(y) P p_0 + \hat{J}(y) P x \,. & (3.13) \end{aligned}$$

Assume a center manifold of the form $y = h(x)$. Following Carr [1981] or Guckenheimer and Holmes [1983], substitution of this expression into the equations of motion yields the following condition on h:

$$Dh(x) \cdot \hat{J(x)}^T (h(x)) Q h(x) - \hat{J}(h(x)) P p_0 - \hat{J}(h(x)) P x = 0 \,. \tag{3.14}$$

Since the first term is of high order h must be identically zero, and by the center manifold theorem both x and y decay exponentially to zero.

■

In contrast to the tridiagonal symmetric Toda equations this does not provide a global stability result, but a local one. (We note also that center manifolds have been used in analyzing the Toda lattice equations – see for example Chu

[1984].) If our flows start sufficiently far from an equilibrium however, blow-up is possible, again in contrast to the Toda lattice flow. (Note however that blow-up can occur in more general Toda lattices – see e.g. Gekhtman and Shapiro [1997], Kodama and Ye [1996] and Bloch and Gekhtman [1998]). Moreover there are more general systems of the type 3.5 that exhibit this kind of asymptotic behavior. Note that the theorem does not require that the systems be Poisson but in some sense the most interesting cases are those which are Poisson. The key property here is that there is a singularity in the Poisson structure at the equilibrium where the system exits the leaf of maximal dimension. We shall return to the general Poisson structure underlying these systems after analyzing some examples. We firstly discuss and simulate below some four-dimensional examples which are of interest. The first two are Poisson but the last is not.

4 Four-dimensional Flows

We consider here some simulations of four-dimensional flows which illustrate the above theory.

4.1 Example 2

We modify Example 1 by taking J to be

$$J(a) = \begin{bmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \\ -a_1 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 \end{bmatrix} \tag{4.1}$$

and

$$H(a,b) = a_1^2 + a_2^2 + b_1^2 + b_2^2 + b_1 b_2 \tag{4.2}$$

as above.
This gives the equations of motion:

$$\begin{aligned} \dot{a_1} &= a_1(2b_1 + b_2) && (4.3) \\ \dot{a_2} &= a_2(b_1 + 2b_2) && (4.4) \\ \dot{b_1} &= -2a_1^2 && (4.5) \\ \dot{b_2} &= -2a_2^2 . && (4.6) \end{aligned}$$

The flow linearized about an equilibrium $(a_1, a_2, b_1, b_2) = (0, 0, b_{10}, b_{20})$ is

$$\begin{aligned} \delta\dot{a}_1 &= (2b_{10} + b_{20})\delta a_1 && (4.7) \\ \delta\dot{a}_2 &= (b_{10} + 2b_{20})\delta a_2 && (4.8) \\ \delta\dot{b}_1 &= 0 && (4.9) \\ \delta\dot{b}_2 &= 0 . && (4.10) \end{aligned}$$

The matrix A of the theorem is in this instance

$$A = \begin{bmatrix} 2b_{10} + b_{20} & 0 \\ 0 & b_{10} + 2b_{20} \end{bmatrix} . \tag{4.11}$$

We then apply the center manifold theory to the equation in the form

$$\begin{aligned} \dot{x_1} &= -2y_1^2 && (4.12) \\ \dot{x_2} &= -2y_2^2 && (4.13) \\ \dot{y_1} &= y_1(2b_{10} + b_{20}) + y_1(2x_1 + x_2) && (4.14) \\ \dot{y_2} &= y_2(b_{10} + 2b_{20}) + y_2(x_1 + 2x_2) . && (4.15) \end{aligned}$$

We want to point out a key difference here with the Toda flow: while the Toda flow is isospectral, i.e. its eigenvalues are conserved along the flow, this is not true here. Thus while in the Toda case one can predict that the flow will evolve so that the Lax matrix L tends to a diagonal matrix with the eigenvalues of $L(0)$ on the diagonal, for the more general flows of the type considered here one knows only that the flow will tend to some equilibrium on the equilibrium manifold (if it is sufficiently close).

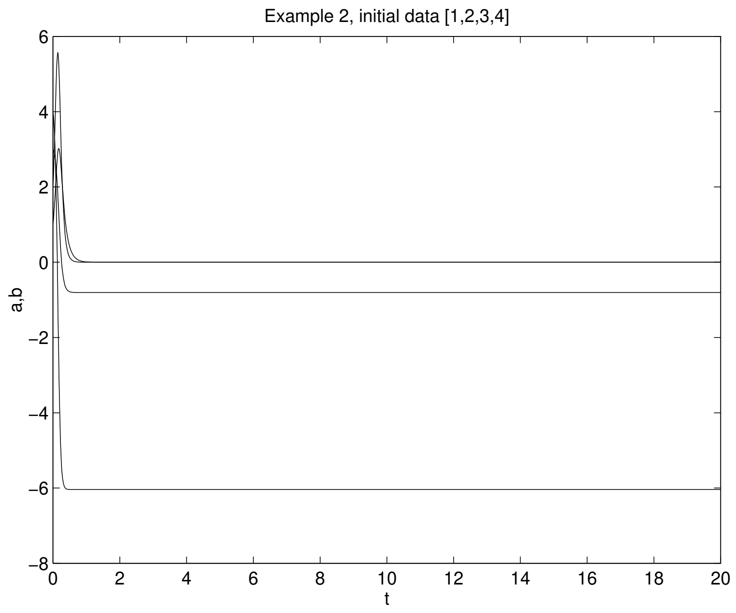

Figure 4.1: Example 2.

4.2 Example 3

Here we take J to be

$$J(q) = \begin{bmatrix} 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & q_2 \\ -q_1 & 0 & 0 & q_2 \\ 0 & -q_2 & -q_2 & 0 \end{bmatrix} \tag{4.16}$$

and

$$H(q,p) = q_1^2 + q_2^2 + p_1^2 + p_2^2 + p_1 p_2 \tag{4.17}$$

as above.
This gives the equations of motion:

$$\begin{aligned} \dot{q_1} &= q_1(2p_1 + p_2) && (4.18) \\ \dot{q_2} &= q_2(2p_1 + 2p_2) && (4.19) \\ \dot{p_1} &= -2q_1^2 + q_2(2p_2 + p_1) && (4.20) \\ \dot{p_2} &= -2q_2^2 - q_2(2p_1 + p_2)\,. && (4.21) \end{aligned}$$

The flow linearized about an equilibrium $(q_1, q_2, p_1, p_2) = (0, 0, p_{10}, p_{20})$ is

$$\begin{aligned} \dot{\delta q_1} &= (2p_{10} + p_{20})\delta q_1 && (4.22) \\ \dot{\delta q_2} &= (2p_{20} + p_{20})\delta q_2 && (4.23) \\ \dot{\delta p_1} &= \delta q_2(2p_{20} + p_{10}) && (4.24) \\ \dot{\delta p_2} &= -\delta q_2(2p_{10} + p_{20})\,. && (4.25) \end{aligned}$$

The matrix A of the theorem is in this instance

$$A = \begin{bmatrix} 2p_{10} + p_{20} & 0 \\ 0 & -(2p_{20} + p_{10}) \end{bmatrix}. \tag{4.26}$$

4.3 Example 4

Finally we take J to be

$$J(q) = \begin{bmatrix} 0 & 0 & 0 & q_1 \\ 0 & 0 & q_1 & 0 \\ 0 & -q_1 & 0 & 0 \\ -q_1 & 0 & 0 & 0 \end{bmatrix} \tag{4.27}$$

and

$$H(q,p) = \frac{1}{2}(q_1^2 + q_2^2 + p_1^2 + p_2^2). \tag{4.28}$$

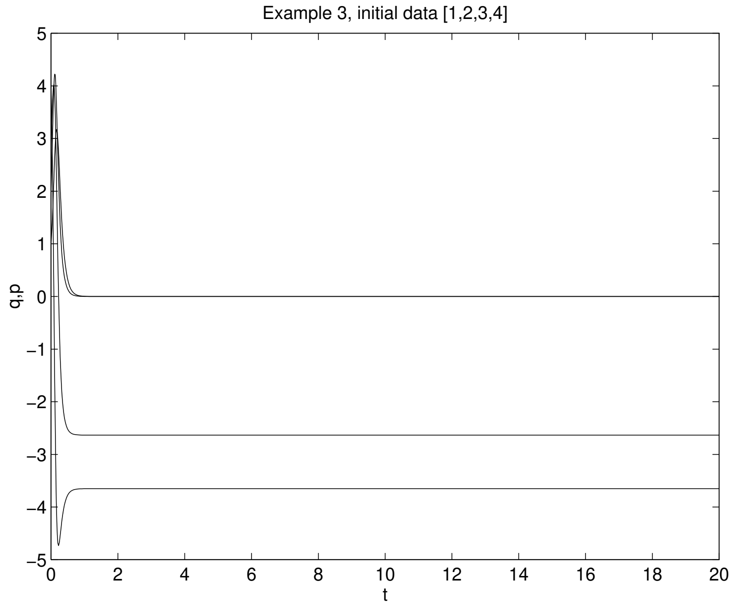

Figure 4.2: Example 3.

This gives the equations of motion:

$$\begin{aligned} \dot{q_1} &= q_1 p_2 && (4.29)\\ \dot{q_2} &= q_1 p_1 && (4.30)\\ \dot{p_1} &= -q_1 q_2 && (4.31)\\ \dot{p_2} &= -q_1^2\,. && (4.32) \end{aligned}$$

This flow exhibits asymptotic stability (but it is not Hamiltonian).

5 Asymptotics for Low-dimensional Systems

The general behavior of the systems of interest here is well illustrated by analyzing some dimensional examples.

5.1 The Two-dimensional Toda Lattice

In the two-dimensional case the matrices in the Lax pair are simply

$$L = \begin{pmatrix} b_1 & a_1 \\ a_1 & -b_1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}$$

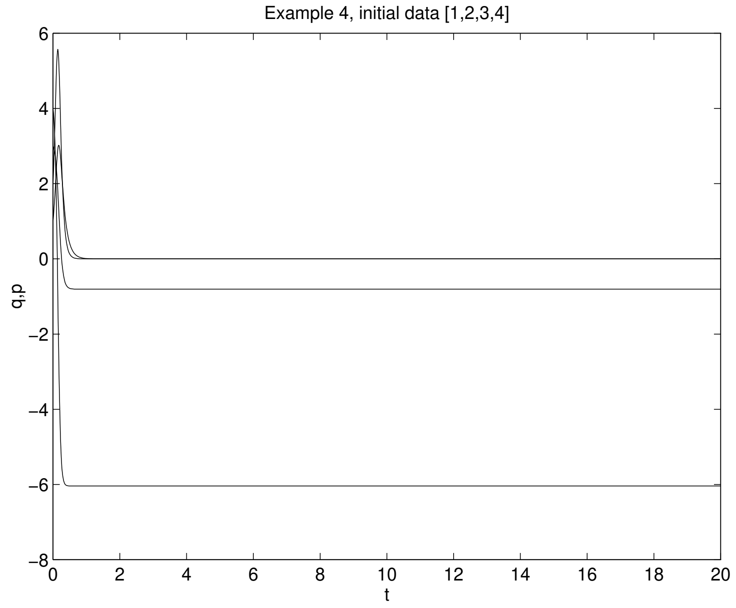

Figure 4.3: Example 4.

and the equations of motion are given by

$$\begin{aligned} \dot{b_1} &= 2a_1^2 \\ \dot{a_1} &= -2a_1 b_1 \,. \end{aligned} \tag{5.1}$$

For the initial data $b_1 = 0$, $a_1 = c$, explicitly carrying out the factorization described in section 2 yields the explicit solution

$$b_1(t) = -c\frac{\sinh 2ct}{\cosh 2ct}\,, \qquad\qquad a_1(t) = \frac{c}{\cosh 2ct}\,. \tag{5.2}$$

5.2 The Chaplygin Sleigh

Here we describe the Chaplygin sleigh, perhaps the simplest mechanical system which illustrates the possible dissipative nature of energy-preserving nonholonomic systems. We compare the sleigh equations to the Toda lattice equations and to those for the three-wave interaction.

The Chaplygin sleigh is discussed for example in Neimark and Fufaev [1972]. (See also the recent paper Ruina [1998] where an interesting connection with a systems with impacts is made.)

Consider then a rigid body in the plane supported at three points, two of which slide freely without friction while the third is a knife edge constraint which allows no motion perpendicular to its edge.

To describe the system we use a coordinate system Oxy fixed in the plane and a coordinate system $A\xi\eta$ fixed in the body with its origin at the point of support of the knife edge and the axis $A\xi$ through the center of mass C of the rigid body. The motion of the body is described by the coordinates x, y and the angle θ between the moving and fixed set of axes. Let m be the mass and J the moment of inertia about the center of mass. Let a be the distance from A to C.

Denote the unit vectors along the axes $A\xi$ and $A\eta$ in the body by $\mathbf{e}_1$ and $\mathbf{e}_2$. The skate constraint can then be expressed as the fact that the velocity at A, is given by $v\mathbf{e}_1$. Setting $\dot{\theta} = \omega$ and using balance of forces we obtain the equations:

$$\begin{aligned} \dot{v} &= a\omega^2 \\ \dot{\omega} &= -\frac{ma^2}{I + ma^2} v\omega \,. \end{aligned} \tag{5.3}$$

We can also derive the Chaplygin sleigh equations from the Lagrange multiplier point of view with the Lagrangian given by

$$L(x, y, \theta) = \frac{1}{2} m \left(\frac{d}{dt} (x + a\cos\theta)^2 + \frac{d}{dt} (y + a\sin\theta)^2 \right) + \frac{1}{2} I \dot{\theta}^2 \,.$$

This set of equations has a family of relative equilibria given by $(v, \omega) | v =$ const, $\omega = 0$. Linearizing about any of these equilibria we find one zero eigenvalue and one negative eigenvalue. In fact the solution curves are ellipses in $v{-}\omega$ plane with the positive v-axis attracting all solutions. Which equilibrium one approaches depends on the initial data.

Normalizing, we have the equations

$$\begin{aligned} \dot{v} &= \omega^2 \\ \dot{\omega} &= -v\omega \,. \end{aligned} \tag{5.4}$$

Scaling time by a factor of two, we have the immediate observation:

Proposition 5.1 *The Chaplygin sleigh equations are precisely equivalent to the two-dimensional Toda lattice equations except for the fact that there is no sign restriction on the variable ω. Hence the system can be written in Lax pair form and solved by the method of factorization.*

The fact that the sign of ω is arbitrary is significant for the Hamiltonian structure and dynamics and we shall return to it below.

We now discuss briefly the solution method given in Neimark and Fufaev [1972] for the purposes of comparison with the factorization method for Toda.

Thus we have

$$\frac{d}{dt}\left(\frac{\dot{\omega}}{\omega}\right) = -\dot{v} = -\omega^2 .$$

Now, multiplying by $\dot{\omega}/\omega$ and integrating we obtain

$$\left(\frac{\dot{\omega}}{\omega}\right)^2 = c^2 - \omega^2 \tag{5.5}$$

for a suitable constant c.

Now define the angle ψ by setting $\omega = c\cos\psi$.

For suitable choice of c, integration gives

$$ct = \int_0^{\psi} \frac{d\theta}{\cos\theta} = \frac{1}{2}\ln\frac{1+\sin\psi}{1-\sin\psi} \tag{5.6}$$

which can be inverted and solved to give precisely the same solution as obtained by the factorization method of Toda.

5.3 The Three-Wave Interaction

Of great interest in general in dynamics is the interaction of nonlinear oscillators and the energy exchange between them. One can view this as an interaction between waves of different frequencies with different resonance conditions. A particularly interesting case is the three-wave or so-called triad interaction – see for example Ablowitz, Funk and Newell [1972] and more recently, Alber, Luther, Marsden and Robbins [1998a,b]. An important special case of this is that where the oscillators are temporarily resonant (or semi-resonant) and where two frequencies are the same and equal to half the frequency of the third. This case is of interest here as we shall see below.

Following Alber et. al. [1998a, b] the general three-wave interaction can be described by the following set of three ordinary differential equations on $\mathbb{C}^3$:

$$\begin{aligned}
\frac{dq_1}{dt} &= i s_1 \gamma_1 q_2 \overline{q}_3 \\
\frac{dq_2}{dt} &= i s_2 \gamma_2 q_1 q_3 \\
\frac{dq_3}{dt} &= s_3 \gamma_3 \overline{q}_1 q_2 .
\end{aligned} \tag{5.7}$$

The γ_i are real numbers such that $\gamma_1 + \gamma_2 + \gamma_3 = 0$ and which, following Alber et. al we shall take to be given by $(\gamma_1, \gamma_2, \gamma_3) = (1, -2, 1)$. The s_i are real numbers of magnitude 1 defining a signature for the system and the type of interaction (so-called decay or explosive).

To integrate the system one makes use of its invariance under the T^3-action, given essentially by multiplying the variables by a phase. Alber et. al. describe a general reduction procedure for this system.

For our purposes it is most straightforward to simply replace the variables by imaginary ones i.e. to replace q_i by iq_i, q_i real. Then the system becomes

$$\begin{aligned}\frac{dq_1}{dt} &= s_1\gamma_1 q_2 q_3 \\ \frac{dq_2}{dt} &= -s_2\gamma_2 q_1 q_3 \\ \frac{dq_3}{dt} &= s_3\gamma_3 q_1 q_2\end{aligned} \tag{5.8}$$

which has conserved quantities

$$K_1 = \frac{1}{2}\left(\frac{q_1^2}{s_1\gamma_1} + \frac{q_2^2}{s_2\gamma_2}\right), \tag{5.9}$$

$$K_2 = \frac{1}{2}\left(\frac{q_2^2}{s_2\gamma_2} + \frac{q_3^2}{s_3\gamma_3}\right), \tag{5.10}$$

$$K_3 = \frac{1}{2}\left(\frac{q_1^2}{s_1\gamma_1} - \frac{q_3^2}{s_3\gamma_3}\right), \tag{5.11}$$

where $K_1 - K_2 = K_3$.

To observe asymptotic behavior we choose $(s_1, s_2, s_3) = (1, -1, 1)$ and make the change of variables $Q_1 = -\sqrt{2}q_1$, $Q_2 = -q_2$ and $Q_3 = -\sqrt{2}q_3$.

Then we obtain the system

$$\begin{aligned}\frac{dQ_1}{dt} &= -Q_2 Q_3 \\ \frac{dQ_2}{dt} &= Q_1 Q_3 \\ \frac{dQ_3}{dt} &= -Q_1 Q_2\end{aligned} \tag{5.12}$$

which has conserved quantities $Q_1^2 + Q_2^2$, $Q_2^2 + Q_3^2$ and $Q_1 - Q_3^2$.

Setting $Q_1 = Q_3$ we obtain

$$\begin{aligned}\dot{Q}_1 &= -Q_1 Q_2 \\ \dot{Q}_2 &= Q_1^2\end{aligned} \tag{5.13}$$

which are our earlier equations. In this case $K_3 = 0$ and $K_1 = K_2$ is the Toda-type Hamiltonian. (Note that this is a conservation law for the three-wave interaction, not the Hamiltonian). This setting corresponds to a degenerate case of the wave surfaces discussed in Alber et. al. In our setting the full wave surface is given naturally by $Q_1^2 + 2Q_2^2 + Q_3^2 = \text{constant}$.

We present some simulations below. Note that the general behavior of the three-wave interaction is periodic while in our special case we get asymptotic behavior. This corresponds to motion along a homoclinic orbit of our wave surface. This special case of the three-wave interaction thus corresponds to the Chaplygin sleigh motion (nonholonomic) or two copies of the Toda flow.

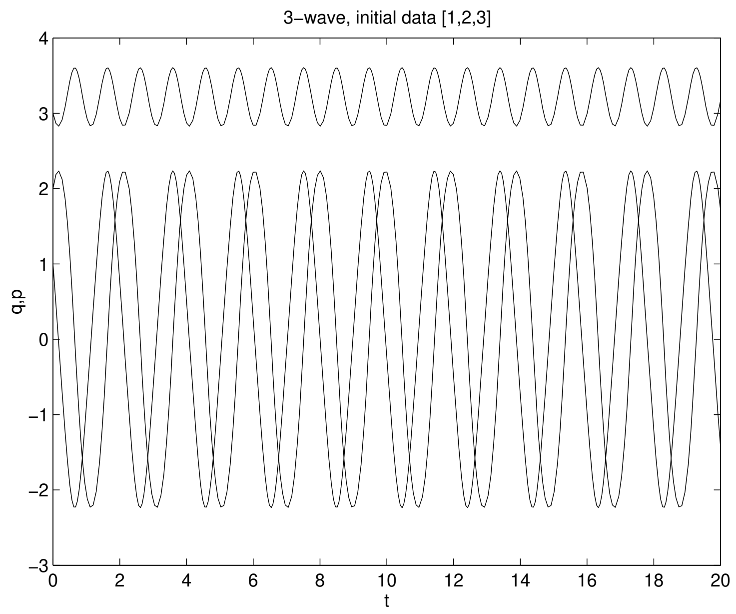

Figure 5.1: General real three-wave trajectories.

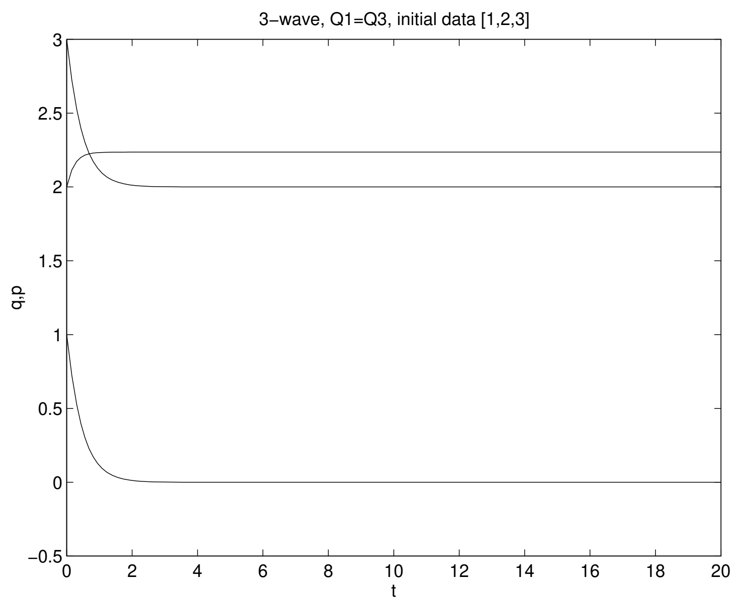

Figure 5.2: Three-wave trajectories for $Q_1 = Q_3$.

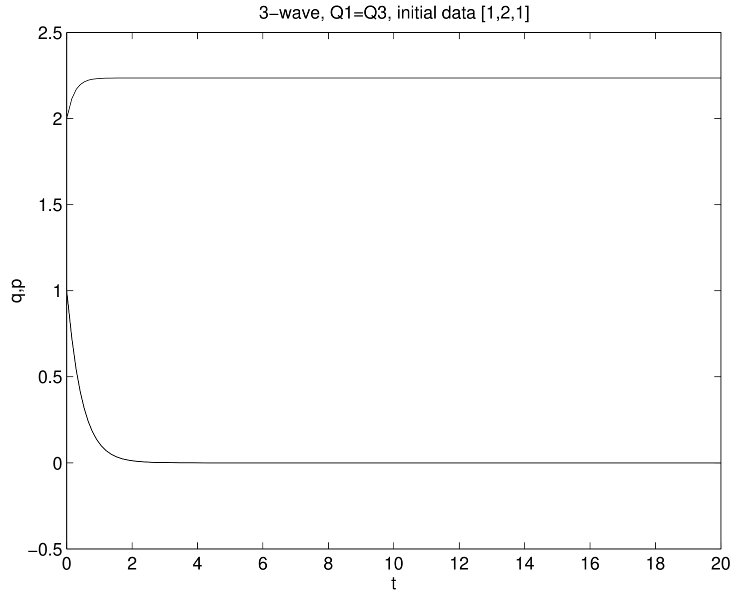

Figure 5.3: Three-wave trajectories for $Q_1 = Q_3$, $Q_1(0) = Q_3(0)$.

5.4 Qualitative Analysis of Low-dimensional Cases

We now compare the asymptotic behavior of the Toda lattice and the Chaplygin sleigh (or fully resonant three-wave interaction). As mentioned earlier the equations of the Toda lattice and the Chaplygin sleigh are identical except for the sign restriction on the variable a_1 (in Toda). But this difference is crucial. At the heart of this comparison is how asymptotic behavior is allowed in these two systems.

Observe firstly that the Toda lattice system has a set of equilibria defined by setting a_i equal to zero in the closure of its phase space. (In the two-dimensional case this set is a line as it is for the Chaplygin sleigh in a different context.) Further, as shown for example in Moser [1974], Symes [1982] or Deift, Nanda and Tomei [1983] and from the double bracket viewpoint in Bloch [1990] and Bloch, Brockett and Ratiu [1992], the matrix L simply tends to a diagonal matrix with the eigenvalues of the matrix defining the initial data ordered by magnitude on the diagonal.

Recall also that the Toda lattice is really a Hamiltonian system as discussed in 2. How is it possible then that one can obtain asymptotic behavior in this Hamiltonian system? Vital here is the fact that the system is integrable and that one is evolving on a level set of the integrals – a Lagrange submanifold of the phase space. By the Arnold-Liouville theorem for integrability, this submanifold is diffeomorphic to a product of circles and lines. For non-periodic Toda it is in fact diffeomorphic to a set of lines and the flow on this set of lines is a gradient flow as shown originally in Moser [1974]. As discussed above the flow is gradient in the Lax pair formulation with respect to the function TrLN and the normal metric.

Key also is the fact that the set defined by $a_i = 0$ is reached only asymptotically but is not actually in the phase space. Thus the system is asymptotically stable only in a generalized sense. Also, at this point the transformation of Flaschka has a singularity leading one to expect the possibility of different qualitative behavior in the system (see e.g. Kuiper and Robbin [1973], Kuiper [1975] and Willems [1980]).

For the two-dimensional Toda the level set is

$$b_1^2 + a_1^2 = \text{const}; \qquad a_1 > 0\,. \tag{5.14}$$

Unlike the harmonic oscillator this is a line, not a circle.

On the other hand for the Chaplygin sleigh one is allowing a_1 (or ω in the sleigh notation) to take any value in $\mathbb{R}$. Hence the flow in the phase plane can asymptotically approach the v-axis from either half-plane, but cannot of course cross this axis – see the figure below. The flow is in fact the union of two flows – the standard Toda lattice flow and the flow with $a_1 < 0$. Note that the energy for the normalized Chaplygin sleigh is given by its Lagrangian and is precisely $1/2(v^2 + \omega^2)$.

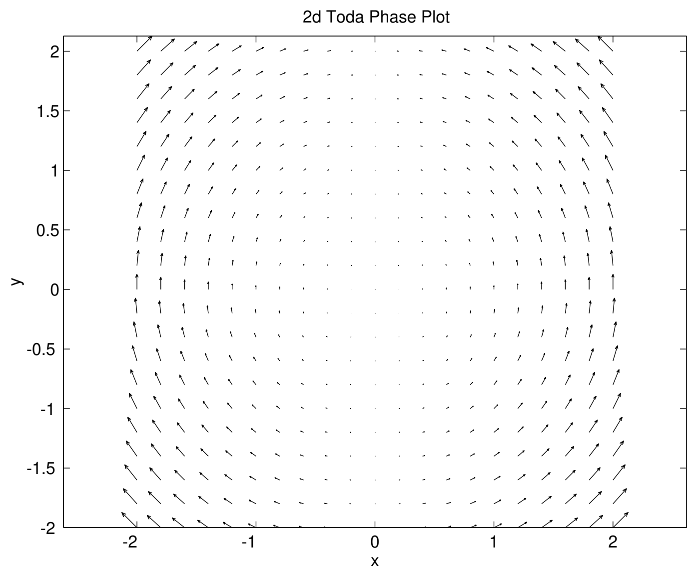

Figure 5.4: 2d Toda phase portrait.

It is also worth considering the Hamiltonian structure of the Toda lattice and its relationship to volume preservation in the present context.

From the Poisson matrix described above, one observes the symplectic form for the Toda lattice in Flaschka's variables is

$$\omega_1 = \frac{1}{a} db \wedge da \tag{5.15}$$

in contrast to the canonical form

$$\omega_2 = db \wedge da\,. \tag{5.16}$$

Since we are in two dimensions, these forms also give us a volume form or measure in the phase space.

We can compute the rate of change of these volume forms along the flow.

The derivative of ω_1 along the flow is obviously zero as can be seen by computing the Lie derivative with respect to X the Toda vector field:

$$\mathcal{L}_X \omega_1 = d(i_X \omega_1) = d(a da + b db) = 0 \tag{5.17}$$

while

$$\mathcal{L}_X \omega_2 = d(i_X \omega_2) = d(a^2 da + ab db) = b db \wedge da \neq 0\,. \tag{5.18}$$

So, in the Toda phase space volume is preserved, but in R^2 with the standard form (which is what really matters for the Chaplygin sleigh and which for *practical* purposes is what matters for Toda also) volume is not preserved.

Summarizing we have

Proposition 5.2 *The flow of the nonholonomic Chaplygin sleigh or the fully resonant three-wave interaction is the union of the flows of the standard and signed two-dimensional Toda lattices. The phase space of the Chaplygin sleigh is the compactification by a line of the union of the phase spaces of two Toda lattice systems.*

Thus we have a nonholonomic system with asymptotically stable equilibria consisting of the union of two Hamiltonian systems with asymptotically stable behavior. The key to interpreting the nonholonomic system in a Hamiltonian fashion is the removal of a line from the phase space. From another point of view the key to the asymptotic behavior may be viewed as a degeneracy in the Poisson structure at the equilibrium as discussed in the general setting above. Further the Poisson manifold in this two-dimensional case is archetypal as discussed in da Silva and Weinstein [1999]. There are two generic leaves off the line, while every point on the line is a leaf of measure zero.

Related comments on the degeneracy of the Poisson structure for the Chaplygin sleigh may be found in Bates [1998]. Further analysis of the phase portraits of nonholonomic systems will be given in a forthcoming publication.

6 Higher-Dimensional Signed Toda, Poisson Structures, and the Higher-Dimensional Suslov Problem

The analysis above extends to the higher-dimensional signed Toda lattices discussed in Tomei [1984]. One simply considers the Toda flows discussed in section 2 but in the case where the off diagonal elements a_i are allowed to be negative or positive. As shown in Tomei [1984] the signs of a_i are conserved along the flow, i.e.

$$\mathrm{sign}(a_i(t) = \mathrm{sign}(a_i(0) \qquad a_i(t) = 0 \,\mathrm{iff}\, a_i(0) = 0\,.$$

However, as in the positive a_i case, all flows asymptote to a diagonal matrix with all $a_i = 0$. Further, any such flow lives on a rather remarkable manifold, consisting of all the Jacobi (tridiagonal symmetric) matrices with fixed eigenvalues but where the a_i are free to assume any sign – see Tomei [1984] and also Davis [1987]. This is because any given flow is isospectral. We can also consider the whole of $\mathbb{R}^{2n}$ coordinatized by b_i, a_i and foliated by such "Jacobi" manifolds. This manifold has a natural Poisson structure given by (2.7). In analogy to the two-dimensional case the submanifold corresponding to all $a_i = 0$ consists of symplectic leaves consisting of discrete points, and each signed Toda flow asymptotes to a unique point on this submanifold which is determined by the initial data – the initial eigenvalues of the matrix L. As shown in Moser [1974] the isospectral set of the Toda lattice is diffeomorphic to $\mathbb{R}^n$, and the same is true for each signed Toda. Each signed Toda flows along $\mathbb{R}^n$ asymptotically towards the boundary with all $a_i = 0$.

However, as we have seen, such behavior is not limited to Toda but can be extended to $\mathbb{R}^{2n}$ endowed with any Poisson structure of the form $J(q)$ with $J(q) = 0$ where one has flows of the form (3.5). In this general nonintegrable case however, one does not flow along a level set of n integrals but rather along the energy level set until it intersects the plane defined by $q_i = 0$.

It is interesting to contrast this behavior with a high-dimensional nonholonomic system, the Suslov problem (see Federov and Suslov [1995]), with dynamics of the form $\dot{z} = J(z)\nabla H(z)$. This system exhibits asymptotic stability and is essentially a generalization of the Chaplygin sleigh. We only give somewhat brief comments here. (A different, non-asymptotic form is analyzed in Zenkov and Bloch [1999]. Further analysis of the phase portraits of nonholonomic systems will be given in a forthcoming joint publication.)

The equations of motion are those of an n-dimensional rigid body with skew-symmetric angular velocity matrix Ω with entries Ω_{ij} and symmetric moment of inertia matrix $I = I_{ij}$. One then introduces the constraints $\Omega_{ij} = 0, i, j \geq 2$.

The resulting nonholonomic equations of motion are:

$$\begin{aligned}
(I_{11}+I_{22})\,\dot\Omega_{12} &= I_{12}\left(\Omega_{13}^2+\Omega_{14}^2+\cdots+\Omega_{1n}^2\right)\\
&\quad -\left(I_{13}\Omega_{13}+I_{14}\Omega_{14}+\cdots+I_{1n}\Omega_{1n}\right)\Omega_{12}\\
(I_{11}+I_{33})\,\dot\Omega_{13} &= I_{13}\left(\Omega_{12}^2+\Omega_{14}^2+\cdots+\Omega_{1n}^2\right)\\
&\quad -\left(I_{12}\Omega_{12}+I_{14}\Omega_{14}+\cdots+I_{1n}\Omega_{1n}\right)\Omega_{13}\\
\cdots\cdots &\quad \cdots\cdots\\
(I_{11}+I_{nn})\,\dot\Omega_{1n} &= I_{1n}\left(\Omega_{12}^2+\Omega_{13}^2+\cdots+\Omega_{1n-1}^2\right)\\
&\quad -\left(I_{12}\Omega_{12}+I_{13}\Omega_{13}+\cdots+I_{1n-1}\Omega_{1n-1}\right)\Omega_{1n}\,. \qquad (6.1)
\end{aligned}$$

This system has the energy integral

$$H=\frac{1}{2}\left((I_{11}+I_{22})\Omega_{12}^2+(I_{11}+I_{33})\Omega_{13}^2+\cdots+(I_{11}+I_{nn})\Omega_{nn}^2\right)\,. \qquad (6.2)$$

.

Defining the momenta $M_{1j}=(I_{11}+I_{jj})\Omega_{1j}$ by the Legendre transform, we can write the system as one of almost Poisson form $\dot M=J(M)\nabla H(M)$ where the almost Poisson matrix (in the angular velocity variables) is

$$J(\Omega)=\begin{bmatrix} 0 & I_{13}\Omega_{12}-I_{12}\Omega_{13} & \cdots & I_{1n}\Omega_{12}-I_{12}\Omega_{1n}\\ I_{12}\Omega_{13}-I_{13}\Omega_{12} & 0 & \cdots & I_{1n}\Omega_{13}-I_{13}\Omega_{1n}\\ \cdots & \cdots & \cdots & \cdots\\ I_{12}\Omega_{1n}-I_1n\Omega_{12} & \cdots & \cdots & 0 \end{bmatrix} \qquad (6.3)$$

Further, the system exhibits asymptotic behavior as indicated by the fact that the function

$$F=(I_{11}+I_{22})I_{12}\Omega_{12}+(I_{11}+I_{33})I_{13}\Omega_{13}+\cdots+(I_{11}+I_{nn})I_{1n}\Omega_{1n} \qquad (6.4)$$

satisfies

$$\dot F=\sum_{i<j}^{n}(I_{1i}\Omega_{1j}-I_{1j}\Omega_1)^2 \qquad (6.5)$$

along the flow and is positive everywhere except at points of the line $\{\Omega_{12}=I_{12}\mu,\cdots,\Omega_{1n}=I_{1n}\mu\}$, $\mu\in\mathbb{R}$.

Thus motion occurs on the energy ellipsoid (a generalization of the Toda/Chaplygin ellipse) and asymptotes to a point on the line intersecting the ellipsoid. It is interesting to comppare to signed Toda. Here we have flow along the energy ellipsoid to a line of equilibria, while in signed Toda we have along the level sets of integrals (a smaller submanifold) to an equilibrium plane (a larger equilibrium manifold). In the two-dimensional case these notions coincide.

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