Gyroscopically Stabilized Oscillators and Heat Baths

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Abstract

In this paper we analyze the stability of a gyroscopic oscillator interacting with a finite- and infinite-dimensional heat bath in both the classical and quantum cases. We consider a finite gyroscopic oscillator model of a particle on a rotating disc and a particle in a magnetic field and we examine stability before and after coupling to a heat bath. The heat bath is modelled in the finite-dimensional setting by a system of independent oscillators with mass. It is shown that if the oscillator is gyroscopically stable, coupling to a sufficiently massive heat bath induces instability even in the finite-dimensional setting. The key mechanism for instability in this paper is thus not induced by damping. The meaning of these ideas in the quantum context is discussed. The model extends the exact diagonalization analysis of an oscillator and field of Ford, Lewis, and O'Connell to the gyroscopic setting. We also discuss the interesting role that damping of Landau type plays in the infinite limit.

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1 Introduction

In this paper we investigate the stability properties of a gyroscopically stabilized system interacting with a finite or infinite-dimensional heat bath. As a standard model for a gyroscopic system, we consider a particle on a rotating disc and a charged oscillator in a magnetic field.

A gyroscopically stabilized system is one that is unstable without gyroscopic forces but becomes stable with the addition of these forces. Infinitesimal dissipative perturbations are known to induce instability in Hamiltonian systems that are gyroscopically stabilized; see Bloch et al. (1994).

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Since the origins of dissipation (*e.g.* friction, viscosity...) lie in the transfer of energy from one form (energy of one subsystem) to another form (that of a second subsystem) of a larger conservative system, it is natural to expect the analogue of the above destabilization phenomenon to be present within the more fundamental context of conservative systems which exhibit internal energy transfer. In Hagerty et al. (1999, 2003), we explore this in the context of a gyroscopic oscillating mechanical system coupled to an extended wave system (infinite string). Due to the coupling, motion within the mechanical system generates waves which can be carried off to infinity. Such *radiation damping* has been studied in models arising in the theory of quantum resonances, ionization type problems and nonlinear waves; for more detail see Soffer and Weinstein (1998a,b, 1999), Kirr and Weinstein (2001), and references therein. There are many interesting papers of course discussing the role of dissipation in quantum systems and systems of quantum oscillators, for example Unruh and Zurek (1989) Caldeira and Leggett (1983), Lebowitz (1994), Jaksic and Pillet (1995), Jaksic and Pillet (1997), Bach et al. (2000), Merkli and Sigal (1999), Derezinski and Jaksic (2002), Derezinski et al. (2003). We cannot begin to give a complete list here (and indeed our focus here is the classical rather than quantum setting) but refer the reader to the literature for other aspects of this problem.

We consider here two models of gyroscopically stabilized systems (a particle stabilized either by rotation or a magnetic field). We show that coupling to a *finite* (i.e. non-thermal) system of oscillators with sufficient coupling strength induces instability. This result is quite striking in the sense that one does not expect a finite reservoir to mimic dissipation and indeed the key mechanism for instability in this paper in not dissipation. We also however consider the infinite-dimensional limit. In this case dissipation does play a role which we discuss. A graphical criterion is given for determining the onset of instability. Further we generalize the analysis to the quantum setting.

We discuss briefly also the quantum setting where we show that while a stable oscillator has positive energy bound states, a gyroscopically stable oscillator has both positive and negative energy bound states. Coupling to the bath with sufficient coupling strength induces unbound states in the gyroscopically stable system.

Our analysis is for a model problem but there are interesting possible applications to systems with similar geometry such as Penning traps which are used to trap charged particles and obtain extremely precise measurements of atomic quantities Brown and Gabrielse (1986).

Our analysis extends the heat bath analysis of Ford et al. (1988) to the gyroscopic setting. We show here that there is a beautiful extension of their graphical (intersection-theoretic) criterion for stability to a more complex class of curves. A brief earlier version of some of these ideas is discussed in Bloch et al. (2001).

In the final section of the paper we discuss gyroscopic oscillators coupled to an infinite heat bath modelled by a string. We discuss in some detail the infinite limit and show that this limit we get an additional mechanism for instability, which we elucidate in a manner analogous to the study of Landau damping. This is related to the work on radiation induced instability discussed in (Hagerty et al. (1999, 2003)). We also discuss other types of infinite-dimensional coupling.

2 Examples of Chetaev systems

In this section we discuss two physical realizations of gyroscopic oscillators: firstly a planar oscillator on a rotating plate and secondly a planar oscillator in a constant magnetic field.

Planar oscillator on a rotating plate

Figure 1 here

One physical model of a gyroscopic system is a planar oscillator on a plate rotating with angular velocity ω. We write the Lagrangian as the kinetic energy minus the potential energy,

$$L = \frac{1}{2}((\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2) - \frac{1}{2}(\alpha x^2 + \beta y^2) \tag{2.1}$$

In the Hamiltonian setting we have

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\alpha x^2 + \beta y^2) + H_\omega, \tag{2.2}$$

where

$$H_\omega = \omega(p_x y - p_y x) \tag{2.3}$$

and the associate momenta are given by

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = \dot{x} - \omega y \\ p_y &= \frac{\partial L}{\partial \dot{y}} = \dot{y} + \omega x. \end{aligned} \tag{2.4}$$

We obtain the equations of motion,

$$\begin{aligned} -\dot{p}_x &= \frac{\partial H}{\partial x} = \alpha x - \omega p_y \\ -\dot{p}_y &= \frac{\partial H}{\partial y} = \beta y + \omega p_x \\ \dot{x} &= \frac{\partial H}{\partial p_x} = p_x + \omega y \\ \dot{y} &= \frac{\partial H}{\partial p_y} = p_y - \omega x\,. \end{aligned} \tag{2.5}$$

Thus we have the following dynamics,

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} + (\alpha - \omega^2)x = 0 \\ \ddot{y} + 2\omega\dot{x} + (\beta - \omega^2)y = 0\,. \end{aligned} \tag{2.6}$$

We comment on this below.

Planar oscillator in a constant magnetic field

Figure 2 here

Another physical realization is the motion of a charged spherical pendulum in a magnetic field whose linearization is that of a charged planar oscillator in a magnetic field. We consider here the motion of such an oscillator (for details of the full nonlinear system see e.g. Hagerty et al. (2003).)

Let $\mathbf{B}$ be a divergence-free vector field. Let $\mathbf{A}$ be the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$. Note if we choose $\mathbf{B}$ to be the constant magnetic field in the direction normal to the plane of oscillation, the vector potential can be chosen as , $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{q}$, where $\mathbf{q} = (x, y, 0)^T$ is the position of the oscillator.

Assume a normalization where the oscillator has unit mass and unit charge and that the speed of light is unity. The Lagrangian, $L : T\mathbb{R}^2 \to \mathbb{R}$, is defined by:

$$\begin{aligned} L(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}\|\dot{\mathbf{q}}\|^2 + \mathbf{A} \cdot \dot{\mathbf{q}} - U(\mathbf{q}) \\ &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \mathbf{A} \cdot (x, y, 0)^T - \frac{1}{2}(\alpha x^2 + \beta y^2). \end{aligned} \tag{2.7}$$

Choosing $\mathbf{B}$ to be of constant strength B, normal to the plane of oscillation we have,

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\alpha x^2 + \beta y^2) + \frac{1}{2}B(x\dot{y} - y\dot{x}). \tag{2.8}$$

The associated Hamiltonian, $H : T^*\mathbb{R}^2 \to \mathbb{R}$, is

$$\begin{aligned} H &= p_x\dot{x} + p_y\dot{y} - L \\ &= \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\alpha x^2 + \beta y^2) + H_B, \end{aligned} \tag{2.9}$$

where

$$H_B = \frac{1}{2}B(p_x y - p_y x) + \frac{1}{8}B^2(x^2 + y^2), \tag{2.10}$$

and the associated momenta are given by

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = \dot{x} - \frac{1}{2}By \\ p_y &= \frac{\partial L}{\partial \dot{y}} = \dot{y} + \frac{1}{2}Bx\,. \end{aligned} \tag{2.11}$$

In the above notation, H_B is the contribution of a magnetic field.

We obtain the equations of motion,

$$\begin{aligned} -\dot{p}_x &= \frac{\partial H}{\partial x} = (\alpha + \frac{1}{4}B^2)x - \frac{1}{2}Bp_y \\ -\dot{p}_y &= \frac{\partial H}{\partial y} = (\beta + \frac{1}{4}B^2)y + \frac{1}{2}Bp_x \\ \dot{x} &= \frac{\partial H}{\partial p_x} = p_x + \frac{1}{2}By \\ \dot{y} &= \frac{\partial H}{\partial p_y} = p_y - \frac{1}{2}Bx\,. \end{aligned} \tag{2.12}$$

This gives the dynamics

$$\begin{aligned} \ddot{x} - B\dot{y} + \alpha x &= 0 \\ \ddot{y} + B\dot{x} + \beta y &= 0\,. \end{aligned} \tag{2.13}$$

Remark 1: If α and β are both negative, the oscillator in a field system is unstable for small B. However, if $B^2 + \alpha + \beta > 2\sqrt{\alpha\beta}$, the oscillator stabilizes, i.e. the eigenvalues are on the imaginary axis – this is what is referred to as *gyroscopic stabilization*. For further details see below and Bloch et al. (1994).

Remark 2: The rotating disc affects the oscillator differently from a magnetic field – compare equations (2.6) and (2.13). While the magnetic field only adds $\dot{q}$ terms, the rotating disc also adds additional q terms to the dynamics. For the equations (2.13), we can see that in the case of a physically stable oscillator with $\alpha, \beta > 0$ for rotation rate ω sufficiently large, the system becomes only gyroscopically stable, i.e. the coefficients $\alpha - \omega^2$ and $\beta - \omega^2$ are negative but the eigenvalues are on the imaginary axis due to the presence of gyroscopic terms.

Remark 3: In either case above, if the system is gyroscopically stable it can be shown that adding a small amount of dissipation to the system renders it unstable (i.e. there are unstable eigenvalues). For a more precise statement and generalizations see section 3.

3 Gyroscopic systems

We recall here some general properties of linear systems with gyroscopic forces. The systems above are examples of such systems.

The general form of a gyroscopic system is

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = 0, \tag{3.1}$$

where $\mathbf{q} \in \mathbb{R}^n$, M is a positive-definite symmetric $n \times n$ matrix, S is skew, and V is symmetric. As in Bloch et al. (1994) we shall call this the *Chetaev system* (see Chetaev (1961)).

We say the system is *gyroscopically stable* if for $S = 0$ the origin is an unstable equilibrium, but for $S \neq 0$, the origin is a spectrally stable equilibrium (i.e. the eigenvalues of the linearized system have non-positive real part). The matrix S is sometimes referred to as a magnetic term which arises from charged oscillators in a magnetic field.

An important property of this system is that it is a normal form for a simple mechanical system about a *relative equilibrium* which is given modulo an abelian group. That is, it is the normal form for a system defined on the cotangent bundle TQ of a configuration space Q where the Lagrangian is given by kinetic minus potential energy. (We recall that a point in TQ is a relative equilibrium with respect to the action of a group if the dynamical orbit through that point remains in the group orbit.) One can obtain a similar normal form in the case of a non-abelian group. (See Bloch et al. (1994) and Simo et al. (1991).) The magnetic term naturally arises in the symplectic form when investigating the quotient space. However, we can obtain the same dynamics from a canonical symplectic form and an augmented Hamiltonian. This can always be done by the momentum shifting lemma (see Marsden and Ratiu (1999)).

If we include the magnetic terms in the symplectic form, the Hamiltonian, $H : T^*Q \to \mathbb{R}$, of a gyroscopic system is the sum of the kinetic energy and potential energy

$$H = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^T V\mathbf{q}. \tag{3.2}$$

The Poisson bracket is given by

$$\{F, K\}_{magnetic} = \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i} - S^{ij}\frac{\partial F}{\partial p_i}\frac{\partial K}{\partial p_j}. \tag{3.3}$$

It also useful to write the system as a Hamiltonian system with respect to the canonical bracket. A simple method of seeing this is a follows:
We wish to write the system (3.1) as a Hamiltonian system with canonical symplectic form

$$\{F,K\}_{canonical} = \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i}, \tag{3.4}$$

and Hamiltonian

$$H_{magnetic} = \frac{1}{2}p^T M^{-1} p + \frac{1}{2}q^T(V+W)q + p^T X q.$$

Equating the resulting equations of motion with the equations (3.1) we see that the skew matrix X and the symmetric matrix W are implicitly defined by

$$XM + MX = -S \tag{3.5}$$

$$W = -XMX. \tag{3.6}$$

Solving equation (3.5) for X, we explicitly have a change of variable from a Hamiltonian system with magnetic terms in the symplectic form to a Hamiltonian system with canonical symplectic form. It can in fact easily be shown using the Fredholm Alternative (see Hagerty et al. (2003) for details), that equation (3.5) has a solution for X.

Classically, the two representations of the Chetaev systems are equally useful. However, the canonical bracket is preferable when quantizing the mechanical system. (See Bloch et al. (1994), Baillieul and Levi (1991), Chetaev (1961) for further physical discussions.)

Stability of Chetaev systems

In the sequel we investigate the stability of perturbed Chetaev systems, but first we summarize the stability of unperturbed Chetaev systems. The stability of the Chetaev system depends both on the signature of the bilinear form associated with a quadratic Hamiltonian and also on the magnetic terms in the symplectic form. In particular, the magnetic terms can stabilize gyroscopic systems with negative eigenvalues of V.

As for gyroscopic stability, the number of negative eigenvalues of the quadratic form plays a crucial part as Chetaev (1961) discusses and which we summarize in the following proposition:

Proposition 3.1 *Consider the canonical gyroscopic system* $M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = 0$, *where* M *is a symmetric positive definite matrix,* S *is a skew-symmetric matrix, and* V *is a symmetric matrix:*

- *If* V *has a odd number of negative eigenvalues (counting multiplicity) then the origin is an unstable equilibrium.*
- *If* V *has an even number of negative eigenvalues (counting multiplicity), we can choose* S *so that the origin is a spectrally stable equilibrium.*

Proof: Consider the associated first order linear operator, L,

$$L = \begin{bmatrix} 0 & I \\ -M^{-1}V & -M^{-1}S \end{bmatrix}. \tag{3.7}$$

Let $p(\lambda)$ be the characteristic polynomial of the matrix L and let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the matrix V.

$$
\begin{aligned}
p(0) &= \det L \\
&= (-1)^n \det(-M^{-1}V) \\
&= \det(M^{-1}) \prod_{i=1}^{n} \lambda_i
\end{aligned}
\tag{3.8}
$$

Suppose that V has an odd number of negative eigenvalues. Due to the positive definiteness of M, we have $p(0) < 0$. We will show that $p(\lambda)$ changes signs as λ increases.

$$
\begin{aligned}
\lim_{\lambda\to\infty} p(\lambda) &= \lim_{\lambda\to\infty} \det\Big(L - \lambda I\Big) \\
&= \lim_{\lambda\to\infty} (-1)^{2n}\lambda^{2n} + O(\lambda^{2n-1}) \\
&= \infty.
\end{aligned}
\tag{3.9}
$$

Hence, if V has an odd number of negative eigenvalues (counting multiplicity) then the characteristic polynomial of L has a positive real root and the origin cannot be gyroscopically stabilized.

Now to prove the second item, suppose that V has an even number of negative eigenvalues (counting multiplicity). Stabilizing a pair of negative eigen-directions at a time, gyroscopic stabilization for $n = 2$ is possible as shown in the introductory examples of Section 2. This completes the proof. ■

Gyroscopically stable systems exhibit interesting instability when perturbed by dissipative forces. Suppose now that V has at least one negative eigenvalue. A key result of Bloch et al. (1994) is that adding small dissipation always yields instability. More precisely it is shown that

Theorem 3.2 *Under the above conditions, if we modify the general Chetaev system by adding a small Rayleigh dissipation term,*

$$
M\ddot{q} + (S + \epsilon R)\dot{q} + Vq = 0 \tag{3.10}
$$

for small $\epsilon > 0$, where R is symmetric and positive definite, then the perturbed linearized equations

$$
\dot{z} = L_\epsilon z,
$$

where $z = (q, p)$ are spectrally unstable, i.e., at least one pair of eigenvalues of L_ϵ is in the right half plane.

This result builds on basic work of Thomson and Tait (1897), Chetaev (1961), and Hahn (1967). We refer to this as *dissipation induced instability.* Some of the radiation induced instabilities that arise in the gyroscopic Lamb model are analogous to the dissipation induced instability. We see that the Hamiltonian of a gyroscopically stabilized Chetaev system is indefinite. In this case, Rayleigh dissipation decreases the value of the Hamiltonian, but this does not bound the motion of the Chetaev system. In particular, the Hamiltonian may decrease to zero, while the displacement and velocities grow exponentially.

4 Oscillator coupled to a bath

We now consider the gyroscopic oscillator coupled to a bath of oscillators via an augmented Lagrangian or Hamiltonian,

$$L = L_0 + L_{bath} \tag{4.1}$$

where

$$\begin{aligned} L_0 &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\alpha x^2 + \beta y^2) + \frac{1}{2}B(x\dot{y} - y\dot{x}) && (4.2) \\ L_{bath} &= \sum_{j=1}^{N} \frac{1}{2}m_j(\dot{x}_j^2 + \dot{y}_j^2) - \frac{1}{2}m_j\omega_j^2((x_j - x)^2 + (y_j - y)^2). && (4.3) \end{aligned}$$

Similarly, the Hamiltonian can be augmented as,

$$H = H_0 + H_{bath} \tag{4.4}$$

where

$$\begin{aligned} H_0 &= \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\alpha x^2 + \beta y^2) + H_B && (4.5) \\ H_{bath} &= \sum_{j=1}^{N} \frac{1}{2}m_j(p_{x_j}^2 + p_{y_j}^2) + \frac{1}{2}m_j\omega_j^2((x_j - x)^2 + (y_j - y)^2), && (4.6) \end{aligned}$$

where H_B is given by equation (2.10) and $0 < \omega_1 < \cdots < \omega_N$. For convenience we define $\omega_{-i} = -\omega_i$. This model extends the model described by Ford et al. (1988) to the gyroscopic setting. As discussed in that paper this model provides a good physical realization of oscillator-bath coupling. For simplicity of the computations here we restrict ourselves to the case $\alpha = \beta$. Similar results hold in the general case. We also restrict ourselves to the generic situation (see the proof for comments on genericity) though again similar results hold in the nongeneric setting.

The theorem below in fact generalizes the results of Ford et al. (1988) in a rather beautiful fashion, leading to the study of a more complex intersection problem.

We can show:

Theorem 4.1 *Consider the system with Lagrangian (4.1) where $\alpha = \beta$ and the equation*

$$(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2) = \sum_{j=1}^{N} m_j \frac{\omega_j^2\omega^2}{\omega^2 - \omega_j^2}. \tag{4.7}$$

Generically we have the following:

The oscillator is stable if there exists $4N + 4$ real frequencies $\omega = \Omega_i, i = \pm 1, \cdots \pm (2N + 2)$ which are solutions to the equation (4.7). Let Ω_1 be the smallest positive solution and let Ω_i be increasing with respect to the index i.

More precisely we have

(i) In the case of instability of the oscillator H_0 ($\alpha + \frac{1}{4}B^2 < 0$), there are only $4N$ real solutions. Additionally, there are 2 pairs of imaginary solutions corresponding to instability.

(ii) *In the case of strong stability of* H_0 ($\alpha > 0$)*, there are* $4N+4$ *real, normal modes maintaining stability via coupling.*

(iii) *In the case of gyroscopic stability of* H_0 ($\alpha + \frac{1}{4}B^2 > 0$)*, there are two possibilities. The values of* m_j *and* ω_j *determine which occurs:*

(a) *If* $\Omega_1 < \omega_1$*, then there are* $4N+4$ *real, normal modes (counting multiplicities) corresponding to stability.*

(b) *If* $\Omega_1 > \omega_1$*, then there are* $4N$ *real, normal modes and* 2 *pairs of imaginary mode corresponding instability in the coupling. This case occurs for large* m_j*.*

Proof. The equations of motion are

$$\ddot{x} - B\dot{y} + \alpha x = \sum_{j=1}^{N} m_j \omega_j^2 (x_j - x) \tag{4.8}$$

$$\ddot{y} + B\dot{x} + \beta y = \sum_{j=1}^{N} m_j \omega_j^2 (y_j - y) \tag{4.9}$$

$$\ddot{x}_j + \omega_j^2 x_j = \omega_j^2 x \tag{4.10}$$

$$\ddot{y}_j + \omega_j^2 y_j = \omega_j^2 y. \tag{4.11}$$

We seek normal mode solutions of the form

$$x(t) = X_0(\omega) e^{i\omega t}; \qquad y(t) = Y_0(\omega) e^{i\omega t} \tag{4.12}$$

$$x_j(t) = X_j(\omega) e^{i\omega t}; \qquad y_j(t) = Y_j(\omega) e^{i\omega t}\,. \tag{4.13}$$

The equations of motion for the normal modes become

$$(-\omega^2 + \alpha) X_0 = iB\omega Y_0 + \sum_{j=1}^{N} m_j \omega_j^2 (X_j - X_0) \tag{4.14}$$

$$(-\omega^2 + \beta) Y_0 = -iB\omega X_0 + \sum_{j=1}^{N} m_j \omega_j^2 (Y_j - Y_0) \tag{4.15}$$

$$(-\omega^2 + \omega_j^2) X_j = \omega_j^2 X_0 \tag{4.16}$$

$$(-\omega^2 + \omega_j^2) Y_j = \omega_j^2 Y_0. \tag{4.17}$$

Eliminating the bath coordinates, we have

$$(-\omega^2 + \alpha) X_0 = iB\omega Y_0 + \sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega_j^2 - \omega^2} X_0 \tag{4.18}$$

$$(-\omega^2 + \beta) Y_0 = -iB\omega X_0 + \sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega_j^2 - \omega^2} Y_0\,. \tag{4.19}$$

We wish to find non-trivial solution to the following linear equation

$$\begin{bmatrix} \omega^2-\alpha+\sum_{j=1}^N m_j \frac{\omega_j^2\omega^2}{\omega_j^2-\omega^2} & iB\omega \\ -iB\omega & \omega^2-\beta+\sum_{j=1}^N m_j \frac{\omega_j^2\omega^2}{\omega_j^2-\omega^2} \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.20}$$

Hence, we have non-trivial normal mode solutions at the zeros of the function

$$f(\omega) = \Big(\omega^2-\alpha+\sum_{j=1}^N m_j \frac{\omega_j^2\omega^2}{\omega_j^2-\omega^2}\Big)\Big(\omega^2-\beta+\sum_{j=1}^N m_j \frac{\omega_j^2\omega^2}{\omega_j^2-\omega^2}\Big) - B^2\omega^2. \tag{4.21}$$

In the case where $\alpha=\beta$, we have

$$(\omega \pm \frac{1}{2}B)^2 - (\alpha+\frac{1}{4}B^2) = \sum_{j=1}^N m_j \frac{\omega_j^2\omega^2}{\omega^2-\omega_j^2} \tag{4.22}$$

Generically equation (4.22) has $4N+4$ distinct complex solutions corresponding to intersections between the two parabolae on the left hand side and the curve on the right. The latter curve has (for ω_j distinct) $2N+1$ branches, the branch through the origin intersecting each parabola twice and each other branch intersecting each parabola once. Thus we obtain the $4N+4$ intersections generically. In the case $B=0$ this number is halved as the parabolae coincide. In addition it is possible for tangencies to occur. Similar results follow in these nongeneric cases and we do not enumerate them here.

We can therefore see the oscillator is stable if there exist $4N+4$ real frequencies which are solutions to the above equation (4.22). Complex solutions lead to instability.

The precise results now follow simply from examining the number of real intersections of the curves on the left and right sides of equation (4.22). For all cases it is helpful to refer to figures 6.3, 6.4, 6.5, and 6.6 for the case $N=1$ and the case $N=2$.

(i) follows since the parabolae in this case have no zeros.

(ii) holds since $\alpha>0$ guarantees that (4.22) has a full set of solutions.

(iii) follows from computing the minimum of $f(\omega)$ in the interval $(0,\omega_1)$. Each parabola intersects the graph of $\sum_{j=1}^N m_j \frac{\omega_j^2\omega^2}{\omega^2-\omega_j^2}$ between any successive singularities, (ω_j,ω_{j+1}), for $j\geq 1$, accounting for $4N$ real modes. However it is the number of modes that exist in $[-\omega_{-1},\omega_1]$ which determines stability. Counting multiplicities, there are $4N+4$ real modes if $\Omega_1<\omega_1$ and only $4N$ real modes if $\Omega_1>\omega_1$.

■

Figure 3 here

Figure 4 here

Figure 5 here

Figure 6 here

Remark: Gyroscopic quantum oscillators

The instability results for the classical gyroscopic oscillator coupled to a heat bath can be extended directly to the quantum setting. This follows from the general quantization procedure for a classical system with quadratic Hamiltonian and gyroscopic terms.

The point is that the eigenvalue computation for the equations of motion in the classical setting is precisely equivalent to that for the Heisenberg equations of motion. Thus for the oscillator coupled to the heat bath the numerical computation in the classical case gives us the correct eigenvalue information for the quantum case also and we can deduce qualitative behavior as for the single gyroscopic oscillator in the previous section. Thus, for sufficiently large coupling, we observe unbound states even when the uncoupled gyroscopic oscillator exhibits bound states.

The quantization of the uncoupled gyroscopic system illustrates that the stability analysis of the quantum system is analogous to the classical normal mode calculation. However there are some subtleties in the interpretation of the stability analysis. We illustrate this briefly with a planar oscillator with gyroscopic forces:

We follow the Dirac formalism as described in Messiah (1991) and Messiah (1990) for example.

In the case of a symmetric planar oscillator, we assume that $\alpha = \beta$ and the Hamiltonian reduces to $(\omega_0 = \omega_{x_0} = \omega_{y_0},)$

$$\begin{aligned} H &= \hbar\omega_0\left(a_0^\dagger a_0 + b_0^\dagger b_0 + 1\right) + \tilde{H}_B, &(4.23)\\ \tilde{H}_B &= i\frac{\hbar B}{2}\left(a_0^\dagger b_0 - a_0 b_0^\dagger\right). &(4.24)\end{aligned}$$

By a standard change of coordinates we have a Hamiltonian in terms of normal modes:

$$\begin{aligned} A_\pm &= \frac{1}{\sqrt{2}}(a_0 \mp i b_0) &(4.25)\\ A_\pm^\dagger &= \frac{1}{\sqrt{2}}(a_0^\dagger \pm i b_0^\dagger), &(4.26)\end{aligned}$$

The eigenvalues of the Hamiltonian, E_{n_1,n_2} can be shown to be

$$E_{n_1,n_2} = \hbar\omega_0(n_1 + n_2 + 1) + \hbar\frac{B}{2}(n_1 - n_2), \tag{4.27}$$

where $n_1, n_2 \in \{0, 1, 2, \cdots\}$. In the case of gyroscopic stability, we have $\frac{B}{2} > \omega_0$, and the energy spectrum is indefinite and unbounded above and below. In the case of pure stability $(\alpha > 0)$, we have $\frac{B}{2} < \omega_0$, and the energy spectrum in positive definite.

In the case of an unstable gyroscopic oscillator, $(\frac{B}{2} + \min\{\alpha, \beta\} < 0)$, there are no bound eigenstates.

Similary in the case of a quantum oscillator coupled to a finite heat bath we can reduce the coupled problem to a system of independent quantum oscillators. Thus the spectrum of the Hamiltonian of the coupled quantum system is positive if and only if the classical system is strongly stable; the spectrum is discrete with both postive and negative eigenvalues if and only if the classical system is gyroscopically stable; and finally, we have unbounded eigenfunctions if and only if the classical system in unstable.

5 Continuum coupling

The above analysis was for a gyroscopic oscillator connected to a *finite* bath of discrete oscillators. It is of interest to investigate stability of the oscillator coupled to a *continuum* of oscillators and to see what new phenomena arise.

In the infinite setting dissipative effects, induced by coupling to the *continuum* bath, become important. Moreover, such dissipative effects induce instability as discussed for general gyroscopic systems in Bloch et al. (1994). In our setting suppose we have a Rayleigh dissipation so that the gyroscopic oscillator equations become

$$\begin{aligned}\ddot{x} - B\dot{y} + \alpha x + \gamma\dot{x} = 0\\ \ddot{y} + B\dot{x} + \beta y + \gamma\dot{y} = 0\end{aligned} \tag{5.1}$$

where $\gamma > 0$. Then the normal mode equation (4.22) becomes

$$(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2) = -i\gamma\omega \tag{5.2}$$

or

$$\omega^2 + \omega(i\gamma \pm B) - \alpha = 0 \tag{5.3}$$

which clearly has complex solutions in general and we obtain instability.

Now the natural limiting Lagrangian for our systems here is

$$L = L_0 + L_{bath}, \tag{5.4}$$

where

$$L_0 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\alpha x^2 + \beta y^2) + \frac{1}{2}B(x\dot{y} - y\dot{x}) \tag{5.5}$$

and

$$L_{bath} \;=\; \int_{\mathbb{R}} \Big(\frac{1}{2}m(\dot{u}^2 + \dot{v}^2) - \frac{1}{2}m\nu^2((u-x)^2 + (v-y)^2)\Big)d\nu\,. \tag{5.6}$$

Here $x(t)$ and $y(t)$ are the distinguished oscillator coordinates and $u(t,\nu)$, $v(t,\nu)$ denote continuum oscillator coordinates of natural frequency ν. Here $m = m(\nu)$ is a mass density: $m(\nu)d\nu/(\int m(\nu')d\nu')$ denotes the proportion of oscillators with natural frequency between ν' and $\nu' + d\nu'$.

The corresponding equations of motion are

$$\ddot{x} - B\dot{y} + \alpha x \;=\; \int_{\mathbb{R}} m\nu^2(u-x)d\nu \tag{5.7}$$

$$\ddot{y} + B\dot{x} + \beta y \;=\; \int_{\mathbb{R}} m\nu^2(v-y)d\nu \tag{5.8}$$

$$\int_{\mathbb{R}} m(\ddot{u} + \nu^2(u-x))d\nu \;=\; 0 \tag{5.9}$$

$$\int_{\mathbb{R}} m(\ddot{v} + \nu^2(v-y))d\nu \;=\; 0\,. \tag{5.10}$$

Assuming mass distributions with finite variance, the last two equation become

$$\ddot{u} + \nu^2 u \;=\; \nu^2 x \tag{5.11}$$

$$\ddot{v} + \nu^2 v \;=\; \nu^2 y. \tag{5.12}$$

To obtain normal mode solutions, we set

$$x(t) = X_0(\omega)e^{i\omega t}; \qquad y(t) = Y_0(\omega)e^{i\omega t} \tag{5.13}$$

$$u(\nu,t) = X_\nu(\omega)e^{i\omega t}; \qquad v(\nu,t) = Y_\nu(\omega)e^{i\omega t}. \tag{5.14}$$

The equations of motion for the normal modes become

$$
\begin{aligned}
(-\omega^2+\alpha)X_0 &= -iB\omega Y_0 + \int_{\mathbb{R}} m\nu^2(X_\nu - X_0) && (5.15)\\
(-\omega^2+\beta)Y_0 &= iB\omega X_0 + \int_{\mathbb{R}} m\nu^2(Y_\nu - Y_0) && (5.16)\\
(-\omega^2+\nu^2)X_\nu &= \nu^2 X_0 && (5.17)\\
(-\omega^2+\nu^2)Y_\nu &= \nu^2 Y_0. && (5.18)
\end{aligned}
$$

Eliminating the bath coordinates, we have

$$
\begin{bmatrix} \omega^2-\alpha+\int_{\mathbb{R}} m\frac{\nu^2\omega^2}{\nu^2-\omega^2}d\nu & -iB\omega \\ iB\omega & \omega^2-\beta+\int_{\mathbb{R}} m\frac{\nu^2\omega^2}{\nu^2-\omega^2}d\nu \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.19}
$$

For nontrivial solution, we require

$$
\Big(\omega^2-\alpha+\int_{\mathbb{R}} m\frac{\nu^2\omega^2}{\nu^2-\omega^2}d\nu\Big)\Big(\omega^2-\beta+\int_{\mathbb{R}} m\frac{\nu^2\omega^2}{\nu^2-\omega^2}d\nu\Big) = B^2\omega^2. \tag{5.20}
$$

In the case where $\alpha=\beta$, we can simplify the dispersion relation,

$$
(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2) = \int_{\mathbb{R}} m(\nu)\frac{\nu^2\omega^2}{\omega^2-\nu^2}d\nu\,. \tag{5.21}
$$

This dispersion relationship in the infinite-dimensional setting requires careful interpretation. As in the finite-dimensional setting the above analyis is simply an analysis of the normal modes of the system. Any complex frequencies will give to an instability. One way to interpret the integral in the dispersion relation is the principle value sense. We then see that the frequencies ω which satisfy the dispersion relation form a discrete set. If m decays fast enough for the principle value of integral to be finite, we have only a finite number of real solutions. Increasing m from zero, the corresponding solutions are the perturbed stable eigenvalues of the discrete system. If there are 4 real modes, then we consider the oscillator to be stable. If there are fewer than 4 real modes, then the motion of the coupled oscillator must include unstable modes. In figure 6.7, we see that there exist enough stable modes to describe the motion of the oscillator. Increasing the mass of the bath can lower the number of real modes. For example, figure 6.8 has no real modes and is hence unstable.

Figure 7 here

Figure 8 here

On the other hand in the infinite setting it is possible to get a further mechanism for instability – damping, as in the Rayleigh damped case discussed above. In the infinite setting one can see this in the analysis of the initial value problem, carried out for example by Laplace transform analysis. The Fourier mode analysis imposes a time reversal symmetry and cannot capture the damping revealed by a Laplace transform analysis as in the case of Landau damping of plasmas (see e.g. Landau (1946), Smereka (1998)). In our setting the system is time reversible and each individual mode may be stable or unstable. However, as in the Landau setting, a superposition of infinitely many modes can damp as they drift out of phase, following the argument of Kampen and Felderhof

(1967). This damping effect leads to instability in the gyroscopic setting even if the individual modes are stable.

We construct an instability as follows. We show that for $\epsilon > 0$ and small there is a solution of

$$(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2) - \epsilon \int_{\mathbb{R}} m(\nu)\frac{\nu^2\omega^2}{\omega^2 - \nu^2}d\nu \ = \ 0, \tag{5.22}$$

$\omega = \omega(\epsilon)$ with negative imaginary part, $\Im\omega < 0$ (so that $e^{i\omega t}$ grows as t increases), and a corresponding finite energy normal mode (5.14).

Let

$$F(\omega,\epsilon) \ \equiv \ (\omega + \frac{\tau_1}{2}B)^2 - (\alpha + \frac{1}{4}B^2) - \epsilon \int_{C_\delta} m(\nu)\frac{\nu^2\omega^2}{\omega^2 - \nu^2}d\nu \ = \ 0, \tag{5.23}$$

where $\tau_1 = \pm 1$ and C_δ is a contour which coincides with the real axis, traversed from $-\infty$ to $+\infty$ for $|\nu| \geq \delta$ and which, for $|\nu| < \delta$, consists of a semicircular arc of radius δ in the upper half plane.

Begin with any one of the four roots, ω_{τ_1,τ_2}, of the unperturbed problem, $\epsilon = 0$:

$$\omega_{\tau_1,\tau_2} = \frac{\tau_1}{2}B + \frac{\tau_2}{2}(\alpha + \frac{1}{4}B^2)^{1/2}, \ \ \tau_1,\tau_2 = \pm 1 \tag{5.24}$$

Note that $F(\omega,\epsilon)$ is analytic for ω varying over the region below the contour C_δ and $\epsilon \in \mathbb{R}$. Furthermore, $F(\omega_{\tau_1,\tau_2},0) = 0$. Using the implicit function theorem, we shall construct a solution of $F(\omega,\epsilon) = 0$, with $\omega(\epsilon)$ near ω_{τ_1,τ_2} and $\Im(\omega(\epsilon)) < 0$ for ϵ near zero. Calculating, we find

$$\begin{aligned} F_\omega(\omega,\epsilon) \ &= \ 2(\omega + \frac{\tau_1}{2}B) + 2\epsilon\omega^3 \int_{C_\delta} m(\nu)\frac{\nu^2}{(\omega^2-\nu^2)^2}\ d\nu \\ &\quad -2\epsilon\omega \int_{C_\delta} m(\nu)\frac{\nu^2}{\omega^2 - \nu^2}\ d\nu \end{aligned} \tag{5.25}$$

and therefore

$$F_\omega(\omega_{\tau_1,\tau_2},0) = \tau_2(\alpha + \frac{1}{4}B^2)^{\frac{1}{2}} \neq 0 \tag{5.26}$$

It follows that for ϵ sufficiently small, there is a function $\omega(\epsilon)$, satisfying $F(\omega(\epsilon),\epsilon) = 0$. Differentiation gives:

$$\begin{aligned} \frac{d\omega}{d\epsilon}(0) \ &= \ -F_\epsilon(\omega_{\tau_1,\tau_2},0)F_\omega^{-1}(\omega_{\tau_1,\tau_2},0) \\ &= \ -(\tau_2(\alpha + \frac{1}{4}B^2)^{\frac{1}{2}})^{-1}\ F_\epsilon(\omega_{\tau_1,\tau_2},0) \end{aligned} \tag{5.27}$$

and therefore the sign of the imaginary part of $\omega(\epsilon)$, for small ϵ, is determined by the sign of

$$F_\epsilon(\omega_{\tau_1,\tau_2},0) \ = \ -\omega^2_{\tau_1,\tau_2} \int_{C_\delta} m(\nu)\frac{\nu^2}{\omega^2_{\tau_1,\tau_2} - \nu^2}d\nu, \tag{5.28}$$

provided it is nonzero. By analyticity of the integrand in (5.28), we can evaluate $F_\epsilon(\omega_{\tau_1,\tau_2},0)$ by taking letting δ tend to zero. We obtain

$$F_\epsilon(\omega_{\tau_1,\tau_2},0) \ = \ -\omega^2_{\tau_1,\tau_2}\ \text{P.V.} \int_{\mathbb{R}} m(\nu)\frac{\nu^2}{\omega^2_{\tau_1,\tau_2} - \nu^2}d\nu + i\pi\omega^3_{\tau_1,\tau_2} m(\omega_{\tau_1,\tau_2})$$

It follows that

$$\Im\left(\frac{d\omega}{d\epsilon}(0)\right) = -(\tau_2(\alpha + \frac{1}{4}B^2)^{\frac{1}{2}})^{-1} \ \pi\omega^3_{\tau_1,\tau_2}m(\omega_{\tau_1,\tau_2}),$$

which can be made negative for appropriate choice of τ_1 and τ_2. And therefore, there exists a solution of $F(\omega(\epsilon), \epsilon) = 0$ for ϵ small with $\Im(\omega(\epsilon)) < 0$. Finally, since $\Im(\omega(\epsilon)) < 0$ we may deform the contour C_δ in the definition of $F(\omega, \epsilon)$ back to the real axis and we have that $\omega(\epsilon)$ actually solves the desired equation (5.22). The corresponding normal mode (5.14) can be constructed algebraically and for mass-densities, $m(\nu)$, which decay sufficiently rapidly as ν tends to infinity, is easily seen to have finite energy. Thus as in the case of Rayleigh dissipation there is a complex root generically which leads to instability.

6 Relationship to other models and final remarks

In this final section we discuss other models of coupling to a thermal reservoir and their relationship to the stability analysis we have conducted in this paper.

One may for example consider an oscillator connected to an infinite heat bath in the form of a string. We dicuss this briefly here. Further details may be found in Hagerty et al. (1999), Hagerty et al. (2003).

One such model is based on the Lamb model (Lamb (1900)) of an oscillator is physically coupled to a string. The vibrations of the oscillator transmit waves into the string and are carried off to infinity. Hence, the oscillator loses energy and is effectively damped by the string.

Let $w(x, t)$ be the displacement of the string at position $x \in \mathbb{R}$ at time t. Let ρ be the mass density of the string and let T be the tension. We can compute the wave speed of transverse oscillations of the free string to be $c = \frac{T}{\rho}$. Assuming a singular mass density at $x = 0$, we can couple the dynamics of an oscillator, q, of mass M, to the otherwise free wave to resolve the singularity,

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= c^2\frac{\partial^2 w}{\partial x^2} \\ M\ddot{q} + Vq &= T[w_x]_{x=0} \\ q(t) &= w(0, t), \end{aligned} \tag{6.1}$$

where $\sqrt{\frac{V}{M}}$ is the frequency of the uncoupled oscillating mass M and $[w_x]_{x=0} = w_x(0+, t) - w_x(0-, t)$ is the jump discontinuity of the slope of the string.

Perturbing the oscillator at time $t = 0$ from its equilibrium position, we can use d'Alembert's solution to the wave equation to solve for w,

$$w = \begin{cases} Ce^{(ct-|x|)\omega} & \text{for } |x| < ct, \\ 0 & \text{for } |x| > ct, \end{cases} \tag{6.2}$$

where $\omega = -\frac{T}{Mc} + i\sqrt{\frac{V^2}{M^2} - (\frac{T}{Mc})^2}$ and C is the size of the initial displacement. For small tension we have damping and oscillator motion, while for large tension, we have pure damping. From the solution of the wave equation, we can compute the jump condition of the oscillator,

$$[w_x]_{x=0} = -2C\omega e^{\omega ct} = -\frac{2}{c}\dot{q}. \tag{6.3}$$

We obtain a reduced form of the dynamics describing the explicit motion of the oscillator subsystem,

$$M\ddot{q} + \frac{2T}{c}\dot{q} + Vq = 0. \tag{6.4}$$

The coupling term arises explicitly as a Rayleigh dissipation term $\frac{2T}{c}\dot{q}$ in the dynamics of the oscillator. For the gyroscopic Lamb coupling, the dissipation term will induce instabilities in gyroscopically stabilized Chetaev systems.

A variant of the Lamb model, is a system which includes gyroscopic terms coupled to the standard wave equation. We have the boundary constraint that the displacement of the gyroscopic oscillator fixes a point of the string. One can show that this local coupling perturbation destabilizes gyroscopically stabilized systems.

We consider a model of a string in $\mathbb{R}^{n+1}$, whose transverse vibrations are independent. Suppose the string lies initially along the $x_{n+1}-$ axis. For simplicity, we denote the x_{n+1} dimension as the z dimension. Coupling the transverse motion to a $n-$dimensional gyroscopic system, we can solve the system

$$\begin{aligned} \frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) &= c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t), \quad z \in \mathbb{R} - \{0\}, t \in \mathbb{R}, \\ M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= T\left[\frac{\partial \mathbf{w}}{\partial z}\right]_{z=0} \\ \mathbf{w}(0,t) &= \mathbf{q}(t), \end{aligned} \tag{6.5}$$

where c is the speed of transverse waves in the string, T is the tension of the string, $\mathbf{w} = \begin{bmatrix} w_1(z,t) & \cdots & w_n(z,t) \end{bmatrix}^T$ is the displacement of the string in the first n dimensions and $[\frac{\partial \mathbf{w}}{\partial z}]_{z=0}$ is the jump discontinuity in the slope of the string. Let us define the initial conditions on the string by $\mathbf{w}_1(z) = \frac{\partial \mathbf{w}}{\partial t}(z,0)$, and $\mathbf{w}_0(z) = \mathbf{w}(z,0)$.

By direct computation we obtain an exact reduced dynamical system for $\mathbf{q}$. This reduction contains explicit dissipation terms reflecting the oscillator/ field coupling. The results on reduction and stability are summarized in the following two results.

Proposition 6.1 *If the initial data $\mathbf{w}_0$ and $\mathbf{w}_1$ have compact support, then for $\pm ct \notin \operatorname{supp}(\mathbf{w}_0) \cup \operatorname{supp}(\mathbf{w}_1)$ the oscillator dynamics of the gyroscopic Lamb model reduces to*

$$M\ddot{\mathbf{q}}(t) + (S + \frac{2T}{c})\dot{\mathbf{q}}(t) + V\mathbf{q}(t) = 0. \tag{6.6}$$

A proof of this proposition may be found in Hagerty et al. (2003). In Bloch et al. (1994), small dissipation is shown to induce instability in gyroscopically stable systems (see Theorem 3.2). In the gyroscopic Lamb model, radiation (energy transfer into an infinite-dimensional system) produces a Rayleigh dissipation term which depends on the tension of the string. Since dissipation induces instability in these system for small dissipation, small tension coupling yields radiation induced instability via an analogous mechanism. We summarize the main result of *radiation induced* instability in gyroscopically stabilized Chetaev systems with a theorem.

Theorem 6.2 *If a gyroscopic mechanical system is gyroscopically stable (i.e. V has a negative eigendirection), then local coupling via the Lamb model induces instability for small coupling parameter T.*

A related model is a field coupling of the mechanical gyroscopic system to the wave equation.

We model force of the wave field on the Chetaev system by the magnitude of the wave field. Coupling of this type is important in various physical models -see Soffer and Weinstein (1998a,b, 1999), and references therein. The interaction between the wave equation and the gyroscopic system is modeled with a coupling parameter κ and with a coupling distribution $\chi(\xi)$; the equations of motion are

$$
\begin{aligned}
&M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = \kappa \int_{\mathbb{R}} \chi(z) w(z,t) dz \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \\
&\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial z^2} = \kappa \chi(z) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q}.
\end{aligned} \tag{6.7}
$$

Again one can show that instability is induced.

Other models of field coupling are also possible and of interest depending on the physical situation of interest. We mention two further models (ignoring gyroscopic forces):

One model is analyzed in Unruh and Zurek (1989) who consider a Lagrangian for an oscillator connected to a free field as follows:

$$
L_1 = \int \left\{ \left[\dot{\phi}^2 - \left(\frac{\partial \phi}{\partial x} \right)^2 \right] + \delta(x) \left(\dot{q}^2 - \Omega_0^2 q^2 - \epsilon q \frac{\partial \phi}{\partial t} \right) \right\} dt dx \,. \tag{6.8}
$$

The equations of motion are (taking the mass to be unity)

$$
\begin{aligned}
\left(\ddot{q} + \Omega_0^2 \right) q &= -\epsilon \frac{\partial \phi}{\partial t}(0,t) \\
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} &= \epsilon \dot{q} \delta(x) \,.
\end{aligned} \tag{6.9}
$$

It is easy to see how damping arises in the model (6.9) and furthermore, see how the system decouples (see Bloch (2003) for related details):

Taking the Fourier transform with respect to x in the wave equation yields

$$
\hat{\phi}(k,t) = \int_{\mathbb{R}} e^{-ikx} \phi(x) dx, \tag{6.10}
$$

$$
\hat{\phi}_{tt}(k,t) + k^2 \hat{\phi}(k,t) = \epsilon \dot{q}(t). \tag{6.11}
$$

Solving the o.d.e. for $\hat{\phi}$, we have

$$
\begin{aligned}
\hat{\phi}(k,t) =& \epsilon \int_0^t \frac{\sin(k(t-s))}{k} q(s) ds \\
&+ \hat{\phi}_{free}(k,t).
\end{aligned} \tag{6.12}
$$

The above equation is the retarded Green's function for the wave equation and ϕ_{free} is the homogeneous solution satisfying the initial conditions. Neglecting ϕ_{free}, we have

$$\begin{aligned}\phi(0,t) &= \frac{\epsilon}{2}\int_0^t \operatorname{sgn}(t-s)\dot{q}(s)ds && (6.13)\\ &= \frac{\epsilon}{2}\int_0^t \dot{q}(s)ds\,. && (6.14)\end{aligned}$$

Thus we obtain

$$\ddot{q}+\Omega_0^2 q = -\frac{\epsilon^2}{2}\,\dot{q}\,. \tag{6.15}$$

If the free term is included one obtains an additional forcing term of $-\epsilon\dot{\phi}_{\rm free}$ on the right hand side.

Finally it is of interest to consider the model in Komech et al. (1999).

In this case one has a scalar wave field $\phi(x)$ in three dimensions coupled to a particle with position q. Ignoring relativistic effects the Lagrangian is of the form

$$L_2 = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\int d^3x\left(\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2 - |\nabla\phi(x,t)|^2\right) + \int d^3x\phi(x,t)\rho(x-q). \tag{6.16}$$

In Komech et al. (1999) $\rho(x)$ is chosen to ensure the energy is bounded from below. We note also the difference between this model and the model (6.7). Here the model is that of oscillator coupled to a scalar field in three dimensions while the model (6.7) is that of an oscillator system coupled via a coupling parameter to n copies of a scalar field of one dimension. This latter model is of interest for analysis of resonances in quantum systems (see Soffer and Weinstein (1999)) while the Komech et al. (1999) model is of interest for certain physical models of a particle interacting with a field.

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Figure Caption Page

Figure 1: **Rotating plate with springs.**

Figure 2: **Inverted spherical pendulum.**

Figure 3: **Gyroscopic system coupled to a heat bath of one oscillator ($N = 1$) maintaining stability.** Stable coupling with $\alpha = -1$, $B = 3$, $m_1 = \frac{1}{4}$, and $\omega_1 = 1$. Note $\Omega_1 < \omega_1$ which implies that there are 8 real solutions. The dashed line is the graph of $\sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega^2 - \omega_j^2}$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 4: **Destabilization of a gyroscopic system coupled to a heat bath of one oscillator ($N = 1$).** Unstable coupling with $\alpha = -1$, $B = 3$, $m_1 = 2$, and $\omega_1 = 1$. Note $\Omega_1 > \omega_1$ which implies that there are only 4 real solutions. The dashed line is the graph of $\sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega^2 - \omega_j^2}$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 5: **Stable gyroscopic system coupled to a heat bath of two oscillators ($N = 2$).** Stable coupling with $\alpha = -1$, $B = 6$, $m_1 = 1$, $\omega_1 = 1$, $m_2 = \frac{1}{8}$, and $\omega_2 = 4$. Notice the system has $4N + 4$ stable normal modes if $\Omega_1 < \omega_1$. The dashed line is the graph of $\sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega^2 - \omega_j^2}$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 6: **Gyroscopic system destabilized via coupling to a heat bath of two oscillators ($N = 2$).** Stable coupling with $\alpha = -1$, $B = 3$, $m_1 = 1$, $\omega_1 = 1$, $m_2 = 1$, and $\omega_2 = 4$. Notice the system has only $4N$ stable normal modes since $\Omega_1 > \omega_1$. The dashed line is the graph of $\sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega^2 - \omega_j^2}$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 7: **Continuum coupling to gyroscopic system.** "Stable" coupling with $\alpha = -1$, $B = 3$, and $m(\nu) = \frac{1}{10} e^{-\frac{|\nu|}{2}}$. The dashed line is the graph of $P.V. \int_{\mathbb{R}} m \frac{\nu^2 \omega^2}{\omega^2 - \nu^2} d\nu$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 8: **Continuum coupling to gyroscopic system.** "Unstable" coupling with $\alpha = -1$, $B = 3$, and $m(\nu) = \frac{1}{2} e^{-\frac{|\nu|}{2}}$. The dashed line is the graph of $P.V. \int_{\mathbb{R}} m \frac{\nu^2 \omega^2}{w^2 - \nu^2} d\nu$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 1:

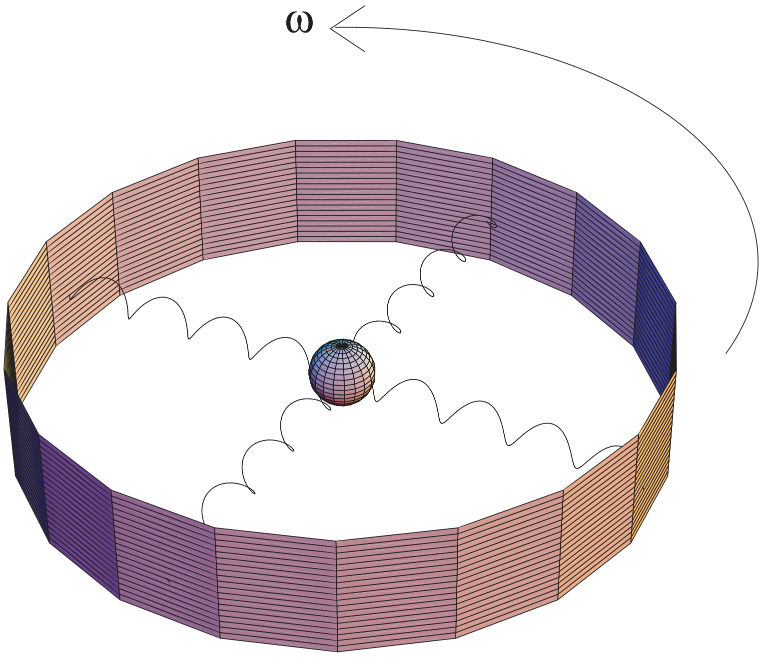

Figure 6.1: **Rotating plate with springs.**

Figure 2:

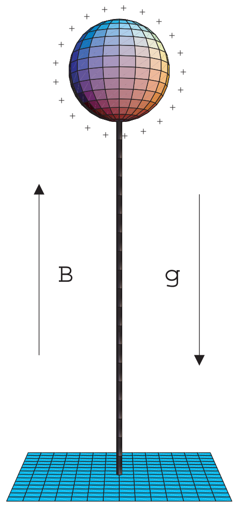

Figure 6.2: **Inverted spherical pendulum.**

Figure 3:

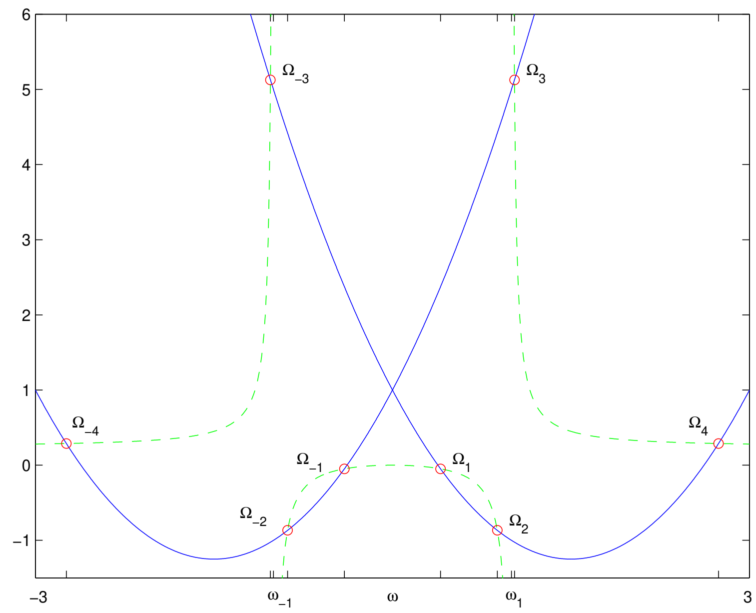

Figure 6.3: **Gyroscopic system coupled to a heat bath of one oscillator ($N = 1$) maintaining stability.** Stable coupling with $\alpha = -1$, $B = 3$, $m_1 = \frac{1}{4}$, and $\omega_1 = 1$. Note $\Omega_1 < \omega_1$ which implies that there are 8 real solutions. The dashed line is the graph of $\sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega^2 - \omega_j^2}$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 4:

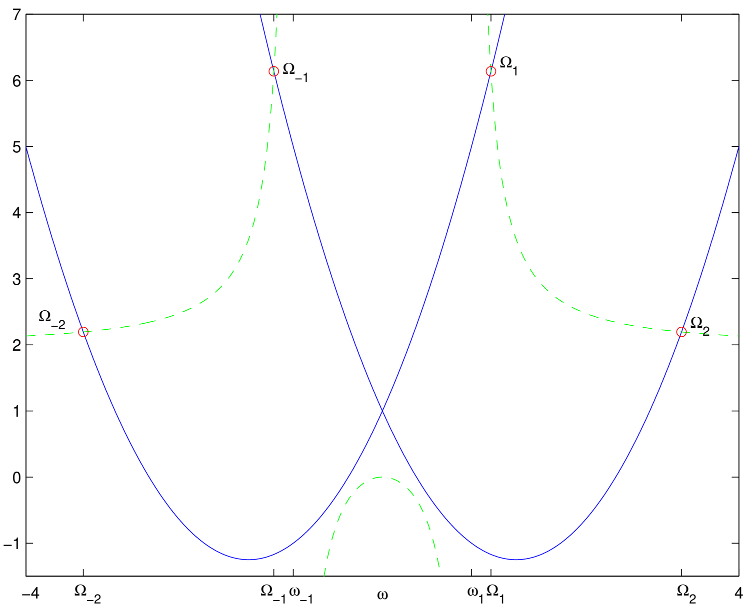

Figure 6.4: **Destabilization of a gyroscopic system coupled to a heat bath of one oscillator ($N = 1$).** Unstable coupling with $\alpha = -1$, $B = 3$, $m_1 = 2$, and $\omega_1 = 1$. Note $\Omega_1 > \omega_1$ which implies that there are only 4 real solutions. The dashed line is the graph of $\sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega^2 - \omega_j^2}$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 5:

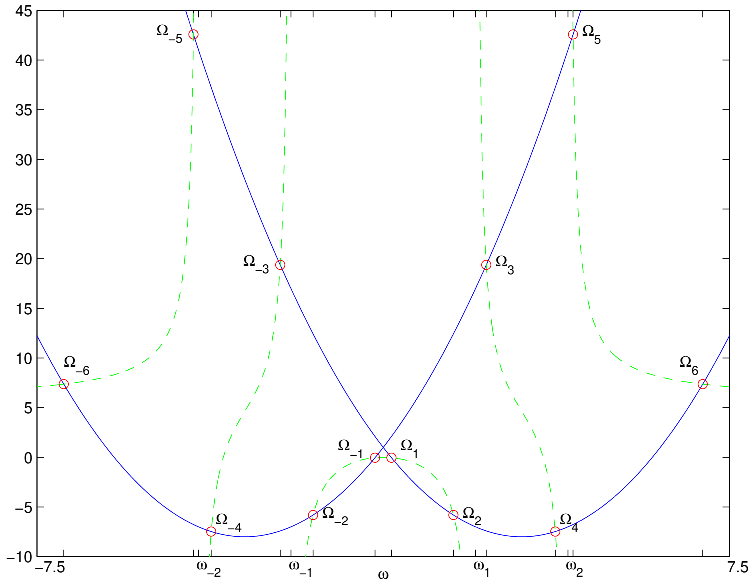

Figure 6.5: **Stable gyroscopic system coupled to a heat bath of two oscillators ($N = 2$).** Stable coupling with $\alpha = -1$, $B = 6$, $m_1 = 1$, $\omega_1 = 1$, $m_2 = \frac{1}{8}$, and $\omega_2 = 4$. Notice the system has $4N + 4$ stable normal modes if $\Omega_1 < \omega_1$. The dashed line is the graph of $\sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega^2 - \omega_j^2}$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 6:

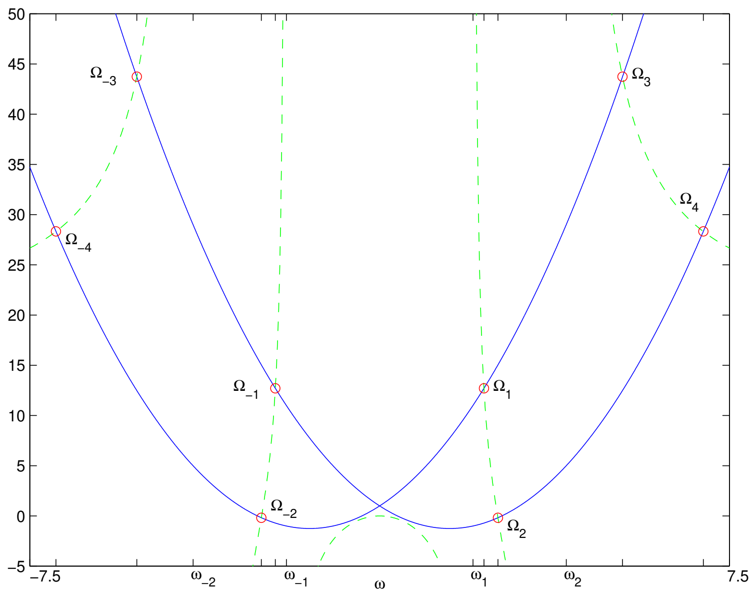

Figure 6.6: **Gyroscopic system destabilized via coupling to a heat bath of two oscillators ($N = 2$).** Stable coupling with $\alpha = -1$, $B = 3$, $m_1 = 1$, $\omega_1 = 1$, $m_2 = 1$, and $\omega_2 = 4$. Notice the system has only $4N$ stable normal modes since $\Omega_1 > \omega_1$. The dashed line is the graph of $\sum_{j=1}^{N} m_j \frac{\omega_j^2 \omega^2}{\omega^2 - \omega_j^2}$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 7:

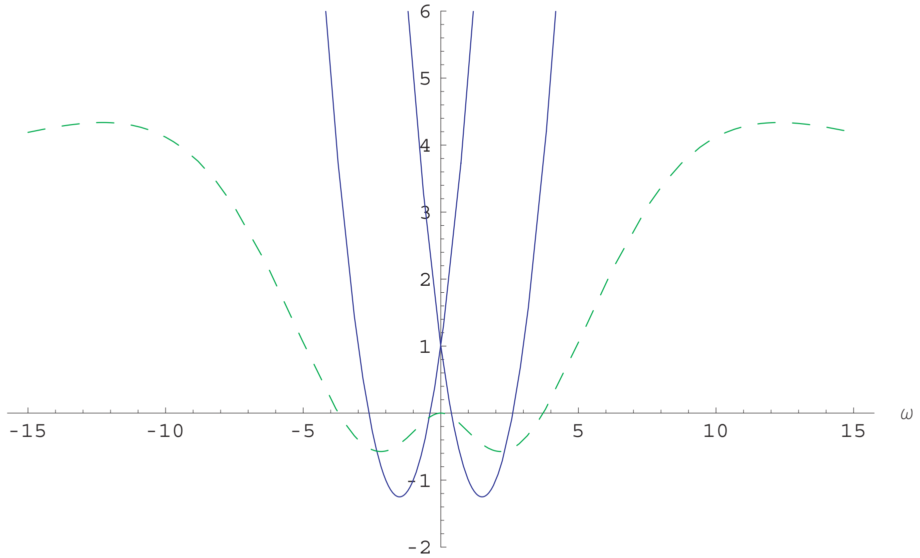

Figure 6.7: **Continuum coupling to gyroscopic system.** "Stable" coupling with $\alpha = -1$, $B = 3$, and $m(\nu) = \frac{1}{10}e^{-\frac{|\nu|}{2}}$. The dashed line is the graph of $P.V. \int_{\mathbb{R}} m \frac{\nu^2\omega^2}{\omega^2-\nu^2} d\nu$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.

Figure 8:

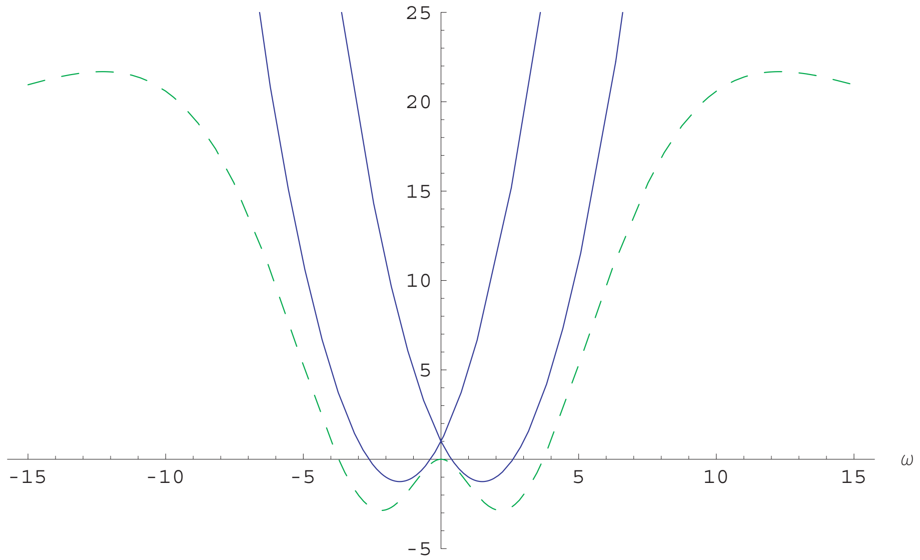

Figure 6.8: **Continuum coupling to gyroscopic system.** "Unstable" coupling with $\alpha = -1$, $B = 3$, and $m(\nu) = \frac{1}{2}e^{-\frac{|\nu|}{2}}$. The dashed line is the graph of $P.V. \int_{\mathbb{R}} m \frac{\nu^2\omega^2}{w^2-\nu^2} d\nu$ while the solid lines are the graphs of the parabolas $(\omega \pm \frac{1}{2}B)^2 - (\alpha + \frac{1}{4}B^2)$.