STABILIZATION OF NONHOLONOMIC SYSTEMS USING ISOSPECTRAL FLOWS*

ANTHONY BLOCH†, SERGEY DRAKUNOV‡, AND MICHAEL KINYON§

Abstract. In this paper we derive and analyze a discontinuous stabilizing feedback for a Lie algebraic generalization of a class of kinematic nonholonomic systems introduced by Brockett. The algorithm involves discrete switching between isospectral and norm-decreasing flows. We include a rigorous analysis of the convergence.

Key words. Nonlinear control, nonholonomic systems, isospectral flows, Lie theory

1. Introduction. In this paper we present a stabilization algorithm for a Lie algebraic generalization of a class of nonholonomic systems originally introduced by Brockett [1981]. These are the systems of the general form $\dot{x} = B(x)u$ in which the dimension of the control vector u is smaller than that of the state vector x, but the system is nonetheless controllable. Such systems are sometimes referred to as kinematic nonholonomic systems (as opposed to dynamic nonholonomic systems which arise from a Lagrangian – see e.g. Bloch and Crouch [1995] or Bloch, Krishnaprasad, Marsden and Murray [1996]). Following standard usage, we will simply refer to kinematic nonholonomic systems as nonholonomic.

A prototypical system in the class we study here is the Heisenberg system or nonholonomic integrator (Brockett [1981], [1983]):

$$\dot{x} = u \tag{1.1}$$

$$\dot{y} = v \tag{1.2}$$

$$\dot{z} = xv - yu, \tag{1.3}$$

where $x, u, y, v, z \in \mathbb{R}$. If we identify the variable z with the skew-symmetric matrix $Y = \begin{pmatrix} 0 & z \\ -z & 0 \end{pmatrix}$, and observe that

$$\begin{pmatrix} x \\ y \end{pmatrix}(u, v) - \begin{pmatrix} u \\ v \end{pmatrix}(x, y) = \begin{pmatrix} 0 & xv - yu \\ -xv + yu & 0 \end{pmatrix},$$

then a generalization of (1.1)-(1.3) immediately suggests itself. This is the $so(n)$ system (Brockett [1981]):

$$\dot{x} = u \tag{1.4}$$

$$\dot{Y} = xu^T - ux^T, \tag{1.5}$$

where x, u are column vectors in $\mathbb{R}^n$ and $Y \in so(n)$, $n \geq 2$. Here $so(n)$ is the Lie algebra of $n \times n$ skew symmetric matrices: $Y^T = -Y$.

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†Department of Mathematics, The University of Michigan, Ann Arbor, MI 48109 (bloch@math.lsa.umich.edu)

‡Department of Electrical Engineering & Computer Science, Tulane University, New Orleans, LA 70118 (drakunov@eecs.tulane.edu)

§Department of Mathematics & Computer Science, Indiana University South Bend, South Bend, IN 46634-7111 (mkinyon@iusb.edu)

The importance of the $so(n)$ system (1.4)-(1.5) is that it is a canonical form for a class of controllable systems of the form $\dot{x} = B(x)u$, $u \in \mathbb{R}^n$, $x \in \mathbb{R}^{n(n+1)/2}$. The class in question is the controllable systems of this type where the first derived algebra of control vector fields spans the tangent space $T\mathbb{R}^{n(n+1)/2}$ at any point. (Recall that if E^0 is the subbundle of the tangent bundle spanned by the control fields, then the first derived algebra is given by $E^1 = E^0 + [E^0, E^0]$.) Brockett showed that such a system can be transformed to the form (1.4)-(1.5) up to a suitable order in the neighborhood of a given point such as the origin (Brockett [1981]). (In the terminology of Bloch *et. al.* [1992], for example, this is a controllable nonholonomic kinematic system of nonholonomy degree 1.)

A different generalization of the Heisenberg system (1.1)-(1.3) is obtained by identifying the vectors $(x, y)^T$ and $(u, v)^T$ with the matrices $X = \frac{1}{\sqrt{2}} \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}$ and $U = \frac{1}{\sqrt{2}} \begin{pmatrix} u & -v \\ -v & -u \end{pmatrix}$, respectively. Then we have

$$[U, X] = UX - XU = \begin{pmatrix} 0 & xv - yu \\ -(xv - yu) & 0 \end{pmatrix}.$$

This suggests the following matrix system occurring in the Lie algebra $sl(n, \mathbb{R})$ of $n \times n$ matrices with trace 0:

$$\dot{X} = U \tag{1.6}$$
$$\dot{Y} = [U, X], \tag{1.7}$$

where $X, U \in sym_0(n, \mathbb{R})$ and $Y \in so(n)$. Here $sym_0(n, \mathbb{R})$ is the space of $n \times n$ real symmetric matrices with trace zero. Note that $sl(n, \mathbb{R}) = sym_0(n, \mathbb{R}) \oplus so(n)$, a direct sum.

The system we study in this paper generalizes both the $so(n)$ system (1.4)-(1.5) and the $sl(n, \mathbb{R})$ system (1.6)-(1.7). Let $\mathfrak{g}$ be a Lie algebra. Assume $\mathfrak{g}$ has a direct sum decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ such that $\mathfrak{h}$ is a Lie subalgebra, $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$. The exact hypotheses will be given in §3; for now, we note that every simple Lie algebra with a Cartan decomposition is of this type. We will consider the following system in $\mathfrak{g}$:

$$\dot{x} = u \tag{1.8}$$
$$\dot{Y} = [u, x] \tag{1.9}$$

where $x, u \in \mathfrak{m}$, $Y \in \mathfrak{h}$.

Clearly the $sl(n, \mathbb{R})$ system (1.6)-(1.7) has the form (1.8)-(1.9). In addition, the $so(n)$ system (1.4)-(1.5) can also be written in this form as we now show. Let $\mathfrak{h} = so(n)$ and let $\mathfrak{m} = \mathbb{R}^n$. For $x, u \in \mathfrak{m}$, define $[u, x] \equiv xu^T - ux^T \in \mathfrak{h}$. For $Y \in \mathfrak{h}$, $x \in \mathfrak{m}$, define $[Y, x] = -[x, Y] \equiv Yx$. Then the Lie algebra $\mathfrak{g} \equiv \mathfrak{m} + \mathfrak{h}$ is isomorphic to $so(n + 1)$. Indeed, this is clear if we make the identifications

$$\mathfrak{h} \cong \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} : Y \in so(n) \right\} \quad \text{and} \quad \mathfrak{m} \cong \left\{ \begin{pmatrix} 0 & -x^T \\ x & 0 \end{pmatrix} : x \in \mathbb{R}^n \right\}.$$

It is easy to check that the desired commutation relations hold, and, with these particular identifications, that the adjoint action of $\mathfrak{h}$ on $\mathfrak{m}$ agrees with the standard action of $so(n)$ on $\mathbb{R}^n$. (Incidentally, that the Heisenberg system (1.1)-(1.3) can be viewed as either an $sl(2,\mathbb{R})$ system or an $so(3)$ system is just a consequence of the fact that $sl(2,\mathbb{R})$ and $so(3)$ are isomorphic Lie algebras.)

We have already noted that the $so(n)$ system (1.4)-(1.5) is a canonical form for those controllable nonholonomic systems for which the first derived algebra $E^1 = E^0 + [E^0, E^0]$ of the control subbundle E^0 spans the tangent space at each point, and for which the dimensions of E^0 and E^1 are arithmetically related in a particular way (see Brockett [1981]). As partial motivation for considering the more general Lie algebra system (1.8)-(1.9), we expect that this system will turn out to be a canonical form for a wider class of controllable nonholonomic systems satisfying conditions which suitably generalize those characterizing the $so(n)$ system, including certain symmetry conditions. In particular, we suspect that (1.8)-(1.9) is a canonical form for nonholonomic systems occurring in certain homogeneous spaces, such as symmetric spaces.

The problem we consider herein is that of finding a stabilizing control for the general system (1.8)-(1.9). Since the dimension of $\mathfrak{m}$, which is where the control u takes its values, is less than the dimension of the state space $\mathfrak{g}$, the system fails Brockett's necessary condition for the existence of a continuous feedback law (Brockett [1983]). Previous work on the stabilization problem for nonholonomic systems such as the Heisenberg and $so(n)$ systems has focused mainly on the development of either smooth dynamic feedback or nonsmooth static feedback. For the former approach, see Coron [1992], and also Pomet [1992] and M'Closkey and Murray [1993]. A discontinuous, discrete-time approach can be found in Bloch, Reyhanoglu and McClamroch [1992] and Kolmanovsky and McClamroch [1993]. Brockett [1993], on the other hand, used a stochastic approach. Other interesting work includes that of Liu and Sussmann, Samson, Sontag, Sordalen, Walsh, Bushnell, Khenouf and Canudas de Wit, Maschke and van der Shaft, and others. The role of differential flatness in these systems is also interesting; see the work of Sira Ramirez and of Fliess *et al.*

In this paper we present a new discontinuous feedback law and an algorithm for its implementation. We also give a rigorous analysis of the convergence of the algorithm. This completely solves the stabilization problem for (1.8)-(1.9), which is more general than the systems considered by others. We stress, however, that our results are new even for the well-known $so(n)$ system (1.4)-(1.5). Our present results are related to our earlier work on the Heisenberg system (1.1)-(1.3) (Bloch and Drakunov [1994], [1995], [1996]), the $so(n)$ system (1.4)-(1.5) (Bloch and Drakunov [1998]), and the general system (1.8)-(1.9) (Bloch, Drakunov and Kinyon [1997]). However, this paper is not so much an extension of our earlier work as it is a completely new approach to the stabilization problem. In addition we provide insights into the natural geometric structure of the problem.

Stabilizing (1.8)-(1.9), even locally, is a nontrivial task, since, as can be easily seen, linearization in the vicinity of the origin $0 \in \mathfrak{g}$ gives the noncontrollable system

$$\begin{aligned} \dot{x} &= u \\ \dot{Y} &= 0. \end{aligned}$$

The main difficulty with (1.8)-(1.9) is the fact that stabilization of x leads to the right hand side of (1.9) being 0. Therefore Y cannot be directly steered to zero when $x = 0$. This simple observation implies that to stabilize the system one needs to make Y converge "faster" than x. The feedback law and algorithm we present in §4 and §5 satisfy this criterion. The idea is as follows: we switch back and forth between flows which decrease one of the variables x and Y in norm while at the same time leaving the other variable as "unchanged" as possible. One of the most remarkable features of our feedback law is what "unchanged" turns out to mean, for not only does the nondecreasing variable remain constant in norm, it turns out to evolve *isospectrally*. For matrix Lie algebras, the meaning of this is clear; for general Lie algebras, the adjoint representation of the variable in question evolves with constant spectrum.

In the case of the $so(n)$ system, we have an additional feature. As will be seen in §4 and §6, in those parts of the algorithm where the skew symmetric matrix Y is decreasing in norm (as x evolves isospectrally), all of the eigenvalues of Y^TY except for the largest one remain constant. The results in §4 suggest that this *partial isospectrality* of Y probably generalizes to other Lie algebras.

We conclude this introduction with an outline of the sequel. In §2, we consider the Heisenberg system (1.1)-(1.3). We introduce our control law for this special case, and we show that it stabilizes the system. As might be expected, the feedback is a prototype for the more general ones to follow.

In §3 we introduce the general Lie algebraic setting. We give the specific hypotheses on the Lie algebra $\mathfrak{g}$, and we show that in a certain sense, these hypotheses cannot be weakened in order to have a stabilization problem that can be solved. We prove certain useful operator identities in $\mathfrak{g}$, and we also verify two crucial inequalities which play a role in the proof of the convergence of the stabilization algorithm. The proof of one of the inequalities, (3.12), is rather delicate and relies on the structure of the Lie algebra $\mathfrak{g}$ in an essential way. This turns out to be somewhat obscured in the setting of specific Lie algebras, and provides another justification for considering the general system (1.8)-(1.9) instead of just, say, the $so(n)$ system.

In §4 we present our discontinuous controls, and we analyze the various cases between which we will be switching in our algorithm. As mentioned above, we show that in each of the main cases, one variable decreases in norm while the other evolves isospectrally. We also verify an equation, (4.22), which, in $so(n)$, is the key to the previously mentioned partial isospectrality of Y in the case when it is the norm-decreasing variable.

In §5 we present the stabilization algorithm, and we give a rigorous proof of its convergence. Again, the estimates are rather delicate, and are most easily seen in the general setting, not in the setting of specific Lie algebras. It is here, as well as in §3, where it will be shown that the generalization to the Lie algebraic setting is not as straightforward as it seems.

Finally, in §6, we apply our results to the $so(n)$ system. We show that stabilization will be achieved in, at most, $\lfloor \frac{n}{2} \rfloor$ iterations of our algorithm. We show some numerics in $so(3)$ to illustrate our results.

2. Stabilization of the Heisenberg System. To illustrate part of our full algorithm for stabilization of the system (1.8)-(1.9), we discuss it in the context of the Heisenberg system or nonholonomic integrator (1.1)-(1.3), which we repeat here for convenience:

$$\dot{x} = u \tag{2.1}$$

$$\dot{y} = v \tag{2.2}$$

$$\dot{z} = xv - yu. \tag{2.3}$$

The usefulness of stabilization for this system may be illustrated by an application to the kinematics of a knife edge in point contact with a plane surface or the motion of a rolling wheel – the simplest form of "mobile robot" (see for example Bloch and Drakunov [1995], Bloch *et al* [1992] and Murray and Sastry [1993]). One may transform these systems to one of Heisenberg type.

Our model for the control of (2.1)-(2.3) is the following (slightly incomplete) control law

$$u = -\alpha x + \beta\, z\, y \tag{2.4}$$

$$v = -\alpha y - \beta\, z\, x, \tag{2.5}$$

where α, and β are positive constants. With this choice, the system (2.1)-(2.3) becomes

$$\dot{x} = -\alpha x + \beta z y \tag{2.6}$$

$$\dot{y} = -\alpha y - \beta z x \tag{2.7}$$

$$\dot{z} = -\beta z(x^2 + y^2). \tag{2.8}$$

Let $V = x^2 + y^2$. Then

$$\dot{V} = 2(x\dot{x} + y\dot{y}) = 2(\beta xzy - \beta yzx - \alpha x^2 - \alpha y^2) = -2\alpha V. \tag{2.9}$$

There are a number of strategies for choosing α and β to stabilize the system. It is clear from (2.8) that if we initially choose $\alpha = 0$ and $\beta > 0$, then for x or y not equal to zero, z will be driven asymptotically to 0, while (2.9) shows that V will remain fixed. On the other hand for $\alpha > 0$ and $\beta = 0$, V will be driven to 0. All stabilization strategies to be discussed later are generalizations of this simple observation.

Note also from (2.8) the following: for $\alpha = 0$ the rate of convergence for the variable z depends on the inital conditions for V. The greater the value of $V|_{t=0}$, the faster will be the convergence. On the other hand, if $V|_{t=0} = 0$, i.e. if the initial state belongs to the z-axis, then the variable z will not converge at all. In order to initialize motion in this case, one may apply any nonzero control for a short period of time. It may be a constant nonzero vector or any other suitable control. One possibility is to use

$$\begin{aligned} u &= -\alpha(x - x_1) \\ v &= -\alpha(y - y_1) \end{aligned}$$

with some x_1, y_1 such that $x_1^2 + y_1^2 \neq 0$. With this control the state will leave the z-axis. Then the control (2.4), (2.5) can be used with $\alpha = 0$, $\beta > 0$. When the state reaches an ϵ-neighborhood of $z = 0$ (observe that the derivative of z in this mode is just a constant times $-z$) we can switch to $\alpha > 0$, $\beta = 0$, thus stabilizing the state to the origin.

Let us note here that because of the switchings, the above strategy assumes that the control input is a discontinuous function of the state variables. Thus the existence of the corresponding solution of the differential equations (off of the z-axis only) should be understood in the sense of the Filippov definition (see Filippov [1988]). In such systems sliding mode behavior (motion along a discontinuity set) is possible which can be used to stabilize the system (see DeCarlo et al [1996], and a generalization of the sliding mode concept in Drakunov and Utkin [1992]). In Bloch and Drakunov ([1994], [1995], [1998]) several methods are given for achieving stability of (2.1)-(2.3) using sliding mode theory.

Similar considerations apply in the general case discussed here, although we will not discuss explicitly sliding mode behavior, only the discrete switching pattern needed to obtain stability.

3. The General Setting. We now consider the general situation. Let $\mathfrak{g}$ be a real semisimple Lie algebra, and let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the Killing form of $\mathfrak{g}$. Assume that $\mathfrak{g}$ has a direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathfrak{h}$ is a compactly imbedded subalgebra and the subspace $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ relative to B. (In the terminology of Helgason [1978], $\mathfrak{g}$ is a semisimple orthogonal symmetric Lie algebra. See this reference, especially Chapter V, for subsequent assertions about Lie algebras.) Under these assumptions, the commutation relations $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$ hold, and the restriction $B|_{\mathfrak{h}\times\mathfrak{h}}$ of the Killing form B to $\mathfrak{h} \times \mathfrak{h}$ is negative definite. In addition, assume that no ideal of $\mathfrak{g}$ is contained in $\mathfrak{h}$. Since the Jacobi identity together with the commutation relations for $\mathfrak{h}$ and $\mathfrak{m}$ imply that $[\mathfrak{m}, \mathfrak{m}]$ is an ideal of $\mathfrak{g}$, it follows that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$. In particular, the representation of $\mathfrak{h}$ on $\mathfrak{m}$ is faithful (because the kernel of this representation is B-orthogonal in $\mathfrak{h}$ to $[\mathfrak{m}, \mathfrak{m}]$).

We will consider stabilization of the system (1.8)-(1.9) in $\mathfrak{g}$, which we repeat here for convenience:

$$\dot{x} = u \tag{3.1}$$
$$\dot{Y} = [u, x] \tag{3.2}$$

where $x, u \in \mathfrak{m}$, $Y \in \mathfrak{h}$. We first show that this system can be analyzed without loss in a more specialized type of Lie algebra.

Under the given hypotheses on $\mathfrak{g}$, there exist B-orthogonal ideals $\mathfrak{g}_+$ and $\mathfrak{g}_-$ with the following properties: (i) $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ (direct sum), (ii) $\mathfrak{h}_\pm = \mathfrak{g}_\pm \cap \mathfrak{h}$ is compactly imbedded in $\mathfrak{g}_\pm$ and contains no ideal of $\mathfrak{g}_\pm$, and (iii) $\mathfrak{g}_+ = \mathfrak{h}_+ \oplus \mathfrak{m}_+$ is of noncompact type and $\mathfrak{g}_- = \mathfrak{h}_- \oplus \mathfrak{m}_-$ is of compact type, where $\mathfrak{m}_\pm = \mathfrak{g}_\pm \cap \mathfrak{m}$. If we let $x = x_+ + x_-$, $u = u_+ + u_-$ and $Y = Y_+ + Y_-$ denote the corresponding decompositions, then the system (3.1)-(3.2) decomposes into the systems

$$\dot{x}_\pm = u_\pm \tag{3.3}$$
$$\dot{Y}_\pm = [u_\pm, x_\pm] \tag{3.4}$$

It follows that to stabilize the system (3.1)-(3.2) in $\mathfrak{g}$, it is enough to stabilize simultaneously the systems (3.3)-(3.4).

We thus assume from now on without loss that $\mathfrak{g}$ is either of noncompact type or of compact type. This implies that the restriction $B|_{\mathfrak{m}\times\mathfrak{m}}$ of B to $\mathfrak{m}$ is positive definite if $\mathfrak{g}$ is of noncompact type, and negative definite if $\mathfrak{g}$ is of compact type.

We should stress here that all simple Lie algebras satisfy our assumptions when $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a Cartan decomposition. Thus the results we obtain in this paper for our general class of Lie algebras apply to many important cases. It is also worth noting that we would gain no generality by weakening our assumptions that $\mathfrak{g}$ is semisimple and that $\mathfrak{h}$ contains no ideal of $\mathfrak{g}$. For example, if we were to assume merely that $\mathfrak{g}$ is effective, that is, that $\mathfrak{h}$ does not contain any elements of the center of $\mathfrak{g}$, then it would follow that $\mathfrak{g}$ would have an additional ideal $\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{m}_0$ of Euclidean type (Helgason [1978]). But in such an ideal, $[u_0, x_0] = 0$, which would imply that Y_0 is constant. Thus stabilization would not be possible. This would still be the case even if we were to keep the semisimplicity, but not the assumption that $\mathfrak{h}$ contains no ideal of $\mathfrak{g}$.

In order to discuss the compact and noncompact cases simultaneously, let

$$\epsilon = \begin{cases} 1 & \text{if } \mathfrak{g} \text{ is of noncompact type} \\ -1 & \text{if } \mathfrak{g} \text{ is of compact type.} \end{cases} \tag{3.5}$$

We will use the inner product on $\mathfrak{g}$ defined by the Killing form:

$$\langle x_1 + Y_1, x_2 + Y_2 \rangle \equiv \epsilon B(x_1, x_2) - B(Y_1, Y_2) \tag{3.6}$$

for $x_1, x_2 \in \mathfrak{m}$, $Y_1, Y_2 \in \mathfrak{h}$. The corresponding norm will be denoted by $\|\cdot\|$.

Now for $x \in \mathfrak{m}$, let

$$M(x) = \epsilon(\text{ad } x)^2|_{\mathfrak{h}}. \tag{3.7}$$

Then $M(x)$ is a nonnegative symmetric operator on $\mathfrak{h}$ relative to the inner product. To see this, note that ad x is B-symmetric if $\mathfrak{g}$ is noncompact and B-skew-symmetric if $\mathfrak{g}$ is compact. In either case, $(\text{ad } x)^2$ is B-symmetric, and thus so is $M(x)$. Note that $(\text{ad } x)^2$ is nonnegative if $\mathfrak{g}$ is noncompact, and nonpositive if $\mathfrak{g}$ is compact, and thus introducing ϵ into the definition of $M(x)$ guarantees that it is a nonnegative operator.

Next, for $Y \in \mathfrak{h}$, let

$$N(Y) = -(\text{ad } Y)^2|_{\mathfrak{m}}. \tag{3.8}$$

Then $N(Y)$ is a nonnegative symmetric operator on $\mathfrak{m}$ relative to the inner product. This follows from the fact that ad Y is B-skew-symmetric.

We will make frequent use of two identities relating the operators $M(x)$ and $N(Y)$. First, the Jacobi identity implies

$$[Y, M(x)Y] = \epsilon[x, N(Y)x] \tag{3.9}$$

for all $x \in \mathfrak{m}$, $Y \in \mathfrak{h}$. Second, the invariance of the Killing form implies

$$\langle Y, M(x)Y \rangle = \|[Y, x]\|^2 = \langle x, N(Y)x \rangle \tag{3.10}$$

for all $x \in \mathfrak{m}$, $Y \in \mathfrak{h}$.

We will also require two estimates arising from $M(x)$ and $N(Y)$. First, since $\mathfrak{m}$ and $\mathfrak{h}$ are each invariant under $(\text{ad } x)^2$ for every $x \in \mathfrak{m}$, we have

$$\begin{aligned} \|x\|^2 &= \epsilon \text{tr}((\text{ad } x)^2) = \epsilon \text{tr}((\text{ad } x)^2|_{\mathfrak{m}}) + \epsilon \text{tr}((\text{ad } x)^2|_{\mathfrak{h}}) \\ &= \epsilon \text{tr}((\text{ad } x)^2|_{\mathfrak{m}}) + \text{tr}(M(x)). \end{aligned}$$

This implies the following inequality:

$$\text{tr}(M(x)) \leq \|x\|^2 \tag{3.11}$$

for all $x \in \mathfrak{m}$.

Our second estimate is the following: there exists a constant $0 < \eta < 1$ such that

$$\text{tr}(N(Y)) > \eta \|Y\|^2 \tag{3.12}$$

for all $Y \in \mathfrak{h}$. This fact, which will be crucial for our discussion of the convergence of our algorithm, is probably part of Lie algebra folklore, but the authors are unaware of any reference. We conclude this section with a proof of this result. Since the discussion that follows is somewhat far afield from our stabilization problem, the reader who is more interested in the control-theoretic aspects might wish to skip ahead to §4.

In addition to the restriction of the Killing form B, the Lie algebra $\mathfrak{h}$ has two other natural invariant forms:

$$B_{\mathfrak{h}}(Y_1, Y_2) = \text{tr}((\text{ad } Y_1)(\text{ad } Y_2)|_{\mathfrak{h}}) \tag{3.13}$$

$$B_{\mathfrak{m}}(Y_1, Y_2) = \text{tr}((\text{ad } Y_1)(\text{ad } Y_2)|_{\mathfrak{m}}), \tag{3.14}$$

for $Y_1, Y_2 \in \mathfrak{h}$. The form $B_{\mathfrak{h}}$ is just the Killing form of $\mathfrak{h}$ itself, while the form $B_{\mathfrak{m}}$ is the natural trace form associated to the representation of $\mathfrak{h}$ on $\mathfrak{m}$. All three invariant forms are related by

$$B(Y_1, Y_2) = B_{\mathfrak{h}}(Y_1, Y_2) + B_{\mathfrak{m}}(Y_1, Y_2) \tag{3.15}$$

for $Y_1, Y_2 \in \mathfrak{h}$. Also notice that

$$\mathrm{tr}(N(Y)) = -B_{\mathfrak{m}}(Y, Y) \tag{3.16}$$

for all $Y \in \mathfrak{h}$.

We have already noted that the restriction of B to $\mathfrak{h}$ is negative definite since $\mathfrak{h}$ is compactly imbedded in $\mathfrak{g}$. In addition, $B_{\mathfrak{m}}$ is negative definite. Indeed, suppose $B_{\mathfrak{m}}(Z, Z) = 0$ for some $Z \in \mathfrak{h}$. Then the nonpositive symmetric operator $(\mathrm{ad}\ Z)^2|_{\mathfrak{m}}$ must be the zero operator (because all of its nonpositive eigenvalues must be 0). Now for all $x \in \mathfrak{m}$, we have $0 = \langle [Z, [Z, x]], x\rangle = -\langle [Z, x], [Z, x]\rangle$, and thus $[Z, x] = 0$. Since $\mathfrak{h}$ acts faithfully on $\mathfrak{m}$, $Z = 0$.

Letting $\mathfrak{z}(\mathfrak{h})$ denote the center of $\mathfrak{h}$, we have

$$\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) + [\mathfrak{h}, \mathfrak{h}], \tag{3.17}$$

a direct sum, which is orthogonal relative to $\langle \cdot, \cdot \rangle$ (that is, relative to B). Indeed, if Z is orthogonal to $[\mathfrak{h}, \mathfrak{h}]$ relative to $\langle \cdot, \cdot \rangle$, then $0 = \langle Z, [Y_1, Y_2]\rangle = -\langle [Y_1, Z], Y_2\rangle$ for all $Y_1, Y_2 \in \mathfrak{h}$. Thus $[Y_1, Z] = 0$ for all $Y_1 \in \mathfrak{h}$, i.e., $Z \in \mathfrak{z}(\mathfrak{h})$. By a similar computation, the subspaces $\mathfrak{z}(\mathfrak{h})$ and $[\mathfrak{h}, \mathfrak{h}]$ are also orthogonal relative to $B_{\mathfrak{h}}$, and it therefore follows from (3.15) that they are orthogonal relative to $B_{\mathfrak{m}}$.

The restriction of $B_{\mathfrak{h}}$ to $[\mathfrak{h}, \mathfrak{h}]$ is negative definite. Indeed, if $B_{\mathfrak{h}}(Z, Z) = 0$, then the nonpositive symmetric operator $(\mathrm{ad}\ Z)^2|_{\mathfrak{h}}$ must be the zero operator. Now for all $Y \in \mathfrak{h}$, we have $0 = \langle [Z, [Z, Y]], Y\rangle = -\langle [Z, Y], [Z, Y]\rangle$, and thus $[Z, Y] = 0$. Therefore $Z \in \mathfrak{z}(\mathfrak{h})$. Since the restriction of $B_{\mathfrak{h}}$ to $[\mathfrak{h}, \mathfrak{h}]$ is just the Killing form of $[\mathfrak{h}, \mathfrak{h}]$ itself (because $[\mathfrak{h}, \mathfrak{h}]$ is an ideal of $\mathfrak{h}$), it follows that $[\mathfrak{h}, \mathfrak{h}]$ is compact semisimple.

Now since the restrictions of both B and $B_{\mathfrak{m}}$ to $[\mathfrak{h}, \mathfrak{h}]$ are negative definite and invariant, it follows that each is a positive multiple of $B_{\mathfrak{h}}$. Thus there exist $c_1, c_2 > 0$ such that

$$B_{\mathfrak{m}}(Y_1, Y_2) = c_1 B_{\mathfrak{h}}(Y_1, Y_2) \quad \text{and} \quad B(Y_1, Y_2) = c_2 B_{\mathfrak{h}}(Y_1, Y_2)$$

for all $Y_1, Y_2 \in [\mathfrak{h}, \mathfrak{h}]$. The relation (3.15) implies $c_2 = 1 + c_1$. Thus let

$$\eta = \frac{c_1}{c_2} = \frac{c_1}{1 + c_1}. \tag{3.18}$$

Then we have the relation

$$B_{\mathfrak{m}}(Y_1, Y_2) = \eta B(Y_1, Y_2) \tag{3.19}$$

for all $Y_1, Y_2 \in [\mathfrak{h}, \mathfrak{h}]$.

For $Y, Z \in \mathfrak{h}$, write $Y = Y_c + Y_s$ and $Z = Z_c + Z_s$ for the decompositions into $\mathfrak{z}(\mathfrak{h})$ and $[\mathfrak{h}, \mathfrak{h}]$ ("c" = center, "s" = semisimple). Using the orthogonality relative to all three forms as well as (3.15) and (3.19), we compute

$$\begin{aligned} B_{\mathfrak{m}}(Y, Z) &= B_{\mathfrak{m}}(Y_c, Z_c) + B_{\mathfrak{m}}(Y_s, Z_s) \\ &= B(Y_c, Z_c) - B_{\mathfrak{h}}(Y_c, Z_c) + \eta B(Y_s, Z_s) \\ &= B(Y_c, Z_c) + \eta B(Y_s, Z_s), \end{aligned} \tag{3.20}$$

where we have used the vanishing of $B_{\mathfrak{h}}$ on $\mathfrak{z}(\mathfrak{h})$ to obtain (3.20). Now since $0 < \eta < 1$ and B is *negative* definite, (3.20) and the orthogonality yield

$$\begin{aligned} B_{\mathfrak{m}}(Y,Z) &< \eta B(Y_c, Z_c) + \eta B(Y_s, Z_s) \\ &= \eta B(Y,Z). \end{aligned} \tag{3.21}$$

If we specialize (3.21) to the case $Y = Z$, we obtain

$$B_{\mathfrak{m}}(Y,Y) < \eta B(Y,Y).$$

Multiplying through by -1 (and noting (3.16)) gives (3.12) for all $Y \in \mathfrak{h}$.

4. Controls. We now present our controls for the system (3.1)-(3.2). These are given by

$$\begin{aligned} u &= -\alpha x + \beta[Y,x] - \gamma[Y,[Y,x]] \\ &= -\alpha x + \beta[Y,x] + \gamma N(Y)x, \end{aligned} \tag{4.1}$$

where $\alpha, \beta, \gamma : \mathfrak{g} \to \mathbb{R}$ are functions. We will assume that $\alpha, \gamma \geq 0$ and $\beta\epsilon \leq 0$.

With (4.1) as our choice of u (and using (3.9)), the system (3.1)-(3.2) becomes

$$\dot{x} = -\alpha x + \beta[Y,x] + \gamma N(Y)x \tag{4.2}$$

$$\begin{aligned} \dot{Y} &= \beta\epsilon M(x)Y + \gamma[N(Y)x, x]. \\ &= \beta\epsilon M(x)Y - \gamma\epsilon[Y, M(x)Y] \end{aligned} \tag{4.3}$$

Using (4.2), we compute

$$\begin{aligned} \frac{d}{dt}\|x\|^2 &= 2\langle x, \dot{x}\rangle \\ &= -2\alpha\langle x, x\rangle + 2\beta\langle x, [Y,x]\rangle + 2\gamma\langle x, N(Y)x\rangle \\ &= -2\alpha\|x\|^2 + 2\gamma\langle x, N(Y)x\rangle. \end{aligned} \tag{4.4}$$

Here the last equality of (4.4) follows because ad Y is B-skew-symmetric. Now let λ_* denote the largest eigenvalue of the symmetric operator $N(Y)$. Then $\langle x, N(Y)x\rangle \leq \lambda_*\|x\|^2$ for all $x \in \mathfrak{m}$ and thus the right hand side of (4.4) is nonpositive if $\lambda_*\gamma \geq \alpha$. In this case $\|x\|$ is nonincreasing, and we also have that $\|x\|$ is constant if $\alpha = \gamma = 0$.

Using (4.3), we compute

$$\begin{aligned} \frac{d}{dt}\|Y\|^2 &= 2\langle Y, \dot{Y}\rangle \\ &= 2\beta\epsilon\langle Y, M(x)Y\rangle - 2\gamma\epsilon\langle Y, [Y, M(x)Y]\rangle \\ &= 2\beta\epsilon\langle Y, M(x)Y\rangle. \end{aligned} \tag{4.5}$$

Since $\beta\epsilon \leq 0$ and $M(x)$ is a nonnegative operator, it follows that the right hand side of (4.5) is nonpositive. Thus $\|Y\|$ is nonincreasing in general, and is constant if $\beta = 0$.

Our stabilization algorithm will be (necessarily) discontinuous, and will require switching of the control (4.1) between the following three cases: (i) $\alpha > 0$, $\beta = \gamma = 0$; (ii) $\alpha = \kappa\lambda_*$, $\gamma = \kappa$ and $\beta = 0$, where, as above, λ_* is the largest eigenvalue of $N(Y)$ and where κ is a positive function; (iii) $\alpha = \gamma = 0$, $\beta\epsilon < 0$. We now discuss the dynamical behavior of the system (4.2)-(4.3) in each of these cases.

Case I. $\alpha > 0$, $\beta = \gamma = 0$:

In this case, the system (4.2)-(4.3) is

$$\dot{x} = -\alpha x \tag{4.6}$$
$$\dot{Y} = 0 \tag{4.7}$$

The dynamics here are quite clear: x is driven to 0 radially while Y remains fixed. The remarks made in §2 regarding the Heisenberg system hold here as well; if Y was not already 0 in the first place, implementing the control (4.1) with these parameter values will render the system unstabilizable. Thus this case will only be used if $Y \equiv 0$.

Case II. $\alpha = \kappa\lambda_*$, $\gamma = \kappa$, $\beta = 0$:

As noted above, $\kappa > 0$. In this case, the control (4.1) has the form

$$u = -\kappa\, (\lambda_* x - N(Y)x), \tag{4.8}$$

while the system (4.2)-(4.3) is

$$\dot{x} = -\kappa(\lambda_* x - N(Y)x) \tag{4.9}$$
$$\dot{Y} = -\kappa\epsilon[Y, M(x)Y] \tag{4.10}$$

In this case, $\|Y\|$ is constant. In addition, (4.10) is a Lax equation in Y. It follows that the spectrum of ad Y is constant. Therefore the spectrum of the operator $N(Y)$ is constant, as are the dimensions of its eigenspaces. In particular, the eigenvalue λ_*, which occurs in (4.9), is constant.

Let $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_s = \lambda_*$ denote those eigenvalues of $N(Y)$ which are distinct (thus $s \leq \dim \mathfrak{m} - 1$). Let $x = x_0 + x_1 + \cdots + x_s$ be the unique decomposition of x into the eigenspaces of $N(Y)$. Then the differential equation (4.9) decouples into the following system of equations in $\mathfrak{m}$:

$$\begin{array}{rcl} \dot{x}_0 & = & -\kappa(\lambda_* - \lambda_0)x_0 \\ \dot{x}_1 & = & -\kappa(\lambda_* - \lambda_1)x_1 \\ & \vdots & \\ \dot{x}_{s-1} & = & -\kappa(\lambda_* - \lambda_{s-1})x_{s-1} \\ \dot{x}_s & = & 0. \end{array} \tag{4.11}$$

Since $\kappa(\lambda_* - \lambda_j) > 0$ for $j = 0, 1, \ldots, s-1$, it follows that $x_j \to 0$ asymptotically. If we let x_* denote the projection of x onto the λ_*-eigenspace of $N(Y)$, that is, $x_* = x_s$, then noting that $x_* \equiv x_*|_{t=0}$ is constant, we conclude that

$$x \to x_*$$

asymptotically.

Note that (4.9)-(4.10) and (3.9) imply the following:

$$\dot{Y} = -\kappa[x, N(Y)x] = [x, \dot{x}]. \tag{4.12}$$

Since x is converging to a λ_*-eigenvector of $N(Y)$, the right hand side of (4.9) is converging to 0 and thus $\dot{x}$ is converging to 0. Therefore (4.12) implies that $\dot{Y}$ is converging to 0.

Summarizing this case, we have that Y evolves isospectrally with constant norm and asymptotically vanishing velocity, while x is driven to x_*, its (constant) projection onto the λ_*-eigenspace of $N(Y)$.

Case III. $\alpha = \gamma = 0$, $\beta\epsilon < 0$:

The system (4.2)-(4.3) for this case is

$$\dot{x} = \beta[Y, x] \tag{4.13}$$
$$\dot{Y} = \beta\epsilon M(x)Y. \tag{4.14}$$

In this case, $\|x\|$ is constant. In addition, (4.13) is a Lax equation in x, and thus ad x has constant spectrum. Therefore the spectrum of the operator $M(x)$ is constant, as are the dimensions of its eigenspaces. Let $0 \le \mu_0 < \mu_1 < \cdots < \mu_r$ denote those eigenvalues of $M(x)$ which are distinct (thus $r \le \dim \mathfrak{h} - 1$). For $Y \in \mathfrak{h}$, let $Y = Y_0 + \cdots + Y_r$ denote the unique decomposition of Y into the eigenspaces of $M(x)$. Then the differential equation (4.14) decouples into the following system of equations in $\mathfrak{h}$:

$$\begin{array}{rcl} \dot{Y}_0 & = & \beta\epsilon\mu_0 Y_0 \\ \dot{Y}_1 & = & \beta\epsilon\mu_1 Y_1 \\ & \vdots & \\ \dot{Y}_r & = & \beta\epsilon\mu_r Y_r \end{array} \tag{4.15}$$

Since $\beta\epsilon\mu_j < 0$ for $j = 1, \ldots, r$, we have that $Y_j \to 0$ asymptotically. If $\mu_0 > 0$, then the same applies to Y_0. Otherwise, if $M(x)$ has $\mu_0 = 0$ as an eigenvalue, then Y_0 remains constant. Thus we have either $Y \to 0$ or $Y \to Y_0$ asymptotically, where $Y_0 \equiv Y_0|_{t=0}$ is constant. In either case, if we let $Y_\#$ denote the projection of Y onto the nullspace of $M(x)$, then noting that $Y_\# \equiv Y_\#|_{t=0}$ is constant, we conclude that

$$Y \to Y_\#$$

asymptotically.

Using system (4.13)-(4.14), we can derive the following equation:

$$\frac{d}{dt} M(x)^n Y = \beta\epsilon[Y, M(x)^n Y] + \beta\epsilon M(x)^{n+1} Y \tag{4.16}$$

for every nonnegative integer n. Indeed, the case $n = 0$ is just (4.14). Using the induction hypothesis, we have for $n > 0$,

$$\begin{array}{rcl} \frac{d}{dt} M(x)^n Y & = & \epsilon[\dot{x}, [x, M(x)^{n-1}Y]] + \epsilon[x, [\dot{x}, M(x)^{n-1}Y]] \\ & & +M(x)(\beta\epsilon[Y, M(x)^{n-1}Y] + \beta\epsilon M(x)^n Y). \end{array} \tag{4.17}$$

Now

$$[\dot{x}, [x, M(x)^{n-1}Y]] = \beta[[Y, x], [x, M(x)^{n-1}Y]] \tag{4.18}$$

and

$$[x, [\dot{x}, M(x)^{n-1}Y]] = \beta[x, [[Y, x], M(x)^{n-1}Y]], \tag{4.19}$$

while applying the Jacobi identity repeatedly gives

$$\begin{array}{rcl} M(x)[Y, M(x)^{n-1}Y] & = & [Y, M(x)^n Y] + \epsilon[[x, Y], [x, M(x)^{n-1}Y]] \\ & & +\epsilon[x, [[x, Y], M(x)^{n-1}Y]]. \end{array} \tag{4.20}$$

Substituting (4.18), (4.19) and (4.20) into (4.17) and simplifying gives (4.16).

Then from (4.16) immediately follows

$$\frac{d}{dt}f(M(x))Y = \beta\epsilon[Y, f(M(x))Y] + \beta\epsilon f(M(x))M(x)Y \tag{4.21}$$

for every real analytic function f. As an interesting special case of this, let $p(\mu)$ be the minimal polynomial of $M(x)$ and assume that $\mu_0 = 0$ is an eigenvalue of $M(x)$ (so that Y does not converge to 0). Then $p(\mu) = \mu q(\mu)$ for some polynomial q. Taking $f = q$ in (4.21) gives

$$\frac{d}{dt}q(M(x))Y = \beta\epsilon[Y, q(M(x))Y]. \tag{4.22}$$

This is a Lax equation, and it follows that the spectrum of $q(M(x))Y$ remains constant. We will see in §6 what we suggested in the Introduction, namely that in the $so(n)$ case, the constancy of the spectrum of $q(M(x))Y$ implies that Y itself evolves partially isospectrally.

Summarizing this case, we have that x evolves isospectrally with constant norm, Y is driven to $Y_{\#}$, its (constant) projection onto the nullspace of $M(x)$, and $q(M(x))Y$ evolves with constant spectrum.

As an aside, it is interesting to compare the system of equations in Case III with the double bracket equations discussed in Brockett [1988] and Bloch, Brockett and Ratiu [1992], for example. In these papers the isospectral flow $\dot{L} = [L, [L, N]]$ was considered, where L, N live in a compact Lie algebra and N is fixed. This flow is the gradient flow of $\langle L, N\rangle$ on an adjoint orbit of the corresponding Lie group with respect to the so-called normal metric. The second equation of system (4.13)-(4.14) is, on the other hand, of the form $\dot{Y} = \beta\epsilon[X, [X, Y]]$. This equation is *not* isospectral (although it is coupled to the isospectral equation (4.13)), and we have a different function, $\langle Y, Y\rangle$, decreasing along its flow. This, of course, is precisely what we want in the present context. We note also that in this system there is an interesting coupling between two equations involving brackets. This is reminiscent of Bloch and Crouch [1996], although there the coupling is between two Lax equations and the overall flow is Hamiltonian.

5. The Stabilization Algorithm. We now describe our feedback strategy. As before, λ_* denotes the largest eigenvalue of the operator $N(Y)$, x_* denotes the projection of x onto the λ_*-eigenspace of $N(Y)$, and $Y_{\#}$ denotes the projection of Y onto the nullspace of $M(x)$. Let $\delta > 0$ be a prescribed error tolerance. In an informal pseudocode, our algorithm can be described as follows.

begin

while $\|Y\| \geq \delta$

1. Let $r = \|x\|$. Implement the control (4.1) with $\alpha = \lambda_*\kappa$, $\gamma = \kappa$ and $\beta = 0$. Then Y evolves isospectrally with constant norm, while x converges to the constant x_*. If $x_* \neq 0$, then go to Step 3.
2. Let z_* denote a fixed λ_*-eigenvector of $N(Y)$ with $\|z_*\| = r\,(1 - 1/\dim\,\mathfrak{m})^{1/2}$. Let $u = -\alpha(x - z_*)$, where $\alpha > 0$. Then x converges to z_* while Y remains constant.
3. Implement the control (4.1) with $\alpha = \gamma = 0$, $\beta\epsilon < 0$. Then x evolves isospectrally with constant norm, while Y converges to the constant $Y_{\#}$.

end while

if $\|x\| \geq \delta$, **then**

4. implement the control (4.1) with $\alpha > 0$, $\beta = \gamma = 0$. Then x will converge to 0 radially, while Y remains 0.

end

In Step 1, if α is a constant, then x will converge to x_* in infinite time; if, for example, $\alpha = 1/\|x - x_*\|$, then x will converge in finite time. Similarly, in Step 3, if β is a constant, then Y will converge to $Y_\#$ in infinite time; if, for example, $\beta = 1/\|Y - Y_\#\|$, then Y will converge in finite time. To establish the convergence claim made in Step 2, we simply note that in this case $x(t)$ has the form $x(t) = f(t)z_*$ where $f(t)$ is a scalar-valued function satisfying $\dot{f} = -\alpha(f - 1)$, $f(0) = 0$. (For instance, if $\alpha > 0$ is constant, we have $f(t) = 1 - e^{-\alpha t}$.) It follows from (3.2) that $\dot{Y} = [u, x] = 0$, so that Y is constant as claimed.

Step 2 is implemented if x converges to 0 in Step 1. One instance where this could happen is if the initial value of x is 0, in which case the first implementation of Step 1 is trivial. More generally, the case where the projection of x onto the λ_*-eigenspace of $N(Y)$ is 0 seems to be the natural higher dimensional analog of the situation in the Heisenberg system (2.1)-(2.3) where the initial value starts on the z-axis. As in Steps 1 and 3, Step 2 can also be implemented in finite time.

The λ_*-eigenspace of $N(Y)$ will, in general, have dimension greater than 1 (since the nonzero eigenvalues of the B-skew-symmetric operator ad Y come in complex conjugate pairs). Thus there is no unique choice of eigenvector z_* in Step 2. Any lexicographic ordering of the eigenvectors relative to a coordinate basis will suffice as a selection scheme. The rationale behind the particular normalization of z_* will be explained below. (Choices of this type occur naturally in stabilizing nonholonomic systems; see Sontag [1998] for comments on this and related robustness issues.)

We will now show that our algorithm successfully stabilizes the system (3.1)-(3.2) by showing that each of $\|x\|$ and $\|Y\|$ can be brought to within the prescribed error tolerance. Note that as soon as the test condition of the while loop fails, that is, as soon as $\|Y\| < \delta$, then the system will be stabilized whether Step 5 needs to be executed or not. Thus we may assume that the initial value of Y satisfies $\|Y\| \geq \delta$ so that the while loop will be executed at least once. If Y ever converges to 0 in Step 3 because $Y_\# = 0$, then the test condition of the while loop will eventually fail. As noted, this is enough to guarantee that the system is stabilizable.

Assume that for every iteration of Step 3, we have $Y_\# \neq 0$. We will show that after finitely many iterations of the while loop, the test condition will fail. In other words, the projection of Y onto the nullspace of $M(x)$ is eventually arbitrarily small in norm. In fact, we will show a stronger result, for when this situation occurs, then it turns out that $\|x\|$ is simultaneously brought to within the error tolerance. Thus as soon as the while loop's test condition fails, the test condition of the if-then statement (Step 5) will also fail, and the system will already be stabilized.

Assume first that Step 3 is about to be executed. Since Step 1 and possibly Step 2 have already been executed, the initial values $x(0) = x_*$ and $Y(0) = Y_*$ satisfy $N(Y_*)x_* = \lambda_* x_*$. As before, let Y_j denote the projection of Y onto the μ_j-eigenspace of $M(x)$. Recall that $Y_\# = Y_0 \equiv Y_0(0)$ throughout Step 3, and that $Y(t) \to Y_\#$

asymptotically. Using the orthogonality of the eigenspaces, we compute

$$\begin{aligned} \|Y_\#\|^2 &= \|Y_*\|^2 - \textstyle\sum_{j=1}^r \|Y_j(0)\|^2 \\ &\le \|Y_*\|^2 - \frac{1}{\sum_{j=0}^r \mu_j} \textstyle\sum_{j=0}^r \mu_j \|Y_j(0)\|^2. \end{aligned} \tag{5.1}$$

Note that we are using $\mu_0 = 0$. Now using the orthogonality once again, we compute

$$\sum_{j=0}^r \mu_j \|Y_j(0)\|^2 = \langle Y_*, \sum_{j=0}^r \mu_j Y_j(0)\rangle = \langle Y_*, M(x_*)Y_*\rangle$$

$$= \langle x_*, N(Y_*)x_*\rangle \tag{5.2}$$

$$= \lambda_* \|x_*\|^2. \tag{5.3}$$

Here we have used (3.10) to obtain (5.2). In addition, using (3.11), we have

$$\sum_{j=1}^r \mu_j \le \mathrm{tr}(M(x_*)) \le \|x_*\|^2. \tag{5.4}$$

Applying (5.3) and (5.4) to (5.1) yields

$$\|Y_\#\|^2 \le \|Y_*\|^2 - \lambda_*. \tag{5.5}$$

Now using (3.12), we have

$$\lambda_* \ge \frac{1}{\dim \mathfrak{m}} \mathrm{tr}(N(Y_*)) > \frac{\eta}{\dim \mathfrak{m}} \|Y_*\|^2. \tag{5.6}$$

Applying (5.6) to (5.5) gives our final estimate for Step 3:

$$\|Y_\#\|^2 < \left(1 - \frac{\eta}{\dim \mathfrak{m}}\right) \|Y_*\|^2. \tag{5.7}$$

Now assume that Step 3 has already been executed and that Step 1 is about to be executed again. Then the initial values $x(0) = x_\#$ and $Y(0) = Y_\#$ in Step 1 satisfy $M(x_\#)Y_\# = 0$. By (3.10), this implies $\langle x_\#, N(Y_\#)x_\#\rangle = 0$. As before, let x_j denote the projection of x into the λ_j-eigenspace of $N(Y)$. Recall that $x_* = x_s \equiv x_s(0)$ throughout Step 1, and that $x(t) \to x_*$ asymptotically. Using the orthogonality of the eigenspaces, we compute

$$\begin{aligned} \|x_*\|^2 &= \|x_\#\|^2 - \textstyle\sum_{j=0}^{s-1} \|x_j(0)\|^2 \\ &\le \|x_\#\|^2 - \frac{1}{\sum_{j=0}^s (\lambda_s - \lambda_j)} \textstyle\sum_{j=0}^s (\lambda_s - \lambda_j) \|x_j(0)\|^2. \end{aligned} \tag{5.8}$$

Using orthogonality again, we compute

$$\sum_{j=0}^s (\lambda_s - \lambda_j)\|x_j(0)\|^2 = \lambda_s \|x_\#\|^2 - \langle x_\#, \sum_{j=0}^s \lambda_s x_j(0)\rangle$$

$$= \lambda_s \|x_\#\|^2 - \langle x_\#, N(Y_\#)x_\#\rangle$$

$$= \lambda_s \|x_\#\|^2. \tag{5.9}$$

Also

$$\sum_{j=0}^{s}(\lambda_s - \lambda_j) = s\lambda_s - \sum_{j=0}^{s-1}\lambda_j. \tag{5.10}$$

Applying (5.9) and (5.10) to (5.8) gives

$$\|x_*\|^2 \le \left(1 - \frac{\lambda_s}{s\lambda_s - \sum_{j=0}^{s-1}\lambda_j}\right)\|x_\#\|^2. \tag{5.11}$$

Finally,

$$\frac{\lambda_s}{s\lambda_s - \sum_{j=0}^{s-1}\lambda_j} \ge \frac{1}{s} \ge \frac{1}{\dim\,\mathfrak{m}}, \tag{5.12}$$

and applying (5.12) to (5.11) gives our final estimate for Step 1:

$$\|x_*\|^2 \le \left(1 - \frac{1}{\dim\,\mathfrak{m}}\right)\|x_\#\|^2. \tag{5.13}$$

Now assume Step 2 is executed because $x = 0$ (that is, $x_* = 0$ in Step 1). Rename $x_* = z_*$ where z_* is the chosen λ_*-eigenvector. Then the normalization of z_* described in Step 2 immediately implies (5.13) holds as an equality.

Define two sequences of real numbers as follows: Let a_j and b_j denote, respectively, the initial values of $\|x\|^2$ and $\|Y\|^2$ prior to the $(j+1)$st iteration of the while loop, where $j = 0, 1, \ldots$. Recall that $\|Y\|$ remains constant during Steps 1 and 2 and $\|x\|$ remains constant during Step 3. Our estimates (5.7) and (5.13) imply that the sequences $\{a_j\}$ and $\{b_j\}$ satisfy

$$a_{j+1} \le \left(1 - \frac{1}{\dim\,\mathfrak{m}}\right)a_j \tag{5.14}$$

$$b_{j+1} < \left(1 - \frac{\eta}{\dim\,\mathfrak{m}}\right)b_j. \tag{5.15}$$

Since

$$0 < 1 - \frac{1}{\dim\,\mathfrak{m}} < 1 - \frac{\eta}{\dim\,\mathfrak{m}} < 1, \tag{5.16}$$

it follows from (5.14)-(5.15) that the sequences $\{a_j\}$ and $\{b_j\}$ each converge to 0. In particular, it is immediate that each of $\|x\|$ and $\|Y\|$ can be brought to within the prescribed error tolerance $\delta > 0$ in finitely many iterations of the while loop.

In summary, we have proven the following result.

Theorem. *The algorithm globally stabilizes the system (3.1)-(3.2).*

We remark while we have used the error tolerance δ above to indicate how the stabilization algorithm works in practise, the formal proof of stability follows from letting δ limit to zero.

6. Example: $so(n)$**.** We consider the $so(n)$ systems (1.4)-(1.5). Let $\mathfrak{g} = so(n+1)$, the Lie algebra of $(n+1)\times(n+1)$ skew-symmetric matrices. As in §1, we identify the Lie subalgebra

$$\mathfrak{h} \equiv \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Y \end{pmatrix} : Y \in so(n) \right\}$$

with $so(n)$ and the subspace

$$\mathfrak{m} \equiv \left\{ \begin{pmatrix} 0 & -x^T \\ x & 0 \end{pmatrix} : x \in \mathbb{R}^n \right\}$$

with $\mathbb{R}^n$, and the adjoint action of $so(n)$ on $\mathbb{R}^n$ is the standard action. Since $\mathfrak{g}$ is of compact type, $\epsilon = -1$. For $x \in \mathbb{R}^n$, the operator $M(x) : so(n) \to so(n)$ is given by

$$M(x)Y = -[x, [x, Y]] = xx^T Y + Y xx^T. \tag{6.1}$$

This satisfies the following minimal polynomial equation:

$$M(x)^2 Y = (x^T x) M(x) Y. \tag{6.2}$$

For $Y \in so(n)$, the operator $N(Y) : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$N(Y)x = Y^T Y x. \tag{6.3}$$

The control (4.1) in this setting is given by

$$u = -\alpha x + \beta Y x + \gamma Y^T Y x, \tag{6.4}$$

for $x \in \mathbb{R}^n$, $Y \in so(n)$, and thus the system (4.2)-(4.3) is

$$\dot{x} = -\alpha x + \beta Y x + \gamma Y^T Y x \tag{6.5}$$

$$\dot{Y} = -\beta(xx^T Y + Y xx^T) + \gamma(Y^T Y xx^T - xx^T Y^T Y). \tag{6.6}$$

In Case III, with $\alpha = \gamma = 0$, we know from (4.22) that the matrix $M(x)Y - (x^T x)Y$ evolves with constant spectrum. Since $x^T x$ remains constant throughout this case, this means that

$$Y - \frac{1}{x^T x} M(x) Y \tag{6.7}$$

has constant spectrum. Assume now that the system is in its initial configuration for Step 3. Then $Y^T Y x = \lambda_* x$ where λ_* is the largest eigenvalue of $N(Y) = Y^T Y$. Observe that

$$\begin{aligned} M(x)Y &= xx^T Y + Y xx^T = xx^T Y + \tfrac{1}{\lambda_*} Y xx^T Y^T Y \\ &= (xx^T + \tfrac{1}{\lambda_*} Y xx^T Y^T) Y. \end{aligned}$$

We claim the symmetric operator $\frac{1}{x^T x}(xx^T + \frac{1}{\lambda_*} Y xx^T Y^T)$ is actually an orthogonal projector onto the subspace of $\mathbb{R}^n$ spanned by x and Yx. Indeed, using the antisymmetry of Y, we compute

$$\begin{aligned} (xx^T + \tfrac{1}{\lambda_*} Y xx^T Y^T)^2 &= (x^T x) xx^T + \tfrac{1}{\lambda_*{}^2} Y xx^T Y^T Y xx^T Y^T \\ &= (x^T x)(xx^T + \tfrac{1}{\lambda_*} Y xx^T Y^T). \end{aligned}$$

This establishes our claim. It follows that the operator given by (6.7) is the composition of the orthogonal projector onto the orthogonal complement of the subspace spanned by x and Yx with Y.

In particular, suppose v is an eigenvector of Y^TY corresponding to a nonzero eigenvalue $\lambda \neq \lambda_*$. Then Yx is also such an eigenvector, and $Yv \pm \lambda^{1/2}iv$ are complex eigenvectors of Y itself corresponding to the eigenvalues $\pm\lambda^{1/2}i$, respectively. Both v and Yv are orthogonal to the subspace spanned by x and Yx. The preceding discussion shows that

$$\begin{aligned}(Y - \tfrac{1}{x^Tx}M(x)Y)v &= Yv \\ (Y - \tfrac{1}{x^Tx}M(x)Y)Yv &= Y^2v = -\lambda v.\end{aligned}$$

It follows that

$$(Y - \frac{1}{x^Tx}M(x)Y)(Yv \pm \lambda^{1/2}iv) = \pm\lambda^{1/2}i(Yv \pm \lambda^{1/2}iv),$$

and thus $\pm\lambda^{1/2}i$ are eigenvalues of $Y - \frac{1}{x^Tx}M(x)Y$. (Similarly, if v is an eigenvector of Y^TY corresponding to the possible eigenvalue 0, then consideration of Yv shows that 0 is an eigenvalue of $Y - \frac{1}{x^Tx}M(x)Y$.) On the other hand, we have

$$\begin{aligned}(Y - \tfrac{1}{x^Tx}M(x)Y)x &= 0 \\ (Y - \tfrac{1}{x^Tx}M(x)Y)Yx &= Y^2x + \lambda_*x = 0.\end{aligned}$$

We see that the eigenvalues of Y other than $\pm{\lambda_*}^{1/2}i$ are also eigenvalues of $Y - \frac{1}{x^Tx}M(x)Y$ with the same multiplicities. It follows that throughout Step 3, the other eigenvalues of Y, and hence Y^TY, will remain constant. (While the preceding discussion seems to assume implicitly that λ_* has multiplicity 1, it is easy to see that the same result applies if there are additional eigenvectors corresponding to λ_*.)

In Step 3, the eigenvalue of Y^TY whose initial value is λ_* is the only one that is evolving nontrivially, and it will in fact converge to 0. To see this, recall that Y converges to $Y_\#$, the projection of Y onto the nullspace of $M(x)$. It follows that $\frac{1}{x^Tx}M(x)Y = \frac{1}{x^Tx}(\mu_0Y_0 + \cdots + \mu_rY_r)$ will converge to 0, for if $Y_\# = Y_0 \neq 0$, then $\mu_0 = 0$. But as we have just seen, $\pm{\lambda_*}^{1/2}i$ are the only nonzero eigenvalues of $\frac{1}{x^Tx}M(x)Y$ at time $t = 0$, and thus they must converge to 0 asymptotically.

These considerations also tell us how many times we can expect the stabilization algorithm to iterate. Indeed, since Y^TY can have at most $\lfloor \frac{n}{2} \rfloor$ distinct positive eigenvalues, stabilization will be achieved in at most $\lfloor \frac{n}{2} \rfloor$ iterations.

Specializing further, let us consider the case $so(3)$. Here Y^TY has only one nonzero eigenvalue, which has multiplicity 2. It follows that after one execution of Step 3, Y will converge to 0. Thus the algorithm will stabilize the system with just one iteration of the while loop.

As a numerical example of this, consider the 6th order system

$$\dot{x} = u \tag{6.8}$$

$$\dot{Y} = xu^T - ux^T \tag{6.9}$$

where $x, u \in R^3$, $Y \in so(3)$ with the following initial conditions:

$$x(0) = \begin{pmatrix} 0.2 \\ 1.1 \\ 1.1 \end{pmatrix}$$

$$Y(0) = \begin{pmatrix} 0 & 0.1 & -0.2 \\ -0.1 & 0 & 3.0 \\ 0.2 & -3.0 & 0 \end{pmatrix}.$$

The spectrum of $Y(0)^T Y(0)$ is $\{9.05, 9.05, 0\}$.

After we apply Step 1 (with $u = -\lambda_* x + Y^T Y x$) over the interval $[0, t_1]$ with $t_1 = 2$ sec, x and Y become respectively

$$x(t_1) = \begin{pmatrix} -0.1076 \\ 1.0966 \\ 1.0726 \end{pmatrix}$$

$$Y(t_1) = \begin{pmatrix} 0 & 0.4359 & 0.1346 \\ -0.4359 & 0 & 2.9738 \\ -0.1346 & -2.9738 & 0 \end{pmatrix}.$$

The spectrum of $Y(t_1)^T Y(t_1)$ remains constant $\{9.05, 9.05, 0\}$ with good accuracy, but as expected, the vector $x(t_1)$ is now the eigenvector corresponding to the eigenvalue $\lambda = 9.05$.

Application of Step 3 (with $u = Yx$) over the interval $[t_1, t_2]$ ($t_2 = 4$ sec) results in the fast decay of $\|Y(t)\|$ to zero as $\|x(t)\|$ remains constant. At the end of this interval, the eigenvalues of $Y(t_2)^T Y(t_2)$ become very small: $\{0.0007, 0.0007, 0\}$.

Finally, Step 5 is executed (with $u = -x$). As expected, x converges to 0 as the value of Y remains unchanged.

Time plots of $\|x(t)\|^2$ and $\|Y(t)\|^2$ are shown in Figure 6.1. The decrease in these magnitudes is clearly seen.

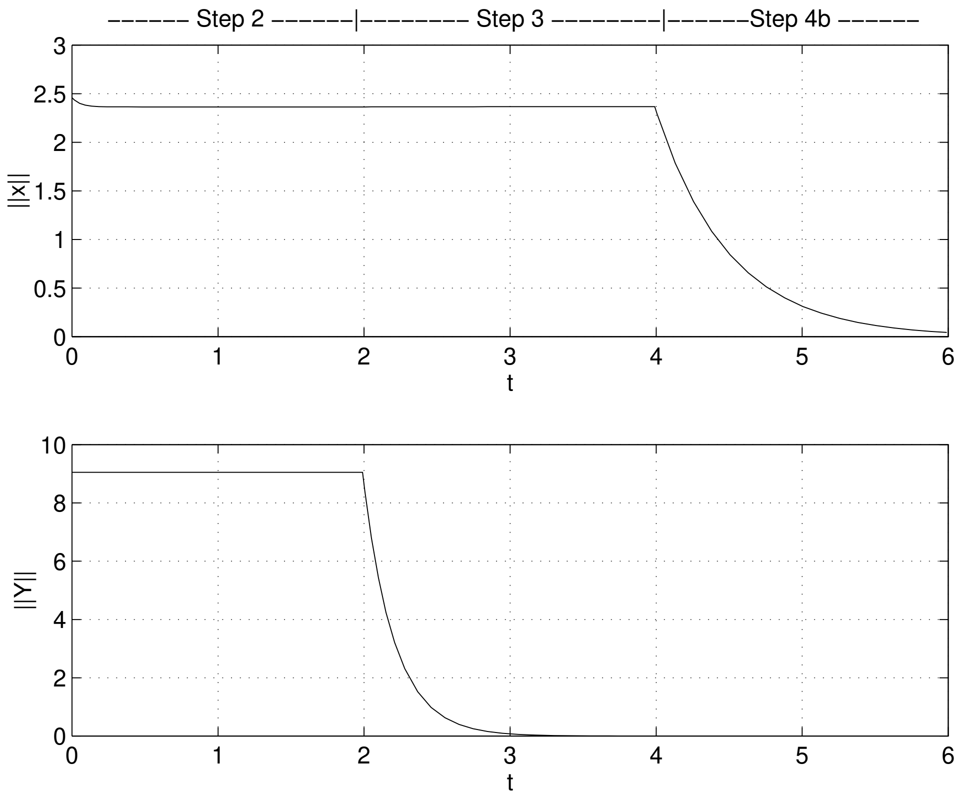

FIG. 6.1. *Numerical Example.*

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