Radiation Induced Instability

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Abstract

In this paper we discuss the stability and instability properties of two classes of conservative dynamical systems which are comprised of a finite dimensional and an infinite-dimensional sub-system. The finite dimensional component is a linear mechanical system with gyroscopic terms. The mechanical system is coupled to a wave equation defined on an infinite spatial domain via two different types of coupling which we denote by Lamb coupling and wave field coupling. We also investigate the effect of dispersion on the coupled system. In particular, we analyze the conditions under which coupling to a wave system induces instability in the finite dimensional system. Analytic results are compared to computer simulations.

1 Introduction

Energy transfer within interconnected mechanical systems is important in many real world settings. When a finite dimensional system is coupled to an infinite dimensional system, energy can be radiated from the finite dimensional system and absorbed by the infinite dimensional system. We call this process *radiation damping.* For example, satellites and space stations have a central rigid body which can radiate energy through flexible components such as solar panels and antennae. Radiation damping can also describe dissipation (*e.g.* friction, viscosity) in a conservative context, where energy dissipates from one form (such as motion of a mechanical system) into another form (such as heat) of a larger conservative system. In general relativity,

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energy may be radiated from a system of masses via gravitational waves (see [23] for details). The Lagrangian or Hamiltonian structure of such systems is also of interest (see [21] or [6] for example).

An early physical model of radiation damping was introduced by Horace Lamb ([18]). In the Lamb model, an oscillator coupled to a string describes the free vibrations of a nucleus in an extended medium. The oscillator transfers energy to the string by generating waves as it moves; see also [12]. A model of a particle coupled to a wave field is studied in [15].

In many linear and nonlinear partial differential equations, it is fruitful to view the dynamics in terms of "particle-like" and field "field-like" components. A decomposition into these types of modes leads to an equivalent description in terms of two coupled subsystems: the first is finite-dimensional and governs the "particle-like" or bound state part of the solution, while the second is infinite dimensional and dispersive. Coupling terms are responsible for how the dynamics of "particles" influence the field and how the dispersive wave-field influences the particle dynamics. An approximate closed equation for the particle dynamics can, in some cases, be derived, where the effect of the dispersive radiation field appears in the form of radiation damping corrections to the finite dimensional dynamics. This approach has been used in the context of the theory of quantum resonances [28, 22], ionization and parametric resonance problems [14, 13], and in the context of the decay of "breather-like" states of nonlinear wave equations [29]. In this paper we explore how, in manner analogous to the mechanism by which dissipation can induce instability [4], radiation damping can induce instability in mechanical systems.

In section 2, we present the stability properties and Hamiltonian formalism of gyroscopic systems, also referred to as Chetaev systems. We also describe the Chetaev system as a normal form of the linearized equations of motion about a relative equilibrium of a simple mechanical system acted on by an abelian group. As we increase the gyroscopic forces in a Chetaev system, it is possible to stabilize (or specifically *gyroscopically* stabilize) an equilibrium of the system. For example, a charged, inverted spherical pendulum can be gyroscopically stabilized by increasing the strength of the ambient magnetic field.

In section 3, we describe a gyroscopic version of the Lamb model coupled to a standard non-dispersive wave equation and to a dispersive wave equation, expanding on the results in [10]. We show that instabilities will arise in certain mechanical systems. However, dispersive wave coupling restricts the oscillator access to low frequency wave modes and allows for a band of stability to exist.

In the dispersionless case, the system is of the form

$$\begin{aligned} \frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) &= c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t), \quad z \in \mathbb{R} - \{0\}, t \in \mathbb{R}, \\ M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= T\left[\frac{\partial \mathbf{w}}{\partial z}\right]_{z=0} \\ \mathbf{w}(0,t) &= \mathbf{q}(t), \end{aligned} \tag{1.1}$$

where c is the speed of transverse waves in the string, T is the tension of the string, $\mathbf{w} = \begin{bmatrix} w_1(z,t) & \cdots & w_n(z,t) \end{bmatrix}^T$ is the displacement of the string in the first n dimensions and $[\frac{\partial \mathbf{w}}{\partial z}]_{z=0}$ is the jump discontinuity in the slope of the string.

In section 4, we introduce a non-local field coupling of a gyroscopic system to a dispersive or a non-dispersive infinite dimensional system. Dispersive waves still allow for a band of stability, but the instabilities produced by the non-dispersive wave coupling have a broader destabilizing effect on mechanical systems. In contrast to the analogous dissipation induced instability, field radiation induces instability in stable, as well as gyroscopically stable, mechanical systems. In this setting, the coupled wave equation and the gyroscopic system takes the form

$$\begin{aligned} M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} &= \kappa \int_{\mathbb{R}} \chi(z) w(z,t) dz \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \\ \ddot{w} - c^2 \frac{\partial^2 w}{\partial z^2} &= \kappa \chi(z) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q} \end{aligned} \tag{1.2}$$

where κ is s coupling parameter and $\chi(\xi)$; is a suitable distribution.

In section 5 we consider a example of a more complicated finite-dimensional mechanical system, a rigid body with rotors, coupled to a wave field. In this case the configuration space of the system includes a nonabelian group, $SO(3)$.

Finally, we include an appendix which discusses some of details of the physical models used in this paper and the numerical techniques used in simulations.

2 Gyroscopic Systems

We recall some general properties of linear systems with gyroscopic forces. The general form of a gyroscopic system is

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = 0, \tag{2.1}$$

where $\mathbf{q} \in \mathbb{R}^n$, M is a positive-definite symmetric $n \times n$ matrix, S is skew, and V is symmetric. As in [4] we shall call this the *Chetaev system* (see [8]). An important property of this system is that it is normal form for a simple mechanical system about a *relative equilibrium* which is given modulo an abelian group; see [20] and section 5.

We say the system is *gyroscopically stable* if for $S = 0$ the origin is an unstable equilibrium, but for $S \neq 0$, the origin is a spectrally stable equilibrium (i.e. the eigenvalues of the linearized system have non-positive real part). The matrix S is sometimes referred to as a magnetic term as it can arise from charged oscillators in a magnetic field.

If we include the magnetic terms in the symplectic form, the Hamiltonian, $H : T^*Q \to \mathbb{R}$, of a gyroscopic system is the sum of the kinetic energy and potential energy

$$H = \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^T V \mathbf{q}. \tag{2.2}$$

The symplectic form, Ω, is defined by the Poisson bracket on the cotangent bundle T^*Q,

$$\Omega(X_F, X_K) = \{F, K\}_{magnetic} = \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i} - S^{ij}\frac{\partial F}{\partial p_i}\frac{\partial K}{\partial p_j} \tag{2.3}$$

where X_G denotes the Hamililtonian vector field with Hamiltonian G.

Proposition 2.1 *The Hamiltonian vector field X_H on the symplectic vector space $(\mathbb{R}^{2n}, \Omega)$ has an equivalent representation $X_{\tilde{H}}$ on the symplectic vector space $(\mathbb{R}^{2n}, \tilde{\Omega})$, where*

$$\begin{aligned} \tilde{H} &= \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^T(V+W)\mathbf{q} + \mathbf{p}^T X\mathbf{q} \\ \tilde{\Omega}(X_F, X_K) = \{F, G\}_{canonical} &= \frac{\partial F}{\partial q^i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q^i}\frac{\partial F}{\partial p_i}, \end{aligned} \tag{2.4}$$

for a skew matrix X and a symmetric matrix W defined implicitly by

$$XM + MX = -S, \qquad W = -XMX. \tag{2.5}$$

Furthermore, the Lyapunov equation (2.5) can be solved using the Fredholm Alternative.

Proof: We obtain the implicit definitions of X and W by matching equations of motion on each manifold. Using the Fredholm Alternative, we show equation (2.5) has a solution for X. Let us define the linear operator on skew matrices $L(X) = XM + MX$ and show that S is orthogonal to the kernel of L^T with the inner product on skew matrices being the trace $\langle A, B\rangle = \text{Trace}(AB)$. Computing L^T, we have $\langle A, L(B)\rangle = \langle L(A), B\rangle = \text{Trace}((MA+AM)B)$. Hence L is self-adjoint with respect to this inner product.

Suppose that Y is in the kernel of L^T, then we have

$$MY + YM = 0, \qquad Y = -M^{-1}YM\,. \tag{2.6}$$

We also have that $MY + (MY)^T = -2YM$, and hence YM is symmetric. Using the commuting property of the trace, we compute that S is orthogonal to Y, $\langle S, Y\rangle = 0$. The Fredholm Alternative is satisfied and we can solve equation (2.5) for X. This completes the proof. ∎

Classically, the two representations of the Chetaev systems are equally useful. However, the classical symplectic form, $\tilde{\Omega}$, is preferable when quantizing the mechanical system. (See [4], [2], [8] for further physical discussions.)

2.1 Examples of Chetaev systems

Two physical models of Chetaev systems which can be gyroscopically stabilized are (i) an oscillator in a magnetic field and (ii) an oscillator on a rotating disk. We derive the dynamics from a Lagrangian and a Hamiltonian perspective.

2.1.1 Planar oscillator in a constant magnetic field

The linearized equations of motion of a charged spherical pendulum in a magnetic field are that of a charged planar oscillator in a magnetic field. The equations of motion form a Chetaev mechanical system (for derivation of the full equations see Appendix A). We describe here the motion of a planar charged oscillator in a magnetic field. Let $\mathbf{B}$ be a divergence-free vector field. Let $\mathbf{A}$ be the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$. Note if we choose $\mathbf{B}$ to be the constant magnetic field in the direction normal to the plane of oscillation, the vector potential can be chosen as , $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{q}$, where $\mathbf{q} = (x, y, 0)^T$ is the position of the oscillator.

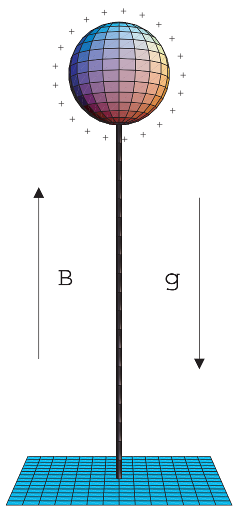

Figure 2.1: **Inverted spherical pendulum.**

Assume the oscillator has unit mass and unit charge and that the speed of light is unity. The Lagrangian, $L : T\mathbb{R}^2 \to \mathbb{R}$, is defined mechanically by the kinetic energy minus the potential energy

$$\begin{aligned} L(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}\|\dot{\mathbf{q}}\|^2 + \mathbf{A}\cdot\dot{\mathbf{q}} - U(\mathbf{q}) \\ &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \mathbf{A}\cdot(x, y, 0)^T - \frac{1}{2}(\alpha x^2 + \beta y^2). \end{aligned} \tag{2.7}$$

Choosing $\mathbf{B}$ to be of constant strength B, normal to the plane of oscillation we have,

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\alpha x^2 + \beta y^2) + \frac{1}{2}B(x\dot{y} - y\dot{x}), \tag{2.8}$$

where the last term is the velocity dependent magnetic term.

The associated Hamiltonian, $H : T^*\mathbb{R}^2 \to \mathbb{R}$, is

$$H = p_x\dot{x} + p_y\dot{y} - L = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\alpha x^2 + \beta y^2) + H_B, \tag{2.9}$$

where

$$H_B = \frac{1}{2}B(p_x y - p_y x) + \frac{1}{8}B^2(x^2 + y^2), \tag{2.10}$$

and the associated momenta are given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - \frac{1}{2}By, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + \frac{1}{2}Bx\,. \tag{2.11}$$

In the above notation, H_B is the contribution of a magnetic field. Since we include the magnetic terms in the Hamiltonian, we use the canonical symplectic form, $\tilde{\Omega}$, to obtain the equations of motion,

$$
\begin{aligned}
-\dot{p}_x = \frac{\partial H}{\partial x} = (\alpha + \frac{1}{4}B^2)x - \frac{1}{2}Bp_y \qquad & - \dot{p}_y = \frac{\partial H}{\partial y} = (\beta + \frac{1}{4}B^2)y + \frac{1}{2}Bp_x \\
\dot{x} = \frac{\partial H}{\partial p_x} = p_x + \frac{1}{2}By \qquad & \dot{y} = \frac{\partial H}{\partial p_y} = p_y - \frac{1}{2}Bx\,.
\end{aligned}
\tag{2.12}
$$

We thus have the dynamics

$$
\begin{aligned}
\ddot{x} - B\dot{y} + \alpha x = 0 \\
\ddot{y} + B\dot{x} + \beta y = 0\,.
\end{aligned}
\tag{2.13}
$$

If α and β are both negative, the oscillator is gyroscopically stabilized (i.e. the eigenvalues are purely imaginary) if $B^2 + \alpha + \beta > 2\sqrt{\alpha\beta}$. Hence, we can increase the strength of the magnetic field to stabilize the oscillator.

2.1.2 Planar oscillator on a rotating plate

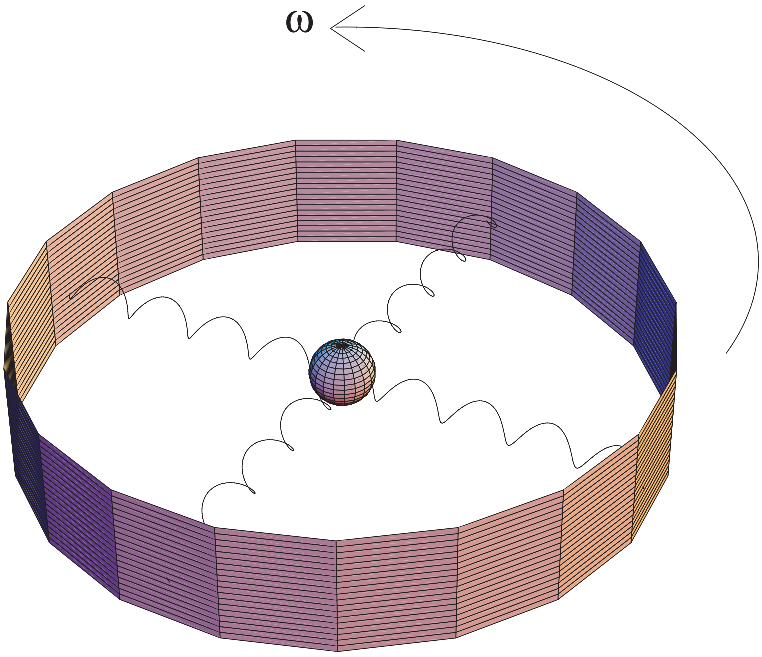

Figure 2.2: **Rotating plate with springs.**

Another physical model of a Chetaev system is a planar oscillator on a plate rotating with angular velocity ω. We write the Lagrangian as the kinetic energy minus the potential energy,

$$
L = \frac{1}{2}((\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2) - \frac{1}{2}(\alpha x^2 + \beta y^2)\,. \tag{2.14}
$$

In the Hamiltonian setting we have

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\alpha x^2 + \beta y^2) + H_\omega, \tag{2.15}$$

$$H_\omega = \omega(p_x y - p_y x) \tag{2.16}$$

and the associate momenta are given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - \omega y, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + \omega x. \tag{2.17}$$

We obtain the equations of motion,

$$\begin{aligned} -\dot{p}_x &= \frac{\partial H}{\partial x} = \alpha x - \omega p_y & -\dot{p}_y &= \frac{\partial H}{\partial y} = \beta y + \omega p_x \\ \dot{x} &= \frac{\partial H}{\partial p_x} = p_x + \omega y & \dot{y} &= \frac{\partial H}{\partial p_y} = p_y - \omega x \end{aligned} \tag{2.18}$$

This gives the dynamics

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} + (\alpha - \omega^2)x &= 0 \\ \ddot{y} + 2\omega\dot{x} + (\beta - \omega^2)y &= 0 \end{aligned} \tag{2.19}$$

Remark 1: The rotating disc affects the oscillator differently from a magnetic field – compare equations (2.19) and (2.13). While the magnetic field only adds $\dot{\mathbf{q}}$ terms, the rotating disc also adds additional $\mathbf{q}$ terms to the dynamics. For the equations (2.13), we can see that in the case of a physically stable oscillator with $\alpha, \beta > 0$ for rotation rate ω sufficiently large, the system becomes only gyroscopically stable, i.e. the coefficients $\alpha - \omega^2$ and $\beta - \omega^2$ are negative but the eigenvalues are on the imaginary axis due to the presence of gyroscopic terms.

Remark 2: In either case above, if the system is gyroscopically stable it can be shown that adding a small amount of dissipation to the system renders it unstable (i.e. there are unstable eigenvalues). For a more precise statement and generalization see section 2.2.

2.2 Stability of Chetaev systems

In this section we first summarize the stability properties of Chetaev systems and then turn to then discuss their dissipative perturbations. The stability of the Chetaev system depends both on the signature of the bilinear form associated with a quadratic Hamiltonian and also on the magnetic terms in the symplectic form. In particular, the magnetic terms can stabilize gyroscopic systems with negative eigenvalues of V.

As for gyroscopic stability, the number of negative eigenvalues of the quadratic form plays a crucial part as discussed in [8]. We summarize this in the following propostion:

Proposition 2.2 *Consider the canonical gyroscopic system $M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = 0$, where M is a symmetric positive definite matrix, S is a skew-symmetric matrix, and V is a symmetric matrix:*

- *If V has a odd number of negative eigenvalues (counting multiplicity) then the origin is an unstable equilibrium.*
- *If V has an even number of negative eigenvalues (counting multiplicity), we can choose S so that the origin is a spectrally stable equilibrium.*

Proof: By standard reduction to a first order system for $(\mathbf{q}, \dot{\mathbf{q}})$ it suffices to consider the first order first order linear operator L,

$$L = \begin{bmatrix} 0 & I \\ -M^{-1}V & -M^{-1}S \end{bmatrix}. \tag{2.20}$$

Let $p(\lambda)$ be the characteristic polynomial of the matrix L and let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the matrix V.

$$p(0) = \det L = (-1)^n \det(-M^{-1}V) = \det(M^{-1}) \prod_{i=1}^{n} \lambda_i \,. \tag{2.21}$$

Suppose that V has an odd number of negative eigenvalues. Due to the positive definiteness of M, we have $p(0) < 0$. Since

$$\lim_{\lambda\to\infty} p(\lambda) = \lim_{\lambda\to\infty} \det\Big(L - \lambda I\Big) = \lim_{\lambda\to\infty} (-1)^{2n}\lambda^{2n} + O(\lambda^{2n-1}) = \infty \tag{2.22}$$

$p(\lambda)$ must change sign and by continuity has a positive root, corresponding to a real and positive eigenvalue of L, an instability.

Hence, if V has an odd number of negative eigenvalues (counting multiplicity) then the characteristic polynomial of L has a positive real root and the origin cannot be gyroscopically stabilized.

Now to prove the second item, suppose that V has an even number of negative eigenvalues (counting multiplicity). By choosing a basis in which V is diagonal, we can introduce a skew matrix S to stabilize each pair of negative eigen-directions, as in the gyroscopic stabilization for $n = 2$ shown in introductory examples. This completes the proof. ■

Gyroscopically stable systems exhibit interesting instability when perturbed by dissipative forces. Suppose now that V has at least one negative eigenvalue. A key result of [4] is that adding small dissipation always yields instability. More precisely it is shown that

Theorem 2.3 *Under the above conditions, if we modify the general Chetaev system by adding a small Rayleigh dissipation term,*

$$M\ddot{\mathbf{q}} + (S + \epsilon R)\dot{\mathbf{q}} + V\mathbf{q} = 0 \tag{2.23}$$

for small $\epsilon > 0$, where R is symmetric and positive definite, then the perturbed linearized equations

$$\dot{z} = L_\epsilon z,$$

where $z = (q, p)$ are spectrally unstable, i.e., at least one pair of eigenvalues of L_ϵ is in the right half plane.

This result builds on basic work of [30], [8], and [11]. We refer to this as *dissipation induced instability*. Some of the radiation induced instabilities that arise in the gyroscopic Lamb model are analogous to the dissipation induced instability. Using the Hamiltonian representation with canonical symplectic form, $\tilde{\Omega}$, we see that the Hamiltonian of a gyroscopically stabilized Chetaev system is indefinite. In this case, with the addition of Rayleigh dissipation, the Hamiltonian does not bound the motion of the Chetaev system. In particular, the displacement and velocities may grow exponentially.

As in section 5.1.2, we present the Chetaev system as a normal form of the linearized equations of motion about the relative equilibrium of an abelian group action. [4] also prove a similar stability result for the non-abelian case, but the abelian result is sufficient for our purposes.

3 Lamb Coupling

In the original Lamb model ([18]), an oscillator is physically coupled to a string. The vibrations of the oscillator transmit waves into the string and are carried off to infinity. Hence, the oscillator loses energy and is effectively damped by the string.

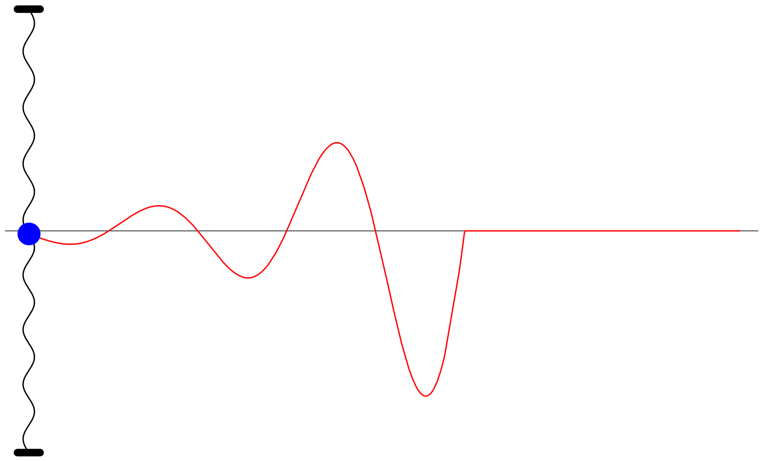

Figure 3.1: **Lamb model of an oscillator coupled to a string.**

Let $w(x,t)$ be the displacement of the string at position $x \in \mathbb{R}$ at time t. Let ρ be the mass density of the string and let T be the tension. We can compute the wave speed of transverse oscillations of the free string to be $c = \frac{T}{\rho}$. Assuming a singular mass density at $x = 0$, we can couple the dynamics of an oscillator, q, of mass M, to the otherwise free wave to resolve the singularity,

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= c^2 \frac{\partial^2 w}{\partial x^2} \\ M\ddot{q} + Vq &= T[w_x]_{x=0} \\ q(t) &= w(0,t), \end{aligned} \tag{3.1}$$

where $\sqrt{\frac{V}{M}}$ is the frequency of the uncoupled oscillating mass M and $[w_x]_{x=0} = w_x(0+,t) - w_x(0-,t)$ is the jump discontinuity of the slope of the string. Note that (3.1) is a Hamiltonian system, as demonstrated for a more complex system in section 3.1.

Perturbing the oscillator at time $t = 0$ from its equilibrium position, we can use d'Alembert's solution to the wave equation to solve for w,

$$w = \begin{cases} Ce^{(ct-|x|)\omega} & \text{for } |x| < ct, \\ 0 & \text{for } |x| > ct, \end{cases} \tag{3.2}$$

where $\omega = -\frac{T}{Mc} + i\sqrt{\frac{V^2}{M^2} - (\frac{T}{Mc})^2}$ and C is the size of the initial displacement. For small tension we have damping and oscillator motion, while for large tension, we have pure damping. From the solution of the wave equation, we can compute the jump condition of the oscillator,

$$[w_x]_{x=0} = -2C\omega e^{\omega ct} = -\frac{2}{c}\dot{q}. \tag{3.3}$$

We obtain a reduced form of the dynamics describing the explicit motion of the oscillator subsystem,

$$M\ddot{q} + \frac{2T}{c}\dot{q} + Vq = 0. \tag{3.4}$$

The coupling term arises explicitly as a Rayleigh dissipation term $\frac{2T}{c}\dot{q}$ in the dynamics of the oscillator. For the gyroscopic Lamb coupling, the dissipation term will induce instabilities in gyroscopically stabilized Chetaev systems.

3.1 Gyroscopic Lamb model

In this section, we investigate a variant of the the Lamb model, a Chetaev system which includes gyroscopic terms coupled to the standard wave equation. We have the boundary constraint that the displacement of the gyroscopic oscillator fixes a point of the string. We show that this local coupling perturbation destabilizes gyroscopically stabilized Chetaev systems.

We begin with a model of a string in $\mathbb{R}^{n+1}$, whose transverse vibrations are independent. Suppose the string lies initially along the $x_{n+1}-$ axis. For simplicity, we denote the x_{n+1} dimension as the z dimension. Coupling the transverse motion to a $n-$dimensional gyroscopic system, we use the method described in [16] to solve the following

$$\begin{aligned} \frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) &= c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t), \quad z \in \mathbb{R} - \{0\}, t \in \mathbb{R}, \\ M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= T\left[\frac{\partial \mathbf{w}}{\partial z}\right]_{z=0} \\ \mathbf{w}(0,t) &= \mathbf{q}(t), \end{aligned}$$

where c is the speed of transverse waves in the string, T is the tension of the string, $\mathbf{w} = \begin{bmatrix} w_1(z,t) & \cdots & w_n(z,t) \end{bmatrix}^T$ is the displacement of the string in the first n dimensions and $[\frac{\partial \mathbf{w}}{\partial z}]_{z=0}$ is the jump discontinuity in the slope of the string. Let us define the initial conditions on the string by $\mathbf{w}_1(z) = \frac{\partial \mathbf{w}}{\partial t}(z,0)$, and $\mathbf{w}_0(z) = \mathbf{w}(z,0)$.

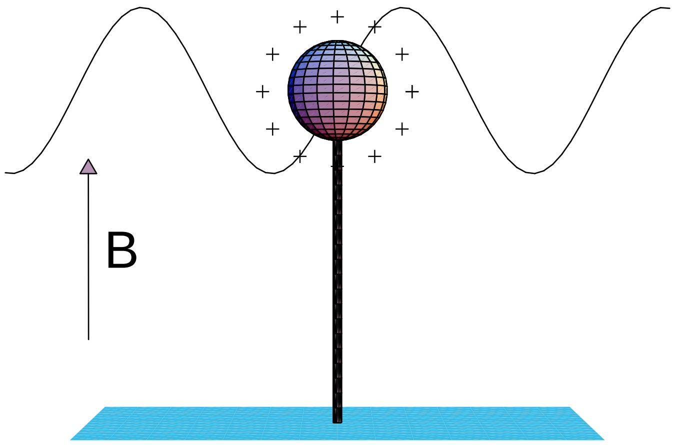

Figure 3.2: **Gyroscopic Lamb coupling to a spherical pendulum.**

By direct computation we obtain an exact reduced dynamical system for $\mathbf{q}$. This reduction contains explicit dissipation terms reflecting the oscillator/ field coupling. The results on reduction and stability are summarized in the following two results. The proofs are given in section 3.1.2.

Proposition 3.1 *If the initial data* $\mathbf{w}_0$ *and* $\mathbf{w}_1$ *have compact support, then for* $\pm ct \notin \mathrm{supp}(\mathbf{w}_0) \cup \mathrm{supp}(\mathbf{w}_1)$ *the oscillator dynamics of the gyroscopic Lamb model reduces to*

$$M\ddot{\mathbf{q}}(t) + (S + \frac{2T}{c})\dot{\mathbf{q}}(t) + V\mathbf{q}(t) = 0. \tag{3.5}$$

In [4], small dissipation is shown to induce instability in gyroscopically stable systems (see Theorem 2.3). In the gyroscopic Lamb model, radiation (energy transfer into an infinite-dimensional system) produces a Rayleigh dissipation term which depends on the tension of the string. Since dissipation induces instability in these system for small dissipation, small tension coupling yields radiation induced instability via an analogous mechanism. We summarize the main result of *radiation induced* instability in gyroscopically stabilized Chetaev systems with a theorem.

Theorem 3.2 *If a gyroscopic mechanical system is gyroscopically stable (i.e.* V *has a negative eigendirection), then local coupling via the Lamb model induces instability for small coupling parameter* T*.*

The ability to describe the coupled system in a conservative manner is one of the key differences between radiation induced instability and dissipation induced instability. We delay the proofs of the above theorems to demonstrate the conservative nature of the gyroscopic Lamb model. While we can obtain the equations of motion from a force calculation, we emphasize the conservative nature of the gyroscopic Lamb model by representing the system as the flow of a Hamiltonian vector field.

3.1.1 Hamilton's equations of motion

For an uncoupled wave equation, the kinetic energy, T_{string}, depends on the velocity of the string while the potential energy, U_{string}, depends on the shape of the string,

$$T_{string} = \sum_{i=1}^{n} \int_{\mathbb{R}} \|\frac{\partial w_i}{\partial t}\|^2 dz, \quad U_{string} = \sum_{i=1}^{n} \int_{\mathbb{R}} c^2 \|\frac{\partial w_i}{\partial z}\|^2 dz. \tag{3.6}$$

Let $Z = \mathcal{H}^1(\mathbb{R}) \times L^2(\mathbb{R})$, where $\mathcal{H}^1(\mathbb{R})$ is the space of square integrable functions whose first derivatives are also square integrable and $L^2(\mathbb{R})$ is the space of square integrable densities. Hence $(w_i, \pi_i) \in Z$, if w_i and $\frac{\partial w_i}{\partial z}$ are each square integrable and $\pi_i = \pi_i' dz$ where π' is a square integrable function on $\mathbb{R}$. We define the canonical symplectic form, Ω_Z, on the vector space Z as

$$\Omega_Z((w_1, \pi_1), (w_2, \pi_2)) = \int_{\mathbb{R}} w_1 \pi_2 - w_2 \pi_1. \tag{3.7}$$

Proposition 3.3 *The gyroscopic Lamb model (3.1) has a Hamiltonian representation with Hamiltonian $(H, (Z, \Omega_Z)^n)$, with*

$$\begin{aligned} H(\mathbf{w}, \pi) &= \int \sum_{i=1}^{n} \frac{1}{2}(\pi_i')^2 + \frac{1}{2} c^2 \|\frac{\partial w_i}{\partial z}\|^2 + \delta(z) H_{osc.}(\mathbf{w}, \pi') dz, \\ H_{osc.}(\mathbf{w}, \pi') &= \frac{1}{2} \pi'^T M^{-1} \pi' + \frac{1}{2} \mathbf{w}^T (V + W) \mathbf{w} + \pi'^T X \mathbf{w} - T \mathbf{w}^T [\mathbf{w}_z]_{z=0}. \end{aligned} \tag{3.8}$$

Proof: The symplectic form induces Hamilton's equations for field theory,

$$\Omega(X_H, \cdot) = dH \tag{3.9}$$

$$\frac{\partial w_i}{\partial t} = \frac{\delta H}{\delta \pi_i}, \quad \frac{\partial \pi_i}{\partial t} = -\frac{\delta H}{\delta w_i}. \tag{3.10}$$

Computing the variations, we have

$$\begin{aligned} \int \frac{\delta H}{\delta \pi_i} \cdot \delta \pi_i dz &= \lim_{\epsilon \to 0} \epsilon^{-1} (H(\mathbf{w}, \pi + \epsilon \delta \pi_i) - H(\mathbf{w}, \pi)) \\ &= \int (\pi_i' + \delta(z) \frac{\partial H_{osc.}}{\partial \pi_i}) \cdot \delta \pi_i dz \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \int \frac{\delta H}{\delta w_i} \cdot \delta w_i dz &= \lim_{\epsilon \to 0} \epsilon^{-1} (H(\mathbf{w} + \epsilon \delta w_i, \pi) - H(\mathbf{w}, \pi)) \\ &= \int (-c^2 \frac{\partial^2 w_i}{dz^2} + \delta(z) \frac{\partial H_{osc.}}{\partial w_i}) \cdot \delta w_i dz. \end{aligned} \tag{3.12}$$

For $z \neq 0$, we have

$$\frac{\partial w_i}{\partial t} = \pi_i', \quad \frac{\partial \pi_i'}{\partial t} = c^2 \frac{\partial^2 w_i}{\partial z^2}, \tag{3.13}$$

or equivalently

$$\frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) = c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t), \quad z \in \mathbb{R} - \{0\}, t \in \mathbb{R}. \tag{3.14}$$

For the singular part of the Hamiltonian at $z = 0$, we define $\mathbf{q}(t) = \mathbf{w}(0,t)$ and $\mathbf{p}(t) = \pi'(0,t)$. Notice that $[\mathbf{w}_z]_{z=0}$ is independent of variations $\delta \mathbf{w}(0)$. We use the induced canonical symplectic form, $\tilde{\Omega}$, on $T^*\mathbb{R}^n$ to obtain the Hamilton's equations of motion

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{\partial H_{osc.}}{\partial \mathbf{p}} = M^{-1}\mathbf{p} + X\mathbf{q} \\ \dot{\mathbf{p}} &= \frac{\partial H_{osc.}}{\partial \mathbf{q}} = (V+W)\mathbf{q} - X\mathbf{p} - T[\mathbf{w}_z]_{z=0}. \end{aligned} \tag{3.15}$$

Differentiating and writing the above first order system as a second order system, we obtain

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = T[\mathbf{w}_z]_{z=0}. \tag{3.16}$$

This completes the proof. ∎

3.1.2 Radiation induced instability in the Lamb model

To prove radiation induced instability, we obtain a reduced form of the Chetaev subsystem. We can use d'Alembert method of solving the wave equation or find the solution via Laplace transforms. We save the latter technique for the more general dispersive case.

Proof of Proposition 3.1 [16] explicitly develops the d'Alembert solution of the wave equation coupled to a simple harmonic oscillator. We use the same method applied to a Chetaev system coupled to a higher dimensional wave equation. The method is to use the d'Alembert method of solving the wave equation on the domain unaffected by the coupling. Inside the domain affected by the coupling, we use continuity and coupling dynamics to solve for the wave equation.

Our model decomposes into n independent 1-dimensional wave equations, we can use d'Alembert decomposition to two traveling waves

$$\mathbf{u}(z,t) = \mathbf{f}_\pm(z - ct) + \mathbf{g}_\pm(z + ct), \qquad \pm z > 0, \tag{3.17}$$

where $\mathbf{f}_\pm, \mathbf{g}_\pm$ are functions on $\mathbb{R}^n$. In components, we have

$$\mathbf{f}_\pm = \left[f_{\pm 1} \cdots f_{\pm 2n}\right]^T, \;\; \mathbf{g}_\pm = \left[g_{\pm 1} \cdot \;\; g_{\pm 2n}\right]^T, \tag{3.18}$$

where $f_i, g_i, i = \pm 1, \cdots, \pm n$ are real-valued functions on $\mathbb{R}$. By the d'Alembert method of solving the wave equation, we have the formulas

$$\mathbf{f}_\pm(z) \;=\; \frac{1}{2}\mathbf{w}_0(z) - \frac{1}{2c}\int_0^z \mathbf{w}_1(v)dv, \qquad \pm z > 0, \tag{3.19}$$

$$\mathbf{g}_\pm(z) \;=\; \frac{1}{2}\mathbf{w}_0(z) + \frac{1}{2c}\int_0^z \mathbf{w}_1(v)dv, \qquad \pm z > 0. \tag{3.20}$$

Since the d'Alembert formula for $|z| \geq c|t|$, is defined in terms of known functions, the method remains valid and we have

$$\mathbf{w}(z,t) = \frac{\mathbf{w}_0(z-ct)+\mathbf{w}_0(z+ct)}{2} + \frac{1}{2c}\int_{z-ct}^{z+ct} \mathbf{w}_1(v)dv. \tag{3.21}$$

To use the d'Alembert formula for $|z| < c|t|$, we need to define $\mathbf{f}_+(z)$ for $z < 0$ and to define $\mathbf{g}_-(z)$ for $z > 0$. ¿From the boundary condition and from continuity of the string, we have

$$\mathbf{f}_\pm(-ct) + \mathbf{g}_\pm(ct) = \mathbf{q}(t), \qquad t > 0, \tag{3.22}$$

$$M\ddot{\mathbf{q}}(t) = -S\dot{\mathbf{q}}(t) - V\mathbf{q}(t) + T[\frac{\partial \mathbf{f}_+}{\partial z}(-ct) + \frac{\partial \mathbf{g}_+}{\partial z}(ct) - \frac{\partial \mathbf{f}_-}{\partial z}(-ct) - \frac{\partial \mathbf{g}_-}{\partial z}(ct)]. \tag{3.23}$$

Differentiating and solving for $\mathbf{f}'_+$ and $\mathbf{g}'_-$, where the $'$ denotes $\frac{\partial}{\partial z}$, we have

$$\begin{aligned} -c\mathbf{f}'_+(-ct) &= \dot{\mathbf{q}}(t) - c\mathbf{g}'_+(ct), \\ c\mathbf{g}'_-(ct) &= \dot{\mathbf{q}}(t) + c\mathbf{f}'_-(-ct), \qquad t > 0. \end{aligned} \tag{3.24}$$

Substituting, we arrive at a differential equation for $\mathbf{q}$ in terms of known functions

$$\begin{aligned} M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= 2T[\mathbf{g}'_+(ct) - \mathbf{f}'_-(-ct) - \frac{1}{c}\dot{\mathbf{q}}(t)], \quad t > 0. \\ &= -\frac{2T}{c}\dot{\mathbf{q}}(t) + T[\mathbf{w}'_0(ct) - \mathbf{w}'_0(-ct) + \frac{1}{c}(\mathbf{w}_1(ct) + \mathbf{w}_1(-ct))]. \end{aligned} \tag{3.25}$$

The effect of coupling to a wave equation (3.25) is manifested by the presence of an explicit dissipative term. If we assume that the initial conditions of the string are such that $\mathbf{w_1}$ and $\mathbf{w_0}$ both have compact support, then for large t we have $\pm ct$ outside the support of $|\mathbf{w_0}| + |\mathbf{w_1}|$ reducing equation (3.25) to

$$\begin{aligned} M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= -\frac{2T}{c}\dot{\mathbf{q}}(t), \\ \pm\, ct \notin \operatorname{supp}(|\mathbf{w_0}| &+ |\mathbf{w_1}|). \end{aligned} \tag{3.26}$$

This completes the proof the of proposition. ■

Proof of Theorem 3.2 Assuming that the initial string has a finite energy, we have that $\mathbf{w}$ and $\mathbf{w_1}$ decay as $|z| \to \infty$. Hence, there are initial conditions of the string which have compact support in an H^1 Sobolev neighborhood of $\mathbf{w}$. Hence, we can use the result from Proposition 3.1. We construct a Lyapunov function W as the sum of the energy of the system and a magnetic term,

$$W(\mathbf{q}) = \frac{1}{2}\dot{\mathbf{q}}^T M\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T V\mathbf{q} + \delta\mathbf{q}^T VM\dot{\mathbf{q}}. \tag{3.27}$$

In [4], W is shown to have a negative eigen-direction for δ sufficiently small if V has a negative eigen-direction, and that $\dot{W}$ is negative definite. Invoking the Lyapunov Instability Theorem, the gyroscopic Lamb model is spectrally unstable if V has a negative eigen-direction. ■

The results from Theorem 3.2 are consistent with Krein signature results on instability (see [1] and [4]), in contrast to the broader instability one sees in the non-local coupling case (see section 4).

3.2 Dispersive gyroscopic Lamb model

In this section, we develop an explicit solution to a dispersive gyroscopic Lamb model, i.e. a gyroscopically stable Chetaev system coupled to a dispersive wave equation. As the dispersive wave equation we choose the Klein Gordon equation with "mass" m:

$$\begin{aligned} \frac{\partial^2 \mathbf{w}}{\partial t^2}(z,t) &= c^2 \frac{\partial^2 \mathbf{w}}{\partial z^2}(z,t) - m^2 \mathbf{w}(z,t), \quad z \in \mathbb{R} - \{0\}, t \in \mathbb{R}, \\ M\ddot{\mathbf{q}}(t) + S\dot{\mathbf{q}}(t) + V\mathbf{q}(t) &= T\Big[\frac{\partial \mathbf{w}}{\partial z}\Big]_{z=0} \\ \mathbf{w}(0,t) &= \mathbf{q}(t). \end{aligned} \tag{3.28}$$

The coupled system has a Hamiltonian, H given by

$$\begin{aligned} H(\mathbf{w},\pi) &= \int \sum_{i=1}^{n} \frac{1}{2}(\pi_i')^2 + \frac{1}{2}c^2 \|\frac{\partial w_i}{\partial z}\|^2 + \frac{1}{2}m^2\|w|^2 + \delta(z) H_{osc.}(\mathbf{w},\pi')dz, \\ H_{osc.}(\mathbf{w},\pi') &= \frac{1}{2}\pi'^T M^{-1}\pi' + \frac{1}{2}\mathbf{w}^T(V+W)\mathbf{w} + \pi'^T X\mathbf{w} - T\mathbf{w}\Big[\mathbf{w}_z\Big]_{z=0}, \end{aligned} \tag{3.29}$$

on the same symplectic vector space $((Z,\Omega_Z)^n)$ as the gyroscopic Lamb model.

Equation (3.28) is linear translation invariant in time, so the initial value problem can be solved by Laplace transform. We denote by

$$\mathcal{L}[f](s) = \int_0^\infty f(t)e^{st}dt.$$

In the transformed space, we have

$$\begin{aligned} (s^2 M + sS + V)\mathcal{L}[\mathbf{q}] &= M(\dot{\mathbf{q}}(0) + s\mathbf{q}(0)) + s\mathbf{q}(0) + T\Big[\mathcal{L}[\mathbf{w}_z]\Big]_{z=0} \\ (s^2+m^2)\mathcal{L}[\mathbf{w}] - c^2\mathcal{L}[\mathbf{w}_{zz}] &= \mathbf{w}_t(z,0) + s\mathbf{w}(z,0), \quad z \neq 0 \\ \mathcal{L}[\mathbf{w}](0,s) &= \mathcal{L}[\mathbf{q}](s). \end{aligned} \tag{3.30}$$

For simplicity of the calculation, we assume that $\mathbf{q}(0) = \mathbf{w}_t(z,0) = \mathbf{w}(z,0) = 0$ and that the perturbation comes from a non trivial velocity of the oscillator at the origin. We can integrate the homogeneous dispersive wave equation on each interval to obtain,

$$\mathcal{L}[\mathbf{w}](z,s) = \begin{cases} \mathbf{a}e^{\frac{\sqrt{s^2+m^2}z}{c}} & \text{for } z < 0, \\ \mathbf{a}e^{-\frac{\sqrt{s^2+m^2}z}{c}} & \text{for } z > 0. \end{cases} \tag{3.31}$$

¿From the boundary constraints, we have $\mathcal{L}[\mathbf{q}] = \mathbf{a}$.

Solving for $\mathcal{L}[\mathbf{w}]$ as function of $\mathcal{L}[\mathbf{q}]$, and computing the jump condition of the string at $z = 0$, we have

$$\begin{aligned} \mathcal{L}[\mathbf{w}_z](z,s) &= -\mathrm{sign}(z)\frac{\sqrt{s^2+m^2}}{c}e^{-\mathrm{sign}(z)\frac{\sqrt{s^2+m^2}z}{c}}\mathcal{L}[\mathbf{q}] \\ \Big[\mathcal{L}[\mathbf{w}_z]\Big]_{z=0} &= -2\frac{\sqrt{s^2+m^2}}{c}\mathcal{L}[\mathbf{q}]. \end{aligned} \tag{3.32}$$

The equations of motion for the coupled Chetaev system decouple from the dispersive wave equation,

$$(s^2 M + sS + V + 2T\frac{\sqrt{s^2+m^2}}{c}I)\mathcal{L}[\mathbf{q}] = M\dot{\mathbf{q}}(0). \tag{3.33}$$

Taking the inverse Laplace transform, and using the identity

$$\mathcal{L}[\frac{J_1(t)}{t}] = \sqrt{s^2+1} - s \tag{3.34}$$

we obtain

$$M\ddot{\mathbf{q}} + (S + \frac{2T}{c}I)\dot{\mathbf{q}} + V\mathbf{q} = -\frac{2Tm}{c}\int_0^t \frac{1}{t-s}J_1(m(t-s))\mathbf{q}(s)ds, \tag{3.35}$$

where J_1 is the Bessel function of the first kind. We can define the approximate identity $J^\epsilon = \frac{J_1(\frac{t}{\epsilon})}{t}$. Since J_1 is an $L^1(\mathbb{R})$ function such that $\int_0^\infty J_1(t)dt = 1$, we can use the approximate identity nature of the convolution to have a necessary condition on stability summarized in a lemma.

Lemma 3.4 *If* $\mathbf{q}$ *is continuous and bounded for all time, then*

$$\lim_{\epsilon\to 0} J^\epsilon * \mathbf{q}(t) = \mathbf{q}(t). \tag{3.36}$$

By the above lemma, we obtain an asymptotic expansion for the perturbed dynamics of the coupled oscillator,

Proposition 3.5 *Consider the dynamical system*

$$M\ddot{\mathbf{q}} + (S + \frac{2T}{c}I)\dot{\mathbf{q}} + (V + \frac{Tm}{c}I)\mathbf{q} = 0. \tag{3.37}$$

As m tends to infinity, the zero equilibrium of the dispersive Lamb model becomes unstable if the zero equilibrium of system (3.37) is unstable.

Proof: In order to produce a contradiction, assume that the dispersive Lamb model is stable as $m \to \infty$. By applying Lemma 3.4 to the reduced dynamics (3.35), we have

$$M\ddot{\mathbf{q}} + (S + \frac{2T}{c}I)\dot{\mathbf{q}} + (V + \frac{Tm}{c}I)\mathbf{q} = O(\frac{1}{\sqrt{m}}). \tag{3.38}$$

Hence, the associated dynamical system (3.37) must also have a stable zero equilibrium. This contradicts the assumption that the system (3.37). Our assumption that the dispersive Lamb model is stable as $m \to \infty$ must be false. This completes the proof. ■

For large values of m, the asymptotic analysis suggests that a gyroscopically stabilized Chetaev system will maintain its stability after dispersive Lamb coupling. The dispersive nature of the wave equation restricts the access of the oscillator from interacting with low frequencies of the wave. This restriction effectively shifts the potential of the oscillator. The potential shift can even stabilize previously unstable oscillators.

The evolution of a Chetaev subsystem mirrors that of stable Chetaev system with dissipation. In the limit as m tends to zero, the asymptotic behavior of the system retains the explicit damping contribution as in the non-dispersive coupling.

4 Non-local field coupling to gyroscopic Chetaev systems

While the Lamb model imposes the boundary constraint that the oscillator is physically attached to the wave field, we now investigate the affect of an ambient field on a Chetaev system without such a constraint. Energy is transmitted between the ambient field and the Chetaev system, not through boundary constraints, but rather through coupling in the Hamiltonian. We analyze the contribution of a non-local distribution on the wave field acting as a force on the oscillator. Mathematically the field acts through a coupling distribution which for simplicity we take for computations to be a Dirac δ-function.

4.1 Non-dispersive, non-local wave field coupling to a Chetaev system

Here we investigate a field coupling of the mechanical gyroscopic system to the wave equation. As before, we have the standard non-dispersive wave equation

$$\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial z^2} = 0. \tag{4.1}$$

We model force of the wave field on the Chetaev system by the magnitude of the wave field. In the spirit of conservation, the coupled system remains Hamiltonian, while each perturbed subsystem may not. Coupling of this type is important in various physical models -see [28], [27], [29], and references therein. The interaction between the wave equation and the Chetaev system is modeled with a coupling parameter κ and with a coupling distribution $\chi(\xi)$; the equations of motion are

$$\begin{aligned} M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} &= \kappa \int_{\mathbb{R}} \chi(z) w(z,t)dz \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \\ \frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial z^2} &= \kappa \chi(z) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q}. \end{aligned} \tag{4.2}$$

For simplicity, we choose $\chi(z)$ to be the Dirac- δ distribution. (So in fact, even though we will continue to use the terminology "non-local coupling" for the type of system discussed in this section, the special case we consider here is in fact also a type of focused, local with respect to the ambient field, coupling without boundary constraints.)

The above coupled system remains Hamiltonian with H defined as

$$H = \frac{1}{2}\left(\int_{\mathbb{R}} w_t^2 + c^2 w_z^2 dz + \dot{\mathbf{q}}^T M \dot{\mathbf{q}} + \mathbf{q}^T V \mathbf{q}\right) - \kappa \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q} \int_{\mathbb{R}} \chi(z) w(z,t) dz. \tag{4.3}$$

Notice that the Hamiltonian is indefinite; hence the energy of the system does not uniformly bound the motion of the Chetaev system and nor the motion of the wave field. Furthermore, the magnetic terms are present in the mechanical symplectic form defined by the Chetaev symplectic form, Ω defined in equation (2.3).

4.1.1 Reduction of non-dispersive system to a finite-dimensional system

Even though the Chetaev system and the wave equation seem intimately coupled, we can decouple the wave motion from that of the Chetaev system as follows. Taking the Fourier transform with respect to z in the wave equation yields

$$\begin{aligned}
\hat{w}(k,t) &= \int_{\mathbb{R}} e^{-ikz} w(z)dz, \\
\hat{w}_{tt}(k,t) + c^2k^2\hat{w}(k,t) &= \kappa \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \mathbf{q}(t), \\
\hat{w}(k,t) &= \kappa \int_0^t \frac{\sin(ck(t-s))}{ck} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q}(s)ds + \hat{w}_{free}(k,t).
\end{aligned} \tag{4.4}$$

The above equation is the retarded Green's function for the wave equation and w_{free} is the homogeneous solution satisfying initial conditions. Computing $w(0,t)$, we have

$$\begin{aligned}
w(0,t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{w}(k,t)dk \\
&= w_{free}(0,t) + \frac{\kappa}{4\pi c} \int_0^t \int_{\mathbb{R}} \frac{2\sin(ck(t-s))}{ck} d(ck) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q}(s)ds \\
&= w_{free}(0,t) + \frac{\kappa}{2c} \int_0^t \operatorname{sgn}(t-s) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q}(s)ds \\
&= w_{free}(0,t) + \frac{\kappa}{2c} \int_0^t \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \mathbf{q}(s)ds
\end{aligned} \tag{4.5}$$

Thus the Chetaev system reduces to,

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = \kappa w_{free}(0,t) \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^T + \frac{\kappa^2}{2c} Ones(n) \int_0^t \mathbf{q}(s)ds, \tag{4.6}$$

where $Ones(n)$ is an n by n matrix with each matrix element equal to one.

4.1.2 Differentiated system

We can interpret equation (4.6) as an integral feedback on the Chetaev system (see Appendix D). To analyze the stability, we investigate the stability of the differentiated system.

Suppose that initially the coupling field is identically zero, and hence $w_{free}(0,t) = 0$ for all time $t > 0$. Since our system decouples, we consider the related system of ordinary differential equations. Let us write the system as a first order system with the variable Q.

$$Q = \begin{bmatrix} \mathbf{q}^T & \dot{\mathbf{q}}^T & \ddot{\mathbf{q}}^T \end{bmatrix}^T. \tag{4.7}$$

Differentiating the reduced system yields

$$\dot{Q} = AQ, \tag{4.8}$$

where

$$A = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ \frac{\kappa^2}{2c}M^{-1}Ones(n) & -M^{-1}V & -M^{-1}S \end{bmatrix} \tag{4.9}$$

Proposition 4.1 *If $\kappa \neq 0$, then the system (4.8) is unstable.*

Proof: Since $\text{Trace}(M^{-1}S) = -\text{Trace}(SM^{-1}) = -\text{Trace}(M^{-1}S)$, $M^{-1}S$ is traceless. For non-zero κ, we compute the rank of A as $rank(A) = 2n+1$. The matrix A has an odd number $(2n+1)$ of non-zero eigenvalues which sum to zero. Since the non-zero eigenvalues satisfy an odd degree polynomial with real coefficients, A has at least one non-zero real eigenvalue. Hence, A has an eigenvalue with positive real part; thus the system is unstable. ■

Example: Assume $n = 2, M = I, S = \begin{bmatrix} 0 & -B \\ B & 0 \end{bmatrix}$, and $V = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$. Computing the characteristic equation of the matrix A, we have

$$p(\mu) = \mu(\mu^5 + (B^2 + \alpha + \beta)\mu^3 - 2\frac{\kappa^2}{2c}\mu^2 + \alpha\beta\mu - \frac{\kappa^2}{2c}(\alpha + \beta)). \tag{4.10}$$

Let μ_1, ... ,μ_5 be the non-zero eigenvalues of A. Then we have

$$\prod_{i=1}^{5} \mu_i = \frac{\kappa^2}{2c}(\alpha + \beta). \tag{4.11}$$

Note the system is gyroscopically stabilized when $B^2 + \alpha + \beta \geq 2\sqrt{\alpha\beta}$. Also the $\text{Trace}(A) = 0$. Hence if $\gamma(\alpha+\beta) \neq 0$, then A has an eigenvalue with positive real part. In particular if $\alpha, \beta < 0$ and $\kappa \neq 0$, then there is a real negative eigenvalues. Thus, there exists a conjugate pair of eigenvalues with positive real part. Hence the differentiated system is unstable.

We can quantify the rate at which the coupled system becomes unstable. If we additional have $\alpha \neq \beta$, we use first order perturbation theory to compute the speed at which the eigenvalues leave the imaginary axis

$$\mu' = \frac{d\mu}{d\gamma}\Big|_{\gamma=0} \tag{4.12}$$

Computing μ', we have

$$\mu' = \frac{2\mu^2 + (\alpha + \beta)}{5\mu^4 + 3(w_0^2 + w_1^2)\mu^2 + w_0^2 w_1^2}, \tag{4.13}$$

where the eigenvalues of the unperturbed system are $\pm iw_0, \pm iw_1$. Simplifying in terms of w_0 and w_1, we have

$$(\pm iw_j)' = \frac{-2w_j^2 + (\alpha + \beta)}{2w_j^2(w_j^2 - w_{1-j}^2)}. \tag{4.14}$$

We compare μ' with the μ' from a purely dissipative perturbation (see the analogous calculation in [19], [4] and references therein), we have

$$(\pm i w_j)' = \frac{(\pm i w_j)'_{dissipative}}{w_j^2}. \tag{4.15}$$

Since we use the differentiated systems we require the existence of higher derivatives. In particular, a C^3 solution to the reduced system (4.6) is a solution to the differentiated system (4.8). Conversely, a solution to the differentiated system (4.8) is a solution to the reduced system (4.6) if initial conditions are satisfied. Satisfying initial condition imposes a mild linear constraint on the system which is easily satisfied. We summarize the result with a theorem,

Theorem 4.2 *The Chetaev system coupled to the standard, non-dispersive wave equation via non-local coupling (4.2), where $\chi(z)$ is the Dirac-δ distribution, yields the reduced finite-dimensional system (4.6). All C^2 solutions of this system are unstable.*

Proof: Let us write the solution to the differentiated problem as

$$\mathbf{q}(t) = \sum_i \lambda_i \mathbf{A}_i e^{\mu_i t}, \tag{4.16}$$

where $\mu_i, \mathbf{A_i}$ are eigenvalues and position components of the eigenvectors for the linear system and λ_i is a scaling chosen to satisfy initial conditions. Evaluation of (4.6) at $t = 0$, with $w_{free} = 0$ yields

$$M\ddot{\mathbf{q}}(0) + S\dot{\mathbf{q}}(0) + V\mathbf{q}(0) = \mathbf{0}. \tag{4.17}$$

We have

$$\ddot{\mathbf{q}}(0) = \sum_i \mu_i^2 \mathbf{A}_i, \qquad \dot{\mathbf{q}}(0) = \sum_i \mu_i \mathbf{A}_i, \qquad \mathbf{q}(0) = \sum_i \mathbf{A}_i. \tag{4.18}$$

Using the fact that μ_i is an eigenvalue of the differentiated system, we have

$$(\mu_i^3 M + \mu_i^2 S + \mu_i V - \gamma Ones(n))\mathbf{A}_i = \mathbf{0}. \tag{4.19}$$

Substituting, the initial condition is satisfied if

$$\gamma Ones(n) \sum_{i,\mu_i \neq 0} \frac{\lambda_i}{\mu_i} \mathbf{A}_i = \mathbf{0}. \tag{4.20}$$

This is just a linear constraint on the initial conditions, which can be satisfied while exhibiting a negative eigen-direction. ■

This instability is unlike the effect of Rayleigh dissipation of the type in gyroscopic Lamb coupling (3.5), i.e. dissipation arising from a term consisting of a positive definite symmetric matrix multiplying velocities. For small coupling, Rayleigh dissipation induces instability only if the Chetaev system is gyroscopically stable (i.e. it is unstable for $B = 0$ but stable for suitably large B). In the non-local field coupling model, instability is always induced. (See Chetaev [1961] and [4] for a discussion of Rayleigh dissipation induced instability).

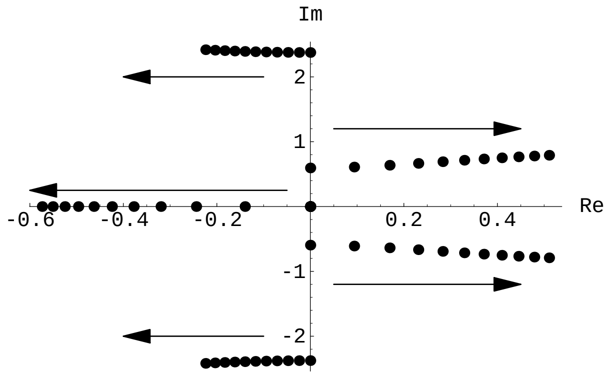

Figure 4.1: **Destabilization of a gyroscopically stabilized system.**

Example 1: For $n = 2$, $\alpha = -1, \beta = -2, c = 1$, and $B = 3$, the origin of the uncoupled system is gyroscopically stabilized. Increasing the coupling parameter, we show that the weaker eigenvalues destabilize as κ increases from 0 to 1 in figure 4.1.

Example 2: As discussed above we have shown that non-local coupling has a broader destabilizing effect than dissipation. In particular, the integral coupling can destabilize a system regardless of whether or not it is gyroscopically stabilized, while Rayleigh dissipative perturbations destabilize a Chetaev system only if it is initially gyroscopically stabilized. For $\alpha = 1, \beta = 2, c = 1$, and $B = 3$, we show numerically how the increasing the coupling parameter destabilizes the system.

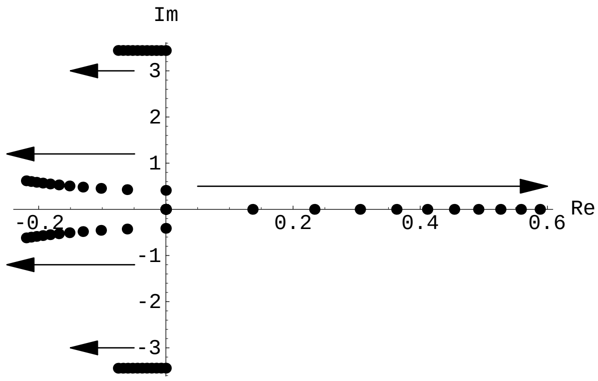

Figure 4.2: **Destabilization of stabilized system.** ($\alpha = 1, \beta = 2, B = 3, c = 1$), κ increases from 0 to 1. A lone eigenvalue escapes from the origin, while the eigenvalues of the original system move into the left half plane.

4.2 Dispersive, non-local wave field coupling to a Chetaev system

In this section, we demonstrate that the nature of the wave field, to which the Chetaev system is coupled, has a significant impact on the stability of the overall system. We now consider the case of a gyroscopic system coupled to the linear Klein-Gordon equation, a standard dispersive wave equation,

$$\begin{aligned} M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} &= \kappa \int_{\mathbb{R}} \chi(z) w(z,t) d\xi \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \\ \frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial^2 w}{\partial z^2} + m^2 w &= \kappa \chi(z) \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \mathbf{q}. \end{aligned} \tag{4.21}$$

In this case, we show that a region of stability is possible for small coupling.

4.2.1 Energy bounds

For some potentials, we can explicitly show that the Hamiltonian is positive definite and hence can be used to bound the motion of the system. In the case that the Hamiltonian is indefinite, our system may or may not be stable. The main stability results of this section we present in a theorem,

Theorem 4.3 *If $|\kappa| < \min(m^2, c^2)$, then the non-local dispersive wave coupling to a mechanical gyroscopic system (4.21), with $\chi(z) = \delta(z)$, is stable if $V - |\kappa| Ones(n)$ is positive definite, where $Ones(n)$ is an $n \times n$ matrix with each entry equal to unity. Explicitly, we have a uniform bound of $\|w\|_{H^1}, \|w_t\|, |\mathbf{q}|$, and $|\dot{\mathbf{q}}|$ in terms of initial energy.*

Proof of Theorem 4.3: We will use energy estimates to give a uniform bound in phase space of the gyroscopic system coupled to the Klein-Gordon equation.

We use the form of the Chetaev system with the magnetic terms in the symplectic form, Ω,. In this case, the Hamiltonian of the coupled system becomes

$$H = \frac{1}{2}\mathbf{p}^T M^{-1} \mathbf{p} + \frac{1}{2}\mathbf{q}^T V \mathbf{q} + \frac{1}{2}\int_{\mathbb{R}} w_t^2 + c^2 w_z^2 + m^2 w^2 dz - \kappa w(0,t) \mathbf{q}_\Sigma, \tag{4.22}$$

where $\mathbf{q}_\Sigma = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \mathbf{q}$. Using the Cauchy-Schwarz inequality, we have the following energy estimate,

$$\begin{aligned} H \geq & \frac{1}{2}\mathbf{p}^T M^{-1} \mathbf{p} + \frac{1}{2}\mathbf{q}^T V \mathbf{q} \\ & + \frac{1}{2}(\|w_t\|_{L_2}^2 + \min(c^2, m^2)\|w\|_{H^1}^2) - \frac{1}{2}|\kappa|(w(0,t)^2 + (\mathbf{q}_\Sigma)^2) \\ = & \frac{1}{2}\mathbf{p}^T M^{-1} \mathbf{p} + \frac{1}{2}\mathbf{q}^T (V - |\kappa| Ones(n)) \mathbf{q} \\ & + \frac{1}{2}(\|w_t\|_{L_2}^2 + \min(c^2, m^2)\|w\|_{H^1}^2) - \frac{1}{2}|\kappa| w(0,t)^2, \end{aligned} \tag{4.23}$$

where $Ones(n)$ is an $n \times n$ matrix with all entries equal to unity. From a standard Sobolev inequality, we have the result that if $w \in C_0^1(\mathbb{R}^2)$, then $w(0,t)^2 \leq \|w\|_{H^1}^2$. Using the Sobolev inequality, we have another bound on the energy,

$$\begin{aligned} H \geq & \frac{1}{2}\mathbf{p}^T M^{-1} \mathbf{p} + \frac{1}{2}\mathbf{q}^T \Big(V - |\kappa| Ones(n)\Big) \mathbf{q} \\ & + \frac{1}{2}\Big(\|w_t\|_{L_2}^2 + (\min(c^2, m^2) - |\kappa|)\|w\|_{H^1}^2\Big). \end{aligned} \tag{4.24}$$

Our Hamiltonian is positive definite if $|\kappa| < \min(c^2, m^2)$ and the matrix $\tilde{H}$ is positive definite, where

$$\tilde{H} = \begin{bmatrix} V - |\kappa| Ones(n) & 0 \\ 0 & M^{-1} \end{bmatrix}. \tag{4.25}$$

Equivalently, H is positive definite if $|\kappa| < \min(c^2, m^2)$ and $V - |\kappa| Ones(n)$ is positive definite. ■

By the same argument we obtain the following theorem

Theorem 4.4 *Suppose $|\kappa| < \frac{1}{n}\min(m^2, c^2)$ and let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of V. Let $\alpha = \min_{i=1,\cdots,n}(|\lambda_i|)$. If $|\kappa| < \alpha$, then the non-local dispersive wave field coupling to the Chetaev system (4.21), with $\chi(z) = \delta(z)$, is stable. Explicitly, we have a uniform bound of $\|w\|_{H^1}, \|w_t\|, |\mathbf{q}|$, and $|\dot{\mathbf{q}}|$ in terms of initial energy.*

Proof: Using the Cauchy-Schwarz inequality in each component of $\mathbf{q}$ in equation (4.23), we have

$$H \geq \frac{1}{2}\mathbf{p}^T M^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^T(V - |\kappa| I)\mathbf{q} \; + \frac{1}{2}\Big(\|w_t\|_{L_2}^2 + (\min(c^2, m^2) - n|\kappa|)\|w\|_{H^1}^2\Big). \tag{4.26}$$

■

4.2.2 Linear algebra and stability of Chetaev systems

By energy estimates, stability is guaranteed if either the matrix $V - |\kappa| Ones(n)$ or the matrix $V - |\kappa| I$ is positive-definite. Dispersion enables purely stable Chetaev systems (stable if $S = 0$) to remain stable for small coupling parameters. In this subsection, we prove some spectral results for these perturbed matrices, to see how large the perturbation can be and guarantee stability. Even though positive definiteness of the Hamiltonian is a sufficient condition for stability, it is not a necessary condition as seen in gyroscopically stabilized Chetaev systems.

Proposition 4.5 *Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of a symmetric $n \times n$ matrix V. Let $\alpha = \min_{i=1,\cdots,n}(|\lambda_i|)$. If $|\kappa| < \frac{\alpha}{n}$, then $W = V - \kappa Ones(n)$ has the same signature as V.*

Proof: Since V is symmetric and $Ones(n)$ is symmetric, W is symmetric and hence has real eigenvalues. Since a change of signature requires a zero eigenvalue of the perturbed matrix, we equivalently show

$$\det\Big(V - \kappa Ones(n)\Big) \neq 0 \quad \forall \kappa \in (0, \frac{\alpha}{n}). \tag{4.27}$$

If V and $Ones(n)$ commute, we can find a change of basis which simultaneously diagonalizes both V and $Ones(n)$. Working in this basis, we have that

$$\begin{aligned} \det\Big(V - \kappa Ones(n)\Big) &= \det\Big(\mathrm{diag}(\lambda_1, \cdots, \lambda_n) - \kappa \mathrm{diag}(\sigma(0, \cdots, 0, n))\Big) \\ &\neq 0 \qquad \forall \kappa \in (0, \frac{\alpha}{n}), \end{aligned} \tag{4.28}$$

where σ is a permutation of n elements. The eigenvalues of W are the eigenvalues of V with exception that one of the eigenvalues is moved to the left by κn units. Hence, if $0 < \kappa < \frac{\alpha}{n}$, then V and W have the same signature as desired.

In the general case, we define a function f of κ as follows:

$$f(\kappa) = \det\Big(V - \kappa Ones(n)\Big). \tag{4.29}$$

Changing bases so that $Ones(n)$ is diagonal, we have

$$f(\kappa) = \det(\tilde{V} - \kappa \mathrm{diag}(n, 0, 0, \cdots, 0)), \tag{4.30}$$

where $\tilde{V}$ does not depend on κ. Hence f is linear in κ.

We now find a bound on the slope of $f(\kappa)$.

$$\begin{aligned} f'(\kappa) = f'(0) &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}(f(\epsilon) - f(0)) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big(\det\Big(V - \epsilon Ones(n)\Big) - \det(V)\Big) \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big(f(0)\det\Big(I - \epsilon(V^{-1}Ones(n))\Big) - f(0)\Big) \\ &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big(f(0)(1 - \epsilon \mathrm{Trace}\Big(V^{-1}Ones(n)\Big) + O(\epsilon^2)) - f(0)\Big) \\ &= -\det(V)\mathrm{Trace}\Big(V^{-1}Ones(n)\Big). \end{aligned} \tag{4.31}$$

Let C be an orthogonal matrix that diagonalizes $Ones(n)$ to $D = \mathrm{diag}(n, 0, 0, \cdots, 0)$.

$$\begin{aligned} \mathrm{Trace}\Big(V^{-1}Ones(n)\Big) &= \mathrm{Trace}\Big(CV^{-1}C^{-1}COnes(n)C^{-1}\Big) \\ &= \mathrm{Trace}\Big(CV^{-1}C^{-1}D\Big) \end{aligned} \tag{4.32}$$

A well-known result of [25] states that the diagonal elements of $CV^{-1}C^{-1}$ lie in the convex hull of the permutations of $\{\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n}\}$. Minimizing over permutations yields

$$|\mathrm{Trace}\Big(CV^{-1}C^{-1}D\Big)| \le \max_{d=(d_1,\cdots,d_n)} \Big(\frac{nd_1}{|\lambda_{\sigma(1)}|} + \sum_{i=2}^{n} \frac{d_2}{|\lambda_{\sigma(i)}|}\Big), \tag{4.33}$$

where σ is a permutation of n elements and $d_i \ge 0$ with $\sum_{i=1}^{n} d_i = 1$. The right hand side of the above equation is a linear function on a convex set and hence attains its maximum at an extreme point,

$$|\mathrm{Trace}\Big(CV^{-1}C^{-1}D\Big)| \le \max_{i\in\{1,\cdots,n\}} \Big(\frac{n}{|\lambda_i|}\Big) \le \frac{n}{\alpha}. \tag{4.34}$$

Let $\beta = \max_{i=1,\cdots,n}(|\lambda_i|)$. Finally, we have a bound on the slope of $f(\kappa)$,

$$\frac{|f(0)|}{\beta/n} \le |f'(\kappa)| \le \frac{|f(0)|}{\alpha/n}. \tag{4.35}$$

Thus for $|\kappa| < \frac{\alpha}{n}$ we have $f(\kappa) \ne 0$; hence, the signatures of V and of $V - \kappa Ones(n)$ are the same, as desired. ■

Corollary 4.6 *Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of a symmetric, positive-definite $n \times n$ matrix V. Suppose $\kappa > \max_{i=1,\cdots,n}(\lambda_i)$. Then $V - \kappa Ones(n)$ has a different signature than the signature of V. In particular, $V - \kappa Ones(n)$ has at least one negative eigenvalue.*

Proof: For $\kappa > \max_{i=1,\cdots,n}(\lambda_i)$, we have Trace$(V - \kappa Ones(n)) < 0$. A symmetric matrix with negative trace has at least one negative eigenvalue. ■

Corollary 4.7 *Let $\lambda_1, \cdots, \lambda_n$ be eigenvalues of a symmetric $n \times n$ matrix V. Suppose $|\lambda_i| > 0$ for $i = 1, \cdots, n$. Let $\alpha = \min_{i=1,\cdots,n}(|\lambda_i|)$ and $\beta = \max_{i=1,\cdots,n}(|\lambda_i|)$. We have the following results:*

- *As $|\kappa|$ increases, the matrix $V - |\kappa|Ones(n)$ changes signature at most once.*
- *If $V - |\kappa|Ones(n)$ changes signature it does so for $|\kappa| \leq \frac{\beta}{n}$.*
- *If $V - |\kappa|Ones(n)$ changes signature, then an odd number of eigenvalues change sign.*

Proof: Linearity of $f(\kappa) = \det\left(V - \frac{\kappa}{n}Ones(n)\right)$ in κ proves the first and third claims. Bounds in the $f'(\kappa)$ prove the second claim. ■

4.2.3 Reduction of dispersive system to a finite dimensional system

Analogously to the computation in section 4.1.1, we eliminate the dynamics of the wave field to obtain reduced dynamics of a perturbed Chetaev system. Taking the Fourier transform of the field equation, we have

$$\ddot{\hat{w}} + \omega^2(k)\hat{w} = \kappa\hat{\chi}\mathbf{q}_\Sigma(t), \tag{4.36}$$

where $\omega = \sqrt{c^2k^2 + m^2}$ and $\mathbf{q}_\Sigma(s) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T \mathbf{q}(s)$. Let $\hat{g}$ be the Green's function for the above equation

$$\ddot{\hat{g}} + \omega^2\hat{g} = \delta(t), \qquad \hat{g} = H(t)\frac{\sin\omega t}{\omega}, \tag{4.37}$$

where $H(t)$ is the Heaviside function. Computing $w(0,t)$ from the Green's function, we have

$$\begin{aligned} w(0,t) &= w_{free}(0,t) + \frac{1}{2\pi}\int_{\mathbb{R}} \hat{w}\ dk \\ &= w_{free}(0,t) + \frac{1}{2\pi}\int_{\mathbb{R}} (\kappa\hat{\chi}\mathbf{q}_\Sigma) * \hat{g}\ dk \\ &= w_{free}(0,t) + \frac{\kappa}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\chi}\mathbf{q}_\Sigma(s)\hat{g}(t-s)\ ds\ dk, \end{aligned} \tag{4.38}$$

where w_{free} is a homogeneous solution to the dispersive wave equation. Assuming that the coupling starts at $t = 0$, we have $\mathbf{q}_\Sigma(t) = 0$ for $t < 0$. We can interchange order of integration,

$$w(0,t) = w_{free}(0,t) + \frac{\kappa}{2\pi}\int_0^t \int_{\mathbb{R}} \hat{\chi}\mathbf{q}_\Sigma(s)\frac{\sin\omega(t-s)}{\omega}\ dk\ ds. \tag{4.39}$$

Choosing χ to be the Dirac-δ distribution and zero initial conditions in the wave field, the reduced dynamics becomes

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = \frac{\kappa^2}{2\pi}\int_0^t \int_{\mathbb{R}} \mathbf{q}_\Sigma(s)\frac{\sin\omega(t-s)}{\omega}\ dk\ ds \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{4.40}$$

Explicitly using the form of the Klein-Gordon coupling, we can integrate the interior integral,

$$\int_{\mathbb{R}} \frac{\sin\omega s}{\omega} dk = 2\int_m^\infty \frac{\sin\omega s}{c\sqrt{\omega^2 - m^2}} d\omega = \frac{\pi}{c} J_0(ms), \tag{4.41}$$

where J_0 is the Bessel function of the first kind. We obtain an integro-differential equation for the reduced dynamics,

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = \frac{\kappa^2}{2c} Ones(n) \int_0^t J_0(m(t-s))\mathbf{q}(s)\ ds. \tag{4.42}$$

From the reduced dynamics (4.42), we can investigate the stability.

4.3 Asymptotic analysis and numerical simulations

For small coupling, we have shown that strongly stable Chetaev systems remain stable for $m \neq 0$ by theorems 4.3 and 4.4. However, the energy bound results yield no information for gyroscopically stabilized Chetaev systems. Using the compact reduced form (4.42), asymptotic analysis suggests that gyroscopic systems maintain their stability under dispersive wave field perturbations. Moreover, the asymptotic analysis also suggests that an unstable system may be stabilized for a band of coupling parameters. In this section, we develop asymptotic expansions for the reduced dynamics (4.42) and compare the results with numerical simulations.

For large values of m the wave field's effect on the Chetaev system appears to be a convolution with an approximate identity, $mJ_0(mt)$. Since $J_0(t)$ is not in $L^1(\mathbb{R})$, showing the approximate identity nature of the convolution becomes non-trivial, but nonetheless true which is summarized in a theorem.

Theorem 4.8 *Consider the integro-differential equation (4.42) with the non-resonant hypothesis* **H1** *and the stability hypothesis* **H2**,

H1: $M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = O(\kappa^2)$,

H2: $\mathbf{q}$ *and* $\dot{\mathbf{q}}$ *are bounded.*

As the dispersion parameter, m, tends to infinity, we have an asymptotic expansion for the dynamics,

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + \left(V - \frac{\kappa^2}{2mc} Ones(n)\right)\mathbf{q} = O(\kappa^2 m^{\frac{-3}{2}}). \tag{4.43}$$

Since the error of the asymptotic expansion is $O(\kappa^2 m^{\frac{-3}{2}})$, the expansion will remain valid even for large coupling parameter κ, provided m is sufficiently large. The hypotheses **H1** and **H2** guarantee the validity of the expansion, but numerical simulations suggest that the expansion is accurate even when the system becomes unstable. In either case, the stability of the dynamical system (4.43) is a necessary condition for the stability of the coupled Chetaev system. If we relax the **H2** hypothesis, the asymptotic expansion holds over any finite time interval.

We also note that the asymptotic expansion is consistent with the results from energy bounds. In particular, we can use the results from section 4.2.2 to investigate the stability of the Chetaev system

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + \Big(V - \frac{\kappa^2}{2mc} Ones(n)\Big)\mathbf{q} = 0. \tag{4.44}$$

For small values of m, we retain the non-dispersive wave field dynamics,

Theorem 4.9 *An asymptotic expansion of the reduced wave field dynamics (4.42), as m tends to zero, is given by*

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = \frac{\kappa^2}{2c} Ones(n) \int_0^t \mathbf{q}(s)ds + O(m). \tag{4.45}$$

We now begin the proofs of the theorems.

Proof of Theorem 4.8: The strategy used in this proof is to reduce the coupling term to a convolution with an approximate identity though integration by parts and substitution of **H1**. To simplify notation, we define the following integrals for Bessel function parameter $\nu \in \{0,1\}$ and $n \in \{0,1,2\}$ derivatives of $\mathbf{q}$,

$$I_{n,\nu} = \int_0^t \frac{J_\nu(m(t-s))}{(t-s)^\nu} \frac{d^n\ \mathbf{q}(s)}{ds^n} ds. \tag{4.46}$$

In the new notation, our reduced system becomes

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} = \frac{\kappa^2}{2c} Ones(n) I_{0,0}. \tag{4.47}$$

Integrating $I_{0,0}$ by parts, and using the fact that $sJ_0(s) = d/ds(sJ_1(s))$ we have

$$\begin{aligned}
I_{0,0} &= \int_0^t J_0(m(t-s))\mathbf{q}(s)ds \\
&= \int_0^t J_0(m(t-s))(t-s)\frac{\mathbf{q}(s)}{t-s} ds \\
&= -\frac{1}{m} J_1(m(t-s))\mathbf{q}(s)\Big|_0^t + \frac{1}{m}\int_0^t J_1(m(t-s))(\dot{\mathbf{q}}(s) + \frac{\mathbf{q}(s)}{t-s})ds \\
&= \frac{J_1(mt)}{m}\mathbf{q}(0) + \frac{1}{m} I_{0,1} + \frac{1}{m^2} J_0(m(t-s))\dot{\mathbf{q}}(s)\Big|_0^t - \frac{1}{m^2} I_{2,0} \\
&= \frac{J_1(mt)}{m}\mathbf{q}(0) + \frac{1}{m^2}(\dot{\mathbf{q}}(t) - J_0(mt)\dot{\mathbf{q}}(0)) + \frac{1}{m} I_{0,1} - \frac{1}{m^2} I_{2,0}.
\end{aligned} \tag{4.48}$$

Similarly to the dispersive Lamb model, we recognize $I_{n,1}$ as an approximate identity convolution with $\frac{d^n\mathbf{q}}{dt^n}$ as m tends to infinity.

With the **H1** non-resonant hypothesis, we solve $I_{0,0}$ to order κ^2 in terms of approximate identity convolutions, $I_{n,1}$ and non-integral terms. In particular, we have

$$\ddot{\mathbf{q}} = -M^{-1}S\dot{\mathbf{q}} - M^{-1}V\mathbf{q} + O(\kappa^2). \tag{4.49}$$

We also integrate by parts to simplify $I_{1,0}$,

$$\begin{aligned} I_{1,0} &= \int_0^t J_0(m(t-s))(t-s)\frac{\dot{\mathbf{q}}(s)}{t-s}ds \\ &= -\frac{1}{m}J_1(m(t-s))\dot{\mathbf{q}}(s)\Big|_0^t + \frac{1}{m}\int_0^t J_1(m(t-s))\Big(\ddot{\mathbf{q}}(s) + \frac{\dot{\mathbf{q}}(s)}{t-s}\Big)ds \\ &= \frac{J_1(mt)}{m}\dot{\mathbf{q}}(0) + \frac{1}{m}I_{1,1} + \frac{1}{m}\int_0^t J_1(m(t-s))\ddot{\mathbf{q}}(s)ds. \end{aligned} \tag{4.50}$$

Employing the **H1** hypothesis, we have

$$\begin{aligned} I_{1,0} &= \frac{J_1(mt)}{m}\dot{\mathbf{q}}(0) + \frac{1}{m}I_{1,1} - \frac{1}{m}\int_0^t J_1(m(t-s))\Big(M^{-1}S\dot{\mathbf{q}}(s) + M^{-1}V\mathbf{q}(s)\Big)ds + O(\kappa^2) \\ &= \frac{J_1(mt)}{m}\dot{\mathbf{q}}(0) + \frac{1}{m}I_{1,1} - \frac{1}{m^2}J_0(m(t-s))M^{-1}S\dot{\mathbf{q}}(s)\Big|_0^t + \frac{1}{m^2}M^{-1}SI_{2,0} \\ &\quad - \frac{1}{m}\int_0^t J_1(m(t-s))M^{-1}V\mathbf{q}(s)ds + O(\kappa^2) \\ &= \Big(\frac{J_1(mt)}{m} + \frac{J_0(mt)}{m^2}M^{-1}S\Big)\dot{\mathbf{q}}(0) - \frac{1}{m^2}M^{-1}S\dot{\mathbf{q}}(t) + \frac{1}{m}I_{1,1} + \frac{1}{m^2}M^{-1}SI_{2,0} \\ &\quad - \frac{1}{m}\int_0^t J_1(m(t-s))M^{-1}V\mathbf{q}(s)ds + O(\kappa^2) \\ &= \Big(\frac{J_1(mt)}{m} + \frac{J_0(mt)}{m^2}M^{-1}S\Big)\dot{\mathbf{q}}(0) - \frac{1}{m^2}M^{-1}S\dot{\mathbf{q}}(t) + \frac{1}{m}I_{1,1} + \frac{1}{m^2}M^{-1}SI_{2,0} \\ &\quad + \frac{1}{m^2}M^{-1}V\Big(J_0(mt)\mathbf{q}(0) - \mathbf{q}(t)\Big) + \frac{1}{m^2}M^{-1}VI_{1,0} + O(\kappa^2). \end{aligned} \tag{4.51}$$

Integrating the non-resonant hypothesis **H1**, we have

$$I_{2,0} = -M^{-1}SI_{1,0} - M^{-1}VI_{0,0} + O(\kappa^2). \tag{4.52}$$

Taking linear combinations of equations (4.48), (4.51), and (4.52), we have an approximation for $I_{0,0}$ in terms of approximate identity convolutions of the form $I_{n,1}$ and non-integral terms. Using **H2** and invoking the convolution lemma 3.4, we have

$$\lim_{m\to\infty} I_{n,1} = \frac{d^n}{dt^n}\mathbf{q}(t). \tag{4.53}$$

In particular, the error in the limit is $O(m^{\frac{-1}{2}})$. Collecting highest order terms in m of the linear combination, we have

$$I_{0,0} = \frac{1}{m}I_{0,1} + O(m^{\frac{-3}{2}}) = \frac{1}{m}\mathbf{q}(t) + O(m^{\frac{-3}{2}}). \tag{4.54}$$

Hence, we have an asymptotic expansion of the perturbed dynamics

$$M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + \Big(V - \frac{\kappa^2}{2mc}Ones(n)\Big)\mathbf{q} = O(\kappa^2 m^{\frac{-3}{2}}), \tag{4.55}$$

as desired. ∎

Proof of Theorem 4.9: Since $\mathbf{q}$ analytically depends on the parameter, m, we have

$$\begin{aligned} M\ddot{\mathbf{q}} + S\dot{\mathbf{q}} + V\mathbf{q} &= \frac{\kappa^2}{2c}Ones(n)\int_0^t J_0(m(t-s))\mathbf{q}(s)ds\Big|_{m=0} + O(m) \\ &= \frac{\kappa^2}{2c}Ones(n)\int_0^t \mathbf{q}(s)ds + O(m), \end{aligned} \tag{4.56}$$

as m tends to zero. ∎

We have already shown the instability of the non-dispersive wave field coupling and the asymptotic suggest that the non-dispersive limit has a similar instability.

4.3.1 One dimensional example, simple harmonic oscillator coupled to the Klein-Gordon equation

We illustrate the asymptotic behavior firstly in the case of a one-dimensional Chetaev system coupled to a dispersive wave equation. Consider a simple harmonic oscillator coupled to the Klein-Gordon equation with the following Hamiltonian system:

$$\begin{aligned} H &= H_0 + \kappa^2 H_1 \\ H_0 &= \frac{1}{2}(\dot{x}^2 + \alpha x^2) + \frac{1}{2}\int_{\mathbb{R}} w_t^2 + c^2 w_z^2 + m^2 q^2 dz. \\ H_1 &= -\frac{1}{\kappa}w(0,t)x. \end{aligned} \tag{4.57}$$

The Hamiltonian equations of motion are

$$\begin{aligned} \ddot{x} + \alpha^2 x &= \kappa w(0,t), \\ \ddot{w} - c^2\frac{\partial^2 w}{\partial z^2} + m^2 w &= \kappa\delta(z)x. \end{aligned} \tag{4.58}$$

We can reduce the coupled system to an integral equation for a perturbed oscillator by solving for $w(0,t)$,

$$\begin{aligned} w(0,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{w}(k,t)dk \\ &= \frac{\kappa}{2\pi}\int_0^t x(s)\int_{\mathbb{R}} \frac{\sin(\omega(t-s))}{\omega}dkds \\ &= \frac{\kappa}{2c}\int_0^t J_0(m(t-s))x(s)ds, \end{aligned} \tag{4.59}$$

where $\omega = \sqrt{c^2k^2 + m^2}$ and J_0 is the Bessel function of the first kind. The perturbed oscillator remains Hamiltonian with the following dynamics:

$$\ddot{x} + \alpha^2 x = \frac{\kappa^2}{2c}\int_0^t J_0(m(t-s))x(s)ds. \tag{4.60}$$

For the large m limit, we have

$$\ddot{x} + (\alpha^2 - \frac{\kappa^2}{2mc})x \simeq 0. \tag{4.61}$$

Figure 4.3 shows a bifurcation diagram of the asymptotic results compared to a numerical solution to the full system. The maximum real part of the spectrum is plotted as a function of κ. The full coupled system is discretized with lattice size Ω and n lattice points for the wave system.

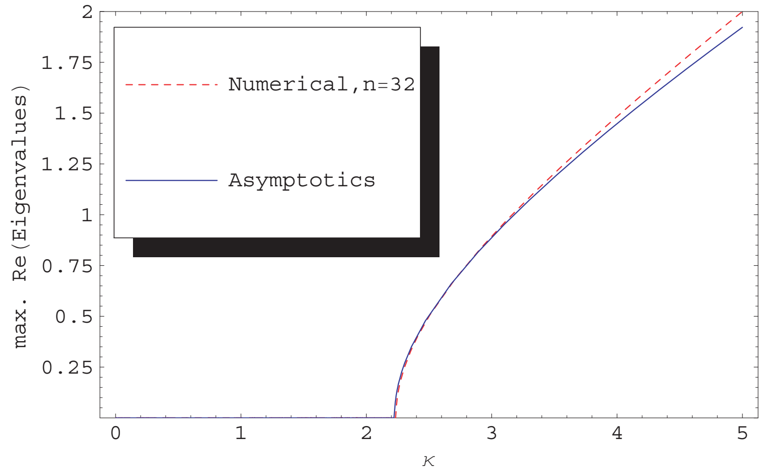

Figure 4.3: **Bifurcation of eigenvalues.** 1-D dispersive wave field coupling: $\Omega = 1$, $n = 32$, $m = 5$, $c = 1$, and $\alpha = 1$.

4.3.2 Two-dimensional gyroscopic system coupled to a dispersive wave equation

The interesting gyroscopic behavior of a Chetaev system is captured in the two dimensional model. We see from section 4.2.2, the large m coupling can effectively move an eigenvalue of V to the left. The continuous nature of the movement allow for bands of stability for small coupling parameter κ and instability for larger coupling. We demonstrate, with numerical results, that stability is maintained for small coupling parameter. See figures 4.4 and 4.5.

The effective movement of an eigenvalue of V allows for an interesting gyroscopic stabilization of an unstable Chetaev system for an interval of positive coupling parameters κ. We realize this dispersive wave field stabilization numerically and asymptotically in figure 4.6.

5 The Non-abelian Setting

In this section we present an example of wave field coupling to a more complicated mechanical system – a rigid body with rotors. The configuration space is this case is the cross product of a nonabelian group, $SO(3)$, and two copies of the circle. This system models a controlled satellite. To analyze this system we need a little more geometry:

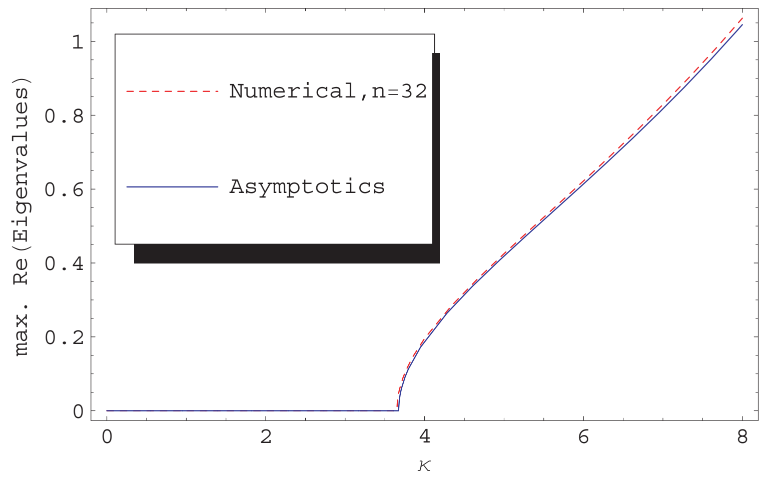

Figure 4.4: **Bifurcation of eigenvalues.** 2-D dispersive wave field coupling to stable Chetaev system: $\Omega = 1$, $n = 32$, $m = 10$, $c = 1$, $\alpha = 1$, $\beta = 2$, and $B = 3$.

5.1 Normal form of principle bundles

Here we present a geometric development of the Chetaev system as a normal form of a group action on a mechanical system. For a more complete development of the normal forms and reduction theory of non-abelian group actions see [4] and [26], from which we highlight some results. We show that the Chetaev system is the normal form of the linearized motion about a relative equilibrium of an abelian group action.

We consider the configuration space, Q, to be a Riemannian manifold with metric $\ll\ ,\ \gg$. Let G be a Lie group which acts freely on Q by isometries. By lifting the group action to the tangent bundle, TQ, or to the cotangent bundle, T^*Q, we have that G acts symplectically. For a mechanical system, the Lagrangian is of the form

$$L(q, v) = \frac{1}{2}\|v\|_q^2 - V(q), \tag{5.1}$$

and the Hamiltonian is of the form

$$H(q, p) = \frac{1}{2}\|p\|_q^2 + V(q), \tag{5.2}$$

where $\|\cdot\|_q$ is the norm on T_qQ or the norm induced on T_q^*Q, and where V is a $G-$invariant potential. Since G acts freely on Q, the projection $\pi : Q \to Q/G$, defines a principle G-bundle.

Recall that the infinitestimal generator of $\xi \in \mathfrak{g}$ on Q is denoted ξ_Q and defined

$$\xi_Q(q) = \left.\frac{d}{dt}\right|_{t=0} \exp(\xi t) \cdot q. \tag{5.3}$$

For each $q \in Q$, we can define the **locked inertia tensor** to be the map $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ defined by $\langle \mathbb{I}\eta, \xi\rangle = \ll \eta_Q(q), \xi_Q(q) \gg$. For coupled rigid bodies, $\mathbb{I}(q)$ is the classical moment of inertia tensor. We can now define a mechanical connection A.

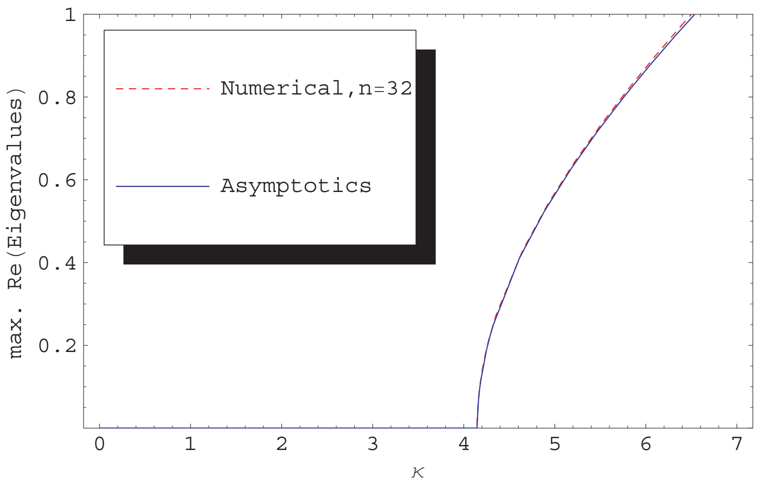

Figure 4.5: **Bifurcation of eigenvalues.** 2-D dispersive wave field coupling to gyroscopically stabilized Chetaev system: $\Omega = 1$, $n = 32$, $m = 10$, $c = 1$, $\alpha = -1$, $\beta = -2$, and $B = 3$.

Definition 5.1 *We define the* **mechanical connection** *on the principle bundle $Q \to Q/G$ to be the map $A : TQ \to \mathfrak{g}$ given by $A(q,v) = \mathbb{I}(q)^{-1}(\mathbf{J}(q,v))$, where $\mathbf{J} : TQ \to \mathfrak{g}^*$ is the* **momentum map** *defined by $\langle \mathbf{J}(q,v), \xi \rangle = \ll v, \xi_Q(q) \gg$. In other words, A is the map that assigns to each (q,v), the corresponding angular velocity of the locked system.*

For $\mu \in \mathfrak{g}^*$, we let G_μ be the isotropy subgroup for the co-adjoint action of G on $\mathfrak{g}^*$,

$$G_\mu = \{g \in G \mid Ad^*_{g^{-1}}\mu = \mu\}. \tag{5.4}$$

The mechanical connection provides a natural decomposition into a horizontal space, hor_q, and a vertical space, ver_q,

$$\begin{aligned} \mathrm{hor}_q &= \{(q,v) \mid \mathbf{J}(q,v) = 0\} \subset T_qQ & (5.5) \\ \mathrm{ver}_q &= \{\xi_Q(q) \mid \xi \in \mathfrak{g}\} = \mathfrak{g} \cdot q \subset T_qQ. & (5.6) \end{aligned}$$

The horizontal space is the space horizontal to $G-$orbits. The vertical space is the kernel of the projection map π.

We obtain the Chetaev system from the reduced dynamics of a Hamiltonian system about a **relative equilibrium**, $z_e = (q_e, v_e)$, defined by the Hamiltonian vector field at z_e pointing in the direction of the group orbit through z_e,

$$X_H(z_e) \in T_{z_e}(G \cdot z_e). \tag{5.7}$$

¿From the Relative Equilibrium Theorem, there exists a $\xi \in \mathfrak{g}$ such that z_e is a critical point of the augmented Hamiltonian, H_ξ

$$H_\xi(z) = H(z) - \langle \mathbf{J}(z) - \mathbf{J}(z_e), \xi \rangle. \tag{5.8}$$

To investigate the stability of the relative equilibrium modulo $G_{\mathbf{J}(z_e)}$, we analyze the second variation of $\delta^2 H_\xi(z_e)$ (the first variation vanishes at z_e) and the corresponding symplectic form

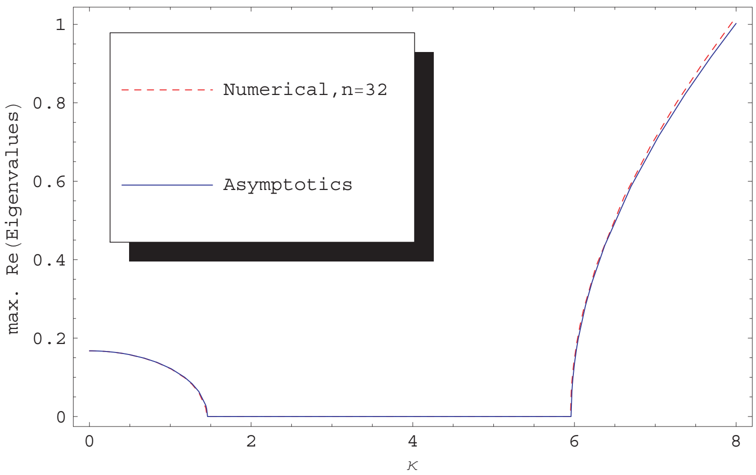

Figure 4.6: **Bifurcation of eigenvalues.** 2-D dispersive wave field coupling to unstable Chetaev system: $\Omega = 1$, $n = 32$, $m = 10$, $c = 1$, $\alpha = \frac{1}{10}$, $\beta = -2$, and $B = 3$.

Ω. Assuming that z_e is a regular point and $\mathbf{J}(z_e)$ is a generic point of $\mathfrak{g}^*$, we can define a rigid-internal splitting such that $\delta^2 H_\xi(z_e)$ block diagonalizes. The rigid component has a co-adjoint orbit representation, while the internal component, or shape-space, consists of $T^*(Q/G)$.

The rigid component is a subset of the vertical space, ver_{z_e} orthogonal to the isotropy Lie algebra, $\mathfrak{g}_\mu$, for the momentum of the relative equilibrium $\mu = \mathbf{J}(z_e)$. Explicitly, we have

$$\mathcal{V}_{RIG} = \{\xi_Q(z_e) \in T_{z_e}Q \mid \langle \mathbb{I}\xi, \eta \rangle = 0 \text{ for all } \eta \in \mathfrak{g}_{\mathbf{J}(z_e)}\}. \tag{5.9}$$

For the internal coordinates or shape-space represents the part of the system which is unaffected by the group action, Q/M. Due to the linearization of the system about the relative equilibrium, the internal variables live in $T_{[z_e]}Q/G$,

$$\mathcal{V}_{INT} = \{\delta q \in T_{z_e}Q \mid [\delta q] \in T_{[z_e]}Q/G\}. \tag{5.10}$$

In this splitting $((r, q, p) \in \mathcal{V}_{RIG} \times \mathcal{V}_{INT} \times \mathcal{V}^*_{INT})$, (see [26], [4] and references therein) $\delta^2 H_\xi$ block diagonalizes

$$\delta^2 H_\xi = \begin{bmatrix} A_\mu & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & M^{-1} \end{bmatrix}, \tag{5.11}$$

with symplectic form, Ω,

$$\Omega = \begin{bmatrix} L_\mu & C & 0 \\ -C^T & S & I \\ 0 & -I & 0 \end{bmatrix}, \tag{5.12}$$

where A_μ is the energy from the co-adjoint orbit block and V is the amended potential.

The linearized Hamiltonian vector field is defined from $\Omega(X_H, \cdot) = dH\cdot$,

$$X_H(r,q,p) = \begin{bmatrix} -L_\mu^{-1}A_\mu & 0 & -L_\mu^{-1}CM^{-1} \\ 0 & 0 & M^{-1} \\ -C^T L_\mu^{-1} A_\mu & -V & -\tilde{S}M^{-1} \end{bmatrix} \begin{bmatrix} r \\ q \\ p \end{bmatrix}, \tag{5.13}$$

where $S = \tilde{S} - C^T L_\mu C$.

5.1.1 Non-abelian example: rigid body with two internal rotors

We consider a rigid body with two symmetric internal rotors. For more detail on the stability of the rigid body see [5] and [4]. The configuration space is $SO(3) \times S^1 \times S^1$, where each of the rotors has a configuration space of S^1. The Lagrangian is a function on the tangent bundle, $T(SO(3) \times S^1 \times S^1)$. Let $\mathbb{I}_{body}$ be the inertia tensor of the rigid body, $\mathbb{I}_{rotor} = \mathrm{diag}(J_1^1, J_2^2, 0)$ the diagonal matrix of rotor inertia about the principle axes, and $\mathbb{I}'_{rotor}$ be the remaining rotor inertia about the other axes. We define the locked inertia tensor, $\mathbb{I}_{lock}$ of the full system as

$$\mathbb{I}_{lock} = \mathbb{I}_{body} + \mathbb{I}_{rotor} + \mathbb{I}'_{rotor} = \mathrm{diag}(B_1, B_2, B_3). \tag{5.14}$$

Let $\mathbb{I}_{lock} - \mathbb{I}_{rotor} = \mathrm{diag}(A_1, A_2, A_3)$. Assume $B_1 > B_2 > B_3$.

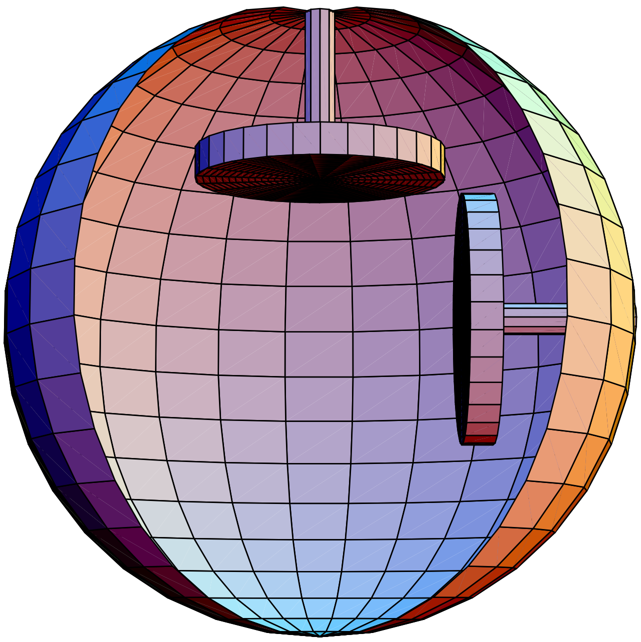

Figure 5.1: **Rigid body with two internal rotors.**

As a simple mechanical system, the Lagrangian is the kinetic energy of the system, i.e. the total kinetic energy of the body plus the total kinetic energy of the rotor.

$$\begin{aligned}L((R,\theta_r),(\Omega,\Omega_r)) &= \frac{1}{2}\Omega\cdot\mathbb{I}_{body}\Omega + \frac{1}{2}\Omega\cdot\mathbb{I}'_{rotor}\Omega + \frac{1}{2}(\Omega+\Omega_r)\cdot\mathbb{I}_{rotor}(\Omega+\Omega_r)\\ &= \frac{1}{2}\Omega\cdot(\mathbb{I}_{lock}-\mathbb{I}_{rotor})\Omega + \frac{1}{2}(\Omega+\Omega_r)\cdot\mathbb{I}_{rotor}(\Omega+\Omega_r),\end{aligned} \tag{5.15}$$

where $R\in SO(3)$ is the attitude of the rigid body relative to an inertial frame, $\theta_r\in S^1\times S^1\times\{0\}$ is the angular configuration of the rotors, $\Omega\in\mathbb{R}^3\cong\mathfrak{so}(3)$ is the vector of body angular velocities, and $\Omega_r\in\mathbb{R}^2\times\{0\}\cong T_{\theta_r}(S^1\times S^1)\times\{0\}$ is the vector of rotor angular velocities about the principle axes with respect to a body fixed frame.

The Euler-Lagrange equations of motion are

$$\begin{aligned}(\mathbb{I}_{lock}-\mathbb{I}_{rotor})\dot{\Omega} &= (\mathbb{I}_{lock}\Omega+\mathbb{I}_{rotor}\Omega_r)\times\Omega,\\ (\mathbb{I}_{lock}-\mathbb{I}_{rotor})\dot{\Omega}_r &= \Omega\times(\mathbb{I}_{lock}\Omega+\mathbb{I}_{rotor}\Omega_r)\\ \dot{R} &= R\Omega\\ \dot{\theta}_r &= \Omega_r.\end{aligned} \tag{5.16}$$

We consider the normal form of the linearized equations of motion about the relative equilibrium $z_e=(q_e,v_e)$ with $q_e=(R,0,0)$ and $v_e=(\Omega^e,\Omega_r^e)$ with $\Omega^e=(0,0,\omega)^T$ and $\Omega_r^e=(0,0,0)^T$. The relative equilibrium, z_e, is a member of an equivalence class of TQ/G, where the action of $G=SO(3)$ can be lifted from acting on Q by rotations of the rigid body to acting on the tangent bundle isometrically. Notice that the associated Hamiltonian vector field X_H on T^*Q evaluated at z_e points in the direction of the group orbit through z_e,

$$X_H(z_e)=(R\Omega^e,0,0,0)\in T_{z_e}(G\cdot z_e). \tag{5.17}$$

We will use the mechanical connection to split the configuration variables into rigid and internal components. After the reduction of the phase space, TQ/G, the rigid component of the splitting is the subset of the rigid tangent space, $\mathfrak{so}(3)$, orthogonal to the isotropy subgroup $G_\mu=S^1$. The internal component is the part of phase space which is not affected by the group action.

Firstly, we define the rigid component of the principle bundle.

$$\begin{aligned}\mathcal{V}_{RIG} &= \{\xi_Q(q_e)\in T_{q_e}Q \mid \langle\mathbb{I}(q)\xi,\eta\rangle=0 \text{ for all } \eta\in\mathfrak{s}^1\subset\mathfrak{so}(3)\}\\ &= \{\xi_Q(q_e)\in T_{q_e}Q \mid d\Omega_3\cdot\xi_Q(q_e)=0\}\\ &= \{((\delta\Omega_1,\delta\Omega_2,0)\times R,0,0,0)\in T_{q_e}Q \mid (\delta\Omega_1,\delta\Omega_2,0)\in\mathfrak{so}(3)\cong\mathbb{R}^3\}\\ &= \{(\delta\Omega_1,\delta\Omega_2)\in\mathbb{R}^2\}.\end{aligned} \tag{5.18}$$

We can realize $\mathfrak{s}^1$ as the Lie algebra to the isotropy subgroup for $\mathbf{J}(z_e)\in\mathfrak{so}(3)^*$ for the co-adjoint action of $SO(3)$ on $\mathfrak{so}(3)^*$. Thus, the rigid variables, $r=(\delta\Omega_1,\delta\Omega_2)$ are even dimensional and represent the co-adjoint orbit block.

In this case, the internal component is the $T(\mathbb{T}^2)$. We can choose the natural configuration for the shape-space, $q=M^{-1}\delta\theta_r$, with associated reduced momenta, $p=\delta\Omega_r$. From the linearized equations of motions, we have

$$\frac{d}{dt}\begin{bmatrix} r \\ q \\ p \end{bmatrix} = \begin{bmatrix} -L_\mu^{-1}A_\mu & 0 & -L_\mu^{-1}CM^{-1} \\ 0 & 0 & M^{-1} \\ -C^T L_\mu^{-1}A_\mu & -V & -\tilde{S}M^{-1} \end{bmatrix}\begin{bmatrix} r \\ q \\ p \end{bmatrix}, \tag{5.19}$$

where

$$\begin{gathered} L_\mu = \begin{bmatrix} 0 & -\frac{1}{\omega} \\ \frac{1}{\omega} & 0 \end{bmatrix} \quad A_\mu = \begin{bmatrix} \frac{B_3-B_1}{A_2} & 0 \\ 0 & \frac{B_3-B_2}{A_1} \end{bmatrix} \\ M^{-1} = \begin{bmatrix} \frac{J_1^1}{A_2} & 0 \\ 0 & \frac{J_2^2}{A_1} \end{bmatrix} \quad \tilde{S} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \\ C = -I \quad V = 0. \end{gathered} \tag{5.20}$$

5.1.2 Abelian group action

If the group G is an abelian group, the co-adjoint orbit block becomes trivial. The isotropy subgroup, G_μ, of the co-adjoint action is all of G. Hence the linearized system about a relative equilibrium contains only the internal components,

$$H_{linear} = \begin{bmatrix} V & 0 \\ 0 & M^{-1} \end{bmatrix}, \tag{5.21}$$

with symplectic form

$$\Omega = \begin{bmatrix} S & I \\ -I & 0 \end{bmatrix}. \tag{5.22}$$

The corresponding second order equations are precisely the Chetaev system

$$M\ddot{q} + S\dot{q} + Vq = 0. \tag{5.23}$$

The abelian case is a classical result dating back to [24].

5.2 Non-abelian example of a wave field coupling: rigid body with internal rotors

In the previous sections, we have coupled a wave equation to the Chetaev system and investigated the stability of the trivial equilibrium. In this section, we demonstrate a similar destabilizing effect for a wave field coupling to the relative equilibrium of a non-abelian group action. In particular, we couple the standard wave equation to a rigid body with internal rotors. We use the same notation as introduced in section 5.1.1. We model internal radiation, that is the field coupling to the internal rotors,

$$\begin{aligned} (\mathbb{I}_{lock} - \mathbb{I}_{rotor})\dot{\Omega} &= (\mathbb{I}_{lock}\Omega + \mathbb{I}_{rotor}\Omega_r) \times \Omega, \\ (\mathbb{I}_{lock} - \mathbb{I}_{rotor})\dot{\Omega}_r &= \Omega \times (\mathbb{I}_{lock}\Omega + \mathbb{I}_{rotor}\Omega_r) + \kappa\int_{\mathbb{R}} \chi(\xi)u(\xi,t)d\xi \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \\ \dot{\theta}_r &= \Omega_r \\ \ddot{u} - c^2 u_{\xi\xi} &= \kappa\chi(\xi)Ones(3)\theta_r. \end{aligned} \tag{5.24}$$

Assuming the distribution χ is the Dirac-δ distribution, the wave equation decouples in precisely the same way as in the abelian case yielding the following equations of motion for the rigid body

$$\begin{aligned}(\mathbb{I}_{lock} - \mathbb{I}_{rotor})\dot{\Omega} &= (\mathbb{I}_{lock}\Omega + \mathbb{I}_{rotor}\Omega_r) \times \Omega,\\ (\mathbb{I}_{lock} - \mathbb{I}_{rotor})\dot{\Omega}_r &= \Omega \times (\mathbb{I}_{lock}\Omega + \mathbb{I}_{rotor}\Omega_r) + \frac{\kappa^2}{2c} Ones(3) \int_0^t \Omega_r(s)ds,\\ \dot{\theta}_r &= \Omega_r,\end{aligned} \tag{5.25}$$

ignoring the free wave homogeneous contribution.

We linearize the system about the relative equilibrium $z_e \in \mathfrak{so}(3) \times T(\mathbb{T}^2)$, where $z_e = ((0,0,\omega),(0,0),(0,0))$. We have the following integral equation in terms of $Q = \begin{bmatrix} r & q & p\end{bmatrix}^T$, with $r = \delta\Omega$, $q = M^{-1}\delta\theta_r$, and $p = \delta\Omega_r$,

$$\dot{Q} = \begin{bmatrix} -L_\mu^{-1}A_\mu & 0 & -L_\mu^{-1}CM^{-1} \\ 0 & 0 & M^{-1} \\ -C^T L_\mu^{-1}A_\mu & -V + \frac{\kappa^2}{2mc}(\mathbb{I}_{lock} - \mathbb{I}_{rotor})^{-1} Ones(3) \int_0^t M \cdot ds & -\tilde{S}M^{-1} \end{bmatrix} Q, \tag{5.26}$$

where

$$\begin{aligned} L_\mu &= \begin{bmatrix} 0 & -\frac{1}{\omega} \\ \frac{1}{\omega} & 0 \end{bmatrix} \quad A_\mu = \begin{bmatrix} \frac{B_3 - B_1}{A_2} & 0 \\ 0 & \frac{B_3 - B_2}{A_1} \end{bmatrix}, \\ M^{-1} &= \begin{bmatrix} \frac{J_1^1}{A_2} & 0 \\ 0 & \frac{J_2^2}{A_1} \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \\ C &= -I, \quad V = 0. \end{aligned} \tag{5.27}$$

Differentiating the system, we reduce the integro-differential equation into a standard ODE,

$$\frac{d}{dt}\begin{bmatrix} \dot{r} \\ q \\ \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -L_\mu^{-1}A_\mu & 0 & 0 & -L_\mu^{-1}CM^{-1} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & M^{-1} \\ -C^T L_\mu^{-1}A_\mu & \frac{\kappa^2}{2mc}(\mathbb{I}_{lock} - \mathbb{I}_{rotor})^{-1} Ones(3) M & -V & -\tilde{S}M^{-1} \end{bmatrix} \begin{bmatrix} \dot{r} \\ q \\ \dot{q} \\ \dot{p} \end{bmatrix}. \tag{5.28}$$

Implicitly, we define the matrix W as

$$\frac{d}{dt}\begin{bmatrix} r \\ q \\ \dot{q} \\ \dot{p} \end{bmatrix} = W \begin{bmatrix} r \\ q \\ \dot{q} \\ \dot{p} \end{bmatrix}. \tag{5.29}$$

We can clearly see that the diagonal in W is zero; thus the sum of the eigenvalues is zero. To show instability we need only show that there exists an eigenvalue with non-zero real part. We can also see that the matrix W is rank 5 under the generic condition

$$\omega^2\kappa^2\Big(A_1(B_2 - B_3)(B_3 - B_2 + J_1^1) + A + 2(B_1 - B_3)(B_3 - B_2 + J_2^2)\Big) \neq 0. \tag{5.30}$$

Generically, W has an odd number of non-zero eigenvalues. The matrix W has real entries, so the complex eigenvalues come in conjugate pairs. There must exist an real non-zero eigenvalue. Thus, our system becomes unstable.

We have presented only one example of the non-abelian coupling, but the instability is completely analogous to the abelian or Chetaev case. We plan to address the general instability of the non-abelian case in future research.

6 Conclusion

We have presented several mechanisms through which internal energy transfer may destabilize a system. One such mechanism models radiation damping, or energy transfer from a finite dimensional subsystem to an infinite dimensional subsystem.

We have generalized the Lamb model of an oscillator coupled to a wave equation to include coupling of Chetaev systems. For gyroscopic Lamb coupling, energy is transferred from the oscillator into the wave field, inducing instability. The radiation induced instability is analogous to dissipation induced instability studied in [4]. When we include dispersion in the gyroscopic Lamb model, there is effectively a stabilizing shift in the potential of the oscillator to balance the damping term.

Similarly to the gyroscopic Lamb model, the wave field coupling can exhibit radiation induced instability. In contrast to the Lamb model, the wave field coupling to an oscillator does not rely on boundary constraints as a mechanism for energy transfer. We investigate a local coupling whose interaction with an oscillator depends on the strength of the field at a point. While the gyroscopic Lamb model induces instability in gyroscopically stabilized oscillators, the wave field coupling induces instability in all oscillators. For large dispersion, the mechanical system has restricted access to low frequency wave modes, allowing a band of stable coupling parameters to exist. Furthermore, largely dispersive wave field coupling also shifts the potential to allow even unstable oscillators to be stabilized.

In addition to dispersive and non-dispersive wave equations, future research plans include investigating the stability of the beam equation coupled to a rigid body. This is of particular interest in modeling antennae on satellites.

Another avenue of research includes coupling multiple oscillators through a wave equation or a beam equation. It might be possible to tune the system so as to trap energy between the oscillators. We can further extend the model to describe a lattice of non-linear, coupled oscillators, such as the Toda lattice, via an infinite-dimensional system.

A Spherical Pendulum

In this section, we derive the linearized equations of motion for the whirling spherical pendulum about the inverted equilibrium.

Consider a rigid spherical pendulum moving in a constant gravitational field. Let θ, ϕ, and r be the standard spherical coordinates for the pendulum.

We compute the equations of motion in a rotating frame. Since our pendulum is rigid, r is constant. The Lagrangian is

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 - U(\mathbf{q}) = \mathbf{\frac{1}{2}m(r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2) - mg}\cos\theta. \tag{A.1}$$

The Euler-Lagrange equation of motion are

$$mr^2\ddot{\theta} - mr^2\dot{\phi}^2\sin\theta\cos\theta - mg\sin\theta = 0 \tag{A.2}$$
$$mr^2(\sin^2\theta\ddot{\phi} + 2\sin\theta\cos\theta\dot{\phi}\dot{\theta}) = 0. \tag{A.3}$$

Expanding the Lagrangian about $\theta = \pi$, the stable equilibrium, and keeping terms to second order in θ yields the following Lagrangian

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\theta^2\dot{\phi}^2) + mg(1 - \frac{1}{2}\theta^2), \tag{A.4}$$

with Euler-Lagrange equations of motion

$$mr^2\ddot{\theta} - mr^2\dot{\phi}^2\theta + mg\theta = 0 \tag{A.5}$$
$$mr^2(\theta^2\ddot{\phi} + 2\theta\dot{\phi}\dot{\theta}) = 0. \tag{A.6}$$

In a rotating coordinate system, we make the change of variables $x = r\theta\cos(\phi - \psi)$, $y = r\theta\sin(\phi - \psi)$, where ψ is the angle of rotation. Let the angular velocity $\omega = \dot{\psi}$ be constant. In the new coordinates the Lagrangian becomes

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m((\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2) + mg(1 - \frac{x^2 + y^2}{2}). \tag{A.7}$$

Our linearized equations of motion are

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} + (g - \omega^2)x &= 0 && \text{(A.8)} \\ \ddot{y} + 2\omega\dot{x} + (g - \omega^2)y &= 0. && \text{(A.9)} \end{aligned}$$

We now physically motivate the rotating frame by modeling a rotor, with moment of inertia I, which rotates the pendulum. The Lagrangian is as follows

$$L(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - mg\cos\theta + \frac{1}{2}I\omega^2. \tag{A.10}$$

The Euler-Lagrange equations of motion remain unchanged with an additional equation

$$mr^2\ddot{\theta} - mr^2\dot{\phi}^2\sin\theta\cos\theta - mg\sin\theta = 0 \tag{A.11}$$
$$mr^2(\sin^2\theta\ddot{\phi} + 2\sin\theta\cos\theta\dot{\phi}\dot{\theta}) = 0 \tag{A.12}$$
$$\dot{\omega} = 0. \tag{A.13}$$

Without a coupling term, a rotating frame is indistinguishable from the inertial frame. The linearization remains identical.

We now wish to stabilize the inverted equilibrium by creating a restoring potential that breaks the symmetry of the motion of the pendulum. Imagine the pendulum with the pivot at the origin. Assume there is an asymmetric potential within the sphere. In particular, we a potential V dependent on θ and $\phi - \psi$, where ψ is the angle of equitorial rotation of the spherical shell and ϕ is the standard angle of the the pendulum.

Assuming that the rotor that is spinning the spherical shell is rotating with a constant angular velocity ω, we compute the Lagrangian

$$L(\theta,\phi,\dot{\theta},\dot{\phi}) = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - mg\cos\theta - V(\theta,\phi-\omega t). \tag{A.14}$$

The Euler-Lagrange equations of motion are

$$\begin{aligned} mr^2\ddot{\theta} - mr^2\dot{\phi}^2\sin\theta\cos\theta - mg\sin\theta + \frac{\partial V(\theta,\phi-\omega t)}{\partial\theta} &= 0 \\ mr^2(\sin^2\theta\ddot{\phi} + 2\sin\theta\cos\theta\dot{\phi}\dot{\theta}) + \frac{\partial V(\theta,\phi-\omega t)}{\partial\phi} &= 0. \end{aligned} \tag{A.15}$$

We work out the equations of motion for an explicit asymmetric potential V

$$V(\theta,\phi-\psi) = -\frac{mr^2\sin^2\theta}{2}(\alpha\cos^2(\phi-\psi) + \beta\sin^2(\phi-\psi)). \tag{A.16}$$

Expanding the Lagrangian about the inverted equilibrium $\theta = \pi$, and in the rotating coordinate system, we have $x = r\theta\cos(\phi-\psi)$, $y = r\theta\sin(\phi-\psi)$ and

$$L = \frac{1}{2}m((\dot{x}-\omega y)^2 + (\dot{y}+\omega x)^2) + mg(1 - \frac{x^2+y^2}{2}) + \frac{m}{2}(\alpha x^2 + \beta y^2). \tag{A.17}$$

Our linearized equations of motion are

$$\ddot{x} - 2\omega\dot{y} + (g - \omega^2 - \alpha)x = 0 \tag{A.18}$$

$$\ddot{y} + 2\omega\dot{x} + (g - \omega^2 - \beta)y = 0. \tag{A.19}$$

We would like to perturb the motion of the spherical pendulum by coupling it to a magnetic field.

Let $\mathbf{B}$ be divergence-free vector field. Let $\mathbf{A}$ be the vector potential, $\mathbf{B} = \nabla\times\mathbf{A}$. Note if $\mathbf{B}$ is a constant magnetic field, we can choose $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{q}$.

Assume the the bob on a spherical pendulum has a charge e and mass m. The Lagrangian, as a function from TS^2 to $\mathbb{R}$, is

$$\begin{aligned} L(\mathbf{q},\dot{\mathbf{q}}) &= \frac{1}{2}m\|\dot{\mathbf{q}}\|^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{q}} - U(\mathbf{q}) && \text{(A.20)} \\ &= \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - mg\cos\theta + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{q}}. && \text{(A.21)} \end{aligned}$$

We now assume that the magnetic field $\mathbf{B}$ is constant, parallel to the gravitational field. The Lagrangian becomes

$$L(\theta,\phi,\dot{\theta},\dot{\phi}) = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - mg\cos\theta + \frac{e}{2c}B\dot{\phi}r^2\sin^2\theta. \tag{A.22}$$

The Euler-Lagrange equations are

$$mr^2\ddot{\theta} - mr^2\sin\theta\cos\theta\dot{\phi}^2 - mg\sin\theta - \frac{e}{c}B\dot{\phi}r^2\sin\theta\cos\theta = 0 \tag{A.23}$$

$$mr^2\sin^2\theta\ddot{\phi} + mr^2\dot{\theta}\sin\theta\cos\theta(\dot{\phi} + \frac{eB}{2mc}) = 0 \tag{A.24}$$

We expand the Lagrangian about $\theta = \pi$ and keep terms to second order in θ. In the coordinates $x = r\theta\cos(\phi - \psi)$, $y = r\theta\sin(\phi - \psi)$ we have the Lagrangian

$$L = \frac{1}{2}m((\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2) \quad + mg - \frac{mg + \frac{eB\omega}{c}}{2}(x^2 + y^2) + \frac{eB}{2c}(x\dot{y} - y\dot{x}). \qquad \text{(A.25)}$$

The angular velocity of rotation ω is constant; hence the Euler-Lagrange equations follow

$$\ddot{x} - (2\omega + \frac{eB}{mc})\dot{y} + (g - \frac{eB\omega}{mc} - \omega^2)x = 0 \qquad \text{(A.26)}$$

$$\ddot{y} + (2\omega + \frac{eB}{mc})\dot{x} + (g - \frac{eB\omega}{mc} - \omega^2)y = 0. \qquad \text{(A.27)}$$

Similarly, linearization about the inverted equilibrium yield the following equations of motion

$$\ddot{x} - (2\omega + \frac{eB}{mc})\dot{y} - (g + \frac{eB\omega}{mc} + \omega^2)x = 0 \qquad \text{(A.28)}$$

$$\ddot{y} + (2\omega + \frac{eB}{mc})\dot{x} - (g + \frac{eB\omega}{mc} + \omega^2)y = 0. \qquad \text{(A.29)}$$

We can increase the magnetic field to *gyroscopically* stabilize the inverted equilibrium. Explicitly, for $B^2 \geq \frac{4gm^2c^2}{e^2}$, the inverted equilibrium is gyroscopically stabilized.

B Network Theory

It is also of interest to discuss the linked Chetaev system and the wave equation from the point of view of network theory and linear systems theory. As have seen, the wave coupling induces instability. It is of interest (see [3]) to analyze infinite-dimensional systems from the point of view of systems theory and in particular to understand asymptotic stability or instability. One problem of interest is the Darlington synthesis problem (see e.g. [7], [17] and references therein) where one shows that one can realize any positive real transfer functions by terminating a lossless 2-port by a 1-ohm resistor. One can then extend this to the infinite-dimensional Hamiltonian setting by terminating by an infinite transmission line.

The situation here is somewhat different but related:

In particular, we view the Chetaev system as a two port connected to a wave system with integral coupling. The previous calculations show that this system can be reduced to a two port with integral feedback.

Consider the two degree of freedom Chetaev system with inputs:

$$\begin{aligned} \ddot{x} - B\dot{y} + \alpha x &= u_x, \\ \ddot{y} + B\dot{x} + \beta y &= u_y, \end{aligned} \qquad \text{(B.1)}$$

Without coupling, we can think of this system as a linear four dimensional first order system with 2 inputs u_x and u_y (to be determined) and two outputs x and y. This makes the Chetaev system a classical two port.

We can also think of the wave field as a first order system:

$$\frac{\partial}{\partial t}\mathbf{w} = A\mathbf{w}, \qquad \text{(B.2)}$$

where $\mathbf{w} = [w, \dot{w}]^T$ and $A = \begin{bmatrix} 0 & 1 \\ m^2 + c^2 \frac{\partial^2}{\partial \xi^2} & 0 \end{bmatrix}$. We think of the input as the second variable $\dot{w}$.

With the integral coupling, we connect the finite and the infinite systems through their inputs. Equivalently, the Chetaev system is given dynamical feedback through the wave equation. In this context, the reduced system in each type of coupling (3.35 and 4.42) becomes a first order system with integral feedback. Moreover, the integral feedback affects the Chetaev system in a not-necessarily Hamiltonian fashion: as we have seen we may only have one unstable eigenvalue.

We remark also that the instead of the mechanical systems discussed here we can physically realize the Chetaev system via coupled LC-circuits. The discussion also extends essentially without change to the n-port case.

C Bessel Functions

In this appendix, we explicitly derive the integral formula,

$$2\int_m^\infty \frac{\sin \omega s}{c\sqrt{\omega^2 - m^2}} d\omega = \frac{\pi}{c} J_0(ms), \tag{C.1}$$

where J_0 is the Bessel function of the first kind.

Let us begin by taking the Laplace transform, denoted $\mathcal{L}[\cdot]$, with respect to s of the right hand side of equation (C.1),

$$\mathcal{L}\Big[2\int_m^\infty \frac{\sin \omega s}{c\sqrt{\omega^2 - m^2}} d\omega\Big] = 2\int_0^\infty \int_m^\infty \frac{\sin \omega s}{c\sqrt{\omega^2 - m^2}} e^{-sz} d\omega ds, \tag{C.2}$$

provided $\mathrm{Re}(z) > 0$. Integrating with respect to s, we have

$$\int e^{-sz} \sin \omega s \; ds = -e^{sz} \frac{\omega \cos \omega z + z \sin \omega s}{\omega^2 + z^2}. \tag{C.3}$$

and

$$\mathcal{L}\Big[2\int_m^\infty \frac{\sin \omega s}{c\sqrt{\omega^2 - m^2}} d\omega\Big] = 2\int_m^\infty \frac{w}{c\sqrt{\omega^2 - m^2}(\omega^2 + z^2)} d\omega. \tag{C.4}$$

With an appropriate substitution, we use the integral

$$\int \frac{w}{\sqrt{\omega^2 - m^2}(\omega^2 + z^2)} d\omega = \frac{\arctan\left(\frac{\sqrt{\omega^2 - m^2}}{\sqrt{m^2 + z^2}}\right)}{\sqrt{m^2 + z^2}}, \tag{C.5}$$

to finish computing the Laplace transform,

$$\mathcal{L}\Big[2\int_m^\infty \frac{\sin \omega s}{c\sqrt{\omega^2 - m^2}} d\omega\Big] = \frac{\pi}{c\sqrt{m^2 + z^2}}. \tag{C.6}$$

To finish deriving equation (C.1), we need only compute the Laplace transform of $J_0(s)$. This computation can be done explicitly with contour integration (see [9]), however we present a more dynamic approach. From the Bessel differential equation, of which $J_0(s)$ is a solution, we have

$$sJ_0''(s) + J_0'(s) + sJ_0(s) = 0, \qquad J_0(0) = 1, \qquad J_0'(0) = 0, \qquad \int_0^\infty J_0(s)ds = 1. \tag{C.7}$$

Taking the Laplace transform of the differential equation, we obtain,

$$-\frac{d}{dz}\Big(z^2\mathcal{L}\Big[J_0\Big] - J_0'(0) - zJ_0(0)\Big) + z\mathcal{L}\Big[J_0\Big] - J_0(0) - \frac{d}{dz}\mathcal{L}\Big[J_0\Big] = 0. \tag{C.8}$$

We solve the above differential equation, with the initial condition $\mathcal{L}\Big[J_0\Big](0) = \int_0^\infty J_0(s)ds = 1$,

$$\mathcal{L}\Big[J_0\Big] = \frac{1}{\sqrt{1+z^2}}. \tag{C.9}$$

A change of variables produces the desired result,

$$\mathcal{L}\Big[\frac{\pi}{c}J_0(ms)\Big] = \frac{\pi}{c\sqrt{m^2+z^2}}. \tag{C.10}$$

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