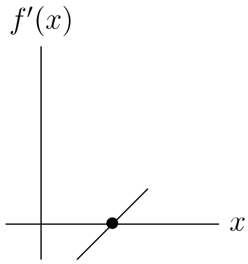
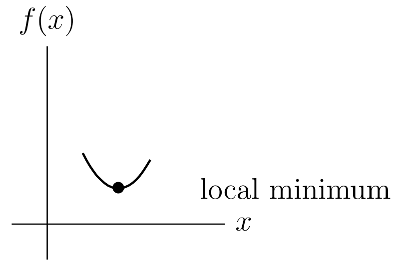
4 Using The Derivative

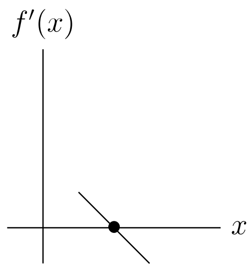
4.1 Using the first and second derivative

Recall the properties of the first and second derivatives:

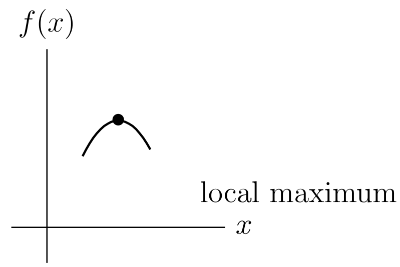


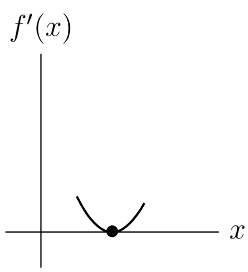
$f(x)$ is ↘ then ↗



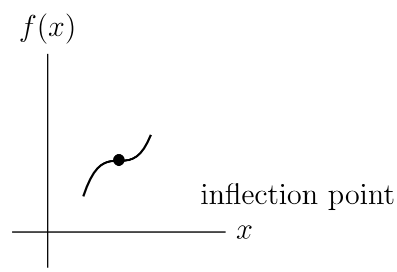


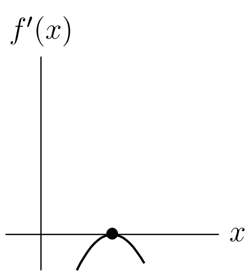
$f(x)$ is ↗ then ↘



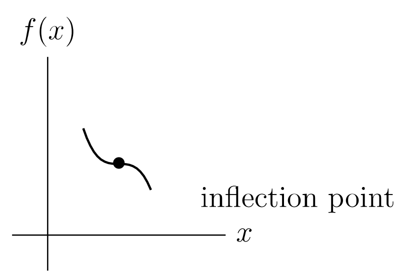


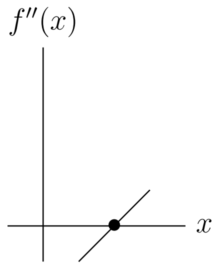
$f(x)$ is ↗ then ↗



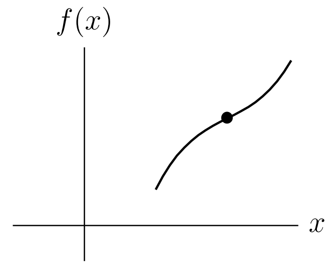


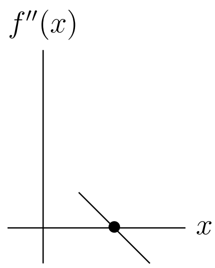
$f(x)$ is ↗ then ↗



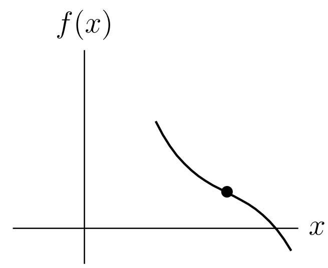


$f(x)$ is $\cap$ then $\cup$





$f(x)$ is $\cup$ then $\cap$



We are in the business of detecting those local extreme for **CTS** functions.
We first define a **Critical Point**: $x = p$ is a critical point of $f(x)$ if $f'(p) = 0$ or undefined.
These points are the candidates for local extrema. We can check each using the **First Derivative test:**

- If $f'(x) > 0$ to the left of p and $f'(x) < 0$ to the right of it then it is a local maximum.
- If $f'(x) < 0$ to the left of p and $f'(x) > 0$ to the right of it then it is a local minimum.

Or using the **Second derivative test:**

- $f'(p) = 0$ and $f''(p) > 0$ then minimum.
- $f'(p) = 0$ and $f''(p) < 0$ then maximum.
- $f'(p) = 0$ and $f''(p) = 0$ then inconclusive.

An inflection point is when $f(x)$ is CTS and changes from being concave up to concave down or the other way around. So if $x = p$ is an inflection point: $f(x)$ CTS at $x = p$, and $f''(p) = 0$ or undefined, and $f''(x)$ changes signs around it.

To sum up - how to find local extrema:

- Find all critical points.
- Sketch a number line and mark where $f'(x)$ is positive or negative.
- Draw a conclusion.

To sum up - how to find inflection points:

- Find all values where $f''(x) = 0$ or undefined.
- Sketch a number line and mark where $f''(x)$ is positive or negative.
- Draw a conclusion.

Example 4.1.1. Find the local extrema of $2x^3 - 9x^2 + 12x - 4$ and inflection points:
Find all critical points:

Draw a number line to get the types of the local points:

$f'(0.5) > 0, \nearrow$ $\quad f'(1.5) < 0, \searrow$ $\quad f'(2.2) > 0, \nearrow$

$x = 1$, $f'(1) = 0$ $\quad x = 2$, $f'(2) = 0$

So we conclude:

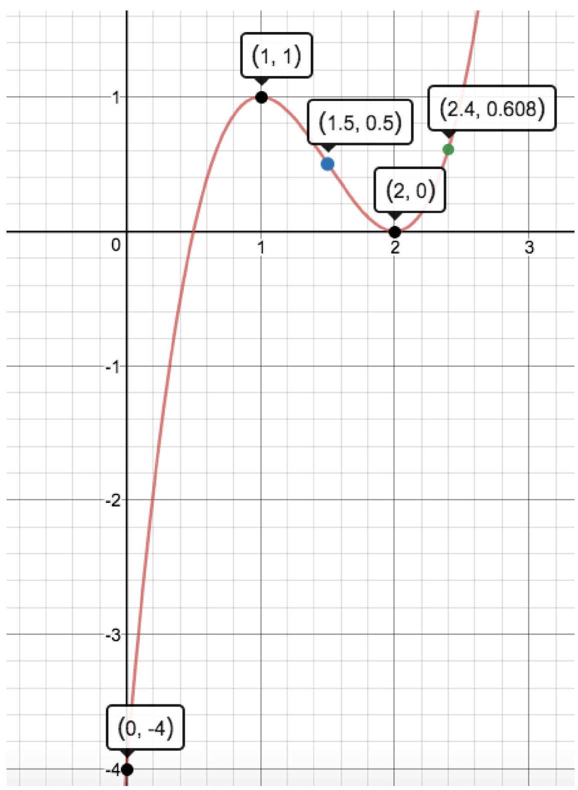
- $f(1) = 1$, local max
- $f(2) = 0$, local min

For the inflection points, $f''(x) = 12x - 18$ so the only candidate for an inflection point is $x = 18/12 = 1.5$.

$f''(1) < 0, \cap$ $\quad f''(2) > 0, \cup$

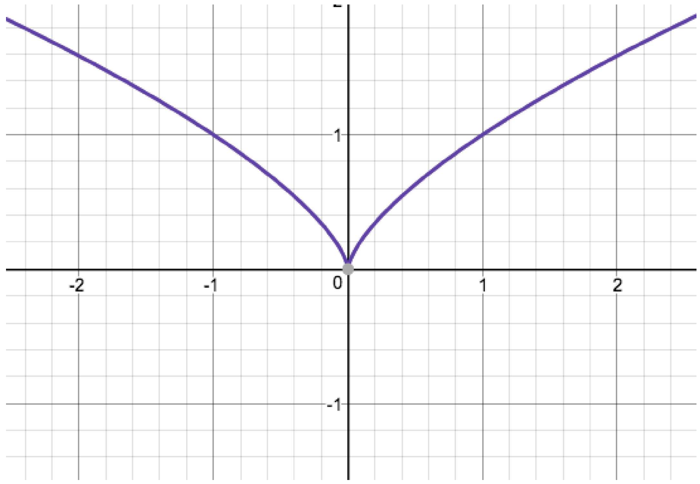
$x = 0$ $\quad x = 1.5$, $f''(1.5) = 0$

So indeed it is an inflection point. Together we get:



Example 4.1.2. Find local extreme of $f(x) = x^{2/3}$.
$f'(x) = (2/3)x^{-1/3}$, it is never 0, but it is undefined for $x = 0$. So this is the only critical point.

$$f'(-1) < 0, f'(1) > 0 \Rightarrow x = 0 \text{ is a local min}$$


1
-1
-2
-1
0
1
2

Exercise 4.1.3.

3. [10 points]
Find the x- and y-coordinates of all local minima, local maxima, and inflection points of the function $f(x)$ defined below. Your answers may involve the positive constant B. You must clearly mark your answers and provide justification to receive credit.

$$f(x) = e^{-18x^2+B}$$

Solution.

3. [10 points]
Find the x- and y-coordinates of all local minima, local maxima, and inflection points of the function $f(x)$ defined below. Your answers may involve the positive constant B. You must clearly mark your answers and provide justification to receive credit.

$$f(x) = e^{-18x^2+B}$$

Solution: We will need both $f'(x)$ (chain rule) and $f''(x)$ (product rule and chain rule).

$$\begin{aligned} f'(x) &= e^{-18x^2+B}(-36x) = -36xe^{-18x^2+B} \\ f''(x) &= -36e^{-18x^2+B} + (-36x)e^{-18x^2+B}(-36x) \\ &= -36e^{-18x^2+B}(1-36x^2) \end{aligned}$$

First we will find the critical points. We know f' is never undefined, and $f'(x) = 0$ only when $x = 0$, since e^{-18x^2+B} is always positive. The y-value for $x = 0$ is $e^{-18 \cdot 0^2+B} = e^B$ Thus, $(0, e^B)$ is the only critical point. Since $f''(0) = -36e^B(1) = -36e^B$ is negative, we know this point is a local maximum.

LOCAL MAXIMUM AT $(0, e^B)$.

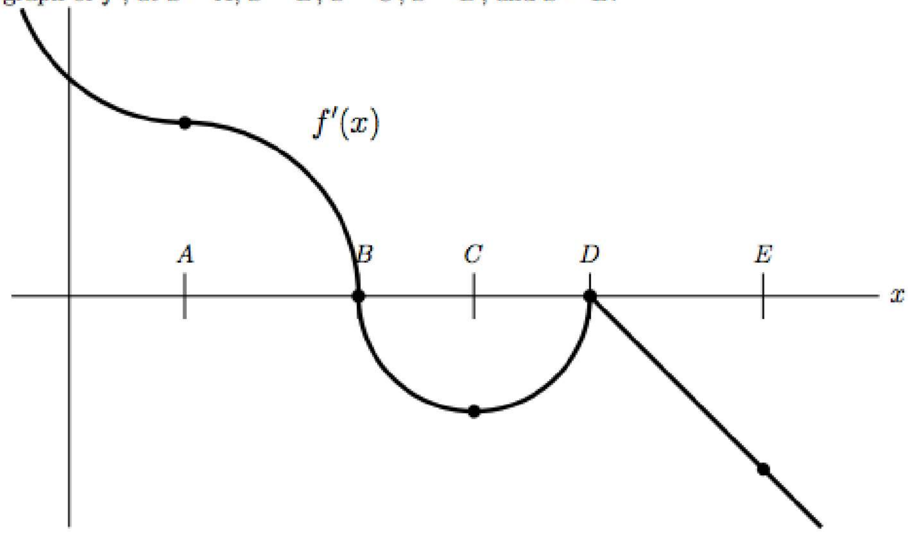
Next we will find potential inflection points. We know f'' is never undefined, and $f''(x) = 0$ when $1 - 36x^2 = 0$, since e^{-18x^2+B} is always positive. Solving, we find that $f''(x) = 0$ when $x = \pm\frac{1}{6}$. Both of these points have a y-value of $e^{-18(\frac{1}{6})^2+B} = e^{-\frac{1}{2}+B}$. We need to test f'' near these x-values to check whether we actually have inflection points.
When $x < -\frac{1}{6}$, $f''(x) > 0$.
When $-\frac{1}{6} < x < \frac{1}{6}$, $f''(x) < 0$.
When $x > \frac{1}{6}$, $f''(x) > 0$.
Since f'' changes sign at both of these points, f changes concavity at both points, so both are inflection points.

INFLECTION POINTS AT $(-\frac{1}{6}, e^{-\frac{1}{2}+B})$ and $(\frac{1}{6}, e^{-\frac{1}{2}+B})$.

□

Exercise 4.1.4.

6. [12 points] The *derivative* of a function f is graphed below. Five points are marked on the graph of f', at $x = A$, $x = B$, $x = C$, $x = D$, and $x = E$.



For each of the following, circle ALL answers which are correct. Each part has at least one correct answer. Pay careful attention to whether each question is asking about f, f', or f''.

a. [2 points] The function f' has a local minimum when ____________.

$x = A$ $\quad$ $x = B$ $\quad$ $x = C$ $\quad$ $x = D$ $\quad$ $x = E$

b. [2 points] The function f is increasing when ____________.

$x = A$ $\quad$ $x = B$ $\quad$ $x = C$ $\quad$ $x = D$ $\quad$ $x = E$

c. [2 points] The function f has a critical point when ____________.

$x = A$ $\quad$ $x = B$ $\quad$ $x = C$ $\quad$ $x = D$ $\quad$ $x = E$

d. [2 points] The global maximum of f on the interval $A \le x \le E$ occurs when ____________.

$x = A$ $\quad$ $x = B$ $\quad$ $x = C$ $\quad$ $x = D$ $\quad$ $x = E$

e. [2 points] The function f has an inflection point when ____________.

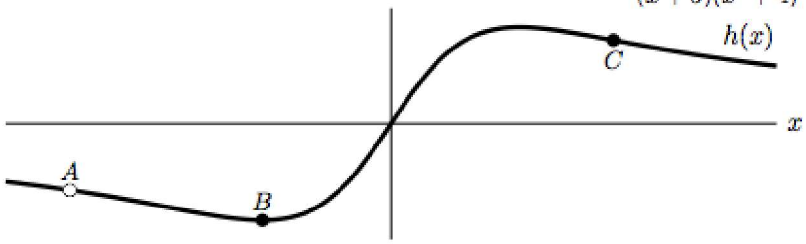
$x = A$ $\quad$ $x = B$ $\quad$ $x = C$ $\quad$ $x = D$ $\quad$ $x = E$

f. [2 points] The function f'' is undefined when ____________.

$x = A$ $\quad$ $x = B$ $\quad$ $x = C$ $\quad$ $x = D$ $\quad$ $x = E$

Exercise 4.1.5.

8. [13 points] Below, there is a graph of the function $h(x) = \dfrac{2x^2 + 10x}{(x+5)(x^2+4)}$.



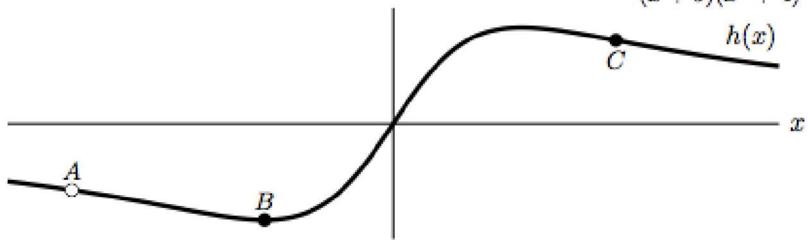
a. [3 points] The point A is a hole in the graph of h. Find the x- and y-coordinates of A.

b. [5 points] The point B is a local minimum of h. Find the x- and y-coordinates of B.

c. [5 points] The point C is an inflection point of h. Find the x- and y-coordinates of C.

Solution.

8. [13 points] Below, there is a graph of the function $h(x) = \dfrac{2x^2+10x}{(x+5)(x^2+4)}$.



a. [3 points] The point A is a hole in the graph of h. Find the x- and y-coordinates of A.

Solution: Simplifying $h(x)$, we have $h(x) = \dfrac{2x(x+5)}{(x+5)(x^2+4)}$. Since the factor $(x+5)$ cancels, the hole occurs when $x = -5$. We look at the limit as x approaches -5 on the cancelled form to get the y-coordinate:

$$\lim_{x\to -5} h(x) = \lim_{x\to -5} \frac{2x}{x^2+4} = \frac{-10}{29},$$

Thus, $A = (-5, \frac{-10}{29})$.

b. [5 points] The point B is a local minimum of h. Find the x- and y-coordinates of B.

Solution: Using the quotient rule on the simplified form of h, we have $h'(x) = \dfrac{4-x^2}{(x^2+4)^2}$. This is never undefined, and it is equal to zero when $4 - x^2 = 0$ or $x = \pm 2$. From the graph, we can see that the local minimum occurs at $x = -2$. The y-coordinate here is $y = \frac{-4}{8} = -\frac{1}{2}$, so $B = (-2, -\frac{1}{2})$.

c. [5 points] The point C is an inflection point of h. Find the x- and y-coordinates of C.

Solution: We use the quotient rule again to find $h''(x) = \dfrac{2x^3-24x}{(x^2+4)^3} = \dfrac{2x(x^2-12)}{(x^2+4)^3}$. This is never undefined, and it is zero when $2x(x^2-12) = 0$, i.e. when $x = 0, \pm 2\sqrt{3}$. From the graph, we see that our x-coordinate must be $+2\sqrt{3}$, and then $y = \frac{4\sqrt{3}}{16} = \frac{\sqrt{3}}{4}$, so $C = (2\sqrt{3}, \frac{\sqrt{3}}{4})$.

□

Exercise 4.1.6.

2. [14 points] The table for the *derivative* of a function h with continuous first derivative is given below. Assume that between each consecutive value of x, the derivative h' is either increasing or decreasing. For each statement below, indicate whether the statement is true, false, or cannot be determined from the information given. No partial credit will be given.

x	-4	-3	-2	-1	0	1	2	3	4
$h'(x)$	2	3	1	-3	-4	-2	0	2	1

a.) The function h has a local maximum on the interval $-2 < x < -1$.

True False Not enough information

b.) The function h is negative on the interval $-1 < x < 1$.

True False Not enough information

c.) The function h is concave up on the interval $0 < x < 4$.

True False Not enough information

d.) The function h is decreasing on the interval $-3 < x < -2$.

True False Not enough information

e.) The function h has an inflection point on the interval $-1 < x < 1$.

True False Not enough information

f.) The derivative function, h', has a critical point at $x = 2$.

True False Not enough information

g.) The second derivative function, h'', is positive on the interval $0 < x < 3$.

True False Not enough information

4.2 Global Optimization

Given a function $f(x)$ we would like to find its global max/min - the points where $f(x)$ attain the biggest/lowest values. Those values can be on local extrema, on endpoints, or DNE.
Case 1: domain is a closed interval and the functions is **CTS**. Then we have the **Extreme Value Theorem** (EVT) that tells us that the global extrema exists and it is one of the local extrema or the endpoints.

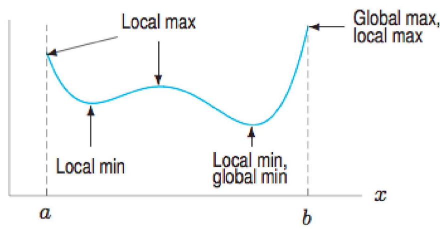


Figure 4.18: Global maximum and minimum on a closed interval $a \leq x \leq b$

In this case:

- Find all all critical points.
- Evaluate $f(x)$ at all critical points and endpoints.
- Take the biggest/lowest value

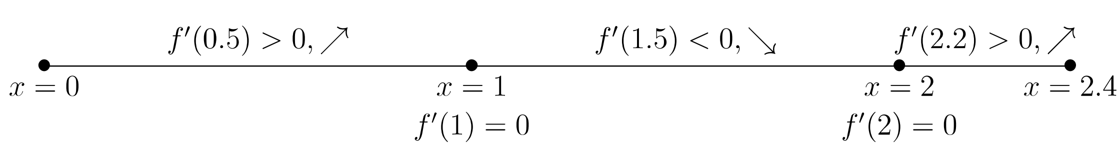
Example 4.2.1. Find the global extrema of $2x^3 - 9x^2 + 12x - 4$ on the domain [0,2.4] and graph the function:
Closed interval so we can use the EVT:
Find all critical points:

$$f'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2) = 0 \Rightarrow x = 1, 2$$

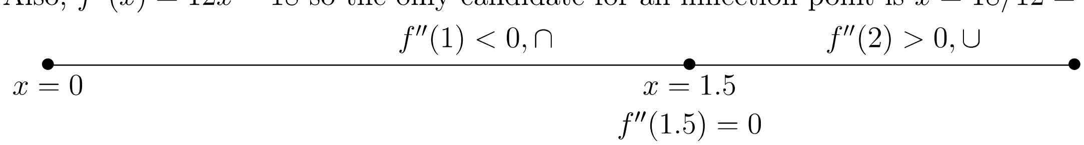
So we need to compare 4 values:

- $f(0) = -4$, end point, global min
- $f(1) = 1$, local max and global max
- $f(2) = 0$, local min
- $f(2.4) = 0.608$, end point.

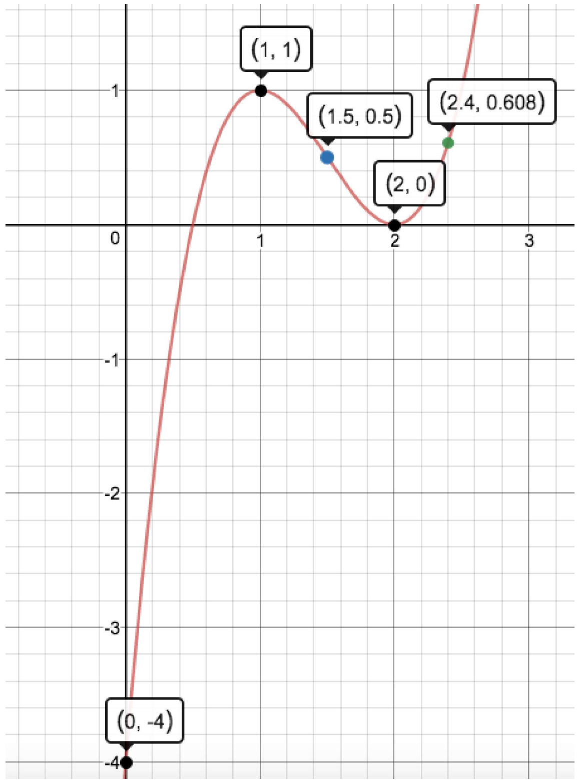
Draw a number line to get the types of the local points (not required for the global extrema but useful for the graph):



Also, $f''(x) = 12x - 18$ so the only candidate for an inflection point is $x = 18/12 = 1.5$.



Together we get:



Case 2: If either the function is not CTS or the domain is not a closed interval we have to:

- Test all critical points.
- Sketch the graph and check:
  - behavior at endpoints and points of discontinuity.
  - behavior at vertical asymptotes, if any,
  - behavior at $\pm\infty$ if part of the domain.

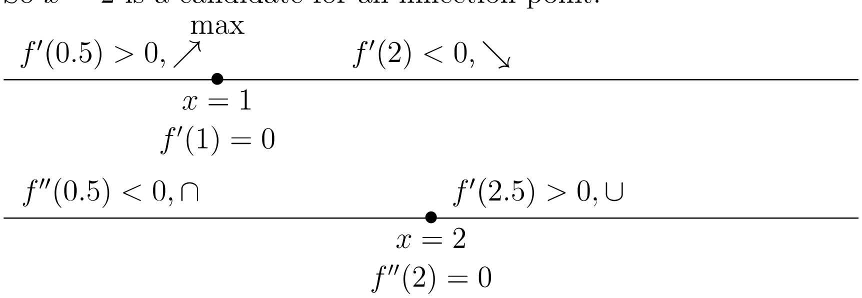
Example 4.2.2. $f(x) = xe^{-x}$. Find global Extrema on $[0, \infty)$.
Since the domain is not a closed interval, we cannot use the EVT. Let us take the derivatives and sketch the graph:

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

So only $x = 1$ is a critical point.

$$f''(x) = -e^{-x} - (1-x)e^{-x} = e^{-x}(x-2)$$

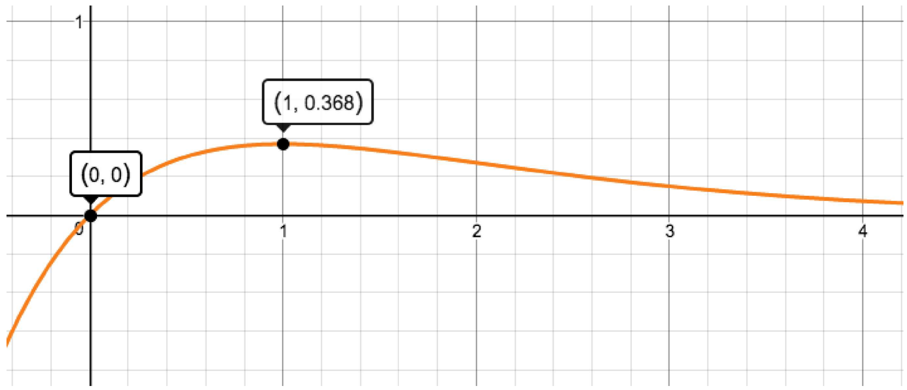
So $x = 2$ is a candidate for an inflection point.



since e^x dominates x, as $x \to \infty$, $f(x) \to 0$, while $f(x) > 0$ for $x > 0$. To sum up the relevant information:

- $(0, f(0)) = (0, 0)$ is a global min.
- $f(x \to \infty) = 0$ but $f(x)$ is positive while doing so.

This is the graph:



Exercise 4.2.3.

7. [15 points] Suppose a is a positive constant and

$$f(x) = 2x^3 - 3ax^2.$$

a. [10 points] Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[-a, \frac{3}{2}a]$. Specify all x values where the maximum and minimum value are achieved.

b. [5 points] Find all inflection points of $f(x)$.

Solution.

7. [15 points] Suppose a is a positive constant and

$$f(x) = 2x^3 - 3ax^2.$$

a. [10 points] Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[-a, \frac{3}{2}a]$. Specify all x values where the maximum and minimum value are achieved.

Solution: Seeking critical points, we take the derivative of f and set it equal to zero.

$$f'(x) = 6x^2 - 6ax = 6x(x - a) = 0.$$

Using this equation we find the critical points to be $x = 0, a$. Now we put the critical points and the endpoints of the interval back into the orginal function and compare the values. We compute $f(-a) = -5a^3$, $f(0) = 0$, $f(a) = -a^3$, $f(\frac{3}{2}a) = 0$.
This means the absolute max of f on this interval is 0 and this value is achieved at $x = 0, \frac{3}{2}a$. The absolute min is $-5a^3$ and this value is achieved at $x = -a$.

b. [5 points] Find all inflection points of $f(x)$.

Solution: Seeking inflection points, we find $f''(x) = 12x - 6a$. Setting this equal to zero we find $x = \frac{a}{2}$. To verify this is an inflection point we test $f''(0) = -6a < 0$ and $f''(a) = 6a > 0$. This means f'' changes sign at $x = \frac{a}{2}$, so it is an inflection point.

□

Exercise 4.2.4.

7. [11 points] Consider the continuous function

$$f(x) = \begin{cases} x \cdot 2^{-x} & 1 \leq x < 3, \\ \frac{1}{2-x} + \frac{11}{8} & 3 \leq x \leq 5. \end{cases}$$

Note that the domain of f is $[1, 5]$.

a. [7 points] Find the x-values of the critical points of f.

b. [4 points] Find the y-values of the global maximum and global minimum of f if they exist, or explain why they don't exist.

Exercise 4.2.5.

8. [11 points] A function $g(t)$ and its derivative are given by

$$g(t) = 10e^{-0.5t}(t^2 - 2t + 2) \qquad \text{and} \qquad g'(t) = -10e^{-0.5t}(0.5t^2 - 3t + 3).$$

a. [2 points] Find the t-coordinates of all critical points of $g(t)$. If there are none, write NONE. For full credit, you must find the exact t-coordinates.

Answer: Critical point(s) at $t =$ ________________

b. [6 points] For each of the following, find the values of t that maximize and minimize $g(t)$ on the given interval. Be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE in the answer blank if appropriate.

(i) Find the values of t that maximize and minimize $g(t)$ on the interval $[0, 8]$.

Answer: Global max(es) at $t =$ ________ Global min(s) at $t =$ ________

(ii) Find the values of t that maximize and minimize $g(t)$ on the interval $[4, \infty)$.

Answer: Global max(es) at $t =$ ________ Global min(s) at $t =$ ________

c. [3 points] Let $G(t)$ be the antiderivative of $g(t)$ with $G(0) = -5$. Find the t-coordinates of all critical points and inflection points of $G(t)$. For each answer blank, write NONE if appropriate. You do not need to justify your answers.

Answer: Critical point(s) at $t =$ ________________

Answer: Inflection point(s) at $t =$ ________________