# Calculus 115 - Notes

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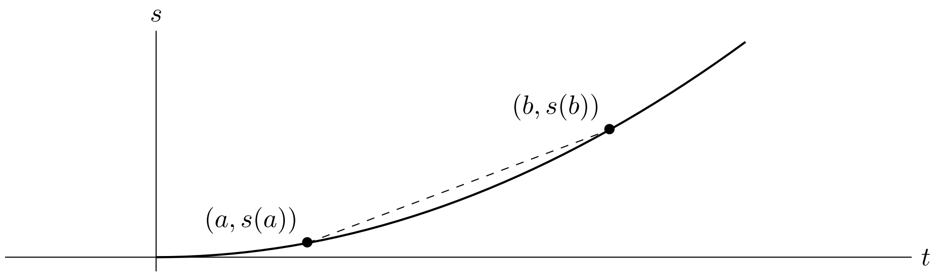
Last update - July 2, 2020

# 1 Derivatives

## 1.1 Speed

Consider position as function of time $s(t)$.



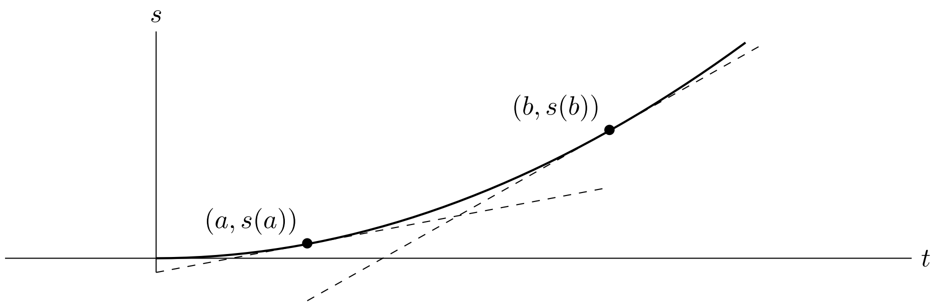

Intuitively we sense that the movement starts slow and gets faster when the time goes by. Mathematically we can quantify this intuition.
The **Average velocity** between $t = a$ and $t = b$ is:

$$\boxed{\frac{s(b) - s(a)}{b - a}}$$

And we know that this is the slop of the secant line, or "rise/run". We sense that the velocity at $a$ is smaller than the velocity at $b$. We define **Instantaneous velocity** at $a$ be letting the point $b$ get closer and closer to $a$ while we record the slope of the secant line. Mathematically:

$$\boxed{\lim_{h\to 0} \frac{s(a+h) - s(a)}{h}}$$

Note that we must do this dynamic procedure from both lines of $a$. Graphically, we end us with the slop of the **tanget line** at $a$. This explains we we "sense" that the velocity at $t = a$ is smaller than at $t = b$.




**Remark 1.1.1.** While velocity can be positive or negative, **Speed** is defined to be the magnitude of the velocity. Suppose the I drive at a constant speed of 50 mph from point A to point B, and then drive from point $B$ to point $A$ at 30 mph. The trip is finished at $t = c$. The average velocity between $t = 0$ and $t = c$ is 0 since $s(0) = 0 = s(c)$. However, the average speed is not. Assume that the distance is 150 mile. Then for 3 hours I drove 50 mph, and for 5 hours I drove 30 mph. $c = 8$ hours. The average is

$$\frac{3 \cdot 50 + 5 \cdot 30}{8} = 300/8 = 37.5mph.$$

In general, the speed between $t = a$ and $t = b$ is:

$$\text{speed} = \frac{\text{total distance covered between time } a \text{ and time } b}{b - a}$$

**2.** [12 points] Angelica Neiring and Simona Koloji decide to enjoy the fall weather by racing each other from the brass block "M" in the center of the Diag along a 2.5 kilometer (2500 meter) route to the Huron River inside the Arb. Let $A(t)$ (respectively $S(t)$) be Angelica's (respectively Simona's) distance along the route (in meters) $t$ seconds after they start racing. Angelica and Simona are both wearing GPS watches that record data about their race. The table of values for the functions $A$ and $S$ below shows some of the resulting data, rounded to the nearest meter. Note that the data is not always recorded at regular intervals.

| $t$ | 0 | 30 | 60 | 66 | 72 | 105 | 114 | 120 | 135 | 168 | 180 | 198 | 300 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A(t)$ | 0 | 55 | 119 | 137 | 156 | 226 | 249 | 265 | 302 | 384 | 415 | 463 | 737 |
| $S(t)$ | 0 | 57 | 120 | 137 | 156 | 225 | 248 | 264 | 303 | 389 | 422 | 473 | 768 |

Use the data above to answer the questions below. Remember to show your work.

**Exercise 1.1.2.**

1. Estimate Angelica's instantaneous velocity 3 minutes into the race.
2. Estimate Simona's instantaneous velocity 120 minutes into the race.
3. Who was ahead after 5 minutes — Angelica / Simona / Cannot be determined?
4. Who was running faster exactly 1 minute into the race — Angelica / Simona / Cannot be determined?
5. In describing the race later, Simona says that her average velocity during the entire race was 2.8 meters per second while Angelica says that after the first 5 minutes, her average velocity for the rest of the race was 3.1 meters per second. Assuming their statements and the table of values above are accurate, who won the race? Or is there not enough information to decide? Explain your reasoning.

*Solution.*

- 3 min = 180 sec. The best estimate is to compute the secant line from the left, from the right and then average. We use a **trick** of calculating the slope using one point to the left and one to the right. It is just as good as an estimation, just quicker. So we use (168,384),(198,463). We get: (463-387)/(198-168)=76/30≈2.53 m/s
- (303-248)/(135-114)=55/21≈2.61 m/s
- At $t = 300$ simona was ahead by 29 meters.
- We can compute the slopes, but it is apparent that the approximations would be too close to give a clear cut answer.
- Based on the average velocity, Simona has finished after (2500)/2.8 = 892.9 seconds. Angelica ran 737 meters at the first 5 minutes. For the rest of the (2500-737) meters , she ran at average velocity of $3.1m/s$ so it took here: (2500-737)/3.1≈568.7 seconds. Add the 5 minutes to it to get a time of 868.7. Thus Angelica has finished the race in about 25 seconds less than Simona.

□

## 1.2 Derivatives

This is a complete analogy of the previous section, only this time we are talking about how functions $f(x)$ change, not necessarily position as a function of time.
For any pair of points on the graph, we have a formula for the slope of the secant line, using rise over run:

$$\frac{f(b)-f(a)}{b-a}$$

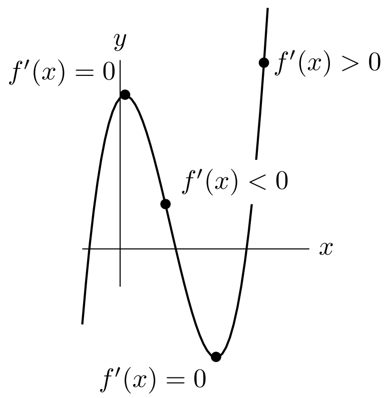
We can talk about instantaneous slope, or the slope of a tangent line, or the derivative at $x=a$, using the same limit procedure of letting the point $b$ be closer and closer to $a$.

> The **derivative of $f$ at $a$**, written $f'(a)$, is defined as
>
> $$\text{Rate of change of } f \text{ at } a = f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$
>
> If the limit exists, then $f$ is said to be **differentiable at $a$**.

Graphically:

- $f'(a)>0$, the function is increasing around $x=a$.
- $f'(a)<0$, the function is decreasing around $x=a$.
- $f'(a)=0$, then the function is neither around $x=a$. This is called a **Critical Point**. Possibly a max/min point.

**A critical point** of $f(x)$ is a point $x=a$ where $f(a)=0$ or $f(a)$ does not exists.




**Example 1.2.1.** Compute algebraically $f'(2)$ for $f(x)=x^2+x+1$.

$$f'(2)=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\frac{4+4h+h^2+2+h+1-4-2-1}{h}=\lim_{h\to 0}\frac{5h+h^2}{h}=\lim_{h\to 0}5+h=5$$

The last inequality is due to the fact that $g(h)=5+h$ is a continuous function, thus we can plug in to get the limit.

### Exercise 1.2.2.

**4.** [9 points] Let $P(v) = \begin{cases} v^2 \sin\left(\frac{1}{v}\right) - v\sin(2) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$

**a.** [5 points]
Use the limit definition of the derivative to write down an explicit expression for $P'(0)$.
Your answer should not include the letter $P$.
Do not attempt to evaluate or simplify the limit.

$P'(0) =$ ________________________________________

**b.** [4 points] Use your answer to (a) to estimate $P'(0)$ to the nearest hundredth.
*Be sure to include enough clear graphical or numerical evidence to justify your answer.*

*Solution.*

**4.** [9 points] Let $P(v) = \begin{cases} v^2 \sin\left(\frac{1}{v}\right) - v\sin(2) & \text{if } v \neq 0 \\ 0 & \text{if } v = 0. \end{cases}$

**a.** [5 points]
Use the limit definition of the derivative to write down an explicit expression for $P'(0)$.
Your answer should not include the letter $P$.
Do not attempt to evaluate or simplify the limit.

$$P'(0) = \lim_{h\to 0} \frac{\left((0+h)^2 \sin\left(\frac{1}{0+h}\right) - (0+h)\sin(2)\right) - 0}{h}$$

**b.** [4 points] Use your answer to (a) to estimate $P'(0)$ to the nearest hundredth.
*Be sure to include enough clear graphical or numerical evidence to justify your answer.*

*Solution:* We plug in small values of $h$ approaching 0. Since the difference quotient is an even function of $h$, we need only check positive values of $h$ (as evenness implies that negative $h$ give precisely the same results).

$h = 0.1$:
$$\frac{0.1^2 \sin(1/0.1) - 0.1\sin(2) - 0}{0.1} \approx -0.964$$

$h = 0.01$:
$$\frac{0.01^2 \sin(1/0.01) - 0.01\sin(2) - 0}{0.01} \approx -0.914$$

$h = 0.001$:
$$\frac{0.001^2 \sin(1/0.001) - 0.001\sin(2) - 0}{0.001} \approx -0.908$$

$h = 0.0001$:
$$\frac{0.0001^2 \sin(1/0.0001) - 0.0001\sin(2) - 0}{0.0001} \approx -0.909$$

We see at this point that the numbers seem to have stabilized to the nearest hundredth at $-0.91$.

□

### Exercise 1.2.3.

1. [11 points] The table below gives several values of a continuous, invertible function $f(x)$. Assume that the domain of both $f(x)$ and $f'(x)$ is the interval $(-\infty, \infty)$.

| $x$ | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-7$ | $-3.5$ | $-2$ | 3 | 4.5 | 6 | 7 | 9 | 19 |

**a.** [3 points] Evaluate each of the following.

(i) $f(f(15))$

**Answer:** $f(f(15)) =$ ____________

(ii) $f^{-1}(3)$

**Answer:** $f^{-1}(3) =$ ____________

(iii) $f^{-1}(2f(12))$

**Answer:** $f^{-1}(2f(12)) =$ ____________

**b.** [2 points] Compute the average rate of change of $f$ on the interval $3 \le x \le 18$.

**Answer:** ____________

**c.** [2 points] Estimate $f'(19)$.

**Answer:** $f'(19) \approx$ ____________

**d.** [2 points] Let $g(x) = f^{-1}(x)$. Estimate $g'(5)$.

**Answer:** $g'(5) \approx$ ____________

**e.** [2 points] Suppose $f'(0) = 2$. Find an equation for the tangent line to the graph of $y = f(x)$ at $x = 0$.

**Answer:** __________________

*Solution.*

**1.** [11 points] The table below gives several values of a continuous, invertible function $f(x)$. Assume that the domain of both $f(x)$ and $f'(x)$ is the interval $(-\infty, \infty)$.

| $x$ | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-7$ | $-3.5$ | $-2$ | 3 | 4.5 | 6 | 7 | 9 | 19 |

**a.** [3 points] Evaluate each of the following.

(i) $f(f(15))$

*Solution:* $f(f(15)) = f(6) = -2$.

**Answer:** $f(f(15)) =$ $-2$

(ii) $f^{-1}(3)$

**Answer:** $f^{-1}(3) =$ 9

(iii) $f^{-1}(2f(12))$

*Solution:* $f^{-1}(2f(12)) = f^{-1}(2(4.5)) = f^{-1}(9) = 21$.

**Answer:** $f^{-1}(2f(12)) =$ 21

**b.** [2 points] Compute the average rate of change of $f$ on the interval $3 \leq x \leq 18$.

*Solution:* This average rate of change is equal to the difference quotient

$$\frac{f(18) - f(3)}{18 - 3} = \frac{7 - (-3.5)}{15} = \frac{10.5}{15} = \frac{7}{10} = 0.7.$$

**Answer:** $10.5/15 = 7/10 = 0.7$

**c.** [2 points] Estimate $f'(19)$.

*Solution:* We approximate $f'(19)$ by the average rate of change of $f$ on the interval $18 \leq x \leq 21$.

$$f'(19) \approx \frac{f(21) - f(18)}{21 - 18} = \frac{9 - 7}{3} = \frac{2}{3}.$$

**Answer:** $f'(19) \approx$ $2/3 \approx 0.67$

**d.** [2 points] Let $g(x) = f^{-1}(x)$. Estimate $g'(5)$.

*Solution:* We approximate $g'(5)$ by the average rate of change of $g(x)$ on the interval $4.5 \leq x \leq 6$.

$$g'(5) \approx \frac{g(6) - g(4.5)}{6 - 4.5} = \frac{f^{-1}(6) - f^{-1}(4.5)}{1.5} = \frac{15 - 12}{1.5} = \frac{3}{1.5} = 2.$$

**Answer:** $g'(5) \approx$ $3/1.5 = 2$

**e.** [2 points] Suppose $f'(0) = 2$. Find an equation for the tangent line to the graph of $y = f(x)$ at $x = 0$.

*Solution:* This is the line with slope $f'(0) = 2$ that passes through the point $(0, f(0)) = (0, -7)$. An equation for this line is $y = 2x - 7$.

**Answer:** $y = 2x - 7$

We can give a general formula for the tangent line of $f(x)$ at a point $x = a$ when $f(x)$ is differentiable there:

$$\boxed{\ell(x) = f'(a)(x - a) + f(a)}$$

□

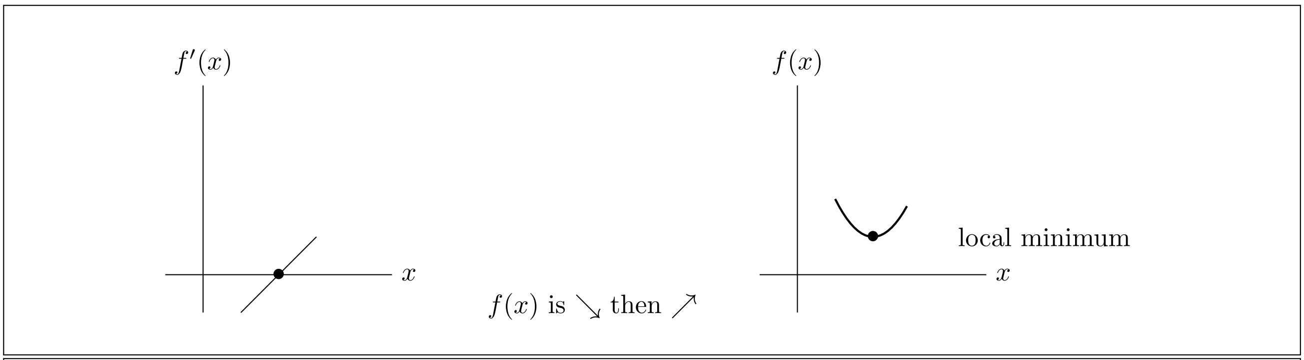
## 1.3 The Derivative Function

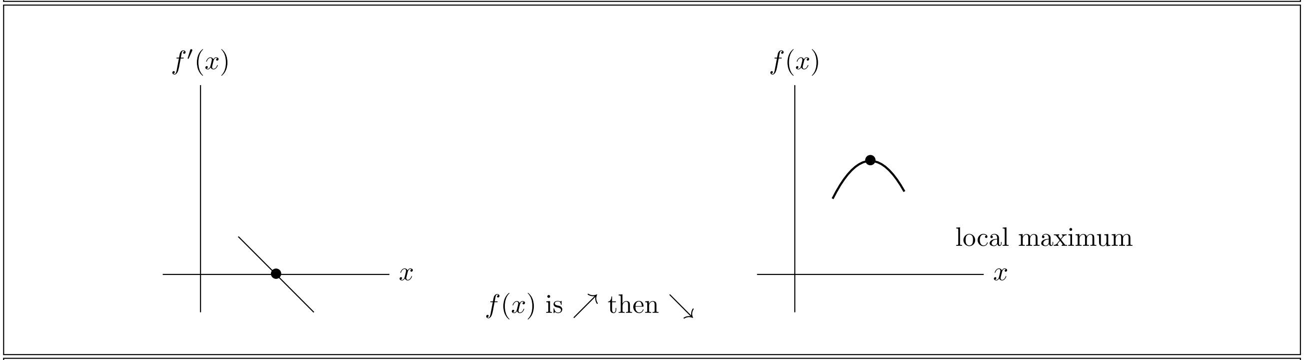
If $f(x)$ is differentiable on an interval, we can define a function assigning to each point its derivative. We get the derivative function:

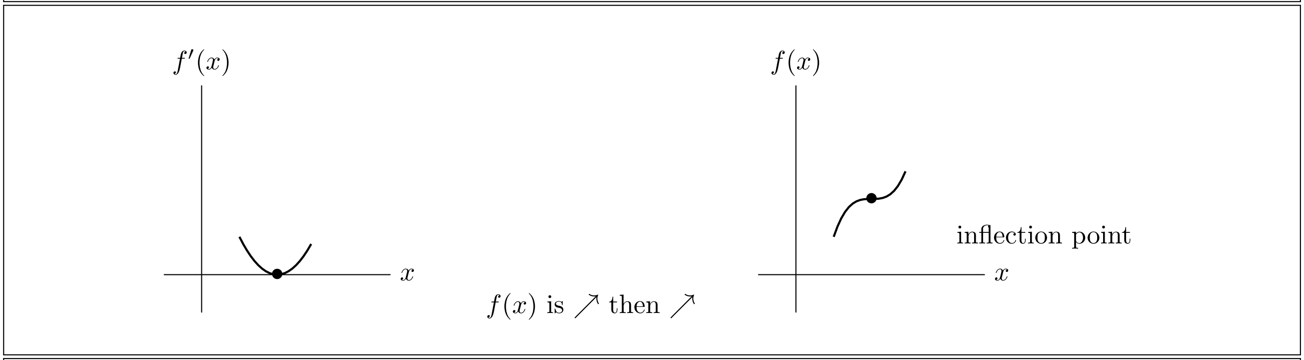
For any function $f$, we define the **derivative function**, $f'$, by

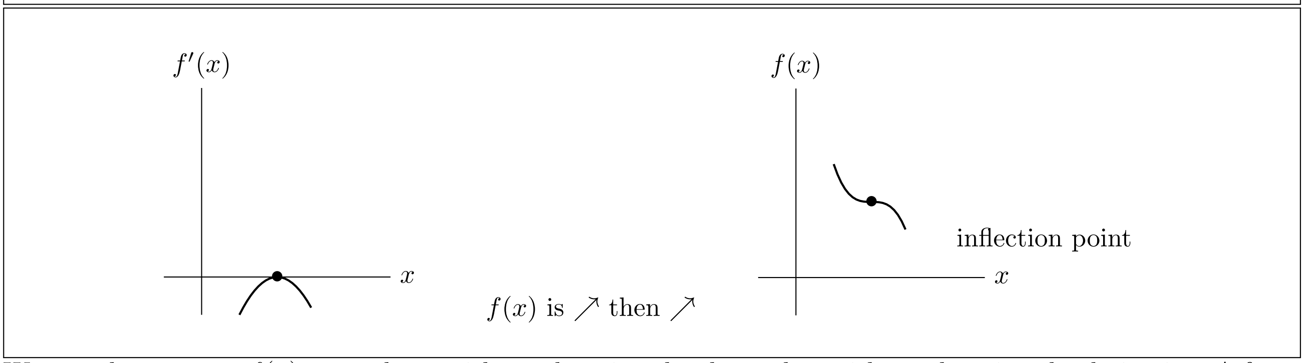
$$f'(x) = \text{Rate of change of } f \text{ at } x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Now we can relate the graph of $f'(x)$ to the graph of $f(x)$:




We can always move $f(x)$ up or down without changing the slope, thus without changing the derivative. A few formulas:

$$f(x) = k \Rightarrow f'(x) = 0$$
$$f(x) = mx + b \Rightarrow f'(x) = m$$
$$g(x) = f(x) + k \Rightarrow g'(x) = f'(x)$$
$$\boxed{f(x) = x^n \Rightarrow f'(x) = nx^{n-1}}$$

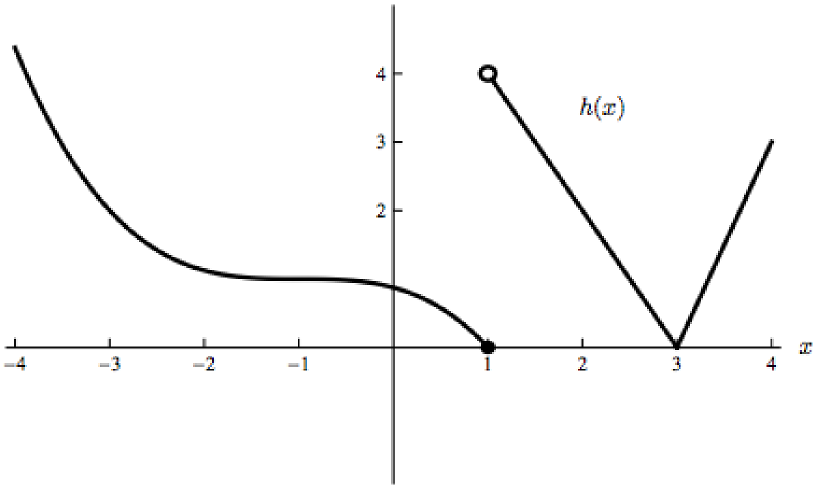
**Remark 1.3.1.** If $f'(a)$ exists then $f(x)$ is continuous at $x = a$. So differentiability implies continuity but NOT the other way around.

When a function is not differentiable?

- Not defined at the point.
- Not continuous at the point.
- Not "smooth" at the point, i.e. has a sharp edge or V-like shape.

**Exercise 1.3.2.**

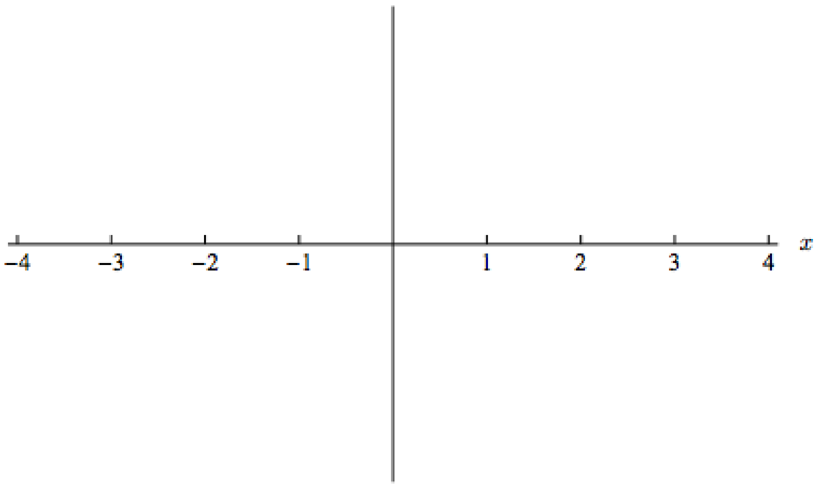
**8.** [10 points] The graph of a function $h(x)$ is given below.




**a.** [1 point] List all $x$-values with $-4 < x < 4$ where $h(x)$ is not continuous. If there are none, write NONE.

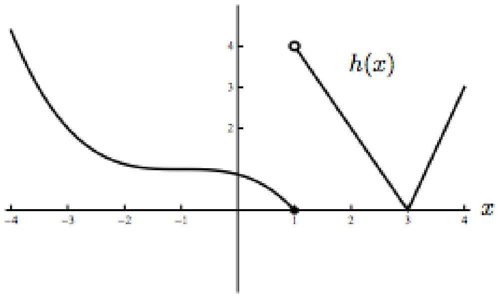
**b.** [1 point] List all $x$-values with $-4 < x < 4$ where $h(x)$ is not differentiable. If there are none, write NONE.

**c.** [8 points] On the axes provided, carefully draw a graph of $h'(x)$. Be sure to label important points or features on your graph.



*Solution.*

**8.** [10 points] The graph of a function $h(x)$ is given below.




**a.** [1 point] List all $x$-values with $-4 < x < 4$ where $h(x)$ is not continuous. If there are none, write NONE.

*Solution:*

$$x = 1$$

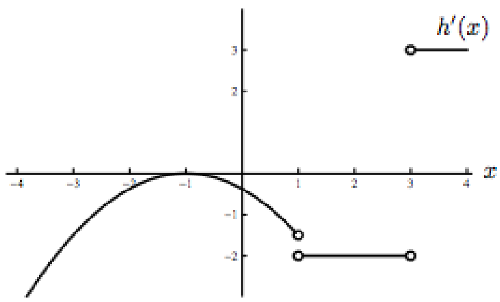
**b.** [1 point] List all $x$-values with $-4 < x < 4$ where $h(x)$ is not differentiable. If there are none, write NONE.

*Solution:*

$$x = 1, 3$$

**c.** [8 points] On the axes provided, carefully draw a graph of $h'(x)$. Be sure to label important points or features on your graph.

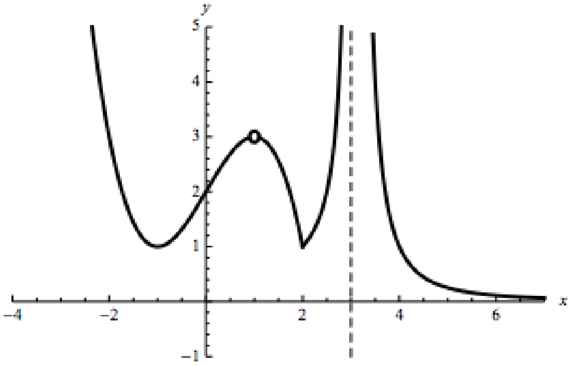
*Solution:*




Note that the derivative does not exists for $x = 1, x - 3$, and the slope to the left of $x = 1$ is a little less extreme than the one to the right of $x = 1$. □

**Exercise 1.3.3.**

**1.** [15 points] The following figure shows the graph of $y = f(x)$ for some function $f$. The dotted line signifies a vertical asymptote.




**a.** [12 points] Using the graph, give the values of each of the following quantities if they exist. Choose your answer in each part from the numbers $0, 1, 2, 3$ or the words "Does not exist." Answers may be used more than once–or not at all.

i) $f(1) =$ ______

ii) $f(2) =$ ______

iii) $f(3) =$ ______

iv) $f'(-1) =$ ______

v) $f'(1) =$ ______

vi) $f'(2) =$ ______

vii) $\lim_{x \to +\infty} f(x) =$ ______

viii) $\lim_{x \to 3} f(x) =$ ______

ix) $\lim_{x \to 2} f(x) =$ ______

x) $\lim_{x \to 1} f(x) =$ ______

xi) $\lim_{x \to -1} f(x) =$ ______

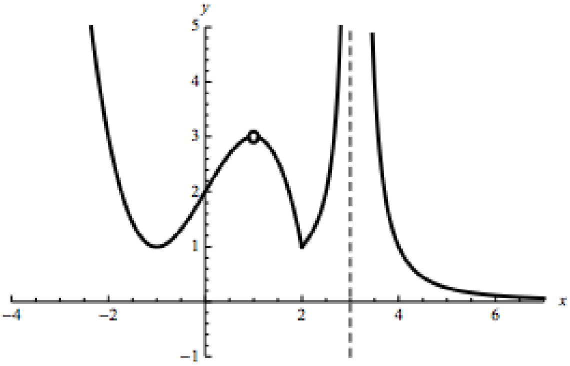
xii) $\lim_{x \to -\infty} f(x) =$ ______

**b.** [3 points] Still looking at the graph, is $f$ continuous at the following $x$ values? (Yes or No)

i) $x = 1$ ______ ii) $x = 2$ ______ iii) $x = 3$ ______

*Solution.*

**1.** [15 points] The following figure shows the graph of $y = f(x)$ for some function $f$. The dotted line signifies a vertical asymptote.




**a.** [12 points] Using the graph, give the values of each of the following quantities if they exist. Choose your answer in each part from the numbers $0, 1, 2, 3$ or the words "Does not exist." Answers may be used more than once–or not at all.

i) $f(1) =$Does not exist.

ii) $f(2) = 1$

iii) $f(3) =$Does not exist.

iv) $f'(-1) = 0$.

v) $f'(1) =$ Does not exist.

vi) $f'(2) =$Does not exist.

vii) $\lim_{x\to+\infty} f(x) = 0$.

viii) $\lim_{x\to 3} f(x) =$Does not exist.

ix) $\lim_{x\to 2} f(x) = 1$.

x) $\lim_{x\to 1} f(x) = 3$.

xi) $\lim_{x\to -1} f(x) = 1$.

xii) $\lim_{x\to-\infty} f(x) =$ Does not exist.

**b.** [3 points] Still looking at the graph, is $f$ continuous at the following $x$ values? (Yes or No)

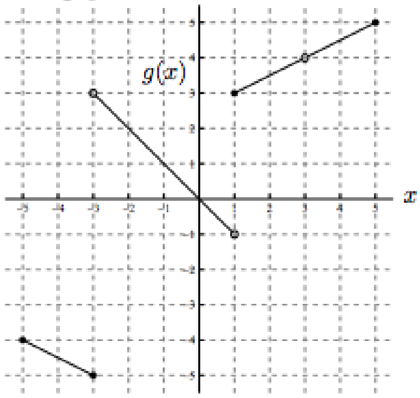
i) $x = 1$, No. ii) $x = 2$, Yes. iii) $x = 3$, No.

□

**Quiz #3**. Please write you name: ______________________ and email: ______________________

**2.** [13 points] Below is a table of values for an invertible, differentiable function $f(x)$ and the graph of a function $g(x)$. Use these to answer the following questions:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $f(x)$ | 8 | 7 | 3 | 2 | 1.5 | 1 |




**a.** [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of $g$.

**b.** [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of $g^{-1}$.

**c.** [8 points] Evaluate the following:

(i) $f(f(5))$

(ii) $g^{-1}(f^{-1}(2))$

(iii) $\lim_{x \to 3} g(x)$

(iv) $g'(1 + f(2))$

**d.** [3 points] Approximate $f'(3)$. (Be sure to show your work.)

*Solution.*

**2.** [13 points] Below is a table of values for an invertible, differentiable function $f(x)$ and the graph of a function $g(x)$. Use these to answer the following questions:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
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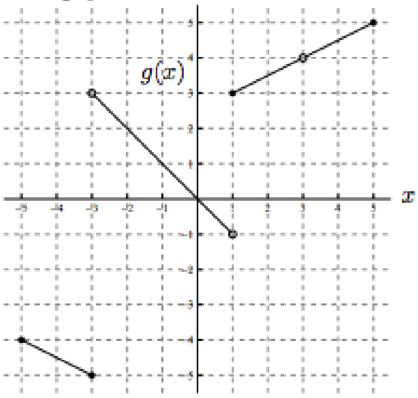
| $f(x)$ | 8 | 7 | 3 | 2 | 1.5 | 1 |




**a.** [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of $g$.

*Solution:* 3

**b.** [1 point] Give one number in the interval $[-5, 5]$ that is *not* in the domain of $g^{-1}$.

*Solution:* Anything in $(-4, -1] \cup \{4\}$. For example, 4.

**c.** [8 points] Evaluate the following:

(i) $f(f(5))$

*Solution:* $f(f(5)) = f(1) = 7$.

(ii) $g^{-1}(f^{-1}(2))$

*Solution:* $g^{-1}(f^{-1}(2)) = g^{-1}(3) = 1$.

(iii) $\lim_{x \to 3} g(x)$

*Solution:* 4, found from graph of $g$.

(iv) $g'(1 + f(2))$

*Solution:* $g'(1 + f(2)) = g'(4) = \frac{1}{2}$ looking at the slope on the graph of $g$ at $x = 4$.

**d.** [3 points] Approximate $f'(3)$. (Be sure to show your work.)

*Solution:* Acceptable answers: $-1$, $-\frac{1}{2}$, $-\frac{3}{4}$. Found approximating the derivative via a difference quotient.

□

## 1.4 Interpretation of The Derivative

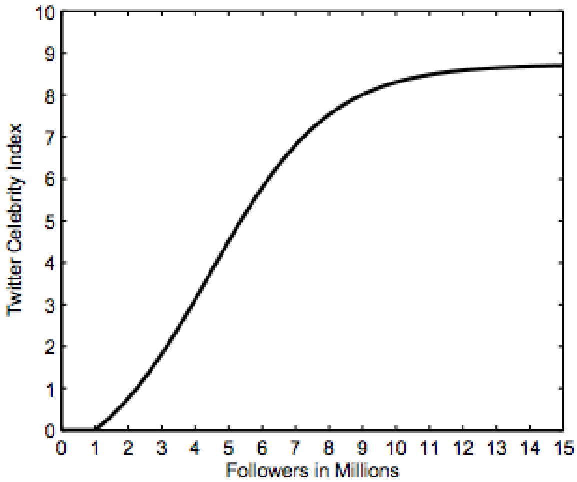
$f'(x)$ represent the instantaneous change of the function. Another notation is

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx}$$

In term of units, it has units of $\boxed{\text{(units of y)/(units of x)}}$. (Note to instructor - the following example should be presented on the board. The next two exercises are group work).

**Exercise 1.4.1.**

**4.** [12 points] The Twitter Celebrity Index (TCI) measures the celebrity of Twitter users; the function $T(x)$ takes the number of followers (in millions) of a given user and returns a TCI value from 0 to 10. Below is a graph of this function.




Use the graph above to help you answer the following questions.

**a.** [3 points] Explain in practical terms what $T(13.72) = 8.67$ means.

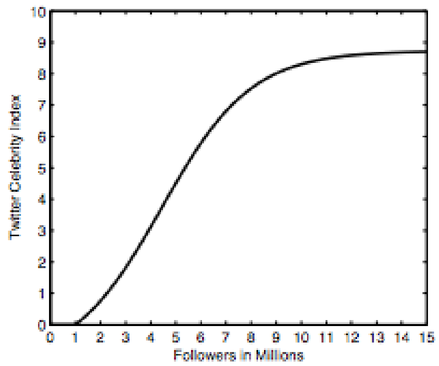
**b.** [3 points] Explain in practical terms what $T^{-1}(4.25) = 4.88$ means.

**c.** [3 points] Explain in practical terms what $T'(10) = 0.2278$ means.

**d.** [3 points] Explain in practical terms what $(T^{-1})'(7.238) = 0.71$ means.

*Solution.*

**4.** [12 points] The Twitter Celebrity Index (TCI) measures the celebrity of Twitter users; the function $T(x)$ takes the number of followers (in millions) of a given user and returns a TCI value from 0 to 10. Below is a graph of this function.




Use the graph above to help you answer the following questions.

**a.** [3 points] Explain in practical terms what $T(13.72) = 8.67$ means.

*Solution:* When a Twitter user has 13.72 million followers, their Twitter Celebrity index is 8.67.

**b.** [3 points] Explain in practical terms what $T^{-1}(4.25) = 4.88$ means.

*Solution:* When a user has a Twitter Celebrity index of 4.25, they have 4.88 million followers.

**c.** [3 points] Explain in practical terms what $T'(10) = 0.2278$ means.

*Solution:* When a Twitter user has 10 million followers, adding $100,000$ followers will increase their celebrity index by roughly .02278.

**d.** [3 points] Explain in practical terms what $(T^{-1})'(7.238) = 0.71$ means.

*Solution:* When a Twitter user has celebrity index 7.238, and increase of .1 to their index corresponds to gaining approximately .071 million (71,000) followers.

□

**Exercise 1.4.2.**

8. [9 points] A certain company's revenue $R$ (in thousands of dollars) is given as a function of the amount of money $a$ (in thousands of dollars) they spend on advertising by $R = f(a)$. Suppose that $f$ is invertible.

   a. [2 points] Which of the following is a valid interpretation of the equation $(f^{-1})'(75) = 0.5$? Circle one option.

   - If the company spends \$75,000 more on advertising, their revenue will increase by about \$500.
   - If the company increases their advertising expenditure from \$75,000 to \$76,000, their revenue will increase by about \$500.
   - If the company wants a revenue of \$75,000, they should spend about \$500 on advertising.
   - If the company wants to increase their revenue from \$75,000 to \$76,000, they should spend about \$500 more on advertising.

   b. [2 points] The company plans to spend about \$100,000 on advertising. If $f'(100) = 0.5$, should the company spend more or less than \$100,000 on advertising? Justify your answer.

   c. [5 points] The company's financial advisor claims that he has a formula for the dependence of revenue on advertising expenditure, and it is

$$f(a) = a\ln(a + 1).$$

   Using this formula, write the *limit definition* of $f'(100)$. You do not need to simplify or evaluate.

*Solution.*

8. [9 points] A certain company's revenue $R$ (in thousands of dollars) is given as a function of the amount of money $a$ (in thousands of dollars) they spend on advertising by $R = f(a)$. Suppose that $f$ is invertible.

a. [2 points] Which of the following is a valid interpretation of the equation $(f^{-1})'(75) = 0.5$? Circle one option.

- If the company spends \$75,000 more on advertising, their revenue will increase by about \$500.
- If the company increases their advertising expenditure from \$75,000 to \$76,000, their revenue will increase by about \$500.
- If the company wants a revenue of \$75,000, they should spend about \$500 on advertising.
- If the company wants to increase their revenue from \$75,000 to \$76,000, they should spend about \$500 more on advertising.

*Solution:* The last option.

b. [2 points] The company plans to spend about \$100,000 on advertising. If $f'(100) = 0.5$, should the company spend more or less than \$100,000 on advertising? Justify your answer.

*Solution:*
They should spend less on advertising, because if they increase their advertising expenditure by \$1000, they will only gain about \$500 in revenue.

c. [5 points] The company's financial advisor claims that he has a formula for the dependence of revenue on advertising expenditure, and it is

$$f(a) = a\ln(a+1).$$

Using this formula, write the *limit definition* of $f'(100)$. You do not need to simplify or evaluate.

*Solution:*

$$f'(100) = \lim_{h\to 0} \frac{(100+h)\ln(100+h+1) - 100\ln(101)}{h}$$

□

**Exercise 1.4.3.**

**2.** [10 points] Louis owns a small soda company and is experimenting with new flavors. Let $b(p)$ model the number of thousands of bottles of bacon-flavored soda sold by his company per month if he charges $p$ cents per bottle. You may assume $b(p)$ is differentiable and invertible.

**a.** [2 points] Give a practical interpretation of the statement $b^{-1}(8) = 150$.

**b.** [3 points] Give a practical interpretation of the statement $(b^{-1})'(4) = -10$.

**c.** [3 points] Write an expression that is equal to the price (in cents) that the company would have to charge per bottle in order to sell twice as many bottles of bacon-flavored soda as it sells at a price of 125 cents per bottle.

**d.** [2 points] Which of the following is a correct formula for a function $h(d)$ that gives the number of thousands of bottles sold per month at a price of $d$ dollars per bottle? (Circle your answer.)

$h(d) = 100b(d)$ $\qquad$ $h(d) = \dfrac{b(d)}{100}$ $\qquad$ $h(d) = b(100d)$ $\qquad$ $h(d) = b\left(\dfrac{d}{100}\right)$

**3.** [5 points] Use the limit definition of the derivative to write an explicit expression for $r'(3)$ where $r(t) = (t+5)^{2t}$. Do not simplify or evaluate the limit. Your answer should not include the letter $r$.

$r'(3) =$ ______________________________

*Solution.*

**2.** [10 points] Louis owns a small soda company and is experimenting with new flavors. Let $b(p)$ model the number of thousands of bottles of bacon-flavored soda sold by his company per month if he charges $p$ cents per bottle. You may assume $b(p)$ is differentiable and invertible.

**a.** [2 points] Give a practical interpretation of the statement $b^{-1}(8) = 150$.

*Solution:* In order to sell 8000 bottles of bacon-flavored soda per month, the company should charge 150 cents per bottle.

**b.** [3 points] Give a practical interpretation of the statement $(b^{-1})'(4) = -10$.

*Solution:* In order to increase the number of bottles sold per month from 4000 to 5000, the company should lower the price about 10 cents.
If the company is currently selling 4000 bottles per month, lowering the price by 10 cents will increase sales by about 1000 bottles per month.
(There are other possible answers.)

**c.** [3 points] Write an expression that is equal to the price (in cents) that the company would have to charge per bottle in order to sell twice as many bottles of bacon-flavored soda as it sells at a price of 125 cents per bottle.

*Solution:* $b^{-1}(2b(125))$

**d.** [2 points] Which of the following is a correct formula for a function $h(d)$ that gives the number of thousands of bottles sold per month at a price of $d$ dollars per bottle? (Circle your answer.)

$h(d) = 100b(d)$ $\quad$ $h(d) = \dfrac{b(d)}{100}$ $\quad$ $\boxed{h(d) = b(100d)}$ $\quad$ $h(d) = b\left(\dfrac{d}{100}\right)$

**3.** [5 points] Use the limit definition of the derivative to write an explicit expression for $r'(3)$ where $r(t) = (t+5)^{2t}$. Do not simplify or evaluate the limit. Your answer should not include the letter $r$.

*Solution:*

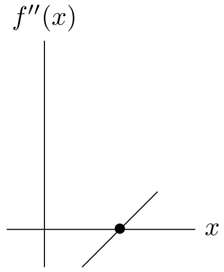
$$r'(3) = \lim_{h\to 0} \frac{(3+h+5)^{2(3+h)} - (3+5)^{2(3)}}{h}$$

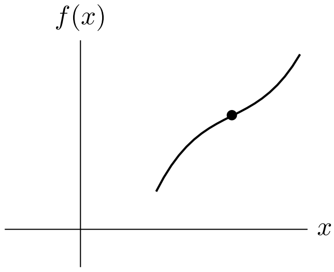
□

## 1.5 The second derivative

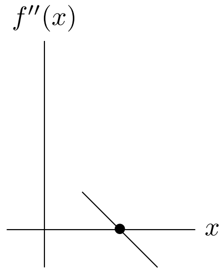
The second derivative is the derivative of the derivative.
Graphically:

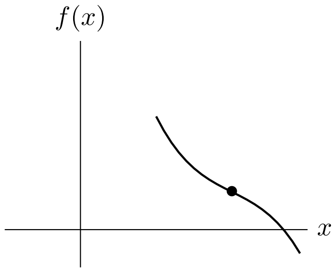
If the graph of $f$ is concave up and $f''$ exists on an interval, then $f'' \geq 0$ there.
If the graph of $f$ is concave down and $f''$ exists on an interval, then $f'' \leq 0$ there.




$f(x)$ is $\cap$ then $\cup$



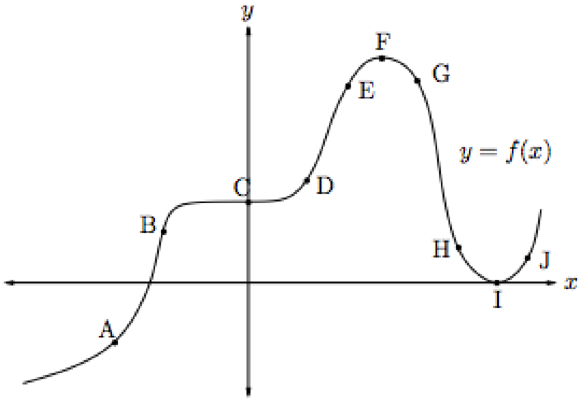




$f(x)$ is $\cup$ then $\cap$

In the context of position $s$ as a function of time $t$, recall that $s'(t) = v(t)$ the velocity. Thus, $s''(t) = v'(t) = a(t)$ the acceleration.
(Note to instructor - the following example should be presented on the board. The next two exercises are group work).

**Exercise 1.5.1.**

**6.** (11 points) The graph of a continuous differentiable function $f$ is given below. Use the graph to answer the following. No explanation necessary.




**(a)** List all labelled points (if any) where $f'$ and $f''$ are both positive.

**(b)** List all labelled points (if any) where $f'$ and $f''$ are both negative.

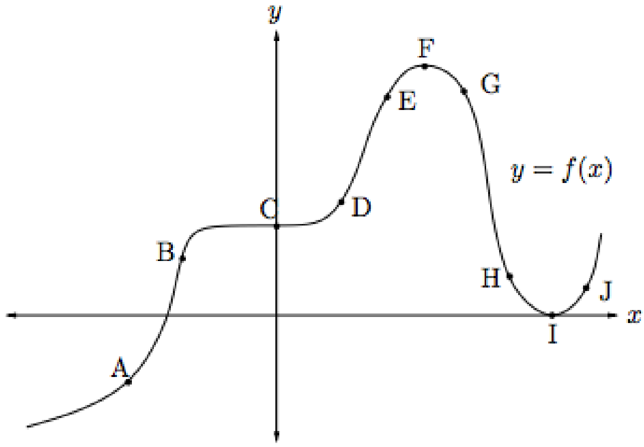
**(c)** List all labelled points (if any) where $f$ and $f'$ are both positive.

**(d)** List all labelled points (if any) where $f$ and $f'$ are both both negative.

**(e)** List all labelled points (if any) where at least two of $f$, $f'$, $f''$ are zero.

*Solution.*

**6.** (11 points) The graph of a continuous differentiable function $f$ is given below. Use the graph to answer the following. No explanation necessary.




**(a)** List all labelled points (if any) where $f'$ and $f''$ are both positive.

$A, D, J$

**(b)** List all labelled points (if any) where $f'$ and $f''$ are both negative.

$G$

**(c)** List all labelled points (if any) where $f$ and $f'$ are both positive.

$B, D, E, J$

**(d)** List all labelled points (if any) where $f$ and $f'$ are both both negative.

none

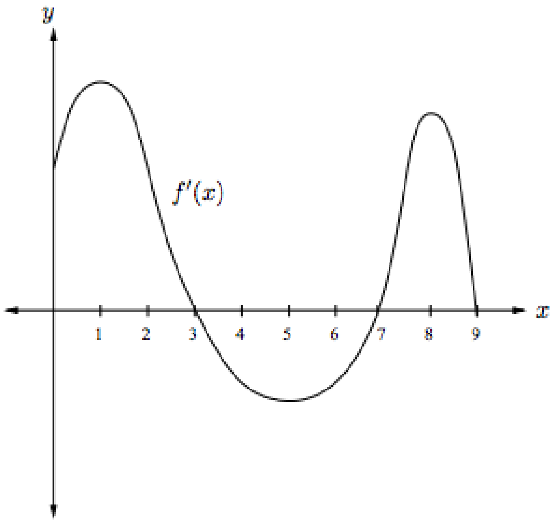
**(e)** List all labelled points (if any) where at least two of $f$, $f'$, $f''$ are zero.

$C, I$

□

**Exercise 1.5.2.**

**9.** (10 points) The graph of $f'(x)$ (i.e., the *derivative* of $f$) is given below. Use the graph to answer the following questions:




**(a)** For which intervals is $f$ increasing?

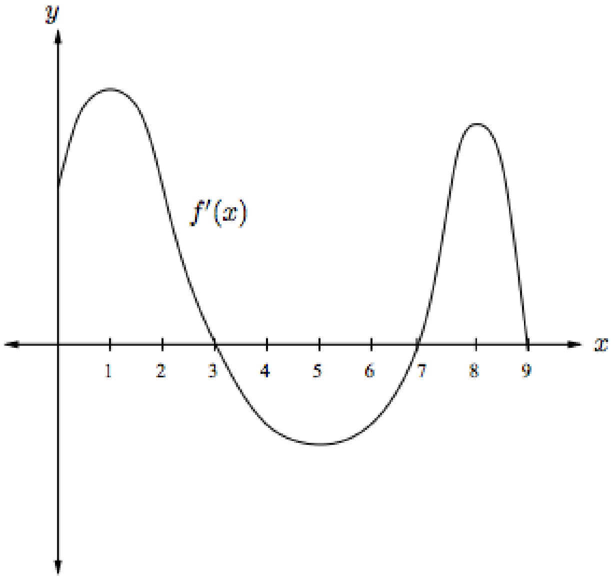
**(b)** For which intervals is $f''$ negative?

**(c)** For which value(s) of $x$ (if any) does $f$ have a local maximum?

**(d)** For which values of $x$ (if any) does $f$ switch from concave up to concave down?

*Solution.*

**9.** (10 points) The graph of $f'(x)$ (i.e., the *derivative* of $f$) is given below. Use the graph to answer the following questions:




**(a)** For which intervals is $f$ increasing?

$f$ is increasing when its derivative is positive, so for $0 < x < 3$ and $7 < x < 9$.

**(b)** For which intervals is $f''$ negative?

$f''$ is negative when $f'$ is decreasing, so for $1 < x < 5$ and $8 < x < 9$.

**(c)** For which value(s) of $x$ (if any) does $f$ have a local maximum?

[Note: This was excluded from grading.] $f$ has a local maximum at a value $a$ when $f'$ is positive for $x < a$ and negative for $x > a$. So $f$ has a local maximum when $x = 3$.

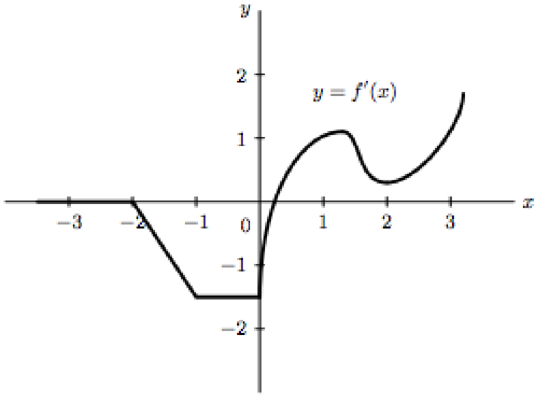
**(d)** For which value(s) of $x$ (if any) does $f$ switch from concave up to concave down?

$f$ will switch from concave up to concave down when the second derivative switches from being positive to being negative, ie., when the derivative switches from increasing to decreasing, so at $x = 1$ and $x = 8$.

□

**Exercise 1.5.3.**

**10.** [10 points] Below is the graph of $f'(x)$, the **derivative** of the function $f(x)$.
Note that $f'(x)$ is zero for $x \leq -2$, linear for $-2 < x < -1$, and constant for $-1 < x < 0$.




For each of the following, circle all of the listed intervals for which the given statement is true over the **entire** interval. If there are no such intervals, circle NONE.
You do not need to explain your reasoning.

**a.** [2 points] $f'(x)$ is increasing.

$-2 < x < -1$ $\quad$ $0 < x < 1$ $\quad$ $1 < x < 2$ $\quad$ $2 < x < 3$ $\quad$ NONE

**b.** [2 points] $f'(x)$ is concave up.

$0 < x < 1$ $\quad$ $1 < x < 2$ $\quad$ $2 < x < 3$ $\quad$ NONE

**c.** [2 points] $f(x)$ is increasing.

$-2 < x < -1$ $\quad$ $-1 < x < 0$ $\quad$ $0 < x < 1$ $\quad$ $1 < x < 2$ $\quad$ $2 < x < 3$ $\quad$ NONE

**d.** [2 points] $f(x)$ is linear but not constant.

$-3 < x < -2$ $\quad$ $-2 < x < -1$ $\quad$ $-1 < x < 0$ $\quad$ $0 < x < 1$ $\quad$ $1 < x < 2$ $\quad$ $2 < x < 3$ $\quad$ NONE

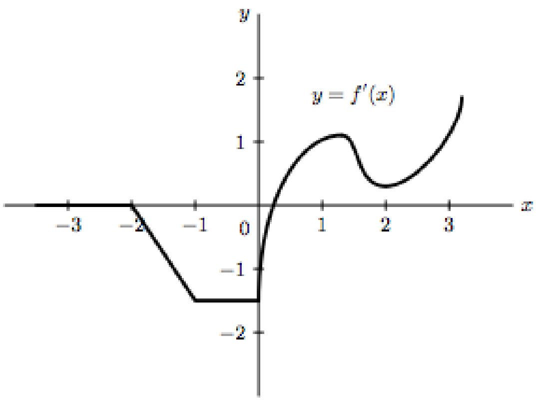
**e.** [2 points] $f(x)$ is constant.

$-3 < x < -2$ $\quad$ $-2 < x < -1$ $\quad$ $-1 < x < 0$ $\quad$ $0 < x < 1$ $\quad$ $1 < x < 2$ $\quad$ $2 < x < 3$ $\quad$ NONE

*Solution.*

**10.** [10 points] Below is the graph of $f'(x)$, the **derivative** of the function $f(x)$.

Note that $f'(x)$ is zero for $x \le -2$, linear for $-2 < x < -1$, and constant for $-1 < x < 0$.




For each of the following, circle <u>all</u> of the listed intervals for which the given statement is true over the **entire** interval. If there are no such intervals, circle NONE.
You do not need to explain your reasoning.

**a.** [2 points] $f'(x)$ is increasing.

$-2 < x < -1$ $\boxed{0 < x < 1}$ $1 < x < 2$ $\boxed{2 < x < 3}$ NONE

**b.** [2 points] $f'(x)$ is concave up.

$0 < x < 1$ $1 < x < 2$ $\boxed{2 < x < 3}$ NONE

**c.** [2 points] $f(x)$ is increasing.

$-2 < x < -1$ $-1 < x < 0$ $0 < x < 1$ $\boxed{1 < x < 2}$ $\boxed{2 < x < 3}$ NONE

**d.** [2 points] $f(x)$ is linear but not constant.

$-3 < x < -2$ $-2 < x < -1$ $\boxed{-1 < x < 0}$ $0 < x < 1$ $1 < x < 2$ $2 < x < 3$ NONE

**e.** [2 points] $f(x)$ is constant.

$-3 < \boxed{x < -2}$ $-2 < x < -1$ $-1 < x < 0$ $0 < x < 1$ $1 < x < 2$ $2 < x < 3$ NONE

□

## 1.6 Differentiablity

At this section we will concentrate on what we already know: We say that a function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists (and finite).

If $f'(a)$ exists then $f(x)$ is continuous at $x = a$. So $\boxed{\text{differentiability implies continuity}}$ but NOT the other way around.

When a function is not differentiable?

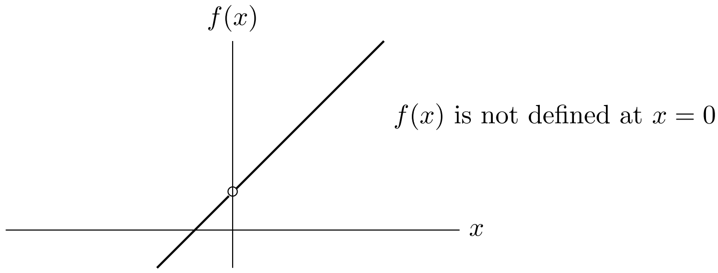
- Not defined at the point.
- Not continuous at the point.
- Not “smooth” at the point, i.e. has a sharp edge or V-like shape.

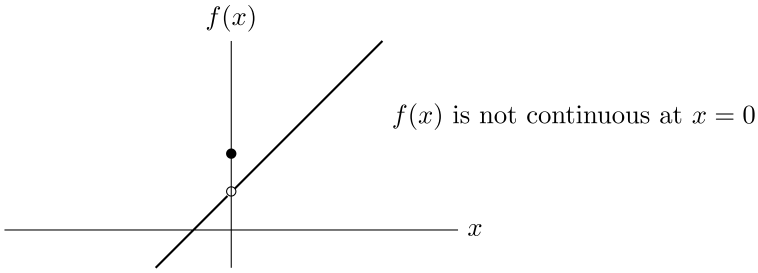
Conversely:

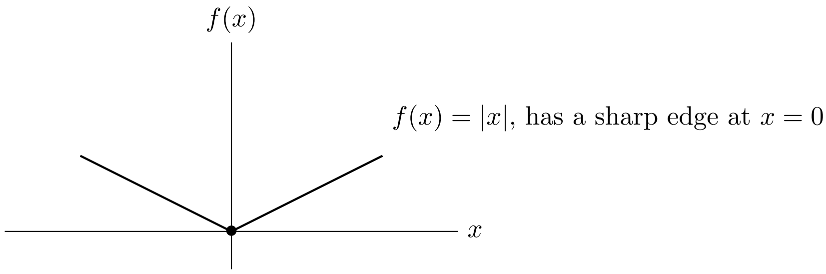
If $f(x)$ is differentiable at $x = a$, i.e. $f'(a)$ exists, then all must be true:

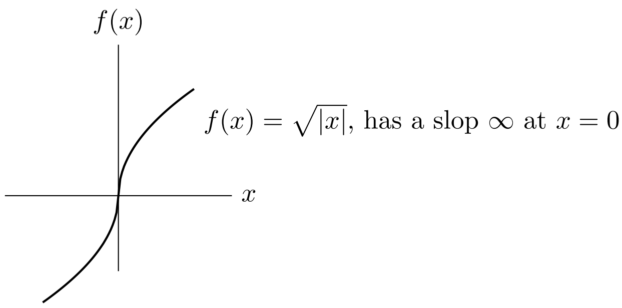
- $f(x)$ is defined at $x = a$.
- $f(x)$ is continuous at $x = a$.
- the graph of $f(x)$ is smooth at $x = a$.

Here are a few examples of functions that are not differentiable at $x = 0$:



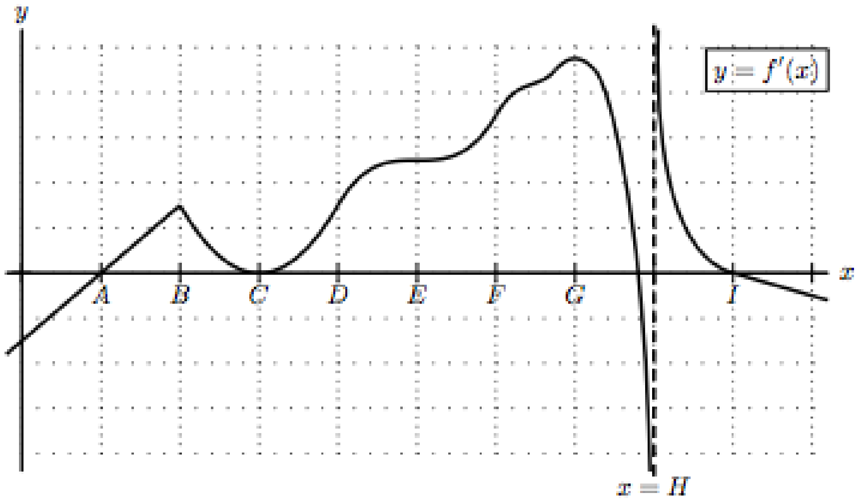



**Exercise 1.6.1.**

**3.** [12 points] The graph of a portion of $y = f'(x)$, the <u>derivative</u> of $f(x)$ is shown below. Note that there is a sharp corner at $x = B$ and that $x = H$ is a vertical asymptote.
The function $f(x)$ is continuous with domain $(-\infty, \infty)$.




For each of the questions below, circle <u>all</u> of the available correct answers.
(Circle NONE if none of the available choices are correct.)

**a.** [2 points] At which of the following six values of $x$ is the function $f(x)$ <u>not</u> differentiable?

$B$ $\quad C$ $\quad E$ $\quad F$ $\quad H$ $\quad I$ $\quad$ NONE

**b.** [2 points] At which of the following six values of $x$ does the function $f'(x)$ appear to be <u>not</u> differentiable?

$A$ $\quad B$ $\quad C$ $\quad D$ $\quad E$ $\quad F$ $\quad$ NONE

**c.** [2 points] At which of the following nine values of $x$ does $f(x)$ have a critical point?

$A$ $\quad B$ $\quad C$ $\quad D$ $\quad E$ $\quad F$ $\quad G$ $\quad H$ $\quad I$ $\quad$ NONE

**d.** [2 points] At which of the following nine values of $x$ does $f(x)$ have a local minimum?

$A$ $\quad B$ $\quad C$ $\quad D$ $\quad E$ $\quad F$ $\quad G$ $\quad H$ $\quad I$ $\quad$ NONE

**e.** [2 points] At which of the following nine values of $x$ is $f''(x) = 0$?

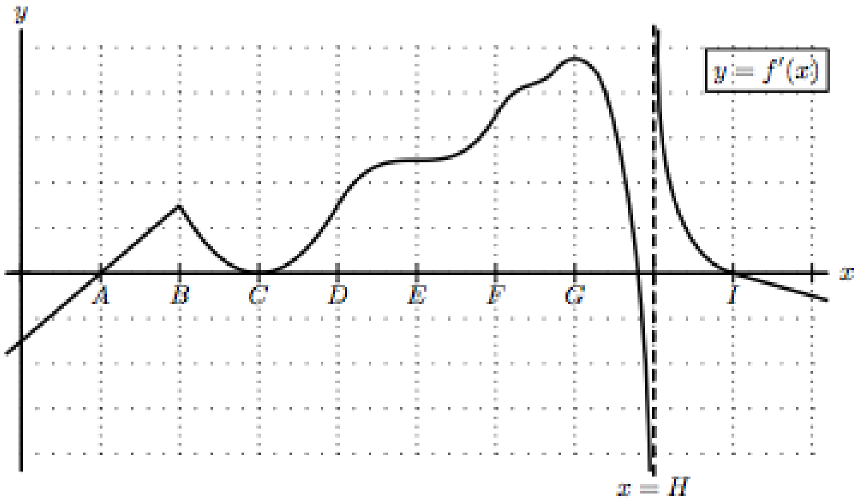
$A$ $\quad B$ $\quad C$ $\quad D$ $\quad E$ $\quad F$ $\quad G$ $\quad H$ $\quad I$ $\quad$ NONE

**f.** [2 points] At which of the following nine values of $x$ does $f(x)$ have an inflection point?

$A$ $\quad B$ $\quad C$ $\quad D$ $\quad E$ $\quad F$ $\quad G$ $\quad H$ $\quad I$ $\quad$ NONE

*Solution.*

**3.** [12 points] The graph of a portion of $y = f'(x)$, the <u>derivative</u> of $f(x)$ is shown below. Note that there is a sharp corner at $x = B$ and that $x = H$ is a vertical asymptote. The function $f(x)$ is continuous with domain $(-\infty, \infty)$.




For each of the questions below, circle <u>all</u> of the available correct answers.
(Circle NONE if none of the available choices are correct.)

**a.** [2 points] At which of the following six values of $x$ is the function $f(x)$ <u>not</u> differentiable?

$B$   $C$   $E$   $F$   [$H$]   $I$   NONE

**b.** [2 points] At which of the following six values of $x$ does the function $f'(x)$ appear to be <u>not</u> differentiable?

$A$   [$B$]   $C$   $D$   $E$   $F$   NONE

**c.** [2 points] At which of the following nine values of $x$ does $f(x)$ have a critical point?

[$A$]   $B$   [$C$]   $D$   $E$   $F$   $G$   [$H$]   [$I$]   NONE

**d.** [2 points] At which of the following nine values of $x$ does $f(x)$ have a local minimum?

[$A$]   $B$   $C$   $D$   $E$   $F$   $G$   [$H$]   $I$   NONE

**e.** [2 points] At which of the following nine values of $x$ is $f''(x) = 0$?

$A$   $B$   [$C$]   $D$   [$E$]   $F$   [$G$]   $H$   $I$   NONE

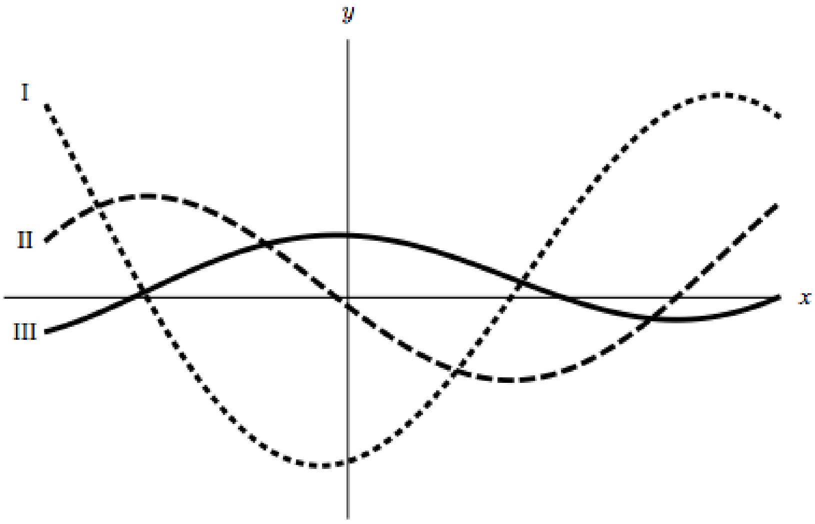
**f.** [2 points] At which of the following nine values of $x$ does $f(x)$ have an inflection point?

$A$   [$B$]   [$C$]   $D$   $E$   $F$   [$G$]   $H$   $I$   NONE

□

**Exercise 1.6.2.**

7. [12 points] On the axes below are graphed $f$, $f'$, and $f''$. Determine which is which, and justify your response with a brief explanation.



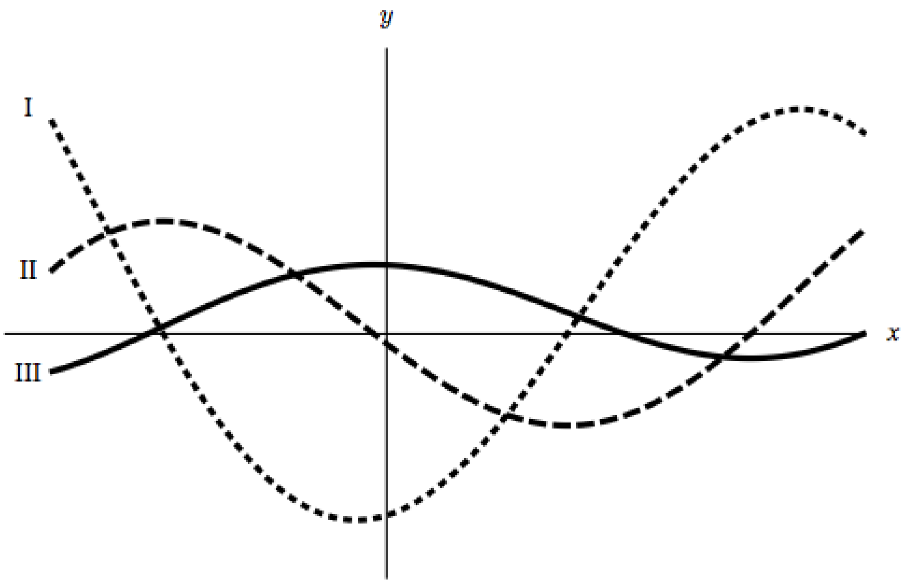

$f$:______

$f'$:______

$f''$:______

*Solution.*

**7.** [12 points] On the axes below are graphed $f$, $f'$, and $f''$. Determine which is which, and justify your response with a brief explanation.




*Solution:* Looking to the far right of the graph, curve **I** has a critical point where it has a slope of zero. At this x-coordinate neither of the other graphs has a root. This means the derivative of **I** is not in this figure, so **I** must be $f''$. Looking to the far left of the graph, **II** has a local maximum where its derivative is zero. Although **III** has a root near the same x-value, **III** changes sign from negative to positive at this point. By the first derivative test, **III** cannot be the derivative of **II**. Thus, by process of elimination, **II** must be $f'$ and **III** must be $f$.

$f$: **III**

$f'$: **II**

$f''$: **I**

□