Calculus 115 - Notes

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1 Shortcuts to differentiation

1.1 Polynomials - derivative formulas

So far, we were dealing with the definition of the derivative. It turns out that there are formulas that can help us with computing the derivative function.

For the power function, $f(x) = x^n$, n is any real number, we have:

$$f'(x) = \boxed{(x^n)' = nx^{n-1}}$$

For example: $(x^2)' = 2x$, $(1/x^3)' = (x^{-3})' = -3x^{-4}$. $(x^e)' = ex^{e-1}$.

The derivative of a constant is zero, because: $f(x) = c = cx^0 \Rightarrow f'(x) = c(x^0)' = c \cdot 0 \cdot x^{-1} = 0$.
The next property, is that the derivative commutes with multiplying by a constance c:

$$(c \cdot f(x))' = cf'(x).$$

For example: $(5x^2)' = 5(x^2)' = 10x$.

The derivative of a constant is zero, because: $f(x) = c = cx^0 \Rightarrow f'(x) = c(x^0)' = c \cdot 0 \cdot x^{-1} = 0$.
Another property, is "the derivative of the sum is the sum of the derivatives":

$$\boxed{(f(x) \pm g(x))' = f'(x) \pm g'(x)}$$

Now, we can take the derivative of any polynomial:

$$f(x) = x^3 + 4x^2 - 5x + 7 \Rightarrow f'(x) = 3x^2 + 6x - 5$$

Example 1.1.1. Let $f(x) = x + \sqrt{x} + 1$. Find the equation of the graph's tangent line at $x = 1$.

So the point is $(1, 3)$. The tangent line will have a slope of $f'(1)$. So

$$f'(x) = 1 + 0.5x^{-0.5} \Rightarrow f'(1) = 1.5$$

So we need the line with slope 1.5 passing through $(1, 3)$:

$$y - 3 = 1.5(x - 1) \Rightarrow y = 1.5x + 1.5 = 1.5(x + 1)$$

1.2 Exponential Function

Here are the formulas:

$$\boxed{(e^x)' = e^x}$$ [1]

$$\boxed{(a^x)' = \ln(a) \cdot a^x}$$

Note that $\ln(a)$ is just a number.

[1] this is the reason why e is so special

Example 1.2.1. What is the derivative of a^{kx}?

$$(a^{kx})' = ((a^k)^x)' = \ln(a^k)a^{kx} = \boxed{k\ln(a)a^{kx}}$$

Skip to Section 3.5 - Trig Functions

Remark 1.2.2.

$$\boxed{(\sin x)' = \cos x}$$
$$\boxed{(\cos x)' = -\sin x}$$
$$\boxed{(\tan x)' = 1/\cos^2 x = 1 + \tan^2 x}$$

Exercise 1.2.3. In groups:

1. $(x^2 + x^\pi)' = 2x + \pi x^{\pi-1}$
2. $(2\cos(x) - \sin(x))' = -2\sin(x) - \cos(x)$
3. $(x^2 + \cos(x))'' = 2 - \cos(x)$
4. $(2^x)'' = \ln(2)^2 2^x$
5. Find a, b, c such that $f(x) = ax^2 + bx + c$ and $g(x) = 2^x$ satisfy: $f(0) = g(0), f'(0) = g'(0), f'(0) = g'(0)$:
   We have $g(0) = 1, g'(0) = \ln(2), g''(0) = \ln^2(2)$.
   We have $f(0) = c, f'(0) = b, f''(0) = 2a \Rightarrow c = 1, b = \ln(2), a = \ln^2(2)/2$

1.3 Product Rule, Quotient Rule

$$(f(x)g(x))' = f' \cdot g + f \cdot g'$$

Example 1.3.1.

1. $(\sin(x)\cos(x)) = \sin(x)'\cos(x) + \sin(x)\cos(x)' = \cos^2(x) - \sin^2(x)$
2. $(x^2)' = (x \cdot x)' = 1x + x \cdot 1 = 2x$
3. $(f(x)^2)' = (f(x) \cdot f(x))' = f'f + ff' = 2f \cdot f'$
4. $\boxed{(f(x)^n)' = nf(x)^{n-1} \cdot f'(x)}$
5. $(\sin^2(x)+\cos^2(x))' = 2\sin(x)\cos(x)+2\cos(x)\cdot -\sin(x) = 0$ This is not surprising since $\sin^2(x)+\cos^2(x) = 1$, it is a constant function.

Quotient rule:

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}}$$

Proof. $f/g = f \cdot g^{-1}$ so we use the product rule:

$$(f \cdot g^{-1})' = f'g^{-1} + f \cdot (g^{-1})' = f'g^{-1} + f \cdot -1 \cdot g^{-2} \cdot g = \frac{f'g - g'f}{g^2}$$

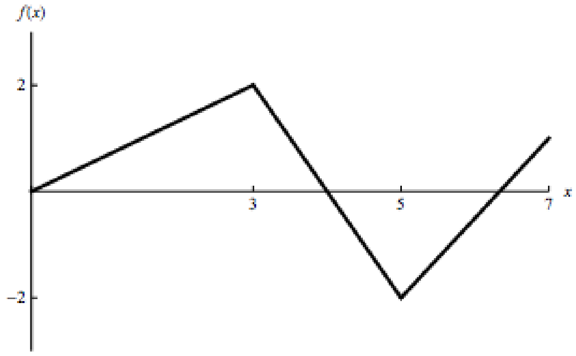
□

One more formula that you need:

$$\boxed{(\ln(x))' = \frac{1}{x} = x^{-1}}$$

Exercise 1.3.2.
Please IGNORE parts i and iv.

1. [10 points] Given below is a graph of a function $f(x)$ and a table for a function $g(x)$.



x	0	1	2	3	4
$g(x)$	4	3	1	2	$\frac{20}{3}$
$g'(x)$	-2	$-\frac{5}{2}$	$\frac{1}{2}$	3	$-\frac{1}{3}$

Give answers for the following or write "Does not exist." No partial credit will be given.

i) $\frac{d}{dx}f(g(x))$ at $x = 0$

ii) $\frac{d}{dx}[f(x)g(x)]$ at $x = 2$

iii) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ at $x = 4$

iv) $\frac{d}{dx}[g(f(x))]$ at $x = 3$

v) $f(g'(3))$

Solution.

i) $\frac{d}{dx}f(g(x))$ at $x = 0$ By the chain rule $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$. At $x = 0$ we have

$$f'(g(0))g'(0) = f'(4) \cdot (-2) = 4.$$

ii) $\frac{d}{dx}[f(x)g(x)]$ at $x = 2$ By the product rule $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$. At $x = 2$ we have

$$f'(2)g(2) + f(2)g'(2) = (2/3)(1) + (4/3)(1/2) = 4/3.$$

iii) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ at $x = 4$ By the quotient rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$. At $x = 4$ we have

$$\frac{f'(4)g(4) - f(4)g'(4)}{[g(4)]^2} = \frac{(-2)(20/3) - (0)(-1/3)}{(20/3)^2} = -3/10.$$

iv) $\frac{d}{dx}[g(f(x))]$ at $x = 3$ We know $g'(f(3)) = 1/2$, so for values of y near $f(3)$, $g(y)$ looks like a line with slope $1/2$. So $g(f(x))$ "looks like" $\frac{1}{2}f(x)+b$ for some constant b, for x near 3. Since $f(x)$ is "pointy" at $x = 3$, $\frac{1}{2}f(x) + b$ looks like a vertically compressed version of this pointy graph (near $x = 3$), which is still pointy. So $g(f(x))$ is also pointy at $x = 3$, hence not differentiable.

v) $f(g'(3))$ By the table, $g'(3) = 3$, so $f(g'(3)) = f(3) = 2$.

□

Exercise 1.3.3.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
$f(x)$	7	6	2	9
$f'(x)$	-2	1	3	2
$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ ____________

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ ____________

Solution.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
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$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ **-15/4**

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ **5**

□

Quiz #4. Please write you name: ________________________ and email: ________________________
Let $f(x) = e^{2x} + \tan(x)$.

1. [20 pt] Where is $f(x)$ continuous?

2. [20 pt] Where is $f(x)$ differentiable?

3. [60 pt]Find the equation of the graph's tangent line at $x = 0$

Solution.
The function is differentiable and continuous as long as it is defined. e^{2x} is defined anywhere whereas $\tan(x) = \sin(x)/\cos(x)$ is defined anywhere expect where where $\cos(x) = 0$, that is for $x = \pi/2, \pi/2 + \pi, \pi/2 + 2\pi, ...$ and $x = -\pi/2, \pi/2 - \pi, -\pi/2 - 2\pi, ...$ We can write that $f(x)$ is defined on $(-\infty, \infty)$ expect for $x = \pi/2 + n\pi$ for any integer n.

$f'(x) = ((e^2)^x)' + \tan'(x) = ln(e^2)e^{2x} + 1 + \tan^2(x) = 2e^{2x} + 1 + \tan^2(x)$. So $f'(0) = 3$. Therefore, the tangent line has slope 3 and passes through $(0, 1)$, so the equation is:

$$y = 3x + 1$$

□

1.4 Chain Rule

The chain rule helps us the take the derivative of a composite function. Say:

$$h(x) = f(g(x))$$

Then $h'(x)$ is calculated in three steps. First, calculate the derivative of $f(\boxed{})$. Then of $g(x)$. Then plug in $g(x)$ in the box and get:

$$\boxed{\frac{d}{dx}f(g(x)) = f'(\boxed{g(x)}) \cdot g'(x)}$$

We can use this formula as long as $g(x) \neq 0$ and both $g'(x)$ and $f'(g(x))$ exist.

Example 1.4.1.

- $(\cos^2(x))' = 2\boxed{cos(x)} \cdot (-\sin(x)) = -2\cos(x)\sin(x)$
- $(e^{x^2})' = e^{\boxed{x^2}} \cdot 2x = 2xe^{x^2}$
- $(\tan(\sin(x)))' = (1 + \tan^2 \boxed{\sin(x)}) \cdot \cos(x)$

Group Work please complete parts i and iv in **??**

Remark 1.4.2. If $g(x)$ in not differentiable at some $x = a$ it does NOT mean that $f(g(x))$ is not differentiable at $g(a)$. Here is an example.
Let $g(x) = |x|$, the absolute value of x. Since it has a sharp edge at $x = 0$, it is not differentiable. Now, take $f(x) = x^2$. Notice: $f(g(x)) = (|x|)^2 = x^2$, which is differentiable everywhere.

1.5 Trig Functions

We already gave the relevant formulas here: **??**

1.6 Inverse Functions

When $f(x)$ is invertible we can define the inverse $g(x) = f^{-1}(x)$. Then $g(f(x)) = x$. Now take the derivative. we get:

$$1 = g'(f(x))f'(x) \Rightarrow \boxed{[f^{-1}(z)]'|_{z=f(x)} = \frac{1}{f'(x)}}$$

Example 1.6.1.

- $f(x) = \tan(x)$. $f^{-1}(1) =?$ ($f^{-1}(z)$ is called $\arctan(z)$). We have $[f^{-1}(z)]'|_{z=f(x)} = \frac{1}{f'(x)}$ so we just need to find x such that $f(x) = 1$. Then the answer will be $1/f'(x)$. $\tan(x) = 1$, so choose $x = \pi/4$. Then $f'(\pi/4) = 1 + \tan^2(\pi/4) = 1 + 1^2 = 2$. So the answer is $1/2$.
- We can do that in general: $\arctan'(z) = \frac{1}{\tan'(x)}$ with $\tan(x) = z$. $\tan'(x) = 1 + \tan^2(x) = 1 + z^2$. Thus

$$\boxed{\arctan'(z) = \frac{1}{1 + z^2}}$$

- $\arcsin'(z) = 1/\cos(x)$ with $\sin(x) = z$. We can write $\cos(x) = \sqrt{1 - \sin^2(x)} = \sqrt{1 - z^2}$. So:

$$\boxed{\arcsin'(z) = \frac{1}{\sqrt{1 - z^2}}}$$

Exercise 1.6.2.

1. [12 points] The table below gives several values of a differentiable function $f(x)$. Assume that both $f(x)$ and $f'(x)$ are invertible. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

x	-3	-2	-1	0	1	2	3
$f(x)$	-8	-4	-1.2	0.5	1.4	1.8	2
$f'(x)$	5	3	2	1.2	0.5	0.3	0.1

a. [2 points] Let $g(x) = 3f(x) + 4$. Find $g'(1)$.

Answer: $g'(1) =$ ________________

b. [2 points] Find $(f^{-1})'(2)$.

Answer: $(f^{-1})'(2) =$ ________________

c. [2 points] Let $h(x) = f(e^x)$. Find $h'(\ln 2)$.

Answer: $h'(\ln 2) =$ ________________

d. [2 points] Let $j(x) = e^{f(x)}$. Find $j'(-2)$.

Answer: $j'(-2) =$ ________________

e. [2 points] Let $k(x) = f(x)f(x-2)$. Find $k'(1)$.

Answer: $k'(1) =$ ________________

f. [2 points] Let $\ell(x) = \frac{f(x)}{f(x+3)}$. Find $\ell'(0)$.

Answer: $\ell'(0) =$ ________________

Solution.

1. [12 points] The table below gives several values of a differentiable function $f(x)$. Assume that both $f(x)$ and $f'(x)$ are invertible. Do not give approximations. If it is not possible to find the value exactly, write NOT POSSIBLE.

x	-3	-2	-1	0	1	2	3
$f(x)$	-8	-4	-1.2	0.5	1.4	1.8	2
$f'(x)$	5	3	2	1.2	0.5	0.3	0.1

a. [2 points] Let $g(x) = 3f(x) + 4$. Find $g'(1)$.

Solution: $g'(x) = 3f'(x)$, so $g'(1) = 3 \cdot 0.5 = 1.5$

Answer: $g'(1) =$ 1.5

b. [2 points] Find $(f^{-1})'(2)$.

Solution: $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$, so $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(3)} = \dfrac{1}{0.1} = 10.$

Answer: $(f^{-1})'(2) =$ 10

c. [2 points] Let $h(x) = f(e^x)$. Find $h'(\ln 2)$.

Solution: $h'(x) = f'(e^x) \cdot e^x$, so $h'(\ln 2) = f'(e^{\ln 2}) \cdot e^{\ln 2} = f'(2) \cdot 2 = 0.3 \cdot 2 = 0.6.$

Answer: $h'(\ln 2) =$ 0.6

d. [2 points] Let $j(x) = e^{f(x)}$. Find $j'(-2)$.

Solution: $j'(x) = e^{f(x)} \cdot f'(x)$, so $j'(-2) = e^{f(-2)} \cdot f'(-2) = e^{-4} \cdot 3.$

Answer: $j'(-2) =$ $3e^{-4}$

e. [2 points] Let $k(x) = f(x)f(x-2)$. Find $k'(1)$.

Solution: $k'(x) = f'(x)f(x-2) + f(x)f'(x-2)$, so
$k'(1) = f'(1)f(1-2) + f(1)f'(1-2) = f'(1)f(-1) + f(1)f'(-1) = 0.5 \cdot (-1.2) + 1.4 \cdot 2 = -0.6 + 2.8 = 2.2.$

Answer: $k'(1) =$ 2.2

f. [2 points] Let $\ell(x) = \frac{f(x)}{f(x+3)}$. Find $\ell'(0)$.

Solution: $\ell'(x) = \dfrac{f'(x)f(x+3) - f'(x+3)f(x)}{(f(x+3))^2}$, so

$$\ell'(0) = \frac{f'(0)f(3) - f'(3)f(0)}{(f(3))^2} = \frac{1.2 \cdot 2 - 0.5 \cdot 0.1}{2^2} = \frac{2.4 - 0.05}{4} = \frac{2.35}{4} = 0.5875.$$

Answer: $\ell'(0) =$ 0.5875

□

Exercise 1.6.3.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
$f(x)$	7	6	2	9
$f'(x)$	-2	1	3	2
$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ ________________

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ ________________

c. [3 points] If $m(x) = g^{-1}(x)$, find $m'(4)$.

$m'(4) =$ ________________

d. [3 points] If $n(x) = f(g(x))$, find $n'(3)$.

$n'(3) =$ ________________

Solution.

8. [12 points] In the following table, both f and g are differentiable functions of x. In addition, $g(x)$ is an invertible function. Write your answers in the blanks provided. You do not need to show your work.

x	2	3	4	5
$f(x)$	7	6	2	9
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$g(x)$	1	4	7	11
$g'(x)$	1	2	3	2

a. [3 points] If $h(x) = \frac{g(x)}{f(x)}$, find $h'(4)$.

$h'(4) =$ **-15/4**

b. [3 points] If $k(x) = f(x)g(x)$, find $k'(2)$.

$k'(2) =$ **5**

c. [3 points] If $m(x) = g^{-1}(x)$, find $m'(4)$.

$m'(4) =$ **1/2**

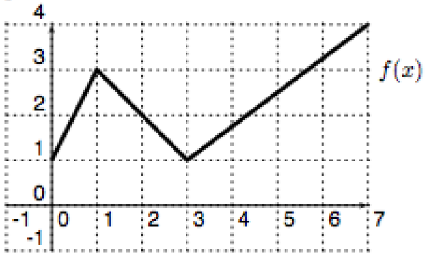
d. [3 points] If $n(x) = f(g(x))$, find $n'(3)$.

$n'(3) =$ **6**

□

Exercise 1.6.4.

2. [12 points]
Use the graph of the function f and the table of values for the function g to answer the questions below.



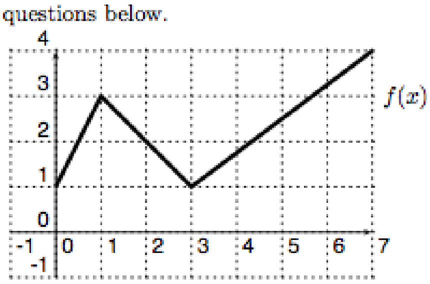
x	1	2	3	4	5	6
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79
g"(x)	-2	-8	-14	-20	-26	-32

a. [6 points] Let $h(x) = \frac{g(x)}{f(2x+3)}$. Find $h'(1)$ or explain why it does not exist.

b. [6 points] Let $k(x) = g(g(x))$. Determine whether k is increasing or decreasing at $x = 2$.

Solution.

2. [12 points]
Use the graph of the function f and the table of values for the function g to answer the questions below.



x	1	2	3	4	5	6
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79
g"(x)	-2	-8	-14	-20	-26	-32

a. [6 points] Let $h(x) = \frac{g(x)}{f(2x+3)}$. Find $h'(1)$ or explain why it does not exist.

Solution: Using the quotient rule and the chain rule, we get

$$\begin{aligned} h'(x) &= \frac{g'(x)f(2x+3) - g(x)f'(2x+3)\cdot 2}{(f(2x+3))^2} \\ h'(1) &= \frac{g'(1)f(5) - g(1)f'(5)\cdot 2}{(f(5))^2} \\ &= \frac{6\cdot 2.5 - 0\cdot 0.75\cdot 2}{(2.5)^2} \\ &= \frac{6}{2.5} = \frac{12}{5} = 2.4 \end{aligned}$$

b. [6 points] Let $k(x) = g(g(x))$. Determine whether k is increasing or decreasing at $x = 2$.

Solution: Using the chain rule, we get

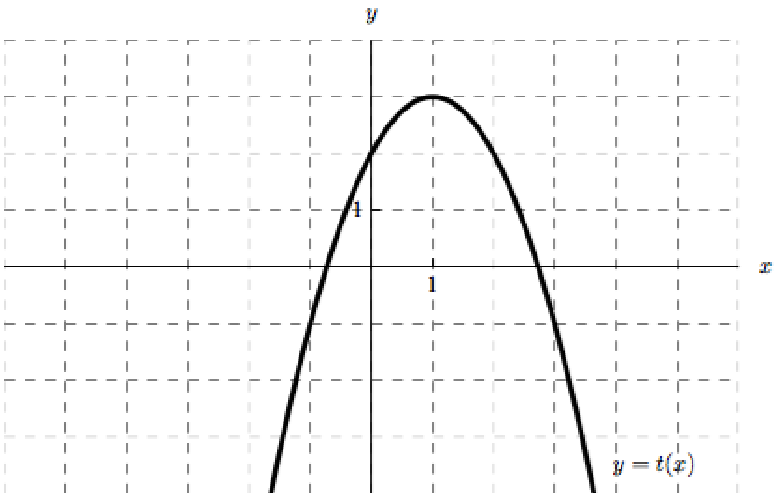
$$\begin{aligned} k'(x) &= g'(g(x))\cdot g'(x) \\ k'(2) &= g'(g(2))\cdot g'(2) \\ &= g'(4)\cdot g'(2) \\ &= (-27)\cdot 1 = -27 \end{aligned}$$

Since $k'(2) < 0$, we know that $k(x)$ is decreasing at $x = 2$.

□

Exercise 1.6.5.

2. [16 points]
Graphed below is a function $t(x)$. Define $p(x) = x^2 t(x)$, $q(x) = t(\sin(x))$, $r(x) = \frac{t(x)}{3x+1}$, and $s(x) = t(t(x))$. For this problem, do not assume $t(x)$ is quadratic.



Carefully estimate the following quantities.

a. [4 points] $p'(-1)$

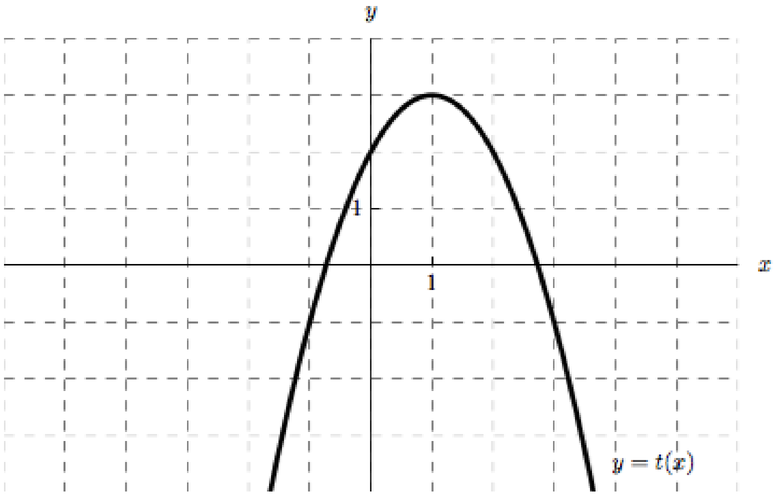
b. [4 points] $q'(0)$

c. [4 points] $r'(3)$

d. [4 points] $s'(0)$

Solution.

2. [16 points]
Graphed below is a function $t(x)$. Define $p(x) = x^2 t(x)$, $q(x) = t(\sin(x))$, $r(x) = \frac{t(x)}{3x+1}$, and $s(x) = t(t(x))$. For this problem, do not assume $t(x)$ is quadratic.



Carefully estimate the following quantities.

a. [4 points] $p'(-1)$

Solution: By the product rule, $p'(x) = 2xt(x) + x^2 t'(x)$. Estimating using the graph, we have

$$p'(-1) = 2(-1)t(-1) + (-1)^2 t'(-1) = (-2)(-1) + 4 = 6.$$

b. [4 points] $q'(0)$

Solution: By the chain rule, $q'(x) = t'(\sin x)\cos x$. Estimating using the graph, we have

$$q'(0) = t'(\sin 0)\cos 0 = t'(0) = 2.$$

c. [4 points] $r'(3)$

Solution: By the quotient rule, $r'(x) = \frac{(3x+1)t'(x) - 3t(x)}{(3x+1)^2}$. Estimating using the graph, we have

$$r'(3) = \frac{(3(3)+1)t'(3) - 3t(3)}{(3(3)+1)^2} = \frac{-40 - 3(-1)}{100} = -\frac{37}{100}$$

d. [4 points] $s'(0)$

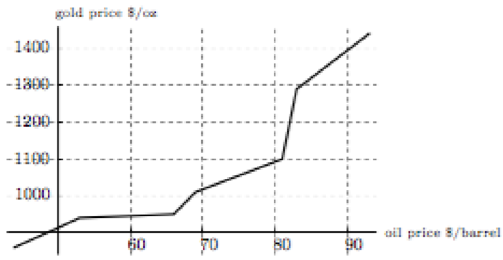
Solution: By the chain rule, $s'(x) = t'(t(x))t'(x)$. Estimating using the graph, we have

$$s'(0) = t'(t(0))t'(0) = t'(2) \cdot 2 = (-2)(2) = -4.$$

□

Exercise 1.6.6.

5. [15 points] The graph to the right shows a function $G(b)$ that approximates the price of an ounce of gold (in dollars) as a function of the cost of a barrel of oil for data between 2009 and 2011.[1]



a. [3 points] Estimate $G'(70)$.

b. [5 points] Recall that G^{-1} is defined to be a function such that $G^{-1}(G(b)) = b$ (or such that $G(G^{-1}(y)) = y$, where y is the price of an ounce of gold). Derive, using the chain rule, a formula for $(G^{-1})'$ in terms of G'.

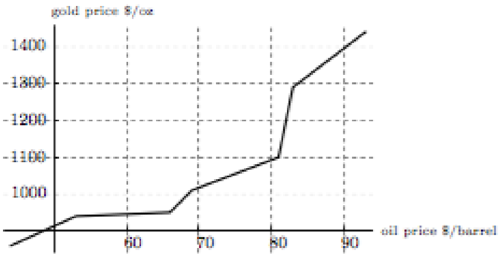
c. [4 points] Using parts (a) and (b), estimate $(G^{-1})'(G(70))$.

d. [3 points] Explain the practical meaning of your result in (c).

[1] Gold prices from <http://www.goldprice.org/>; oil from <http://en.wikipedia.org/wiki/Price_of_petroleum>.

Solution.

5. [15 points] The graph to the right shows a function $G(b)$ that approximates the price of an ounce of gold (in dollars) as a function of the cost of a barrel of oil for data between 2009 and 2011.[1]



a. [3 points] Estimate $G'(70)$.

Solution: From the graph, it appears that between $b = 70$ and $b = 80$, G increases by about 70 as b increases about 10. Thus we estimate that $G'(70) \approx 7$ \$/oz per \$/barrel.

b. [5 points] Recall that G^{-1} is defined to be a function such that $G^{-1}(G(b)) = b$ (or such that $G(G^{-1}(y)) = y$, where y is the price of an ounce of gold). Derive, using the chain rule, a formula for $(G^{-1})'$ in terms of G'.

Solution: We know that $G^{-1}(G(b)) = b$. Thus $\frac{d}{db}G^{-1}(G(b)) = 1$. Differentiating the left-hand side of this using the chain rule, we have $\frac{d}{db}G^{-1}(G(b)) = (G^{-1})'(G(b)) \cdot G'(b) = 1$. Thus $(G^{-1})'(G(b)) = 1/G'(b)$.
Alternately, if we start with $G(G^{-1}(y)) = y$, we have $\frac{d}{dy}G(G^{-1}(y)) = 1$. Applying the chain rule to the left-hand side, we have $G'(G^{-1}(y)) \cdot (G^{-1})'(y) = 1$, so that $(G^{-1})'(y) = 1/G'(G^{-1}(y))$. (Obviously, with $y = G(b)$, this is the same as the previous expression.)

c. [4 points] Using parts (a) and (b), estimate $(G^{-1})'(G(70))$.

Solution: Using part (b), we have $(G^{-1})'(G(70)) = 1/G'(70) = 1/7$ \$/barrel per \$/oz.

d. [3 points] Explain the practical meaning of your result in (c).

Solution: $(G^{-1})'(G(70)) = 0.14$ indicates that when the price of oil is 70 \$/barrel, the price of a barrel of oil goes up by about \$0.14 if the price of gold goes up by \$1.

[1] Gold prices from <http://www.goldprice.org/>; oil from <http://en.wikipedia.org/wiki/Price_of_petroleum>.

□

Exercise 1.6.7. Take a practice Gateway exam.

1.7 Implicit functions

Sometimes we are presented with an equation that includes functions. What we can is the take the derivative of both sides an thus get an equation for the derivative.

Example 1.7.1. Suppose we have

$$y^2x^3 = 4$$

And we need to find the derivative of $y(x)$ when $x = 1$. Then we take the derivative of both sides. Remember that $y(x)$ is a function so we need to use the chain rule.

$$2yy'x^3 + 3y^2x^2 = 0$$

This is an equation for the derivative. Plug in $x = 1$. We get:

$$2y(1)y'(1) + 3y^2(1) = 0 \Rightarrow y'(1) = -(3/2)y(1)$$

But what is $y(1)$? we have the original equation. We can plug in $x = 1$ and get:

$$y^2(1) = 4 \Rightarrow y(1) = \pm 2$$

So $y'(1) = \mp 3$. To get a specific value, we need more information. For example, whether $y(x)$ is increasing or decreasing when $x = 1$.

Exercise 1.7.2.

5. [13 points] The equation below implicitly defines a hyperbola.

$$x^2 - y^2 = 2x + xy + y + 2.$$

a. [5 points] Find $\frac{dy}{dx}$.

b. [4 points] Consider the two points $(4, 2)$ and $(2, -1)$. Show that one of these points lies on the hyperbola defined above, and one does not.

c. [4 points] For the point in part (b) which lies on the hyperbola, find the equation of the line which is tangent to the hyperbola at this point.

Solution.

5. [13 points] The equation below implicitly defines a hyperbola.

$$x^2 - y^2 = 2x + xy + y + 2.$$

a. [5 points] Find $\frac{dy}{dx}$.

Solution: We use implicit differentiation:

$$2x - 2y\frac{dy}{dx} = 2 + \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + \frac{dy}{dx} + 0.$$

Then solve for $\frac{dy}{dx}$:

$$\begin{aligned} 2x - 2 - y &= \frac{dy}{dx}(x + 2y + 1) \\ \frac{dy}{dx} &= \frac{2x - 2 - y}{x + 2y + 1} \end{aligned}$$

b. [4 points] Consider the two points $(4, 2)$ and $(2, -1)$. Show that one of these points lies on the hyperbola defined above, and one does not.

Solution: For the point $(4, 2)$, $x^2 - y^2 = 4^2 - 2^2 = 12$ and $2x + xy + y + 2 = 2(4) + 4(2) + 2 + 2 = 20$ are not equal, so $(4, 2)$ IS NOT on the hyperbola.

For the point $(2, -1)$, $x^2 - y^2 = 2^2 - (-1)^2 = 3$ and $2x + xy + y + 2 = 2(2) + 2(-1) - 1 + 2 = 3$ are equal, so $(2, -1)$ IS on the hyperbola.

c. [4 points] For the point in part (b) which lies on the hyperbola, find the equation of the line which is tangent to the hyperbola at this point.

Solution: From part (a),

$$\frac{dy}{dx} = \frac{2x - 2 - y}{x + 2y + 1},$$

so

$$\frac{dy}{dx}\Big|_{(x,y)=(2,-1)} = \frac{2(2) - 2 - (-1)}{2 + 2(-1) + 1} = \frac{3}{1} = 3.$$

Then the equation of the tangent line is $y = 3(x - 2) - 1$ or $y = 3x - 7$.

□

Exercise 1.7.3.

5. [12 points] Suppose a curve in the plane is given by the equation

$$\sin(\pi xy) = y - 1.$$

a. [3 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

b. [5 points] Calculate $\frac{dy}{dx}$.

c. [4 points] Find the equation for the tangent line to the curve at the point $(1, 1)$.

Solution.

5. [12 points] Suppose a curve in the plane is given by the equation

$$\sin(\pi xy) = y - 1.$$

a. [3 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

Solution: At $(1, 1)$, the right hand side is $\sin(\pi) = 0$ and the left hand side is $1 - 1 = 0$. Therefore the point is on the curve since the right and left hand sides are equal.

b. [5 points] Calculate $\frac{dy}{dx}$.

Solution: Taking the derivative with respect to x of the equation, we have

$$\pi \cos(\pi xy) \cdot (y + x\frac{dy}{dx}) = \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{\pi y \cos(\pi xy)}{1 - \pi x \cos(\pi xy)}.$$

c. [4 points] Find the equation for the tangent line to the curve at the point $(1, 1)$.

Solution: The slope of the tangent line to the curve is

$$\frac{dy}{dx}(1,1) = \frac{\pi \cos(\pi)}{1 - \pi \cos(\pi)} = \frac{-\pi}{1+\pi}.$$

The equation for the tangent line is

$$y - 1 = \frac{-\pi}{1+\pi}(x - 1).$$

□

Exercise 1.7.4.

1. [12 points] The following questions relate to the implicit curve $2x^2 + 4x - x^2y^2 + 3y^4 = -1$.

   a. [6 points] Calculate $\frac{dy}{dx}$.

   b. [2 points] Q is the only point on the curve that has a y-coordinate of 1. Find the x-coordinate of Q.

   c. [4 points] Find the equation of the tangent line to the curve at Q.

Solution.

1. [12 points] The following questions relate to the implicit curve $2x^2 + 4x - x^2y^2 + 3y^4 = -1$.

a. [6 points] Calculate $\frac{dy}{dx}$.

Solution: Differentiating both sides with respect to x, we get

$$4x + 4 - 2xy^2 - 2x^2y\,\frac{dy}{dx} + 12y^3\,\frac{dy}{dx} = 0.$$

Moving all terms with no $\frac{dy}{dx}$ to the other side and factoring out $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx}(12y^3 - 2x^2y) = 2xy^2 - 4x - 4.$$

So

$$\frac{dy}{dx} = \frac{2xy^2 - 4x - 4}{12y^3 - 2x^2y} = \frac{xy^2 - 2x - 2}{6y^3 - x^2y}.$$

b. [2 points] Q is the only point on the curve that has a y-coordinate of 1. Find the x-coordinate of Q.

Solution: Plugging $y = 1$ into the equation for the curve gives us

$$2x^2 + 4x - x^2 + 3 = -1.$$

Moving all the terms to the left, we get

$$x^2 + 4x + 4 = 0.$$

This factors as $(x + 2)^2 = 0$, so $x = -2$.

c. [4 points] Find the equation of the tangent line to the curve at Q.

Solution: To find the slope, we plug in $x = -2$ and $y = 1$ to $\frac{dy}{dx}$.

$$\text{slope} = \frac{-2 + 4 - 2}{6 - 4} = 0.$$

Thus, the tangent line is the horizontal line passing through Q, which has equation $y = 1$.

□

1.8 SKIP

1.9 Linear Approximation

Let $y = f(x)$ be a function. The tangent line approximation of $f(x)$ at the point $(a, f(a)$ is:

$$\boxed{y = \ell(x) = f(a) + f'(a)(x - a)}$$

This is also the tangent line. We call it "approximation" because we can use the tanget line to approximate the values of $f(x)$ in the vicinity of $x = a$. For a point b near a we have:

$$f(b) \approx f(a) + f'(a)(b - a)$$

Example 1.9.1. Suppose that we are given that $f(2) = 5$ and $f'(2) = 3$. Approximate $f(2.1)$.
We use the tangent line approximation.

$$f(2.1) \approx f(2) + f'(2)(2.1 - 2) = 5 + 3(0.1) = 5.3$$

Graphically:

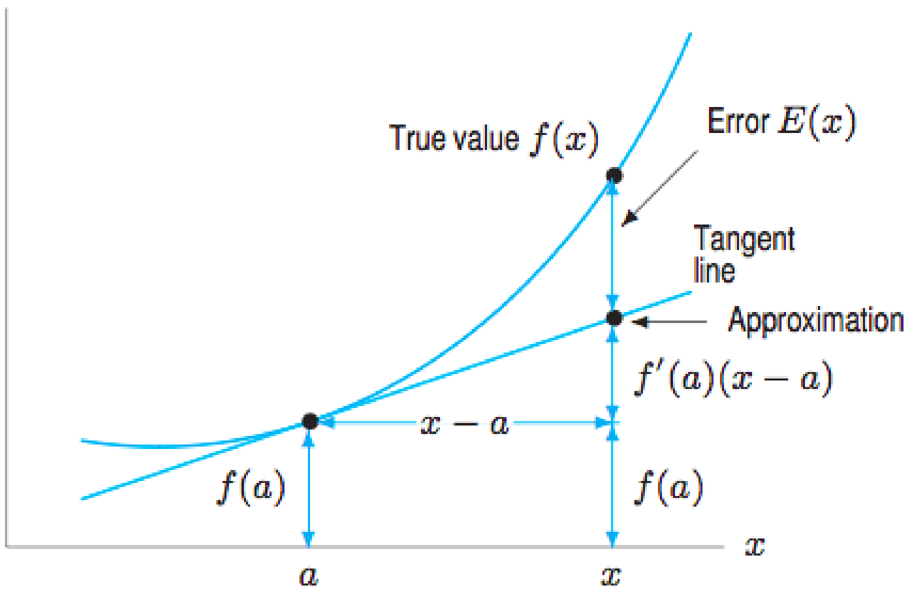


Figure 3.40: The tangent line approximation and its error

The error is defined as:

$$\boxed{E(x) = f(x) - \ell(x) = f(x) - f(a) - f'(a)(a - x)}$$

Remark 1.9.2.
- In order to to give the value of the error at the point $x = b$ we need to know the actual value of $f(x)$ at $x = b$.
- We have a way to evaluate the error if we know the second derivative at the point $x = a$

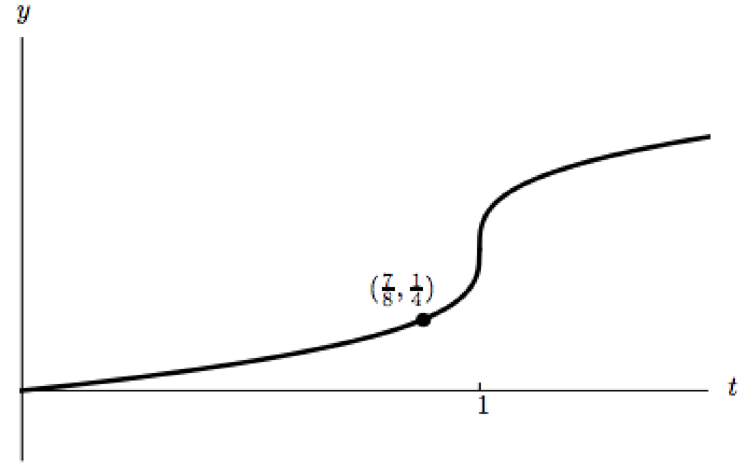
$$E(b) \approx \frac{f''(a)}{2}(b - a)^2$$

- Graphically we can see that if $f''(a) > 0$ and the function concave up, then the error is positive. So we underestimate using the tanget line approximation. Otherwise - we have an overestimate.

$f''(a) > 0$	underestimation
$f''(a) < 0$	overestimation
$f''(a) = 0$	can't tell

Exercise 1.9.3.

6. [15 points] Given below is the graph of a function $h(t)$. Suppose $j(t)$ is the local linearization of $h(t)$ at $t = \frac{7}{8}$.



a. [5 points] Given that $h'(\frac{7}{8}) = \frac{2}{3}$, find an expression for $j(t)$.

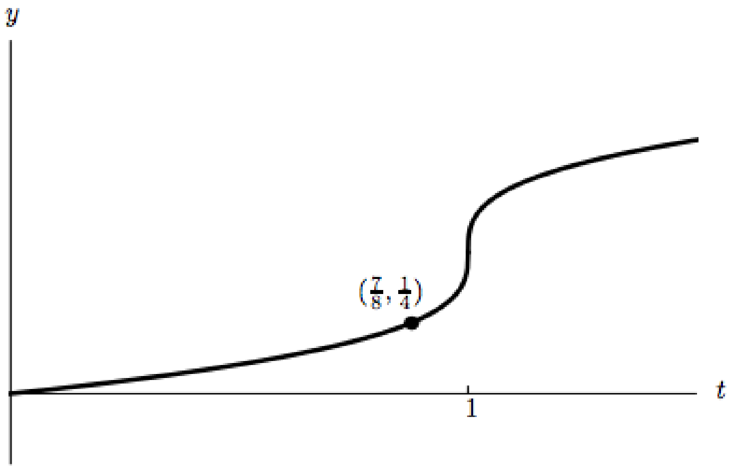
b. [4 points] Use your answer from (a) to approximate $h(1)$.

c. [3 points] Is the approximation from (b) an over- or under-estimate? Explain.

d. [3 points] Using $j(t)$ to estimate values of $h(t)$, will the estimate be more accurate at $t = 1$ or at $t = \frac{3}{4}$? Explain.

Solution.

6. [15 points] Given below is the graph of a function $h(t)$. Suppose $j(t)$ is the local linearization of $h(t)$ at $t = \frac{7}{8}$.



a. [5 points] Given that $h'(\frac{7}{8}) = \frac{2}{3}$, find an expression for $j(t)$.

Solution: The local linearization is the tangent line to the curve. We know this line has slope $h'(\frac{7}{8}) = \frac{2}{3}$ and it goes through the point $(\frac{7}{8}, \frac{1}{4})$, so it has equation

$$y - \frac{1}{4} = \frac{2}{3}(t - \frac{7}{8})$$

using point slope form. Solving for y we have $y = \frac{2}{3}t - \frac{1}{3}$. So $j(t) = \frac{2}{3}t - \frac{1}{3}$. stuff

b. [4 points] Use your answer from (a) to approximate $h(1)$.

Solution: Since $j(t)$ approximates $h(t)$ for t-values near $\frac{7}{8}$, we have

$$h(1) \approx j(1) = \frac{2}{3}(1) - \frac{1}{3} = \frac{1}{3}.$$

c. [3 points] Is the approximation from (b) an over- or under-estimate? Explain.

Solution: The approximation in (b) is an underestimate. The function $h(t)$ is concave up at $t = 7/8$ which means the graph lies above the local linearization for t-values near $7/8$. Since we are using the local linearization to estimate the function value, our estimate will be less than the actual function value.

d. [3 points] Using $j(t)$ to estimate values of $h(t)$, will the estimate be more accurate at $t = 1$ or at $t = \frac{3}{4}$? Explain.

Solution: The estimate at $t = 3/4$ will be more accurate. This can be seen by drawing the tangent line and measuring the vertical distance between the estimated value and the function value at the t values $3/4$ and 1. The line is much closer to the function at $t = 3/4$ than it is at $t = 1$.

□

Exercise 1.9.4.

6. [10 points] Calvin is stuck in the desert, and he needs to build a cube out of cactus skins to hold various supplies. He wants his cube to have a volume of 8.1 cubic feet, but he needs to figure out the side length to cut the cactus skins the right size. He has forgotten his trusty calculator, so he decides to figure out the side length of his cube using calculus.

a. [5 points] Find a local linearization of the function $f(x) = (x+8)^{1/3}$ at $x = 0$.

b. [3 points] Use your linearization to approximate $(8.1)^{1/3}$.

c. [2 points] Should your approximation from part **b.** be an over-estimate or an under-estimate? Why?

Solution.

6. [10 points] Calvin is stuck in the desert, and he needs to build a cube out of cactus skins to hold various supplies. He wants his cube to have a volume of 8.1 cubic feet, but he needs to figure out the side length to cut the cactus skins the right size. He has forgotten his trusty calculator, so he decides to figure out the side length of his cube using calculus.

a. [5 points] Find a local linearization of the function $f(x) = (x + 8)^{1/3}$ at $x = 0$.

Solution: The derivative is $f'(x) = \frac{1}{3}(x + 8)^{-2/3}$. To find the local linearization we compute $f'(0) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$ and $f(0) = 2$. The equation for the tangent line to f at $x = 0$ is $y - 2 = \frac{1}{12}x$. So the local linearization of f near $x = 0$ is

$$L(x) = \frac{1}{12}x + 2.$$

b. [3 points] Use your linearization to approximate $(8.1)^{1/3}$.

Solution: We need to approximate $(8.1)^{1/3} = f(0.1)$. According to our local linearization,

$$f(0.1) \approx L(0.1) = \frac{1}{12}(0.1) + 2 = \frac{241}{120}.$$

c. [2 points] Should your approximation from part **b.** be an over-estimate or an under-estimate? Why?

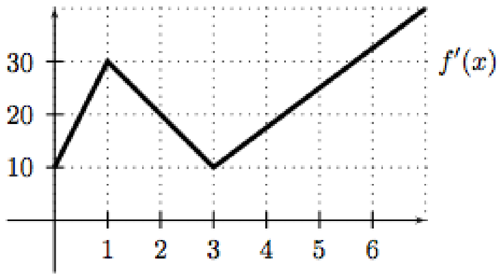
Solution:
The second derivative of f is $f''(x) = -\frac{2}{9}(x + 8)^{-5/3}$. For values of x near 0, the second derivative will be negative which means f is concave down near 0. This means our estimate is an overestimate.

□

Please ignore part c in the next one.

Exercise 1.9.5.

2. [12 points]
Use the graph of the function f' and the table of values for the function g to answer the questions below. Each problem requires only a small amount of work, but you must show it.



x	-20	-10	0	10	20	30
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79

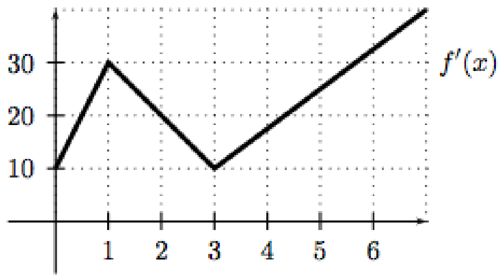
a. [3 points] Write a formula for the local linearization of g near $x = 10$ and use it to approximate $g(10.1)$.

b. [3 points] Using the table, estimate $g''(-10)$.

c. [3 points] If $f(3) = 30$, find the exact value of $f(1)$.

Solution.

2. [12 points]
Use the graph of the function f' and the table of values for the function g to answer the questions below. Each problem requires only a small amount of work, but you must show it.



x	-20	-10	0	10	20	30
g(x)	0	4	0	-18	-56	-120
g'(x)	6	1	-10	-27	-50	-79

a. [3 points] Write a formula for the local linearization of g near $x = 10$ and use it to approximate $g(10.1)$.

Solution:

$$\begin{aligned} g(x) &\approx g(10) + g'(10)(x-10) \\ g(10.1) &\approx g(10) + g'(10)(10.1-10) \\ &= -18 + (-27)(0.1) = -20.7 \end{aligned}$$

b. [3 points] Using the table, estimate $g''(-10)$.

Solution: Note that $g''(x) \approx \dfrac{\Delta g'(x)}{\Delta x}$. We will use the two x-values closest to -10: $x = -20$ and $x = 0$.

$$g''(10) \approx \frac{g'(0) - g'(-20)}{0-(-20)} = \frac{-10-6}{0-(-20)} = \frac{-16}{20} = \frac{-4}{5} = -0.8$$

c. [3 points] If $f(3) = 30$, find the exact value of $f(1)$.

Solution: By the Fundamental Theorem of Calculus,

$$f(3) - f(1) = \int_1^3 f'(x)dx.$$

This integral represents the area under $f'(x)$ and above the x-axis between $x = 1$ and $x = 3$. Using basic geometry, this area is 40. Thus, $f(3) - f(1) = 40$, so $f(1) = f(3) - 40 = 30 - 40 = -10$.

□

Quadratic Approximation - Not in the book, only in class
Suppose $f(x) = \cos(x)$ and we would like to use a linearization at $x = 0$. Notice that $\cos(0) = 1$ and $\cos'(0) = -\sin(x)|_{x=0} = 0$ so the linear approximation is just $y = 1$. This is not good. So the have the quadratic approximation at $x = a$:

$$\boxed{Q(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$

Notice: $Q(x)$ is the best quadratic polynomial one can fit to $f(x)$ at $x = a$. Also:

- $f(a) = Q(a)$
- $f'(a) = Q'(a)$
- $f''(a) = Q''(a)$

Example 1.9.6. Let $f(x) = \cos(x)$. Find the quadratic approximation at $x = 0$.
So we have $\cos'(0) = -\cos(x)|_{x=0} = -1$. We get:

$$Q(x) = 1 - \frac{1}{2}x^2$$

Approximate $\cos(\pi/6)$. We get $1 - 0.5(\pi/6)^2 = 0.863$. The real value is $\sqrt{3}/2 = 0.866$.

Exercise 1.9.7.

5. [12 points] In Srebmun Foyoj, Maddy and Cal are eating lava cake. Let $T(v)$ be the time (in seconds) it takes Maddy to eat a v cm^3 serving of lava cake. Assume $T(v)$ is invertible and differentiable for $0 < v < 1000$. Several values of $T(v)$ and its first and second derivatives are given in the table below.

v	10	15	60	100	150	200	300
$T(v)$	11	22	84	194	393	513	912
$T'(v)$	2.4	1.9	1.8	3.6	3.7	0.9	17.5
$T''(v)$	-0.11	-0.08	0.05	0.04	-0.04	-0.05	0.59

Remember to show your work carefully.

a. [4 points] Use an appropriate linear approximation to estimate the amount of time it takes Maddy to eat a 64 cm^3 serving of lava cake. *Include units.*

Answer: ____________________

b. [4 points] Use the quadratic approximation of $T(v)$ at $v = 200$ to estimate $T(205)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Answer: $T(205) \approx$ ____________________

c. [4 points] Let $C(v)$ be the time (in seconds) it takes Cal to eat a v cm^3 serving of lava cake, and suppose $C(v) = T(\sqrt{v})$. Let $L(v)$ be the local linearization of $C(v)$ at $v = 100$. Find a formula for $L(v)$. Your answer should not include the function names T or C.

Answer: $L(v) =$ ____________________

Solution.

5. [12 points] In Srebmun Foyoj, Maddy and Cal are eating lava cake. Let $T(v)$ be the time (in seconds) it takes Maddy to eat a v cm^3 serving of lava cake. Assume $T(v)$ is invertible and differentiable for $0 < v < 1000$. Several values of $T(v)$ and its first and second derivatives are given in the table below.

v	10	15	60	100	150	200	300
$T(v)$	11	22	84	194	393	513	912
$T'(v)$	2.4	1.9	1.8	3.6	3.7	0.9	17.5
$T''(v)$	-0.11	-0.08	0.05	0.04	-0.04	-0.05	0.59

Remember to show your work carefully.

a. [4 points] Use an appropriate linear approximation to estimate the amount of time it takes Maddy to eat a 64 cm^3 serving of lava cake. *Include units.*

Solution: The closest point in the table to $v = 64$ is $v = 60$, so this is the appropriate choice for the tangent line approximation. Based on the table, the line will go through $(60, 84)$ and have slope 1.8, so it must be $L(v) = 84 + 1.8(v - 60)$. Plugging in 64 for v, we get an estimate of 91.2 seconds.

Answer: 91.2 seconds

b. [4 points] Use the quadratic approximation of $T(v)$ at $v = 200$ to estimate $T(205)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Solution: Let $Q(v)$ be the quadratic approximation of $T(v)$ at $v = 200$. Then

$$Q(v) = T(200)+T'(200)(v-200)+\frac{T''(200)}{2}(v-200)^2 = 513+0.9(v-200)+\frac{-0.05}{2}(v-200)^2.$$

So the resulting approximation of $T(205)$ is given by

$$T(205) \approx Q(205) = 513+0.9(205-200)-\frac{0.05}{2}(205-200)^2 = 513+4.5-0.625 = 516.875.$$

Answer: $T(205) \approx$ 516.875

c. [4 points] Let $C(v)$ be the time (in seconds) it takes Cal to eat a v cm^3 serving of lava cake, and suppose $C(v) = T(\sqrt{v})$. Let $L(v)$ be the local linearization of $C(v)$ at $v = 100$. Find a formula for $L(v)$. Your answer should not include the function names T or C.

Solution: We know $L(v) = C(100)+C'(100)(v-100)$. We also know $C(100) = T(10) = 11$. So we need to find $C'(100)$.

Since $C(v) = T(\sqrt{v})$, we apply the chain rule and see that $C'(v) = \frac{1}{2\sqrt{v}}T'(\sqrt{v})$. Using the table above, we then find that $C'(100) = \frac{1}{20}T'(10) = \frac{2.4}{20} = 0.12$.

So $L(v) = 11 + 0.12(v - 100)$.

Answer: $L(v) =$ $11 + 0.12(v - 100)$

□

1.10 MVT

Theorem 3.7: The Mean Value Theorem

If f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, then there exists a number c, with $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In other words, $f(b) - f(a) = f'(c)(b - a)$.

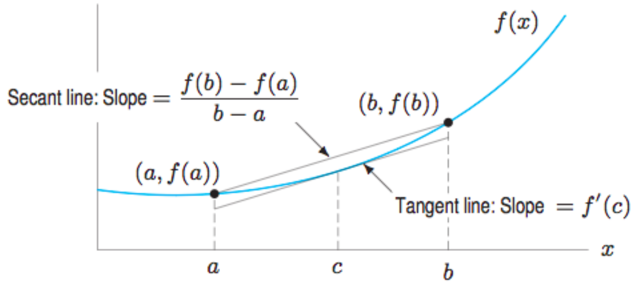
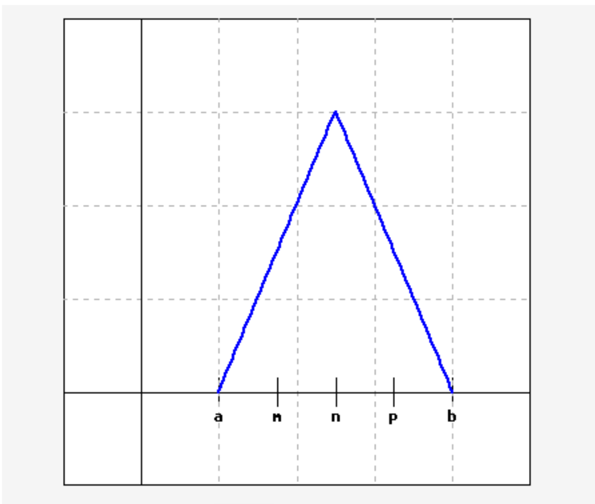


Figure 3.44: The point c with $f'(c) = \frac{f(b)-f(a)}{b-a}$

This theorem tells us that we can take a CTS and differentiable function, draw a secant line, and then it is guaranteed that for some middle point we can find a tangent line with the same slope as the secant line.

Example 1.10.1. Consider the following graph:



Dose this function satisfy the hypotheses of the MVT? NO!
Indeed, the secant line between a and b has a slope 0, but no point on the graph has derivative 0.
On the other hand, if we "smooth" the function up, then MVT applies. The tip will have a derivative of 0 — similar to downward facing parabola.