Notes on Self Concordant Barrier Functions

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Topic 1

Self-Concordant Barrier Function Parameters: A Result for the Conic Hausdorff Metric

1 Background on Cones

1.1 The Descent Cone

Definition 1.1 (Cone)**.** *A cone is a set $K \subseteq \mathbb{R}^n$ that is positively homogeneous: $K = \tau K$ for all $\tau > 0$. A convex cone is a cone which is also a convex set.*

Definition 1.2 (Descent Cone)**.** *The descent cone $\mathcal{D}(f, x)$ associated with a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, and a point $x \in \mathbb{R}^d$ is the set:*

$$\mathcal{D}(f, x) = \bigcup_{\tau > 0} \{u \in \mathbb{R}^n : f(x + \tau u) \leq f(x)\}.$$

It is worth noting that the descent cone of a convex function is itself a convex cone. While we have this guarantee, there are examples where the descent cone may not be closed.

1.2 Polyhedral Cones

Definition 1.3 (Polyhedral Cone)**.** *A cone C is polyhedral if it can be expressed as the intersection of a finite number of half spaces, i.e.*

$$C = \bigcap_{i=1}^{N} \{x \in \mathbb{R}^n : \langle u_i, x \rangle \geq 0\} \text{ for some } u_i \in \mathbb{R}^n.$$

We will denote the set of all closed convex cones in $\mathbb{R}^n$ by $\mathcal{C}_n$.

Definition 1.4 (Polar)**.** *The polar cone associated with a general cone in $\mathbb{R}^n$ is the closed convex cone*

$$K^\circ = \{v \in \mathbb{R}^n : \langle v, x \rangle \leq 0 \text{ for all } x \in K\}.$$

Fact 1.5. *The polar P° of a polyhedral cone P is itself polyhedral.*

2 A New Metric

2.1 Angular and S Distance

Definition 1.6 (Angular Distance)**.** *The angular distance between two non zero vectors x and y*

$$\text{dist}_S(x, y) = \arccos\left(\frac{\langle x, y \rangle}{||x||||y||}\right) \text{ for } x, y \in \mathbb{R}^n \backslash \{0\}.$$

Definition 1.7 (S Distance). *Using the above definition as well as the conventions that:* $dist_S(\mathbf{0},\mathbf{0}) = 0$ *and* $dist_S(\mathbf{0},x) = dist_S(x,\mathbf{0}) = \pi/2$ *for* $x \neq 0$*, we can define the S distance between two cones* C, C' *as*

$$\text{dist}_S(C, C') = \inf_{x\in C, y\in C'} \text{dist}_S(x, y)$$

given that $C, C' \neq \{\mathbf{0}\}$*. In this case, we use the notions above to define distances involving* $\{\mathbf{0}\}$*.*

Definition 1.8 (Angular Expansion). *The angular expansion* $T_S(C,\alpha)$ *of a cone* $C \in \mathcal{C}_n$ *by an angle* $0 \le \alpha \le 2\pi$ *is the set*

$$T_S(C,\alpha) = \{x \in \mathbb{R}^n : \text{dist}_S(x, y) \le \alpha \textit{ for some } y \in C\}$$

Fact 1.9. *The angular expansion of a convex cone is not necessarily convex for all* $\alpha > 0$*. An easy to see example of this phenomenon is to consider* C *a proper subspace which has an angular expansion that is never (i.e. for all* α*) convex.*

Combining all of the above we can define a metric on $\mathcal{C}_n$ as follows:

Definition 1.10 (Conic Hausdorff Metric). *For* $C_1, C_2 \in \mathcal{C}_n$ *we define:*

$$\text{dist}_H(C_1, C_2) = \inf \{\alpha \ge 0 : T_S(C_1,\alpha) \supseteq C_2 \textit{ and } T_S(C_2,\alpha) \supseteq C_1\}.$$

Fact 1.11. *The metric space* $(\mathcal{C}_n, \text{dist}_H)$ *equipped with the conic Hausdorff metric is a compact metric space. Moreover polarity is a local isometry on* $\mathcal{C}_n$*, that is: For* $\alpha < \pi/2, \text{dist}_H(C_1, C_2) = \alpha$ *implies* $\text{dist}_H(C_1^\circ, C_2^\circ) = \alpha$*.*

We now come to the most important fact of this section:

Fact 1.12. *Let* $C \in \mathcal{C}_n$ *be a closed convex cone. For each* $\varepsilon > 0$*, there is a polyhedral cone* $C_\varepsilon \in \mathcal{C}_n$ *that satisfies* $\text{dist}_H(C, C_\varepsilon) < \varepsilon$*. That is to say, the set of polyhedral cones is dense in* $\mathcal{C}_n$*.*

3 Relation to Barrier Parameters

Theorem 1.13. *Let a cone* $C \in \mathcal{C}_n$ *have optimal barrier paramater* $\vartheta(C) = K$*. Then for an arbitrary cone* D *with* $\text{dist}_H(C, D) < \delta$*, we have* $\vartheta(D) \sim K \pm O(n\delta^{3/2})$*).*

In [1] it was shown that for any polyhedral cone C, we can calculate the characteristic (and thus universal barrier) function for C in an algorithmic way needing only a simplicial decomposition of the dual cone C^*. Using the above result along with the density of the polyhedral cones in $\mathcal{C}_n$ we can produce a "good" approximation to a desired barrier function for a cone D, by using a close approximation via polyhedral cones and a corresponding barrier for these cones. Moreover, our approximate barrier function has certain guarantees in terms of optimality of the barrier paramater.

References

[1] Osman Güler. Barrier functions in interior point methods. *Mathematics of Operations Research*, 21(4):860–885, 1996.