Boolean Cube Partition Function
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Anthony Della Pella

1 Background Information

We set out to generalize a result of Barvinok [2] which showed that the partition function of the Boolean cube defined therein is zero free when polynomial coefficients are taken in a certain circular domain centered on the real axis. We show that a similar result holds for a thin strip stretched along the real axis. Our main result is presented as Theorem 5 below.

2 Useful Preliminaries

We recall definitions pertaining to functions on the boolean cube and their partition function. First, we have $f : \{-1, 1\}^n \to \mathbb{C}$ a polynomial (indexed by subsets I of $[n] = \{1, \ldots, n\}$) of the form:

$$f(x) = \sum_{I:I\subseteq[n]} \alpha_I \mathbf{x}^I$$

where $\mathbf{x}^I = \prod_{i\in I} x_i$. For a given f we can define quantities:

$$L(f) = \max_{i=1,\ldots,n} \sum_{I:i\in I} |\alpha_I|$$

and

$$\deg(f) = \max_{I:\alpha_I\neq 0} |I|.$$

We define the partition function of the boolean cube for a given polynomial f as:

$$\mathbb{E}\left[e^f\right] = \frac{1}{2^n} \sum_{x\in\{-1,1\}^n} e^{f(x)}.$$

We recall that a face of the boolean cube $F \subset \{-1, 1\}^n$ is determined by the choice of some subset of indices as well as signs (± 1) for each chosen index where we fix the index to have that sign and then vary all other indices over ± 1. In notation if $I_+(F)$ represents the subset of indices to have positive values, and $I_-(F)$ the subset of indices to have negative values, we have:

$$F = \{(x_1, \ldots, x_n) \text{ where } x_i = 1 \text{ for } i \in I_+(F) \text{ and } x_i = -1 \text{ for } i \in I_-(F)\} \subset \{-1, 1\}^n.$$

Moreover, the dimension of a face is the number of "free" indices (those without a specification) and the codimension is the ambient dimension n minus the dimension of the face. For a given face F and a subset $J \subset I_+(F) \cup I_-(F)$ we define the sign of the subset to be the quantity $\text{sign}_F(J) = \prod_{j \in J} x_i$.

Finally, we can fix subsets of free indices in a face. For a subset $J \subset [n]$ we denote by $\{-1,1\}^J$ the set of all points $x = (x_j : j \in J)$ where x_j is fixed as either ± 1 (and all other x_i are varied over $\{-1,1\}$). For a face $F \subset \{-1,1\}^n$ of the boolean cube we define $I(F)$ as the free indices of F, that is the elements of $[n]$ not fixed in the definition of F. We can then define for $J \subset I(F)$, F^ϵ where $\epsilon \in \{-1,1\}^J$ to be:

$$F^\epsilon = \{x \in F, x = (x_1, \dots, x_n) : x_j = \epsilon_j \text{ for } j \in J \subset I(F)\} \subset F.$$

Taking a conditional expectation with respect to a face of the boolean cube, we have:

$$\mathbb{E}\big[e^f \bigm| F\big] = \frac{1}{2^{\dim(F)}} \sum_{x \in F} e^{f(x)}.$$

We now recall a key geometric lemma from [1].

Lemma 1. *Let $u_1, \dots, u_n \in \mathbb{C}$ be non-zero complex numbers viewed as vectors in the plane such that the angle between any two doesn't exceed $\pi/2$. Let*

$$v = \sum_{j=1}^n \alpha_j u_j \text{ and } w = \sum_{j=1}^n \beta_j u_j$$

where $|1 - \operatorname{Re} \alpha_j| \le \delta, |1 - \operatorname{Re} \beta_j| \le \delta$, and $|\operatorname{Im} \alpha_j| \le \tau, |\operatorname{Im} \beta_j| \le \tau$ for $j = 1, \dots, n$ and some $0 \le \delta < 1$ and $0 \le \tau < 1 - \delta$. Then, $v, w \ne 0$ and the angle between v and w doesn't exceed:

$$2 \arctan \delta + 2 \arcsin \frac{\tau}{1 - \delta}.$$

Lemma 2. *Let $u_1, \dots, u_n \in \mathbb{C}$ be as in the lemma, and suppose $a > 0$. Then, setting*

$$v = \sum_{j=1}^n \alpha_j u_j \text{ and } w = \sum_{j=1}^n \beta_j u_j$$

we have the $v, w \neq 0$ and the angle between v and w doesn't exceed $2\arctan\delta + 2\arcsin\frac{\tau}{1-\delta}$ provided $|1 - a\operatorname{Re}\alpha_j| \leq \delta, |1 - a\operatorname{Re}\beta_j| \leq \delta$, and $|a\operatorname{Im}\alpha_j| \leq \tau, |a\operatorname{Im}\beta_j| \leq \tau$ where $0 \leq \delta < 1$ and $0 \leq \tau < 1 - \delta$.

Proof. Note that we can write

$$v = \sum_{j=1}^{n} a\alpha_i \left(\frac{1}{a}\right) u_j$$

and

$$w = \sum_{j=1}^{n} a\beta_i \left(\frac{1}{a}\right) u_j.$$

The vectors $\left(\frac{1}{a}\right) u_j$ satisfy the conditions of Lemma 1 and by assumption $|1 - \operatorname{Re} a\alpha_j| \leq \delta, |1 - \operatorname{Re} a\beta_j| \leq \delta$ and $|\operatorname{Im} a\alpha_j| \leq \tau, |\operatorname{Im} a\beta_j| \leq \tau$ so applying Lemma 1 to v and w written as above we obtain the desired result. $\square$

The following definition will be useful to save on bulky notation:

Definition 3. *Let $\zeta = s + it$ be a complex number with $0 < s < 0.5$, and $-0.2 < t < 0.2$. Then, we define $\delta'(\zeta) = \dfrac{2e^{|s|}}{e^s + e^{-s}} - 1$ and $\tau'(\zeta) = \dfrac{2e^{|s|}\sin(|t|)}{\cos(t)(e^s + e^{-s})}$. Further, we define the quantity*

$$\sphericalangle(\zeta) = 2\arctan(\delta'(\zeta)) + 2\arcsin\left(\frac{\tau'(\zeta)}{1 - \delta'(\zeta)}\right).$$

Corollary 4. *Suppose that $u_1, \ldots, u_n$ are as above, and fix some complex number z. Then, if α_i is chosen so that $\alpha_i = \exp(a_i) = \exp(\pm z)$, while $\beta_i = \exp(b_i)$ where $b_i = \mp z$, we have the angle between u and v not exceeding:*

$$\sphericalangle(z) = 2\arctan(\delta'(z)) + 2\arcsin\left(\frac{\tau'(z)}{1 - \delta'(z)}\right).$$

Proof. Set

$$a = \frac{2}{(e^{\operatorname{Re} z} + e^{-\operatorname{Re} z})\cos(\operatorname{Im} z)}$$

with $a > 0$. Then, we apply Lemma 2 with a and note that:

$$\begin{aligned}|1 - a \operatorname{Re} \alpha_i| &\leq \left|1 - \frac{2e^{\operatorname{Re} a_i}}{e^{\operatorname{Re} a_i} + e^{-\operatorname{Re} a_i}}\right| \\ &\leq \frac{2e^{|\operatorname{Re} a_i|}}{e^{\operatorname{Re} a_i} + e^{-\operatorname{Re} a_i}} - 1 \\ &= \delta'(z),\end{aligned}$$

while

$$\begin{aligned}|a \operatorname{Im}(\alpha_i)| &\leq \frac{2e^{|\operatorname{Re} a_i|} \sin(|\operatorname{Im} a_i|)}{\cos(\operatorname{Im} a_i)\,(e^{\operatorname{Re} a_i} + e^{-\operatorname{Re} a_i})} \\ &= \tau'(z).\end{aligned}$$

Since similar inequalities hold for β_i, we can apply Lemma 2 to obtain our result. □

3 The Main Theorem

Theorem 5. *Let n be a positive integer. Consider the domain in $\mathbb{C}^{2^n}$, denoted $\mathcal{U}(\delta, \tau)$, and defined to be the set*

$$\left\{ f(x) = \sum_{I \subset [n]} \alpha_I \mathbf{x}^I : \{-1, 1\}^n \to \mathbb{C} \,\middle|\, \forall j \in [n], \sum_{J \subset [n] : j \in J} |\operatorname{Re} \alpha_J| < \frac{\delta}{\deg(f)} \text{ and } \sum_{J \subset [n] : j \in J} |\operatorname{Im} \alpha_J| < \frac{\tau}{\deg(f)} \right\}.$$

For a fixed $0 < \delta < \pi/4$, set $\tau = \frac{1}{50}(\frac{\pi}{4} - \delta)^2$. Then, those polynomials $f \in \mathcal{U}(\delta, \tau)$ have the property that $\mathbb{E}\left[e^f\right]$ is nonzero.

As an expository remark, we mention here that this is similar to a theorem in the prequel [2] where instead a circular zero free region was obtained.

Proof. We will use descending induction on the codimension r of faces F of the boolean cube. We use three Statements:

- Statement 1.r: Let F be a face of codimension r. Then:

  $$\mathbb{E}\left[e^f \mid F\right] \neq 0$$

  for all $f \in \mathcal{U}(\delta, \tau)$.

- Statement 2.r: Let F be a face of codimension $r-1$. For every element $j \in I(F)$, we denote by $j^+ \in \{-1,1\}^{\{j\}}$ the set of points with coordinate j fixed to $+1$ and similarly for j^-. Then, for any $f \in \mathcal{U}(\delta,\tau)$, the angle between:

$$\mathbb{E}\Big[e^f \;\Big|\; F^{j^+}\Big] \neq 0 \quad\text{and}\quad \mathbb{E}\Big[e^f \;\Big|\; F^{j^-}\Big] \neq 0 \tag{1}$$

  doesn't exceed $\frac{\pi}{2}$. Moreover, choosing $J \subset I(F)$, we have for any $\epsilon, \phi \in \{-1,1\}^J$, the same statement as above for

$$\mathbb{E}\big[e^f \;\big|\; F^{\epsilon}\big] \neq 0 \quad\text{and}\quad \mathbb{E}\big[e^f \;\big|\; F^{\phi}\big] \neq 0, \tag{2}$$

  where now f is a polynomial in $\mathcal{U}(\delta,\tau)$ with $|J| \leq \deg(f)$.

- Statement 3.r: Let $f, g \in \mathcal{U}(\delta,\tau)$ be two polynomials which differ only on the coefficient corresponding to $\mathbf{x}^J$ where the signs of the coefficients, α_J and β_J, corresponding to $\mathbf{x}^J$ are flipped. Then, for any face F of codimension r, the angle between

$$\mathbb{E}\big[e^f \;\big|\; F\big] \neq 0 \text{ and } \mathbb{E}[e^g \mid F] \neq 0$$

  doesn't exceed $\sphericalangle(\alpha_J) = 2\arctan\delta'(\alpha_J) + 2\arcsin\frac{\tau'(\alpha_J)}{1-\delta'(\alpha_J)}$.

We now show that each Statement holds for $r = n$ *i.e.* faces with dimension 0.

- Statement 1.n: This holds trivially since $\mathbb{E}\big[e^f \;\big|\; F\big] = e^{f(x)}$ where $x \in F$ is the single element in the face. We can write:

$$e^{f(x)} = \prod_{I \subset [n]} e^{\alpha_I \mathbf{x}^I} \tag{3}$$

  which is certainly nonzero provided $f \in \mathcal{U}(\delta,\tau)$.

- Statement 2.n: Since the codimension of F is $n-1$, we note that F^{j^+} and F^{j^-} each correspond to one vertex on the cube, call it $\mathbf{x}$. Thus, we can write:

$$\mathbb{E}\Big[e^f \;\Big|\; F^{j^+}\Big] = \prod_{I \subset [n]: j \notin I} e^{\alpha_I \mathbf{x}^I} \prod_{J \subset [n]: j \in J} e^{\alpha_J \mathbf{x}^J}$$

  and

$$\mathbb{E}\Big[e^f \;\Big|\; F^{j^-}\Big] = \prod_{I \subset [n]: j \notin I} e^{\alpha_I \mathbf{x}^I} \prod_{J \subset [n]: j \in J} e^{-\alpha_J \mathbf{x}^J}.$$

  Each expectation is nonzero (being a product of nonzero exponential terms). Moreover, by dividing the two expressions, we note that the angle between the two expectations is at most $\frac{2\tau}{\deg(f)} < \frac{\pi}{2}$.

- Statement 3.n: Let F be a face of codimension n, that is a point in the cube (call it $\mathbf{x}$). Then, Statement 3.n holds by utilizing the representation Eq. (3). Namely, for $f(x) = \sum_{I\subset[n]} \alpha_I \mathbf{x}^I$ and $g = \sum_{\substack{I\subset[n]\\ I\neq J}} \alpha_I \mathbf{x}^I + \beta_J \mathbf{x}^J$, (where $\beta_J = -\alpha_J$) we have

$$\mathbb{E}\big[e^f \mid F\big] = \prod_{I\subset[n]} e^{\alpha_I \mathbf{x}^I} \tag{4}$$

while

$$\mathbb{E}[e^g \mid F] = e^{\beta_J \mathbf{x}^J} \prod_{\substack{I\subset[n]\\ I\neq J}} e^{\alpha_I \mathbf{x}^I} \tag{5}$$

where the products differ only by a factor $e^{\alpha_J - \beta_J}$. We claim then that the angle between the expectations is smaller than $2(\operatorname{Im}\alpha_J)$ which in turn is bounded above by $2\sphericalangle(\alpha_J)$ whenever $f \in \mathcal{U}(\delta,\tau)$.

Indeed, letting $\alpha_J = s + it$, we have:

$$\begin{aligned}\tau'(s+it) &= \frac{2e^{|s|}\sin(|t|)}{\cos(t)\,(e^s + e^{-s})}\\ &\geq \frac{\sin(|t|)}{\cos(t)}\end{aligned}$$

which for small $t < \pi/2$,

$$= \tan(|t|).$$

Similarly, we have:

$$\begin{aligned}1 - \delta'(s+it) &= 2 - \frac{2e^{|s|}}{e^s + e^{-s}}\\ &\leq 1.\end{aligned}$$

Thus, by considering the arcsin term of $\sphericalangle(\alpha_J)$ we have:

$$\begin{aligned}2\operatorname{Im}(\alpha_J) &= 2t\\ &\leq 2\arcsin(\tan(|t|))\\ &\leq 2\arctan(\delta'(s+it)) + 2\arcsin\left(\frac{\tau'(s+it)}{1-\delta'(s+it)}\right)\\ &= \sphericalangle(s+it)\\ &= \sphericalangle(\alpha_J)\end{aligned}$$

where it is worth remarking that $1-\delta'(s+it)$ is non-negative since $e^{|s|} \le e^s+e^{-s}$. This proves the claim.

With the base case of our induction verified, we will show that Statements $1.r, 2.r$, and $3.r$ imply Statements $1.(r-1), 2.(r-1)$, and $3.(r-1)$.

Statement $1.(r-1)$ follows from an application of $2.r$ to a decomposition of $\mathbb{E}\big[e^f \bigm| F\big]$. In particular, suppose that F is a face of codimension $r-1$. Then, we have for some $j \in I(F)$: $2\,\mathbb{E}\big[e^f \bigm| F\big] = \mathbb{E}\Big[e^f \Bigm| F^{j^+}\Big] + \mathbb{E}\Big[e^f \Bigm| F^{j^-}\Big]$ where as before, $j^+ \in \{-1,1\}^{\{j\}}$ is the set of points with a 1 in the j^{th} entry and similarly for j^-. By $1.r$ each summand is itself nonzero and the angle between them is bounded by $\frac{\pi}{2}$ via $2.r$, thus the sum cannot possibly be zero.

For Statement $3.(r-1)$ we note that there are two possible cases for $\mathbb{E}\big[e^f \bigm| F\big]$ and $\mathbb{E}[e^g \mid F]$. Let J be the coefficient on which f and g differ. If $J \subset I_+(F) \cup I_-(F)$ then $\operatorname{sign}_F(J)$ is well defined and we can factor out the J coefficient which is equal up to sign in f and g as follows:

$$\begin{aligned}
\mathbb{E}\big[e^f \bigm| F\big] &= \sum_{x\in F} e^{f(x)} \\
&= \sum_{x\in F} e^{\sum_{I\subset[n]} \alpha_I \mathbf{x}^I} \\
&= \sum_{x\in F} e^{\sum_{I\subset[n]:I\neq J} \alpha_I \mathbf{x}^I} e^{\alpha_J \mathbf{x}^J} \\
&= e^{\alpha_J \operatorname{sign}_F(J)} \sum_{x\in F} e^{\sum_{I\subset[n]:I\neq J} \alpha_I \mathbf{x}^I}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[e^g \mid F] &= \sum_{x\in F} e^{g(x)} \\
&= \sum_{x\in F} e^{\sum_{I\subset[n]} \alpha_I \mathbf{x}^I} \\
&= \sum_{x\in F} e^{\sum_{I\subset[n]:I\neq J} \alpha_I \mathbf{x}^I} e^{\beta_J \mathbf{x}^J} \\
&= e^{\beta_J \operatorname{sign}_F(J)} \sum_{x\in F} e^{\sum_{I\subset[n]:I\neq J} \alpha_I \mathbf{x}^I}
\end{aligned}$$

Then, dividing the common parts we have the angle between the two is at most $|\operatorname{Im}\alpha_J| + |\operatorname{Im}\beta_J| = 2|\operatorname{Im}\alpha_J|$ and noting that $f \in \mathcal{U}(\delta,\tau)$ we get the desired result.

Contrarily, if J is not fixed by F, there is some nonempty, and maximal subset of J say $\hat{J}$ of elements whose sign is not fixed in F, *i.e.* $\hat{J} \subset I(F)$. Then we can write:

$$\mathbb{E}\big[e^f \mid F\big] = \frac{1}{2^{r+|\hat{J}|}} \sum_{\epsilon \in \{-1,1\}^{\hat{J}}} \sum_{x \in F^\epsilon} e^{\sum_{I:I\neq J} \alpha_I \mathbf{x}^I} e^{\alpha_J \mathbf{x}^J}.$$

Similarly,

$$\mathbb{E}[e^g \mid F] = \frac{1}{2^{r+|\hat{J}|}} \sum_{\epsilon \in \{-1,1\}^{\hat{J}}} \sum_{x \in F^\epsilon} e^{\sum_{I:I\neq J} \alpha_I \mathbf{x}^I} e^{\beta_J \mathbf{x}^J}.$$

We now note that each ϵ leads to a complex number $u_\epsilon = \sum_{x\in F^\epsilon} e^{\sum_{I:I\neq J} \alpha_I \mathbf{x}^I}$ multiplied by either an e^{α_J} factor or an e^{β_J} factor (up to the sign of $\mathbf{x}^J$). Since by Statement 2.r the u_ϵ differ in angle no more than $\frac{\pi}{2}$ (taking for instance a function $h : \{-1,1\}^n \to \mathbb{C}$ which agrees with $f \equiv g$ on all other coefficients than J and is 0 for coefficient J) by use of Statement 2.r. We can apply Corollary 4 and obtain that the angle between $\mathbb{E}\big[e^f \mid F\big]$ and $\mathbb{E}[e^g \mid F]$ doesn't exceed $\sphericalangle(\alpha_J) = 2\arctan \delta'(\alpha_J) + 2\arcsin \frac{\tau'(\alpha_J)}{1-\delta'(\alpha_J)}$.

We now prove Statement 2.$(r-1)$ assuming the higher codimension statements $1, 2$, and 3 hold. Let F be a face of codimension $r-2$. For any $f = \sum_{I\subset[n]} \alpha_I \mathbf{x}^I \in \mathcal{U}(\delta,\tau)$ and any $J \subset I(F)$, we can define a family of subsets $\mathcal{F} \subset \mathcal{P}([n])$, where $K \in \mathcal{F}$ precisely when $K \cap J \neq \emptyset$ and $\alpha_K \neq 0$. Moreover, (for any $\epsilon, \phi \in \{-1,1\}^J$) we can write:

$$\mathbb{E}\big[e^f \mid F^\epsilon\big] = \sum_{x\in F^\epsilon} \prod_{I:J\cap I=\emptyset} e^{\alpha_I \mathbf{x}^I} \prod_{K:J\cap K\neq\emptyset} e^{\alpha_K \mathbf{x}^K}$$

and

$$\mathbb{E}\big[e^f \mid F^\phi\big] = \sum_{x\in F^\phi} \prod_{I:J\cap I=\emptyset} e^{\alpha_I \mathbf{x}^I} \prod_{K:J\cap K\neq\emptyset} e^{\alpha_K \mathbf{x}^K}.$$

We now note that the rightmost product in both expressions is over subsets $K \in \mathcal{F}$. Consider an algorithm which remedies the discrepancy between the signs of the coefficients of f between F^ϕ and F^ϵ. For each each $K \in \mathcal{F}$ we can define a function h which agrees with f in all coefficients except that corresponding to K. For the coefficient corresponding to K, h will have the sign corresponding to whether the sign (for fixed $\mathbf{x} \in F$) of $\mathbf{x}^K$ differs between ϕ and ϵ or not. In the case that the sign does flip, we let the coefficient corresponding to K of h have the opposite sign of the coefficient K in f. By Statement 3.(s) (with $s \geq r$) we note that the rotation between any two expectations in our process is at most $\sphericalangle(\alpha_K)$. After we have made the signs of every coefficient corresponding to $K \in \mathcal{F}$ correct we have rotated an angle at most $\sum_{K\in\mathcal{F}} \sphericalangle(\alpha_K)$.

We let $\alpha_K = s + it$ and compute the the Maclaurin series (as an upper bound) for $\arctan(\delta'(\alpha_K))$ and an elementary upper bound on $\arcsin\left(\frac{\tau'(\alpha_K)}{1-\delta'(\alpha_K)}\right)$ assuming $f \in \mathcal{U}(\delta, \tau)$.

$$\arctan(\delta'(s+it)) = \arctan\left(\frac{2e^{|s|}}{e^s + e^{-s}} - 1\right) \tag{6}$$

$$\leq |s| - \frac{2|s|^3}{3} + \frac{2|s|^5}{3} + O(|s|^8) \tag{7}$$

$$\leq |s|. \tag{8}$$

Also,

$$\arcsin\left(\frac{\tau'(\alpha_K)}{1-\delta'(\alpha_K)}\right) \leq \arcsin\left(\frac{1.7\sin(|t|)}{0.3\cos(t)}\right)$$

which follows from upper and lower bounds on $\delta'(s+it)$ and $1-\delta'(s+it)$ respectively. The Maclaurin series here ($5.\overline{666}|t| + 32.21\overline{666}|t|^3 + O(|t|^5)$) is upper bounded by $6t$ for $|t| < \frac{\pi}{200}$. With these upper bounds, we have that the sum

$$\sum_{K\in\mathcal{F}} \sphericalangle(\alpha_K) \tag{9}$$

is bounded above by

$$\sum_{K\in\mathcal{F}} 2|\mathrm{Re}(\alpha_K)| + 12|\mathrm{Im}(\alpha_K)| \leq \deg(f)\left(\frac{2\delta}{\deg(f)} + \frac{12\tau}{\deg(f)}\right). \tag{10}$$

We see that choosing $\delta < \frac{\pi}{4}$ suffices to obtain the required $\frac{\pi}{2}$ bound needed for the induction to proceed since $\tau = \frac{1}{50}(\frac{\pi}{4} - \delta)^2$ in our definition of $\mathcal{U}(\delta, \tau)$.

4 Sharpening for the Degree 2 Case

We notice that the bound for $\arctan(\delta'(s+it))$ in Eq. (6) is weak – namely the linear behavior of $\arctan(\delta')$ in the real part (s) of the argument is suboptimal as s

increases. In particular, for large values of $|s|$ the linearization $|s|$ is too crude. It is better to use a linearization centered at 0.43: $0.717792(s-0.43)+0.385085$. For $\deg(f)=2$, we note that the family $\mathcal{F}$ is small, and so we can bound Eq. (9) with our linearization at 0.43. In this case, we can choose $\delta = .44$ since:

$$\begin{aligned}\sum_{K\in\mathcal{F}} 2|\operatorname{Re}(\alpha_K)| &\le 4(0.717792(|s|-0.43)+0.385085)\\ &= 4(0.392263)\\ &\le 4(0.3926...)\\ &= 4\left(\frac{\pi}{8}\right)\end{aligned}$$

in Theorem 5 and still obtain Eq. (9) with a proper tuning of τ. The theorem for the degree 2 case follows.

Theorem 6. *Let n be a positive integer. Consider the domain in $\mathbb{C}^{2^n}$, denoted $\mathcal{U}(\delta,\tau)$, and defined to be the set*

$$\left\{ f(x)=\sum_{I\subset[n]}\alpha_I\mathbf{x}^I : \{-1,1\}^n\to\mathbb{C} \,\middle|\, \deg(f)=2 \text{ and } \forall j\in[n], \sum_{J\subset[n]:j\in J}|\operatorname{Re}\alpha_J|<\frac{\delta}{2} \text{ and } \sum_{J\subset[n]:j\in J}|\operatorname{Im}\alpha_J|<\frac{\tau}{2}\right\}.$$

For a fixed $0<\delta<0.44$, set $\tau=\frac{1}{50}(0.44-\delta)^2$. Then, those polynomials $f\in\mathcal{U}(\delta,\tau)$ have the property that $\mathbb{E}\left[e^f\right]$ is nonzero.

□

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References

[1] Alexander Barvinok. *Combinatorics and complexity of partition functions*, volume 276. Springer, 2016.

[2] Alexander Barvinok. Computing the partition function of a polynomial on the boolean cube. In *A Journey Through Discrete Mathematics*, pages 135–164. Springer, 2017.