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TESTING FOR DENSE SUBSETS IN A GRAPH VIA THE PARTITION FUNCTION*

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Abstract. For a set S of vertices of a graph G, we define its density $0 \leq \sigma(S) \leq 1$ as the ratio of the number of edges of G spanned by the vertices of S to $\binom{|S|}{2}$. We show that, given a graph G with n vertices and an integer $m \ll n$, the partition function $\sum_S \exp\{\gamma m \sigma(S)\}$, where the sum is taken over all m-subsets S of vertices and $0 < \gamma < 1$ is fixed in advance, can be approximated within relative error $0 < \epsilon < 1$ in quasi-polynomial $n^{O(\ln m - \ln \epsilon)}$ time. We discuss numerical experiments and observe that for the random graph $G(n, 1/2)$ one can afford a much larger γ, provided the ratio n/m is sufficiently large.

Key words. graph, density, partition function, algorithm, complex zeros

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1. Introduction and main results.

Let $G = (V, E)$ be an undirected graph, without loops or multiple edges. For a nonempty subset $S \subset V$ of vertices, we define the *density* $\sigma(S)$ as the fraction of the pairs of vertices of S that span an edge of G:

$$\sigma(S) = \frac{\left|\binom{S}{2} \cap E\right|}{\binom{|S|}{2}},$$

where $\binom{S}{2}$ is the set of all unordered pairs of vertices from S. Hence $0 \leq \sigma(S) \leq 1$ for all subsets, $\sigma(S) = 0$ if S is an *independent set* and $\sigma(S) = 1$ if S is a *clique*.

We are interested in the following general problem: given a graph $G = (V, E)$ with $|V| = n$ vertices and an integer $m \leq n$, estimate the highest density of an m-subset $S \subset V$. This is, of course, a hard problem: for example, testing whether a given graph contains a clique of a given size, or even estimating the size of the largest clique within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$, fixed in advance, is already an NP-hard problem [12], [15]. Moreover, modulo some plausible complexity assumptions, it is hard to approximate the highest density of an m-subset for a given m, within a constant factor, fixed in advance [5]. The best known efficient approximation achieves the factor of $n^{1/4}$ in quasi-polynomial $n^{O(\ln n)}$ time [6]. There are indications that the factor $n^{1/4}$ might be hard to beat [7]. We note that the most interesting case is when m grows and $n \gg m$, since the highest density of an m-subset can be computed in polynomial time up to an additive error of $\epsilon n^2/m^2$ for any $\epsilon > 0$, fixed in advance [10] (and if m is fixed in advance, the densest m-subset can be found by the exhaustive search in polynomial time).

1.1. Partition function.

In this paper, we approach the problem of finding the densest, or just a reasonably dense subset, via computing the *partition function*

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$$\text{den}_m(G;\gamma) = \binom{n}{m}^{-1} \sum_{\substack{S \subset V: \\ |S|=m}} \exp\{\gamma m \sigma(S)\}, \tag{1.1}$$

where $\gamma > 0$ is a parameter. We are interested in computing (approximating) $\text{den}_m(G;\gamma)$ efficiently. The *exponential tilting*, $\sigma(S) \longmapsto \exp\{\gamma m \sigma(S)\}$ (see, for example, section 13.7 of [14]), puts greater emphasis on the sets of higher density. Let us consider the set $\binom{V}{m}$ of all m-subsets of V as a probability space with the uniform measure. By the Markov inequality, for any $0 < \sigma_0 < 1$, we have

$$\sigma_0 + \frac{\ln \mathbf{P}(\sigma(S) \ \geq \ \sigma_0)}{\gamma m} \ \leq \ \frac{\ln \text{den}_m(G;\gamma)}{\gamma m} \ \leq \ \max_{\substack{S \subset V: \\ |S|=m}} \sigma(S), \tag{1.2}$$

so the larger γ we can afford, the better approximation for the densest m-subset we get. In particular, if we could choose $\gamma \gg \ln n$, then from (1.2) we could approximate the highest density of an m-subset within an arbitrarily small additive error.

The partition function (1.1) was introduced in [3], where an algorithm of quasi-polynomial $n^{O(\ln m - \ln \epsilon)}$ complexity was constructed to compute (1.1) within relative error $0 < \epsilon < 1$, when $\gamma = 0.07$ and when $\gamma = 0.27$, under additional assumptions that $n \geq 8m$ and $m \geq 10$. It follows from (1.2) that if the probability to hit an m-subset S of density at least σ_0 at random is $e^{-o(m)}$, then we can certify the existence of an m-subset of density at least $\sigma_0 - o(1)$ in quasi-polynomial time, just by computing (1.1). It is also shown in [3] that by successive conditioning, one can find in quasi-polynomial time an m-subset S with density at least as high as certified by the value of (1.1).

In this paper, we present an algorithm, which, for any $0 < \gamma < 1$, fixed in advance, and a given $0 < \epsilon < 1$, computes the value of (1.1) within relative error ϵ in quasi-polynomial $n^{O(\ln m - \ln \epsilon)}$ time, provided $n > \omega(\gamma) m$ for some constant $\omega(\gamma) > 1$. This improvement from $\gamma = 0.27$ to $\gamma = 1$ makes the algorithm competitive in some situations where it was not competitive before. Suppose, for example, we want to separate efficiently the graphs that have sufficiently many m-cliques from the graphs that are sufficiently far from having a single m-clique. Below we show that for $\gamma < 0.5$ our algorithm is inferior to a simple test based on the Kruskal–Katona Theorem, while for $\gamma > 0.5$ the former can cover a greater range than the latter.

Example 1 (Testing graphs for m-cliques). Let us fix two numbers $0 < \delta < 1$ and $\alpha > 0$ and consider the following two mutually exclusive conditions.

CONDITION 1.1. *For every $S \subset V$ such that $|S| = m$ we have $\sigma(S) \leq 1 - \delta$*

and

CONDITION 1.2. *If $S \subset V$ is a random subset, sampled uniformly from the set $\binom{V}{m}$ of all m-sets of vertices, then the probability that S is a clique is at least $e^{-\alpha m}$.*

Suppose further, we are presented with a graph $G = (V, E)$ and told that either Condition 1.1 or Condition 1.2 holds. Our goal is to decide which one. This is somewhat in the spirit of "property testing" [11].

We observe that if Condition 1.1 holds, then $\text{den}_m(G;\gamma) \leq e^{\gamma m(1-\delta)}$ and if Condition 1.2 holds, then $\text{den}_m(G;\gamma) \geq e^{(\gamma-\alpha)m}$. Consequently, if

$$\alpha \ < \ \gamma\delta \tag{1.3}$$

and we can approximate $\text{den}_m(G;\gamma)$ efficiently, we can efficiently tell Condition 1.1 and Condition 1.2 apart.

An anonymous referee to [3] noticed that another, much simpler, algorithm can be inferred from the Kruskal–Katona Theorem. Let $|V| = n$. If Condition 1.1 holds, then $|E| \leq (1-\delta)\binom{n}{2}$. The Kruskal–Katona Theorem (see, for example, section 5 of [8]) implies that if Condition 1.2 holds, then for every k such that $\binom{k}{m} \leq e^{-\alpha m}\binom{n}{m}$, we must have $|E| \geq \binom{k}{2}$, the model case being a graph G consisting of a k-clique and $n-k$ isolated vertices. A computation shows that as $n \longrightarrow \infty$, we can tell Condition 1.1 and Condition 1.2 apart just by counting the edges of G, provided

$$\alpha < -\frac{1}{2}\ln(1-\delta). \tag{1.4}$$

Comparing (1.3) and (1.4), we observe that the algorithm based on computing the partition function $\mathrm{den}_m(G;\gamma)$ is not competitive as long as $\gamma < 0.5$, which is the case in [3], but becomes competitive at least for small values of δ as soon as $\gamma > 0.5$. Numerical estimates show that as long as we can choose $\gamma > 0$ arbitrarily close 1, the condition (1.3) serves a wider range of α than the condition (1.4) provided $\delta < 0.7968$.

We still don't know, however, if (1.1) can be efficiently computed for *any* $\gamma > 0$, fixed in advance, and as we remarked above, it is unlikely that (1.1) can be efficiently computed for $\gamma \gg \ln n$. Our numerical experiments seem to indicate that we can afford a substantially larger γ. This can be partially explained by the fact that for the Erdős–Rényi random graph $G(n, 0.5)$ indeed a much larger γ can be used with high probability; see Theorem 1.4 below.

The improvement from $\gamma = 0.27$ to an arbitrary $\gamma < 1$ required the addition of some new ideas to the technique of [3]. The approach of [3] and of this paper are based on the "interpolation method" [4]. As applied to our case, the idea of the method is to consider $\mathrm{den}_m(G;z)$ for a *complex* parameter z. We can efficiently approximate $\mathrm{den}_m(G;z)$ at $z=\gamma$ if there is a connected open set $U \subset \mathbb{C}$, not dependent on m or G, such that $0 \in U$, $\gamma \in U$ and $\mathrm{den}_m(G;z) \neq 0$ for all $z \in U$. In [3], the set U is a disc centered at $z=0$, whereas in the current paper it is a thin neighborhood of the interval $[0,\gamma]$, which allows us to reach larger γ, but also requires a more refined analysis to establish zero-freeness. We give some more details now.

1.2. Multivariate partition function. Given $n \times n$ symmetric complex matrix $Z = (z_{ij})$ and $2 \leq m \leq n$, we define

$$P_m(Z) = \sum_{\substack{S \subset \{1,\ldots,n\} \\ |S|=m}} \exp\left\{ \sum_{\substack{\{i,j\} \subset S \\ i \neq j}} z_{ij} \right\}. \tag{1.5}$$

Note that the diagonal entries of Z are irrelevant, so we assume that $z_{ii} = 0$ for all i.

Given a graph $G = (V, E)$ with set $V = \{1, \ldots, n\}$ of vertices and $\gamma > 0$, we define $Z_0 = (z_{ij})$ by

$$z_{ij} = \begin{cases} \frac{\gamma}{m-1} & \text{if } \{i,j\} \in E \\ -\frac{\gamma}{m-1} & \text{if } \{i,j\} \notin E \end{cases}$$

and observe that

$$\begin{aligned} P_m(Z_0) &= \sum_{\substack{S \subset \{1,\ldots,n\} \\ |S|=m}} \exp\left\{ m\gamma\sigma(S) - \frac{\gamma m}{2} \right\} \\ &= \exp\left\{-\frac{\gamma m}{2}\right\}\binom{n}{m}\mathrm{den}_m(G;\gamma). \end{aligned} \tag{1.6}$$

Hence to compute (1.1) it suffices to compute $P_m(Z_0)$. We compute $P_m(Z_0)$ by interpolation; see [3], [4]. For that, it suffices to show that $P_m(Z) \neq 0$ in some neighborhood of a path connecting the zero matrix to Z_0 in the space of complex matrices.

We prove the following result.

THEOREM 1.3. *For any $0 < \delta < 1$ there exist $\eta = \eta(\delta) > 0$ and $\omega = \omega(\delta) > 1$ such that if $n \geq \omega m$, then $P_m(Z) \neq 0$ for any $n \times n$ symmetric complex matrix $Z = (z_{ij})$ such that*

$$|\Re z_{ij}| \leq \frac{\delta}{m-1} \quad \textit{and} \quad |\Im z_{ij}| \leq \frac{\eta}{m-1} \quad \textit{for all} \quad 1 \leq i \neq j \leq n.$$

We prove Theorem 1.3 in sections 2 and 3. Using Theorem 1.3, in section 4 we present an algorithm of quasi-polynomial $n^{O(\ln m)}$ complexity to compute $P_m(Z_0)$ and hence $\operatorname{den}_m(G;\gamma)$ for any $0 < \gamma < 1$, fixed in advance.

In [3] it was established that $P_m(Z) \neq 0$ in a polydisc

$$\mathcal{D}_{m,n} = \left\{ Z = (z_{ij}) : \ |z_{ij}| \leq \frac{0.27}{m-1} \quad \text{for all} \quad 1 \leq i \neq j \leq n \right\}$$

provided $n \gg m$ and m is large enough. In Theorem 1.3, we establish that $P_m(Z) \neq 0$ in a more "economical" domain, "stretched" along the real part of the complex space of matrices. This allows us to improve the constant γ for which $\operatorname{den}_m(G;\gamma)$ is still efficiently computable.

In section 5, we discuss some results of our numerical experiments, which seem to indicate that we can afford an essentially bigger δ in Theorem 1.3. This can be partially explained by the fact that for the Erdős–Rényi random graph $G(n, 0.5)$ this is indeed the case. Namely, we prove the following result in section 6.

THEOREM 1.4. *Let us choose positive integers n and $2 \leq m \leq n$. For $n \times n$ symmetric matrix $W = (w_{ij})$ of independent random variables, where*

$$\mathbf{P}(w_{ij} = 1) = \mathbf{P}(w_{ij} = -1) = \frac{1}{2},$$

we define the polynomial

$$h_W(z) = \binom{n}{m}^{-1} \sum_{\substack{S \subset \{1,\ldots,n\} \\ |S|=m}} \prod_{\{i,j\} \subset S} (1 + z w_{ij}).$$

Let $r > 0$ and $\tau > 1$ be real numbers. If $n \geq 2m^2 \left(1 + r^2\right)^m + 2m$, then the probability that $h_W(z)$ has a root in the disc $|z| < r/\sqrt{2\tau}$ does not exceed $1/\tau$.

In particular, if $n \gg m^2$, then with high probability $h_W(z)$ has no roots in the disc $|z| < c/\sqrt{m}$, for an arbitrary large $c > 0$, fixed in advance. Similarly, if $\ln n \gg m$, then with high probability $h_W(z)$ has no roots in the disc $|z| < c$ for an arbitrary large $c > 0$, fixed in advance.

The polynomial $h_W(z)$ is easily translated into the partition function $\operatorname{den}_m(G;\gamma)$, where G is the graph with set $V = \{1, \ldots, n\}$ of vertices and two vertices $\{i, j\}$ span an edge if and only if $w_{ij} = 1$: for $0 < \alpha < 1$, we have

$$h_W(\alpha) = (1-\alpha)^{\binom{m}{2}} \operatorname{den}_m(G;\gamma) \quad \text{where} \quad \gamma = \frac{m-1}{2} \ln \frac{1+\alpha}{1-\alpha}. \tag{1.7}$$

Consequently, with high probability we can can approximate $\text{den}_m(G;\gamma)$ in quasi-polynomial time for γ as large as $\gamma=\sqrt{m}$ provided $n \gg m^2$ and as large as $\gamma = m$ provided $\ln n \gg m$. Since the graphs we experimented on were to a large degree random (but not necessarily Erdős–Rényi $G(n, 0.5)$), we may have obtained overly optimistic numerical evidence.

As is easily seen, $\mathbf{E}\, h_W(\alpha) = 1$ and from our proof in section 6 it follows that $h_W(\alpha)$ is strongly concentrated. For example, in the regime of $n = \Omega(m^2)$ and $\alpha = 1/\sqrt{m}$, we have $\mathbf{var}\, h_W(\alpha) = O(1)$. This concentration, however, does not allow us to predict with high probability the value of $h_W(\alpha)$ with the precision that the interpolation technique based on Theorem 1.4 allows for.

In section 6, we also discuss what may happen if G is a random graph $G(n, 0.5)$ with a planted m-clique.

2. Preliminaries. We consider the partition function P_m of section 1.2 within a family of partition functions, which will allow us to prove Theorem 1.3 by induction.

2.1. Functionals $P_\Omega(Z)$. Let us fix integers n and $2 \leq m \leq n$. For a subset $\Omega \subset \{1, \ldots, n\}$ and $n \times n$ complex symmetric matrix $Z = (z_{ij})$, we define

$$P_\Omega(Z) = \sum_{\substack{S \subset \{1,\ldots,n\}: \\ |S|=m, \Omega \subset S}} \exp\left\{ \sum_{\substack{\{i,j\}\subset S \\ i \neq j}} z_{ij} \right\},$$

where we agree that $P_\Omega(Z) = 0$ if $|\Omega| > m$. In other words, we restrict the sum (1.5) defining $P_m(Z)$ onto subsets S containing a given set Ω. In particular,

$$P_\Omega(Z) = P_m(Z) \quad \text{if} \quad \Omega = \emptyset.$$

The induction will be built on the following straightforward formulas:

$$P_\Omega(Z) = \frac{1}{m - |\Omega|} \sum_{j \in \{1,\ldots,n\}\setminus\Omega} P_{\Omega\cup\{j\}}(Z) \quad \text{provided} \quad |\Omega| < m \tag{2.1}$$

and for $i \neq j$, we have

$$\frac{\partial}{\partial z_{ij}} P_\Omega(Z) = \begin{cases} P_\Omega(Z) & \text{if } i, j \in \Omega, \\ P_{\Omega\cup\{j\}}(Z) & \text{if } i \in \Omega, j \notin \Omega, \\ P_{\Omega\cup\{i\}}(Z) & \text{if } i \notin \Omega, j \in \Omega, \\ P_{\Omega\cup\{i,j\}}(Z) & \text{if } i, j \notin \Omega. \end{cases} \tag{2.2}$$

We will often consider complex numbers as vectors in the plane, by identifying $\mathbb{C} = \mathbb{R}^2$ and measuring, in particular, angles between nonzero complex numbers. We will use the following geometric lemma.

LEMMA 2.1. *Let $u_1, \ldots, u_n \in \mathbb{C}$ be nonzero complex numbers such that the angle between any two does not exceed θ for some $0 < \theta < \pi/2$. Suppose that*

$$\Im\left(\sum_{j=1}^n u_j\right) = 0 \quad \textit{and} \quad \sum_{j=1}^n |u_j| = c.$$

Then

$$\sum_{j=1}^n |\Im\, u_j| \ \leq \ c \sin\frac{\theta}{2}.$$

Proof. Scaling u_j, if necessary, without loss of generality we assume that $c = 1$.

Without loss of generality, we assume that $\arg u_j \neq 0$ for $j = 1, \ldots, n$. Indeed, if $\arg u_j = 0$ for some j, we can remove the vector from the collection, which would make the sum

$$\sum_{j=1}^{n} |u_j| \tag{2.3}$$

only smaller. Rescaling $u_j \longmapsto \tau u_j$ for some real $\tau > 1$, we make (2.3) equal to 1 and increase

$$\sum_{j=1}^{n} |\Im\, u_j|\,. \tag{2.4}$$

Reflecting the vectors u_j in the coordinate axes if necessary, without loss of generality we may assume that $\Re\, u_1 \geq 0$ and $\Im\, u_1 > 0$. Hence there is a vector, say u_2, such that $\Im\, u_2 < 0$. We necessarily have $\Re\, u_2 \geq 0$, since otherwise the angle between u_1 and u_2 exceeds $\pi/2$. Then for any vector u_j, we must have $\Re\, u_j \geq 0$, since otherwise one of the angles formed by u_j with u_1 or u_2 will exceed $\pi/2$.

Hence without loss of generality, we assume that $\Re\, u_j > 0$ for $j = 1, \ldots, n$. Let

$$\alpha = \max_{j=1,\ldots,n} \arg u_j,$$

so that

$$0 \ < \ \alpha \ < \ \theta,$$

and let

$$-\beta = \min_{j=1,\ldots,n} \arg u_j \ < \ 0.$$

Then $\alpha + \beta \leq \theta$.

Let

$$J_+ = \{j : \ \arg u_j > 0\} \quad \text{and} \quad J_- = \{j : \ \arg u_j < 0\}\,.$$

Next, without loss of generality, we assume that $\arg u_j = \alpha$ for all $j \in J_+$ and that $\arg u_j = -\beta$ for all $j \in J_-$. Indeed, suppose that $\arg u_1 = \alpha_1$, where $0 < \alpha_1 < \alpha$. We can modify

$$u_1 \longmapsto \frac{\sin \alpha_1}{\sin \alpha} e^{i(\alpha - \alpha_1)} u_1$$

(we rotate and shrink u_1 so as to make its argument equal to α and leave $\Im\, u_1$ intact). The sum (2.3) gets smaller while all other conditions and the sum (2.4) remain intact. Rescaling $u_j \longmapsto \tau u_j$ for some real $\tau > 1$, we make (2.3) equal to 1 and increase (2.4), while keeping other constraints of the lemma intact. The case of $\arg u_j > -\beta$ for some $j \in J_-$ is handled similarly.

Next, without loss of generality, we assume that $\alpha + \beta = \theta$. Indeed, if $\alpha + \beta < \theta$, we can rotate and scale vectors u_j as above, so that the sum (2.4) increases while all other conditions are satisified.

Now, let

$$u_+ = \sum_{j \in J_+} u_j \quad \text{and} \quad u_- = \sum_{j \in J_-} u_j.$$

Then $\arg u_+ = \alpha$, $\arg u_- = -\beta$, $\Im\,(u_+ + u_-) = 0$, $|u_+| + |u_-| = 1$, and (2.4) is equal to $|\Im\, u_+| + |\Im\, u_-|$.

Denoting $a = |u_+|$ and $b = |u_-|$, we have $a + b = 1$ and $a \sin\alpha - b \sin\beta = 0$, from which

$$a = \frac{\sin\beta}{\sin\alpha + \sin\beta} \quad \text{and} \quad b = \frac{\sin\alpha}{\sin\alpha + \sin\beta},$$

and so

$$|\Im u_+| + |\Im u_-| = \frac{2\sin\alpha\sin\beta}{\sin\alpha + \sin\beta}.$$

Now, the function

$$\alpha \longmapsto \frac{1}{\sin\alpha} \quad \text{for} \quad 0 \leq \alpha \leq \frac{\pi}{2}$$

is convex and hence the minimum of

$$\frac{\sin\alpha + \sin\beta}{\sin\alpha\sin\beta} = \frac{1}{\sin\alpha} + \frac{1}{\sin\beta}$$

on the interval $\alpha + \beta = \theta$, $\alpha, \beta \geq 0$, is attained at $\alpha = \beta = \theta/2$. The proof now follows. □

We need another geometric lemma.

Lemma 2.2. *Let $u_1, \ldots, u_n \in \mathbb{C}$ be nonzero complex numbers such that the angle between any two does not exceed θ for some $0 \leq \theta < 2\pi/3$. Let $u = u_1 + \cdots + u_n$. Then*

$$|u| \ \geq \ \left(\cos\frac{\theta}{2}\right) \sum_{k=1}^{n} |u_k|.$$

Proof. This is Lemma 3.1 of [3] and Lemma 3.6.3 of [4]. □

3. Proof of Theorem 1.3. We identify the space of $n \times n$ zero-diagonal complex symmetric matrices $Z = (z_{ij})$ with $\mathbb{C}^{\binom{n}{2}}$. Given $\delta \geq \eta > 0$, we define a domain $\mathcal{U}(\delta, \eta) = \mathcal{U}_{n,m}(\delta, \eta) \subset \mathbb{C}^{\binom{n}{2}}$ by

$$\mathcal{U}(\delta, \eta) = \left\{ Z = (z_{ij}) : \ |\Re z_{ij}| \ \leq \ \frac{\delta}{m-1} \quad \text{and} \quad |\Im z_{ij}| \ \leq \ \frac{\eta}{m-1} \right\}.$$

If $Z' = \left(z'_{ij}\right)$ and $Z'' = \left(z''_{ij}\right)$ are two matrices from $\mathcal{U}(\delta, \tau)$, then

$$\left|z'_{ij} - z''_{ij}\right| \ \leq \ \frac{\sqrt{(2\delta)^2 + (2\eta)^2}}{m-1} \ \leq \ \frac{2\sqrt{2}\delta}{m-1} \quad \text{for all} \quad i, j.$$

We will prove by descending induction on $|\Omega|$ that $P_\Omega(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta, \eta)$ and that, moreover, a number of stronger conditions are met. The induction is based on the following two lemmas that describe how $P_\Omega(Z)$ changes when only the entries in the ith row and column of Z change. The first lemma deals with the case of $i \in \Omega$.

Lemma 3.1. *Let us fix $\Omega \subset \{1, \ldots, n\}$ such that $|\Omega| < m$. Suppose that for any $Z \in \mathcal{U}(\delta, \eta)$ and any $j, k \notin \Omega$, we have $P_{\Omega \cup \{j\}}(Z) \neq 0$, $P_{\Omega \cup \{k\}}(Z) \neq 0$ and the angle between the two nonzero complex numbers does not exceed θ for some $0 < \theta \leq \pi/2$. Then*

- *Part 1: We have*

$$P_\Omega(Z) \neq 0 \quad \textit{for all} \quad Z \in \mathcal{U}(\delta, \eta).$$

- *Part 2: Suppose additionally that $\Omega \neq \emptyset$, and let us fix an $i \in \Omega$. Let $Z', Z'' \in \mathcal{U}(\delta, \eta)$ be two matrices that differ only in the coordinates $z_{ij} = z_{ji}$ for $j \neq i$. Then*
$$\left| \frac{P_\Omega(Z')}{P_\Omega(Z'')} \right| \leq e^{6\delta},$$
and the angle between $P_\Omega(Z') \neq 0$ and $P_\Omega(Z'') \neq 0$ does not exceed
$$2\delta \tan \frac{\theta}{2} + 5\eta.$$

Proof. It follows from (2.1) and Lemma 2.2 that

$$|P_\Omega(Z)| \geq \frac{\cos(\theta/2)}{m - |\Omega|} \sum_{j \notin \Omega} \left| P_{\Omega \cup \{j\}}(Z) \right| \geq \frac{1}{(m-1)\sqrt{2}} \sum_{j \notin \Omega} \left| P_{\Omega \cup \{j\}}(Z) \right|. \tag{3.1}$$

In particular, Part 1 follows.

To prove Part 2, let us choose a branch of $\ln P_\Omega(Z)$ for $Z \in \mathcal{U}(\delta, \eta)$. For $0 \leq t \leq 1$, let $Z(t) = tZ'' + (1-t)Z'$. Then

$$\begin{aligned} \ln P_\Omega(Z'') - \ln P_\Omega(Z') &= \int_0^1 \frac{d}{dt} \ln P_\Omega\left(Z(t)\right) \, dt \\ &= \int_0^1 \sum_{j:\ j \neq i} \left(z''_{ij} - z'_{ij}\right) \frac{\partial}{\partial z_{ij}} \ln P_\Omega(Z) \Big|_{Z = Z(t)} \, dt. \end{aligned}$$

Using (2.2), we conclude that

$$\frac{\partial}{\partial z_{ij}} \ln P_\Omega(Z) = \begin{cases} 1 & \text{if } j \in \Omega, \\ P_{\Omega \cup \{j\}}(Z) / P_\Omega(Z) & \text{if } j \notin \Omega, \end{cases}$$

and hence

$$\begin{aligned} \ln P_\Omega(Z'') - \ln P_\Omega(Z') = &\sum_{j \in \Omega, j \neq i} \left(z''_{ij} - z'_{ij}\right) \\ &+ \int_0^1 \sum_{j \notin \Omega} \left(z''_{ij} - z'_{ij}\right) \frac{P_{\Omega \cup \{j\}}(Z(t))}{P_\Omega(Z(t))} \, dt. \end{aligned} \tag{3.2}$$

Using (3.1), we get from (3.2) that

$$\begin{aligned} |\Re \ln P_\Omega(Z'') - \Re \ln P_\Omega(Z')| &\leq 2\delta + (m-1)\sqrt{2} \max_{j \notin \Omega} \left| z''_{ij} - z'_{ij} \right| \\ &\leq 2\delta + 4\delta = 6\delta, \end{aligned}$$

and hence

$$\left| \frac{P_\Omega(Z')}{P_\Omega(Z'')} \right| \leq e^{6\delta},$$

as claimed.

From (2.1), for all $Z \in \mathcal{U}(\delta, \eta)$ we have that

$$\sum_{j \notin \Omega} \frac{P_{\Omega \cup \{j\}}(Z)}{P_\Omega(Z)} = m - |\Omega|$$

is real, while from (3.1), we conclude that

$$\sum_{j \notin \Omega} \left| \frac{P_{\Omega \cup \{j\}}(Z)}{P_\Omega(Z)} \right| \ \leq \ \frac{m - |\Omega|}{\cos(\theta/2)} \ \leq \ \frac{m-1}{\cos(\theta/2)}.$$

Applying Lemma 2.1 with $u_j = P_{\Omega \cup \{j\}}(Z)/P_\Omega(Z)$, we conclude that

$$\sum_{j \notin \Omega} \left| \Im \, \frac{P_{\Omega \cup \{j\}}(Z)}{P_\Omega(Z)} \right| \ \leq \ (m-1) \tan \frac{\theta}{2}.$$

Therefore, from (3.2),

$$\begin{aligned} |\Im \ln P_\Omega(Z'') - \Im \ln P_\Omega(Z')| \ &\leq 2\eta + (m-1) \tan \frac{\theta}{2} \max_{j \notin \Omega} \left| \Re\, z''_{ij} - \Re\, z'_{ij} \right| \\ &\qquad + (m-1)\sqrt{2} \max_{j \notin \Omega} \left| \Im\, z''_{ij} - \Im\, z'_{ij} \right| \\ &\leq 2\delta \tan \frac{\theta}{2} + 5\eta. \end{aligned}$$

Hence the angle between $P_\Omega(Z'')$ and $P_\Omega(Z')$ does not exceed $2\delta \tan \frac{\theta}{2} + 5\eta$, as claimed. □

The second lemma shows that $P_\Omega(Z)$ does not change much if only the entries of Z in the ith row and column are changed for some $i \notin \Omega$, assuming that $n \gg m$.

LEMMA 3.2. *Let us fix an $\Omega \subset \{1, \ldots, n\}$, $|\Omega| \leq m-1$. Suppose for any $i, j \notin \Omega$ and all $Z \in \mathcal{U}(\delta, \eta)$ we have $P_{\Omega \cup \{i\}}(Z) \neq 0$, $P_{\Omega \cup \{j\}}(Z) \neq 0$ and the angle between the two complex numbers does not exceed $\pi/2$ and that*

$$\left| \frac{P_{\Omega \cup \{i\}}(Z)}{P_{\Omega \cup \{j\}}(Z)} \right| \ \leq \ \lambda$$

for some $\lambda \geq 1$.

In addition, suppose that if $|\Omega| \leq m-2$, then for any distinct $i, j, k \notin \Omega$ and all $Z \in \mathcal{U}(\delta, \eta)$ we have $P_{\Omega \cup \{i,j\}}(Z) \neq 0$, $P_{\Omega \cup \{i,k\}}(Z) \neq 0$, and the angle between the two complex numbers does not exceed $\pi/2$.

Let us fix an $i \notin \Omega$, and let $Z', Z'' \in \mathcal{U}(\delta, \eta)$ be two matrices that differ only in the coordinates $z_{ij} = z_{ji}$ for $j \neq i$. Then

$$\left| \frac{P_\Omega(Z')}{P_\Omega(Z'')} \right| \ \leq \ \exp\left\{ \frac{10\delta\lambda m}{n-1} \right\},$$

and the angle between $P_\Omega(Z') \neq 0$ and $P_\Omega(Z'') \neq 0$ does not exceed

$$\frac{10\delta\lambda m}{n-1}.$$

Proof. It follows from Lemma 3.1 that $P_\Omega(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta, \eta)$.

Arguing as in the proof of Lemma 3.1, we introduce $Z(t) = tZ'' + (1-t)Z'$ and write

$$\ln P_\Omega(Z'') - \ln P_\Omega(Z') = \int_0^1 \sum_{j:\ j \neq i} \left(z''_{ij} - z'_{ij} \right) \frac{\partial}{\partial z_{ij}} \ln P_\Omega(Z) \Big|_{Z = Z(t)} \, dt.$$

From (2.2), we write

$$\ln P_\Omega(Z'') - \ln P_\Omega(Z') = \int_0^1 \sum_{j\in\Omega} \left(z''_{ij} - z'_{ij}\right) \frac{P_{\Omega\cup\{i\}}\big(Z(t)\big)}{P_\Omega\big(Z(t)\big)} + \sum_{j\notin\Omega, j\neq i} \left(z''_{ij} - z'_{ij}\right) \frac{P_{\Omega\cup\{i,j\}}\big(Z(t)\big)}{P_\Omega\big(Z(t)\big)}\, dt. \tag{3.3}$$

Suppose first that $|\Omega| \leq m-2$. From (2.1), we have

$$P_{\Omega\cup\{i\}}(Z) = \frac{1}{m-|\Omega|-1} \sum_{j\notin\Omega, j\neq i} P_{\Omega\cup\{i,j\}}(Z).$$

Applying Lemma 2.2, we get that

$$\sum_{j\notin\Omega, j\neq i} \left|P_{\Omega\cup\{i,j\}}(Z)\right| \ \leq\ (m-1)\sqrt{2}\left|P_{\Omega\cup\{i\}}(Z)\right| \tag{3.4}$$

for all $Z \in \mathcal{U}(\delta, \eta)$.

Since by (2.1) we also have

$$P_\Omega(Z) = \frac{1}{m-|\Omega|} \sum_{j\notin\Omega} P_{\Omega\cup\{j\}}(Z)$$

applying Lemma 2.2, we conclude that

$$\sum_{j\notin\Omega} \left|P_{\Omega\cup\{j\}}(Z)\right| \ \leq\ (m-|\Omega|)\sqrt{2}\left|P_\Omega(Z)\right|.$$

Hence for all $i \notin \Omega$, we have

$$\left|P_{\Omega\cup\{i\}}(Z)\right| \ \leq\ \frac{\lambda(m-|\Omega|)\sqrt{2}}{n-|\Omega|}\left|P_\Omega(Z)\right| \ \leq\ \frac{\lambda m\sqrt{2}}{n}\left|P_\Omega(Z)\right|. \tag{3.5}$$

Combining (3.5) and (3.4), we get

$$\sum_{j\notin\Omega, j\neq i} \left|P_{\Omega\cup\{i,j\}}(Z)\right| \ \leq\ \frac{2\lambda m(m-1)}{n}\left|P_\Omega(Z)\right|. \tag{3.6}$$

Combining (3.3), (3.4), (3.5), and (3.6), we get

$$\begin{aligned} \left|\ln P_\Omega(Z'') - \ln P_\Omega(Z')\right| &\leq \frac{2\sqrt{2}\delta}{m-1}\cdot\frac{\lambda|\Omega|(m-|\Omega|)\sqrt{2}}{n-|\Omega|} + \frac{2\sqrt{2}\delta}{m-1}\cdot\frac{2\lambda m(m-1)}{n} \\ &\leq \frac{4\delta\lambda m}{n-1} + \frac{4\sqrt{2}\delta\lambda m}{n} \ \leq\ \frac{10\delta\lambda m}{n-1}. \end{aligned}$$

If $|\Omega| = m-1$, then from (3.3) and (3.5), we get

$$\left|\ln P_\Omega(Z'') - \ln P_\Omega(Z')\right| \ \leq\ \frac{2\sqrt{2}\delta}{m-1}\cdot\frac{\lambda m\sqrt{2}}{n} \ \leq\ \frac{4\delta\lambda m}{n-1},$$

which concludes the proof. □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Given $0 < \delta < 1$, we choose $0 < \theta < \pi/2$ so that

$$2\delta \tan\frac{\theta}{2} \ < \ \theta.$$

We then choose $\eta > 0$ such that

$$2\delta \tan\frac{\theta}{2} + 5\eta \ < \ \theta.$$

We choose

$$\lambda > e^{6\delta}$$

and choose $\omega > 1$ so that

$$2\delta \tan\frac{\theta}{2} + 5\eta + \frac{10\delta\lambda m}{n-1} \ \leq \ \theta \quad \text{and} \quad \exp\left\{6\delta + \frac{10\delta\lambda m}{n-1}\right\} \leq \lambda$$

whenever $n \geq \omega m$.

Suppose that $n \geq \omega m$. We prove by descending induction on $r = m, m-1, \ldots, 1$ that if $\Omega_1, \Omega_2 \in \{1, \ldots, n\}$ are two sets such that $|\Omega_1| = |\Omega_2| = r$ and $|\Omega_1 \Delta \Omega_2| = 2$, then for all $Z \in \mathcal{U}(\delta, \eta)$ we have $P_{\Omega_1}(Z) \neq 0$, $P_{\Omega_2}(Z) \neq 0$, the angle between $P_{\Omega_1}(Z)$ and $P_{\Omega_2}(Z)$ does not exceed θ while the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed λ.

Assume that $r = m$. Without loss of generality, we assume that $\Omega_1 = \Omega \cup \{1\}$ and $\Omega_2 = \Omega \cup \{2\}$ for some $\Omega \subset \{3, \ldots, n\}$ such that $|\Omega| = m - 1$. We have

$$P_{\Omega_1}(Z) = \exp\left\{\sum_{\{i,j\} \subset \Omega} z_{ij}\right\} \exp\left\{\sum_{i \in \Omega} z_{1i}\right\} \quad \text{and}$$

$$P_{\Omega_2}(Z) = \exp\left\{\sum_{\{i,j\} \subset \Omega} z_{ij}\right\} \exp\left\{\sum_{i \in \Omega} z_{2i}\right\}.$$

Clearly, $P_{\Omega_1}(Z) \neq 0$, $P_{\Omega_2}(Z) \neq 0$, the angle between $P_{\Omega_1}(Z)$ and $P_{\Omega_2}(Z)$ does not exceed $2\eta \leq \theta$ while the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed $e^{2\delta} \leq \lambda$.

Suppose now that the statements hold for all subsets $\Omega \subset \{1, \ldots, n\}$ of cardinality at least $r + 1$ for some $r \leq m - 1$, and let $\Omega_1, \Omega_2 \subset \{1, \ldots, n\}$ be two subsets of cardinality $r \geq 1$ such that $|\Omega_1 \Delta \Omega_2| = 2$. Again, without loss of generality, we assume that $\Omega_1 = \Omega \cup \{1\}$ and $\Omega_2 = \Omega \cup \{2\}$ for some $\Omega \subset \{3, \ldots, n\}$ such that $|\Omega| = r - 1$. Then we observe that $P_{\Omega_2}(Z) = P_{\Omega_1}(Z')$, where

$$z'_{1i} = z'_{i1} = z_{2i} = z_{i2} \quad \text{and} \quad z'_{2i} = z'_{i2} = z_{1i} = z_{i1} \quad \text{for} \quad i \neq 1, 2,$$

while all other entries of Z and Z' coincide. Applying Lemma 3.1 and Lemma 3.2 and the induction hypothesis to sets $\Omega_1 \cup \{j\}$ for $j \notin \Omega_1$ and $\Omega_1 \cup \{j, k\}$ for $j, k \notin \Omega_1$, we conclude that the angle between $P_{\Omega_1}(Z) \neq 0$ and $P_{\Omega_2}(Z) \neq 0$ does not exceed

$$2\delta \tan\frac{\theta}{2} + 5\eta + \frac{10\delta\lambda m}{n-1} \ \leq \ \theta,$$

while the ratio of $|P_{\Omega_1}(Z)|$ and $|P_{\Omega_2}(Z)|$ does not exceed

$$\exp\left\{6\delta + \frac{10\delta\lambda m}{n-1}\right\} \leq \lambda.$$

This proves that $P_{\{i\}}(Z) \neq 0$ for all $i \in \{1, \ldots, n\}$ and all $Z \in \mathcal{U}(\delta, \eta)$ and that the angle between $P_{\{i\}}(Z) \neq 0$ and $P_{\{j\}}(Z) \neq 0$ does not exceed θ for all $i, j \in \{1, \ldots, n\}$. From (2.1) we conclude that $P_m(Z) = P_{\emptyset}(Z) \neq 0$ for all $Z \in \mathcal{U}(\delta, \eta)$. □

4. Computing the partition function. Here we show how to compute the density partition function $\mathrm{den}_m(G; \gamma)$. First, we make a change of coordinates to convert the partition function $P_m(Z)$ of section 1.2 into a multivariate polynomial.

4.1. A polynomial version of $P_m(Z)$. For an $n \times n$ complex symmetric matrix $W = (w_{ij})$ with zero diagonal, we define

$$p_m(W) = \binom{n}{m}^{-1} \sum_{\substack{S \subset \{1, \ldots, n\} \\ |S| = m}} \prod_{\substack{\{i,j\} \subset S \\ i \neq j}} (1 + w_{ij}).$$

Hence $p_m(W)$ is a polynomial of degree $\binom{m}{2}$ in the entries w_{ij} and, assuming that $|w_{ij}| < 1$ for all i, j, we can write

$$p_m(W) = \binom{n}{m}^{-1} P_m(Z), \quad \text{where} \quad Z = (z_{ij}) \quad \text{and} \quad z_{ij} = \ln(1 + w_{ij})$$

(we choose the standard branch of the logarithm in the right half-plane of $\mathbb{C}$). Theorem 1.3 implies that for every $0 < \delta < 1$ there is $\eta = \eta(\delta) > 0$ and $\omega = \omega(\delta) > 1$ such that

$$\text{(4.1)} \qquad p_m(W) \neq 0 \quad \text{whenever} \quad \begin{aligned} |\Re \ln(1 + w_{ij})| &\leq \frac{\delta}{m-1}, \\ |\Im \ln(1 + w_{ij})| &\leq \frac{\eta}{m-1}, \quad \text{and} \\ n &\geq \omega m. \end{aligned}$$

To compute $\mathrm{den}_m(G; \gamma)$ for a given $0 < \gamma < 1$ and a given graph $G = (V, E)$, we define

$$\text{(4.2)} \qquad w_{ij} = \begin{cases} \exp\left\{\frac{\gamma}{m-1}\right\} - 1 & \text{if } \{i,j\} \in E, \\ \exp\left\{-\frac{\gamma}{m-1}\right\} - 1 & \text{if } \{i,j\} \notin E. \end{cases}$$

Then, by (1.6), we have

$$\text{(4.3)} \qquad \mathrm{den}_m(G; \gamma) = \exp\left\{\frac{\gamma m}{2}\right\} p_m(W).$$

The interpolation method is based on the following simple lemma.

Lemma 4.1. *Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ be a univariate polynomial, and suppose that $g(z) \neq 0$ provided $|z| < \beta$, where $\beta > 1$ is some real number. Let us choose a branch of $f(z) = \ln g(z)$ in the disc $|z| < \beta$, and let*

$$T_r(z) = f(0) + \sum_{k=1}^{r} \frac{f^{(k)}(0)}{k!} z^k$$

be the Taylor polynomial of f of degree r computed at $z = 0$. Then

$$|f(1) - T_r(1)| \leq \frac{\deg g}{\beta^r (\beta - 1)(r + 1)}.$$

Proof. This is Lemma 2.2.1 of [4]; see also Lemma 1.1 of [3]. □

The gist of Lemma 4.1 is that to approximate $f(1)$ within an additive error ϵ; it suffices to compute the Taylor polynomial of $f(z)$ at 0 of degree $r = O_\beta(\ln \deg g - \ln \epsilon)$, where the implicit constant in the "O" notation depends on β alone. We would like to apply Lemma 4.1 to the univariate polynomial

$$h(z) = \binom{n}{m}^{-1} \sum_{\substack{S \subset \{1,\dots,n\} \\ |S|=m}} \prod_{\substack{\{i,j\} \subset S \\ i \neq j}} (1 + z w_{ij}), \tag{4.4}$$

where w_{ij} are defined by (4.2). Indeed, the value we are ultimately interested is $h(1) = p_m(W)$. However, Lemma 4.1 requires that $h(z) \neq 0$ in a disc of some radius $\beta > 1$, whereas (4.1) only guarantees that $h(z) \neq 0$ for z in a neighborhood of the interval $[0,1] \subset \mathbb{C}$. To remedy this, we compose h with a polynomial $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\phi(0) = 0$, $\phi(1) = 1$ and ϕ maps the disc $|z| < \beta$ for some $\beta > 1$ inside the prescribed neighborhood of $[0,1] \subset \mathbb{C}$. We then apply Lemma 4.1 to the composition $g(z) = h((\phi(z))$. The following lemma provides an explicit construction of ϕ.

LEMMA 4.2. *For $0 < \rho < 1$, we define*

$$\alpha = \alpha(\rho) = 1 - e^{-\frac{1}{\rho}}, \quad \beta = \beta(\rho) = \frac{1 - e^{-1-\frac{1}{\rho}}}{1 - e^{-\frac{1}{\rho}}} \ > \ 1,$$

$$N = N(\rho) = \left\lfloor \left(1 + \frac{1}{\rho}\right) e^{1+\frac{1}{\rho}} \right\rfloor, \quad \sigma = \sigma(\rho) = \sum_{k=1}^{N} \frac{\alpha^k}{k}, \quad \textit{and}$$

$$\phi(z) = \phi_\rho(z) = \frac{1}{\sigma} \sum_{k=1}^{N} \frac{(\alpha z)^k}{k}.$$

Then $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ is a polynomial of degree N such that $\phi(0) = 0$, $\phi(1) = 1$,

$$-\rho \ \leq \ \Re\, \phi(z) \ \leq \ 1 + 2\rho, \quad \textit{and} \quad |\Im\, \phi(z)| \ \leq \ 2\rho,$$

provided $|z| \leq \beta$.

Proof. This is Lemma 2.2.3 of [4]. □

Lemma 4.1 also requires the derivatives $f^{(k)}(0)$ of $f(z) = \ln g(z)$ at $z = 0$. Those, however, can be easily computed from the derivatives $g^{(k)}(0)$, as described in section 2.2.2 of [4]; see also section 2.1 of [3]. We briefly sketch how.

4.2. Computing derivatives. Suppose that $f(z) = \ln g(z)$ as in Lemma 4.1. Then

$$f'(z) = \frac{g'(z)}{g(z)} \quad \text{and} \quad g'(z) = f'(z) g(z).$$

Differentiating the product $k-1$ times, we obtain

$$g^{(k)}(0) = \sum_{j=0}^{k-1} \binom{k-1}{j} f^{(k-j)}(0) g^{(j)}(0) \quad \text{for} \quad k = 1, \dots, r. \tag{4.5}$$

We interpret (4.5) as a system of linear equations in variables $f^{(k)}(0)$ for $k = 1, \dots, r$ with coefficients $g^{(k)}(0)$ for $k = 0, \dots, r$. This is a triangular system of linear equations with nonzero entries $g^{(0)}(0) = g(0)$ on the diagonal, that can be solved in $O(r^2)$ time, provided the values of $g^{(k)}(0)$ are known.

To supply the last ingredient of the algorithm, we show how to compute $h^{(k)}(0)$ for $k = 0, \ldots, r$, where h is the polynomial defined by (4.4). This is also done in [3], but we reproduce it here for completeness.

We have

$$h^{(k)}(0) = \binom{n}{m}^{-1} \sum_{\substack{S \subset \{1,\ldots,n\} \\ |S|=m}} \sum_{\{i_1,j_1\},\ldots,\{i_k,j_k\} \subset S} w_{i_1 j_1} \ldots w_{i_k j_k},$$

where the inner sum is taken over all ordered collections of distinct unordered pairs $\{i_1, j_1\}, \ldots, \{i_k, j_k\} \subset S$. For such a collection, say I, let $\nu(I)$ be the number of distinct vertices among $i_1, j_1, \ldots, i_k, j_k$. Then there are exactly $\binom{n-\nu(I)}{m-\nu(I)}$ different m-subsets S containing the edges from I, and we can rewrite the above sum as

$$h^{(k)}(0) = \binom{n}{m}^{-1} \sum_{I=(\{i_1,j_1\},\ldots,\{i_k,j_k\})} \binom{n-\nu(I)}{m-\nu(I)} w_{i_1 j_1} \ldots w_{i_k j_k}, \tag{4.6}$$

where the sum is taken over all ordered collections of k unordered pairs $\{i_s, j_s\}$. It is clear now that $h^{(k)}(0)$ can be computed in $n^{O(k)}$ time by the exhaustive enumeration of all possible collections of k pairs.

In section 5 we present faster formulas for computing $h^{(2)}(0)$ and $h^{(3)}(0)$ that we used for our numerical experiments.

4.3. The algorithm. Let us fix $0 < \gamma < 1$. Below we summarize the algorithm for computing $\operatorname{den}_m(G; \gamma)$ within relative error $0 < \epsilon < 1$, by which we understand computing $\ln \operatorname{den}_m(G; \gamma)$ within additive error ϵ. We assume that $m \geq 4$ and that $n \geq \omega m$ for some $\omega = \omega(\gamma) > 1$, to be specified below.

Given a graph $G = (V, E)$ with set $V = \{1, \ldots, n\}$ of vertices, and an integer $m \leq n$, we compute the $n \times n$ symmetric matrix $W = (w_{ij})$ by (4.2). Since $m \geq 4$, we have $|w_{ij}| \leq 0.4$ for all i, j.

Our goal is to compute $p_m(W) = h(1)$, where h is the univariate polynomial defined by (4.4). We note that $\deg h = \binom{m}{2}$.

Let us choose $1 > \delta > \gamma$, and let $\eta = \eta(\delta) > 0$ and $\omega = \omega(\delta) > 1$ be the numbers of Theorem 1.3 and in (4.1). We find $\rho = \rho(\delta) > 0$ such that

$$|\Re \ln(1 + z w_{ij})| \leq \frac{\delta}{m-1} \quad \text{and} \quad |\Im \ln(1 + z w_{ij})| \leq \frac{\eta}{m-1}$$

as long as

$$-\rho \leq \Re z \leq 1 + \rho \quad \text{and} \quad |\Im z| \leq \rho. \tag{4.7}$$

Indeed, if $z \in [0, 1]$, then

$$-\frac{\gamma}{m-1} \leq \ln(1 + z w_{ij}) \leq \frac{\gamma}{m-1},$$

and for $|z| \leq 2$, we have

$$\left| \frac{d}{dz} \ln(1 + z w_{ij}) \right| = \left| \frac{w_{ij}}{1 + z w_{ij}} \right| \leq \frac{10}{m-1},$$

so the desired ρ can indeed be found.

It follows by (4.1) that $h(z) \neq 0$ as long as $n \geq \omega m$ and (4.7) holds.

Using Lemma 4.2, we construct a polynomial $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ of some degree $N = N(\rho) = N(\delta)$ such that $\phi(0) = 0$, $\phi(1) = 1$ and

$$-\rho \leq \Re\,\phi(z) \leq 1+\rho \quad \text{and} \quad |\Im\,\phi(z)| \leq \rho$$

as long as $|z| \leq \beta$ for some $\beta = \beta(\rho) = \beta(\delta) > 1$. We define

$$g(z) = h(\phi(z)),$$

and our goal is to compute $g(1) = h(\phi(1))$. We note that

$$\deg g \leq N \deg h = N\binom{m}{2}.$$

We choose a branch of $f(z) = \ln g(z)$ for z satisfying (4.7).

Using Lemma 4.1, we find an integer $r = O_\rho(\ln m - \ln \epsilon) = O_\delta(\ln m - \ln \epsilon)$ such that

$$|T_r(1) - f(1)| \leq \epsilon,$$

where $T_r(z)$ is the Taylor polynomial of $f(z)$ of degree r, computed at $z = 0$. The implicit constant in the "O" notation depends only on ρ, which in turn depends only on δ. Hence our goal is to compute $T_r(1)$, for which we need to compute $f^{(k)}(0)$ for $k = 1, \ldots, r$. As in section 4.2, we reduce it in $O(r^2)$ time to computing $g^{(k)}(0)$ for $k = 1, \ldots, r$. Note that

$$g(0) = h(\phi(0)) = h(0) = 1.$$

Let $\phi_r(z)$ be the truncation of the polynomial $\phi(z)$ obtained by discarding all monomials of degree higher than r. Similarly, let $h_r(z)$ be the truncation of the polynomial $h(z)$, obtained by discarding all monomial of degree higher than r. We compute $h_r(z)$ as in section 4.2 in $n^{O(r)}$ time. Finally, we compute the truncation of the composition $h_r(\phi_r(z))$. A fast (polynomial in r) way to do it, is to use Horner's method: assuming that

$$h_r(z) = \sum_{k=0}^{r} b_k z^k,$$

we successively compute

$$b_r\phi_r(z) + b_{r-1}, \quad (b_r\phi_r(z) + b_{r-1})\,\phi_r(z) + b_{r-2},$$
$$((b_r\phi_r(z) + b_{r-1})\,\phi_r(z) + b_{r-2})\,\phi_r(z) + b_{r-3}, \ldots$$

discarding on the way all monomials of degree higher than r. In the end, we have computed $g^{(k)}(0)$ for $k = 0, \ldots, r$, and hence $f^{(k)}(0)$ for $k = 0, \ldots, r$, and hence $T_m(1)$ approximating $f(1) = \ln h(1)$ within additive error ϵ. From (4.3), we compute

$$\text{den}_m(G; \gamma) = \exp\left\{\frac{\gamma m}{2}\right\} h(1)$$

within relative error $\epsilon > 0$.

5. Remarks on the practical implementation. We implemented a *much* simplified version of the algorithm. Given a graph $G = (V, E)$ with set $V = \{1, \ldots, n\}$ of vertices and an integer $2 \leq m \leq n$, we define the $n \times n$ matrix $= (w_{ij})$ by

$$w_{ij} = \begin{cases} \alpha & \text{if } \{i,j\} \in E \\ -\alpha & \text{if } \{i,j\} \notin E, \end{cases}$$

where $0 < \alpha < 1$ is a parameter.

We consider the polynomial $h(z)$ defined by (4.4) and let $f(z) = \ln h(z)$.

Our goal is to approximate $f(1) = \ln h(1)$, and hence

$$\begin{aligned} h(1) &= \sum_{\substack{S \subset \{1,\ldots,n\} \\ |S|=m}} (1+\alpha)^{\binom{m}{2}\sigma(S)} (1-\alpha)^{\binom{m}{2}(1-\sigma(S))} \\ &= (1-\alpha)^{\binom{m}{2}} \operatorname{den}_m(G;\gamma), \quad \text{where} \quad \gamma = \frac{m-1}{2} \ln \frac{1+\alpha}{1-\alpha}. \end{aligned}$$

We approximate $f(1)$ by the degree r Taylor polynomial of $f(z)$ computed at $z = 0$. The results of [3] suggest that for $\alpha = O(1/m)$, we should get a reasonable approximation if we use $r \sim \ln m$. The results of our numerical experiments suggest that we get reasonable approximations if we use $\alpha = \Omega(1)$ and $r = 2$ or $r = 3$. In short, on the examples we tested, the quality of approximation was more consistent with the quality of the Taylor polynomial approximation of $\ln(1 \pm \alpha)$.

More precisely, we ran the algorithm typically with parameters $n = 50, 100$ and $m = 10$, although occasionally we chose n as large as $n = 300$. For the parameters $n = 50$ and $m = 10$ we were able to compare our approximation with the exact value. Typically, choosing $\alpha = 0.5$ or lower produced an approximation of $f(1)$ within 1% accuracy. For $\alpha = 0.7$, the accuracy went down to $10\% - 20\%$ and for $\alpha > 0.7$ the approximation was not accurate. For higher values of n, where the exact value of $f(1)$ was unavailable, we compared the approximations obtained for $r = 2$ and $r = 3$. If the approximations were close to each other, we considered it as an indication that they are also close to the true value of $f(1)$. Again, we observed that up to $\alpha = 0.5$, the approximations agreed but were beginning to essentially differ at $\alpha = 0.7$ and higher. For the graphs, we used the Erdős–Rényi models $G(n, 0.5)$, $G(n, 0.4)$, those graphs with planted cliques of size m, and occasionally manually constructed "random-looking" graphs.

We provide below the explicit formulas for the approximations up to degree 3, in case the reader will be interested to do some numerical experiments. We interpret w_{ij} as weights on the edges of a complete graph with n vertices. Borrowing an idea from [13], we express the derivatives $f^{(k)}(0)$ in terms of various sums associated with *connected* subgraphs, since it improves the computational complexity of the algorithm. We remark, however, that it looks unlikely that the methods of [13] can be pushed to improve the complexity of our algorithm in the general situation from quasi-polynomial to genuinely polynomial, since we work with graphs of unbounded degrees.

It is convenient to introduce the following sums:

$$A_1 = \sum_{\{i,j\}} w_{ij},$$

where the sum is taken over all unordered pairs $\{i, j\}$ of distinct indices;

$$B_1 = \sum_{\{i,j\}} w_{ij}^2, \quad B_2 = \sum_{j,\{i,k\}} w_{ij} w_{jk},$$

where in the formula for B_1 the sum is taken oven all unordered pairs $\{i, j\}$ of distinct indices and in B_2 the sum is taken over all pairs consisting of an index j and an unordered pair $\{i, k\}$, so that all three indices are distinct; and

$$C_1 = \sum_{\{i,j\}} w_{ij}^3, \quad C_2 = \sum_{(i,j,k)} w_{ij}^2 w_{jk}, \quad C_3 = \sum_{\{i,j,k\}} w_{ij} w_{jk} w_{ki},$$
$$C_4 = \sum_{(i,j,k,l)} w_{ij} w_{jk} w_{kl}, \quad C_5 = \sum_{\{j,k,l\},i} w_{il} w_{ij} w_{ik},$$

where in C_1 the sum is taken over all unordered pairs $\{i, j\}$ of distinct indices, in C_2 the sum is taken over all ordered triples (i, j, k) of distinct indices, in C_3 the sum is taken over all unordered triples of distinct integers, in C_4 the sum is taken over all ordered 4-tuples (i, j, k, l) of distinct indices, and in C_5 the sum is taken over all pairs consisting of an index i and an unordered triple $\{j, k, l\}$ so that all four indices $\{i, j, k, l\}$ are distinct.

5.1. First-order approximation. Clearly, $h(0) = 1$. From (4.6), we have

$$h'(0) = \binom{n}{m}^{-1} \binom{n-2}{m-2} \sum_{\{i,j\} \subset \{1,\ldots,n\}} w_{ij} = \frac{m(m-1)}{n(n-1)} A_1.$$

Since $f(0) = \ln h(0) = 0$ and $f'(0) = h'(0)/h(0) = h'(0)$, we obtain the first order approximation

$$f(1) \approx h'(0),$$

where $h'(0)$ is defined as above. The complexity of computing the first order approximation is $O(n^2)$.

5.2. Second-order approximation. From (4.6), we have

$$h''(0) = \binom{n}{m}^{-1} \sum_{I=(\{i_1,j_1\},\{i_2,j_2\})} \binom{n-\nu(I)}{m-\nu(I)} w_{i_1 j_1} w_{i_2 j_2}.$$

Here $\nu(I) = 4$ if the pairs $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are pairwise disjoint and $\nu(I) = 3$ if they share exactly one index. Hence we can write

$$\begin{aligned} h''(0) &= \binom{n}{m}^{-1} \left(2\binom{n-3}{m-3} B_2 + \binom{n-4}{m-4} \left(A_1^2 - 2B_2 - B_1\right) \right) \\ &= 2\frac{m(m-1)(m-2)}{n(n-1)(n-2)} B_2 + \frac{m(m-1)(m-2)(m-3)}{n(n-1)(n-2)(n-3)} \left(A_1^2 - 2B_2 - B_1\right). \end{aligned}$$

Since

$$f''(0) = h''(0) - \left(h'(0)\right)^2,$$

we obtain the second order approximation:

$$f(1) \approx f'(0) + \frac{1}{2} f''(0) = h'(0) - \frac{1}{2}\left(h'(0)\right)^2 + \frac{1}{2} h''(0),$$

where $h'(0)$ and $h''(0)$ are defined as above. The complexity of computing the second order approximation is $O(n^3)$.

5.3. Third-order approximation. From (4.6), one can deduce that

$$\begin{aligned}h'''(0) = & 6\frac{m(m-1)(m-2)}{n(n-1)(n-2)}C_3 + \frac{m(m-1)(m-2)(m-3)}{n(n-1)(n-2)(n-3)}(6C_5+3C_4) \\ & + 6\frac{m(m-1)(m-2)(m-3)(m-4)}{n(n-1)(n-2)(n-3)(n-4)}(A_1B_2 - 3C_5 - 3C_3 - C_4 - C_2) \\ & + \frac{m(m-1)(m-2)(m-3)(m-4)(m-5)}{n(n-1)(n-2)(n-3)(n-4)(n-5)}\Big(A_1^3 + 12C_3 - 6A_1B_2 \\ & + 12C_5 + 3C_4 + 6C_2 - 3A_1B_1 + 2C_1\Big).\end{aligned}$$

Since we have

$$f'''(0) = h'''(0) - 2f''(0)h'(0) - f'(0)h''(0) = 2(h'(0))^3 - 3h'(0)h''(0) + h'''(0),$$

we obtain the third order approximation approximation

$$\begin{aligned}f(1) &\approx f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f'''(0) \\ &= h'(0) - \frac{1}{2}(h'(0))^2 + \frac{1}{2}h''(0) + \frac{1}{3}(h'(0))^3 - \frac{1}{2}h'(0)h''(0) + \frac{1}{6}h'''(0).\end{aligned}$$

The complexity of computing the third order approximation is $O\left(n^4\right)$.

6. Proof of Theorem 1.4 and concluding remarks. We got the idea of the proof from [9], where a similar question about complex zeros of the permanents of matrices with independent random entries was treated.

Proof of Theorem 1.4. Applying Jensen's formula (see, for example, section 5.3 of [1]), we obtain

$$\ln|h_W(0)| = \sum_{s=1}^{N} \ln\frac{|a_{s,W}|}{r} + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|h_W\left(re^{i\theta}\right)\right|\, d\theta, \tag{6.1}$$

where $a_{s,W}, s = 1, \ldots, N$ are the roots of the polynomial $h_W(z)$ in the disc $|z| < r$, and we assume that $h_W(z)$ has no zeros on the circle $|z| = r$ (since there are only finitely many values of r with roots on the circle $|z| = r$, this assumption is not restrictive). We have

$$\ln|h_W(0)| = 0,$$

and furthermore, applying Jensen's inequality, we bound:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \ln\left|h_W\left(re^{i\theta}\right)\right|\, d\theta &= \frac{1}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi} \ln\left|h_W\left(re^{i\theta}\right)\right|^2\, d\theta \\ &\leq \frac{1}{2}\ln\left(\frac{1}{2\pi}\int_0^{2\pi}\left|h_W\left(re^{i\theta}\right)\right|^2\, d\theta\right).\end{aligned} \tag{6.2}$$

For a fixed $\theta \in [0, 2\pi]$, we compute the expectation

$$\begin{aligned}\mathbf{E}\,\left|h_W\left(re^{i\theta}\right)\right|^2 &= \binom{n}{m}^{-2} \sum_{\substack{S_1, S_2 \subset \{1,\ldots,n\} \\ |S_1|=|S_2|=m}} \mathbf{E}\left(\prod_{\{j,k\}\subset S_1} \left(1+re^{i\theta}w_{jk}\right) \right. \\ &\qquad \left. \times \prod_{\{j,k\}\subset S_2} \left(1+re^{-i\theta}w_{jk}\right)\right) \\ &= \binom{n}{m}^{-2} \sum_{\substack{S_1, S_2 \subset \{1,\ldots,n\} \\ |S_1|=|S_2|=m}} \left(1+r^2\right)^{\binom{|S_1\cap S_2|}{2}}.\end{aligned}$$

A subset $S \subset \{1,\ldots,n\}$ of cardinality $l=|S| \leq m$ can be represented as the intersection $S = S_1 \cap S_2$ of m-subsets S_1, S_2 in $\binom{n-l}{m-l}\binom{n-m}{m-l}$ ways. Hence

$$\mathbf{E}\,\left|h_W\left(re^{i\theta}\right)\right|^2 = \binom{n}{m}^{-2} \sum_{l=0}^{m} \binom{n}{l}\binom{n-l}{m-l}\binom{n-m}{m-l}\left(1+r^2\right)^{\binom{l}{2}}. \tag{6.3}$$

To bound (6.3), we consider the ratio of the $(l+1)$st term to the lth term:

$$\begin{aligned}\frac{n-l}{l+1}\cdot\frac{m-l}{n-l}\cdot\frac{m-l}{n-2m+l+1}\cdot\left(1+r^2\right)^l &= \frac{(m-l)^2\left(1+r^2\right)^l}{(l+1)(n-2m+l+1)} \\ &\leq \frac{m^2(1+r^2)^m}{n-2m+1}.\end{aligned}$$

In particular, if

$$n \ \geq \ 2m^2(1+r^2)^m + 2m, \tag{6.4}$$

the ratio does not exceed 1/2, and hence we can bound the sum (6.3) by

$$\mathbf{E}\,\left|h_W\left(re^{i\theta}\right)\right|^2 \ \leq \ 2\binom{n}{m}^{-2}\binom{n}{m}\binom{n-m}{m} \ \leq \ 2.$$

Integrating over θ, we conclude that if (6.4) holds, then

$$\mathbf{E}\left(\frac{1}{2\pi}\int_0^{2\pi} \left|h_W\left(re^{i\theta}\right)\right| \ d\theta\right) \ \leq \ 2.$$

By the Markov inequality, for any $\tau \geq 1$, we get

$$\mathbf{P}\left(\frac{1}{2\pi}\int_0^{2\pi} \left|h_W\left(re^{i\theta}\right) \ d\theta\right| \ \geq \ 2\tau\right) \ \leq \ \frac{1}{\tau}.$$

Consequently, from (6.1) and (6.2), we have

$$\mathbf{P}\left(\sum_{s=1}^{N} \ln\frac{|a_{s,W}|}{r} \ \leq \ -\frac{1}{2}\ln 2\tau\right) \ \leq \ \frac{1}{\tau},$$

and the proof follows. □

An anonymous referee asked what happens if G is a random graph $G(n, 0.5)$ with a planted m-clique. The most interesting asymptotic regime is when $m^2 \ll n \leq m^{O(1)}$ and m grows; see [2] for results and references. Here we are interested in a polynomial time algorithm which, with high probability, tells G from $G(n, 0.5)$. A quasi-polynomial time algorithm is readily available (by an exhaustive search for a clique of size at least $3\log_2 n$, say). Our proof of Theorem 1.4 does not seem to extend to random graphs with a planted clique. We note, however, that if the radius of zero-free region is roughly the same $r = \Omega(1/\sqrt{m})$ as in Theorem 1.4 or even weaker, $r = \Omega(m^{-1+\epsilon})$ for some $\epsilon > 0$, we do obtain a desired polynomial time algorithm. Indeed, in the latter case, we can choose $\gamma = m^{\epsilon'}$ with some $0 < \epsilon' < \epsilon$. If G is a graph with a planted m-clique, we have

$$\mathrm{den}_m(G;\gamma) \;\geq\; \exp\left\{m^{1+\epsilon'} - O(m\ln m)\right\};$$

cf. (1.2). If G is a random graph $G(n, 0.5)$, our proof Theorem 1.4 implies that

$$\mathrm{den}_m(G;\gamma) \;\leq\; \exp\left\{\frac{m^{1+\epsilon'}}{2} + O(1)\right\}$$

with high probability; cf. (1.7). Note that by choosing $\epsilon' < \epsilon$, we choose γ sufficiently "deep" inside the purported zero-free region, and hence we can get a genuinely polynomial, as opposed to a quasi-polynomial, algorithm by computing a constant, as opposed to logarithmic, number of terms in the Taylor polynomial approximation; cf. Lemma 4.1.

REFERENCES

[1] L. V. Ahlfors, *Complex Analysis. An introduction to the theory of analytic functions of one complex variable*, 3rd ed., McGraw-Hill Book Co., New York, 1978.

[2] M. Alon, M. Krivelevich, and B. Sudakov, *Finding a large hidden clique in a random graph*, Random Structures Algorithms, 13 (1998), pp. 457–466.

[3] A. Barvinok, *Computing the partition function for cliques in a graph*, Theory Comput., 11 (2015), pp. 339–355, https://doi.org/10.4086/toc.2015.v011a013.

[4] A. Barvinok, *Combinatorics and Complexity of Partition Functions*, Springer, Cham, 2016.

[5] A. Bhaskara, *Finding dense structures in graphs and matrices*, Ph.D. dissertation, Princeton University, 2012, https://www.cs.utah.edu/~bhaskara/files/thesis.pdf.

[6] A. Bhaskara, M. Charikar, E. Chlamtac, U. Feige, and A. Vijayaraghavan, *Detecting high log-densities – an $o(n^{1/4})$ approximation for densest k-subgraph*, in Proceedings of the 2010 ACM International Symposium on Theory of Computing, New York, 2010, ACM, pp. 201–210.

[7] A. Bhaskara, M. Charikar, V. Guruswami, A. Vijayaraghavan, and Y. Zhou, *Polynomial integrality gaps for strong sdp relaxations of* `densest k-subgraph`, in Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, New York, 2012, ACM, pp. 388–405.

[8] B. Bollobás, *Combinatorics: Set systems, Hypergraphs, Families of Vectors and Combinatorial Probability*, Cambridge University Press, Cambridge, 1986.

[9] L. Eldar and S. Mehraban, *Approximating the permanent of a random matrix with vanishing mean*, in Proceedings of the 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS 2018), Los Alamitos, CA, 2018, IEEE, pp. 23–34.

[10] A. Frieze and R. Kannan, *Quick approximation to matrices and applications*, Combinatorica, 19 (1999), pp. 175–220.

[11] O. Goldreich, *Introduction to Property Testing*, Cambridge University Press, Cambridge, 2017.

[12] J. Håstad, *Clique is hard to approximate within $n^{1-\epsilon}$*, Acta Mathematica, 182 (1999), pp. 105–142.

[13] V. Patel and G. Regts, *Deterministic polynomial-time approximation algorithms for partition functions and graph polynomials*, SIAM J. Comput., 46 (2017), pp. 1893–1919.

[14] C. Terrell, *Mathematical Statistics: A Unified Introduction*, Springer-Verlag, New York, 1999.

[15] D. Zuckerman, *Linear degree extractors and the inapproximability of max clique and chromatic number*, Theory Comput., 3 (2007), pp. 103–128, https://doi.org/10.4086/toc.2007.v003a006.