Counting Cliques Using Polynomials

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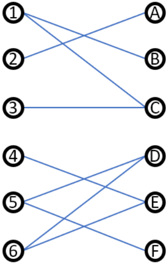
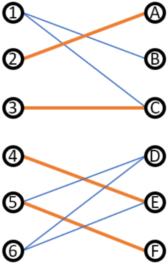
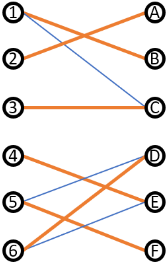
Bipartite Matchings



(a) A Bipartite Graph G



(b) A Matching of G



(c) A Perfect Matching of G

Bipartite Matching as a Polynomial

We can create the *biadjacency matrix* of a bipartite graph by having the $(i.j)$ entry be a 1 if there is an edge between i (on the left) and j (on the right), and 0 otherwise.

Permanent

Given a real (or complex) matrix $A = (a_{ij})$ of dimension $n \times n$, we define the permanent:

$$\mathrm{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If A is a biadjacency matrix with $a_{ij} \in \{0, 1\}$, then $\mathrm{Per}(A)$ counts the number of perfect matchings in the bipartite graph G with biadjacency matrix A. This is a known #P problem in complexity theory.

The Interpolation Method

Lemma (Barvinok)

Let $g(z)$ be a complex polynomial of degree d and suppose $g(z) \neq 0$ for all $|z| \leq \beta$ where $\beta > 1$. Choose a branch of $f(z) = \ln g(z)$ for $|z| \leq 1$, and consider the n^{th} order Taylor polynomial of f:

$$p_n(z) = f(0) + \sum_{k=1}^{n} \left(\frac{\mathrm{d}^k}{\mathrm{d}z^k} f(z) \Big|_{z=0} \right) \frac{z^k}{k!}.$$

Then,

$$|f(z) - p_n(z)| \leq \frac{d}{(n+1)\beta^n(\beta - 1)} \text{ for all } |z| \leq 1.$$

The Interpolation Method

- The "interpolation method" is then applying this lemma to a sufficient polynomial representing our combinatorial problem. For the permanent/bipartite matching problem, we use the function $g(z) = \mathrm{Per}(J + z(A - J))$ where J is the all 1s matrix.
- We will see another function below when we study the problem of counting cliques in a graph.
- Note: the Taylor polynomial from the previous slide depends on the absence of zeros in our polynomial $g(z)$ in a region around the origin (in the complex plane). This is to ensure that $lng(z)$ is well defined!

Permanent Approximation Result

Theorem (Barvinok)

Fix $0 < \delta < 0.5$*. Then, there is* $\gamma = \gamma(\delta) > 0$ *such that for any* $\epsilon > 0$*, and positive integer* n*, there is a polynomial* $p = p_{n,\delta,\epsilon}$ *in the entries of an* $n \times n$ *complex matrix* A *satisfying*

$$\deg p \leq \gamma(\ln n - \ln \epsilon)$$

and

$$|\ln \mathrm{Per}(A) - p(A)| \leq \epsilon$$

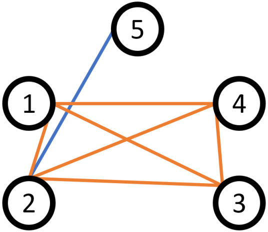
provided

$$|1 - a_{ij}| \leq \delta.$$

The proof relies on the interpolation method which "smoothes out" the computation allowing approximation.

Cliques in Graphs

We consider the following graph and its (regular) adjacency matrix A.



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Notice the clique (complete graph) of size 4 on the vertices 1,2,3 and 4.

Dense Subsets of a Graph

Let $G = (V, E)$ be an undirected simple graph. For a non-empty subset $S \subseteq V$ of vertices, define the density $\sigma(S)$ as

$$\sigma(S) = \frac{\left|\binom{S}{2} \cap E\right|}{\binom{|S|}{2}}$$

Partition Function of Cliques

We define the partition function of cliques as:

$$\mathrm{den}_m(G;\gamma) = \binom{n}{m}^{-1} \sum_{S\subseteq V, |S|=m} \exp\{\gamma m \sigma(S)\}.$$

The exponential tilting of $\sigma(S)$ puts greater emphasis on sets of higher density. As γ grows this partition function approximates the density of the densest subset.

Definition

A "Partition Function" for Cliques

Let $Z = (z_{ij})_{i,j=1}^{n}$ be an $n \times n$ real or complex matrix. Then for an integer $1 < m \leq n$, we define the polynomial

$$P_m(Z) = \sum_{\substack{S \subset [n] \\ |S| = m}} \exp\left\{ \prod_{\substack{i,j \subset S \\ i \neq j}} z_{ij} \right\}.$$

Note that $P_m(Z_0) = \exp\{-\gamma m/2\} \binom{n}{m} \mathrm{den}_m(G;\gamma)$ if $Z_0 = (z_{ij})$ satisfies:

$$z_{ij} = \begin{cases} \gamma/(m-1) & \text{if } \{i,j\} \in E \\ -\gamma/(m-1) & \text{if } \{i,j\} \notin E. \end{cases}$$

We approximate $P_m(Z_0)$ using the interpolation method by showing $P_m(Z) \neq 0$ in a suitable region.

Polynomial Version of $P_m(Z)$

Polynomial Version

For W an $n \times n$ complex matrix with zero diagonal, write:

$$p_m(W) = \binom{n}{m}^{-1} \sum_{\substack{S \subseteq [n] \\ |S|=m}} \prod_{\substack{i,j \in S \\ i \neq j}} (1 + w_{ij})$$

so that for $z_{ij} = \ln(1 + w_{ij})$, $\binom{n}{m} p_m(W)$ coincides with $P_m(Z)$.

From $p_m(W)$, we can define $h(z)$:

$$h(z) = \binom{n}{m}^{-1} \sum_{\substack{S \subset [n] \\ |S|=m}} \prod_{\substack{\{i,j\} \subset S \\ i \neq j}} (1 + z w_{ij}).$$

We note that $h(1) = p_m(W)$.

Non-Vanishing Theorem

Theorem (Barvinok,ADP)

For any $0 < \delta < 1$ there exist $\eta = \eta(\delta)$ and $\omega = \omega(\delta) > 1$ such that if $n \geq \omega m$ then $P_m(Z) \neq 0$ for any $n \times n$ symmetric complex matrix Z such that

$$|\mathrm{Re}(z_{ij})| \leq \frac{\delta}{(m-1)} \text{ and } |\mathrm{Im}(z_{ij})| \leq \frac{\eta}{(m-1)}.$$

- The interpolation trick applied to $h(z)$ then allows us to (for a fixed $0 < \gamma < 1$) efficiently approximate $\mathrm{den}_m(G; \gamma)$ within relative error ϵ. In particular, we have an algorithm running in quasi-polynomial $n^{O(\ln m)}$ complexity to compute $P_m(Z_0)$ for any $0 < \gamma < 1$ fixed in advance.

References

1. A.Barvinok, *Combinatorics and Complexity of Partition Functions*, Springer, 2017
2. A.Barvinok, A.Della Pella, *Testing for Dense Subsets in a Graph via the Partition Function*

Thank You

Many thanks to the organizers for setting up the Mathematics Continued Conference, and to you for your attention. Questions?