# Bergman Kernel Zeroes For A Range Of Weights

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9/8/14

## 1 Introduction to Bergman Spaces

**Definition 1.** *We call the set* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ *the disk, and we define the Bergman space* $\mathbb{A}^p(\mathbb{D}) = L^p(\mathbb{D}) \cap \mathcal{O}(\mathbb{D})$ *where* $L^p$ *is the usual Lebesgue space, on the disk and* $\mathcal{O}$ *is the set of all functions holomorphic (analytic) in the disk.*

The Bergman space is equipped with the $L^p$ function norm, that is

$$\|f\| = \left[\int_{\mathbb{D}} |f(z)|^p dA\right]^{\frac{1}{p}}$$

where $dA$ is the standard area measure. One of the interesting properties of the Bergman spaces is that they come equipped with a reproducing kernel function. This function is discussed in detail below. For further details see the books [3] or [4]. The space $\mathbb{A}^2$ is equipped with an inner product given by:

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA.$$

Furthermore, there is a function $K(z, w)$ on $\mathbb{D}\times\mathbb{D}$ with the reproducing property that:

$$f(z) = \int_{\mathbb{D}} f(w)K(z, w)dA(w)$$

where $z$ is some element of $\mathbb{D}$. We then call this function $K$ the kernel of $\mathbb{A}^2$ denoted $B_\lambda(z, w)$ where $\lambda$ is some weight function introduced into the inner product.

## 2 The Theorem's of Rouche and Montel

Informally, Montels theorem tells us that any uniformly bounded family of holomorphic functions defined on an open subset of the complex plane, $\mathbb{C}$ is normal. Here, normal means that the functions have a certain amount of robustness to them, a notion made precise in the following:

**Definition 2** (Normal Family)**.** *A set $\mathcal{F}$ of continuous functions f defined on a complete metric space $X$ with range $Y$, another complete metric space is called normal if every sequence of functions in $\mathcal{F}$ contains a subsequence uniformly converging on compact sets to a continuous function $g$ from $X$ to $Y$. Formally, for arbitrary sequences of functions in $\mathcal{F}$, there is a subsequence $\{f_n(x)\}_n^\infty$ and a continuous function $g(x)$ from $X$ to $Y$ such that the following holds for every compact subset $K$ contained in X:*

$$\lim_{n\to\infty} \sup_{x\in K} d(f_n(x), f(x)) = 0.$$

Rouche's theorem is useful in that it tells us where zeroes occur (up to some region) in analytic functions. While it is almost a direct result of the argument principle, it is powerful in that the hypotheses are easily met and are present in many applications.

**Theorem 1** (Rouche's theorem)**.** *Let $C$ denote a simple closed contour, and suppose that*
*(i) two functions $f(z)$ and $g(z)$ are analytic both on $C$, and inside the region contained within $C$,*
*(ii) $|f(z)| > |g(z)|$ at each $z_0 \in C$. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeroes (including multiplicities) inside the region defined by $C$.*

For a proof of Rouche's theorem, see [2].

# 3 Results

## 3.1 Determining Valid 'A' for Which the Bergman Kernel $B_\lambda(z, w)$ Has Zeroes

We set out to find the range of values $A$ for which the weighted Bergman Kernel, $B_\lambda(z, w)$ has zeroes for weights $\lambda$ of the form:

$$\lambda(r) = \begin{cases} A+1, & 0 \le r \le \frac{1}{4} \\ 1, & \frac{1}{4} < r \le 1 \end{cases}.$$

In order to accomplish this, I adapt the proof in [1] which showed that for A = 17, the corresponding Bergman Kernel does indeed contain a zero in $\mathbb{D} \times \mathbb{D}$. The proof essentially has 3 components once the proper arrangement of the kernels terms are given, notice that:

$$(1 - z\bar{w})^2 B_\lambda(z,w) = (1 - z\bar{w})^2 \sum_{k=0}^{\infty} \alpha_k (z\bar{w})^k$$

$$= \alpha_0 + \alpha_1(z\bar{w}) - 2\alpha_0(z\bar{w}) + \sum_{k=2}^{\infty} (\alpha_k - 2\alpha_{k-1} + \alpha_{k-2})(z\bar{w})^k$$

$$(\text{where } \alpha_k = \frac{1}{2\pi \int_0^1 r^{2k+1}\lambda(r)dr} = \frac{16^{k+1}(2k+2)}{(2 * \pi)(16^{k+1} + A)})$$

which we then define to be $L(t)+S(t)$ for linear part, and sum part respectively.

The problem then becomes to find suitable $A$ for which $L(t)$ has zeroes, then show that for values of $|t| = 1$, $L(t) > S(t)$ and lastly use Rouch's theorem to conclude that $B_\lambda(z,w)$ has zeroes.

Since we have:

$$\max_{|t|=1-\epsilon} |S(t)| < \sum_{k=2}^{\infty} |\alpha_k - 2\alpha_{k-1} + \alpha_{k-2}|$$

we need only show that

$$\min_{|t|=1-\epsilon} |L(t)| > \sum_{k=2}^{\infty} |\alpha_k - 2\alpha_{k-1} + \alpha_{k-2}|$$

for $\epsilon$ smaller than some value.

Just as was done in [1], we show that $\alpha_k - \alpha_{k-1} < \alpha_{k-1} - \alpha_{k-2}$ at least for $k \geq 2$ allowing us to recognize the series as telescoping. I have shown that $\alpha_k - \alpha_{k-1} < \alpha_{k-1} - \alpha_{k-2}$ holds by considering

$$(\alpha_k - \alpha_{k-1})\frac{(16^{k-1} + A)}{(16^{k-1} + A)} \tag{1}$$

and

$$(\alpha_{k-1} - \alpha_{k-2})\frac{(16^{k+1} + A)}{(16^{k+1} + A)} \tag{2}$$

leaving us with common denominators, making the proof of the inequality easier. Now, the numerator of (1) is $(A \times 16^{k-1})(2 \times 16^{k+2} + 2 \times 16^2 \times 15k + 2 \times 16 \times 15k \times A + 16^2 \times 2 \times A)$ when we get rid of terms shared by (2). Similarly, the numerator of (2) is $(A \times 16^{k-1})(2 \times 16^k + 2 \times 16^{k+1} \times 15k + 2 \times 15k \times A + 2 \times A)$ when we remove terms found exactly in (1). By comparing numerators, we see (2) dominates (1) for sufficiently large $k$. Also, there can be values of $k$ where the two equations are equal, so for all $k$ greater than or equal to the maximum of these points of incidence we have our desired inequality. By setting (2) equal to (1) with $k = 2$ we find that $A = \frac{3328}{47}$ is our maximal value on $A$, that is the point at which (2) becomes less than or equal to (1) for $k = 2$. Clearly for positive $A < \frac{3328}{47}$, our inequality holds for $k \geq 2$. Since $\alpha_k - \alpha_{k-1} < \alpha_{k-1} - \alpha_{k-2}$ the series is telescoping, converging to

$$\alpha_1 - \alpha_0 - \lim_{k\to\infty} (\alpha_k - \alpha_{k-1}) = (\alpha_1 - \alpha_0) - 2.$$

Note that our limit is independent of $A$ relying only on the value of $\frac{1}{4}$ and $k$. Since

$$\min_{|t|=1-\epsilon} |L(t)| = (\alpha_1 - 3\alpha_0) - \epsilon(\alpha_1 - 2\alpha_0)$$

we need to show that

$$-2\alpha_0 - \epsilon(\alpha_1 - 2\alpha_0) > -2$$

which holds true for $A > 16 - \epsilon/4$.

Lastly we show that $L(t)$ has zeroes for $t \in \mathbb{D}$. Consider

$$\alpha_0 + (\alpha_1 - 2\alpha_0)t = \frac{16(2)}{16 + A} + (\frac{16^2(4)}{16^2 + A} - \frac{4(16)}{16 + A})t$$

which when set equal to 0 tells us that

$$t = \frac{-256 - A}{30A}$$

which has $|t| < 1$ for $A > 256/29$. Then only for $A \in [16, 3328/47]$ the Bergman Kernel $B_\lambda(z, w)$ has a zero in $\mathbb{D} \times \mathbb{D}$, as these values of $A$ permit a zero of $L(t)$ in $|t| < 1 - \epsilon$ and $|L(t)| > |S(t)|$ on $|t| = 1 - \epsilon$ when $\epsilon$ is chosen to be small enough. Rouch's theorem allows us to conclude that $L(t) + S(t) = (1 - z\bar{w})^2 B_\lambda(z, w)$ has a zero in $\mathbb{D} \times \mathbb{D}$ since $(1 - z\bar{w})^2$ doesn't vanish here.

# 4 Computational Results

**Definition 3** (truncation). *Given a Bergman kernel function* $B_\lambda(z, w) = \sum_{n=0}^{\infty} \alpha_n (z\bar{w})^n$ *we call the* $K^{th}$ *degree polynomial* $\sum_{n=0}^{K} \alpha_n (z\bar{w})^n$ *the* $K-$*truncation of the kernel function. We denote the* $K-$*truncation of* $B_\lambda(z, w)$ *as* $|K|B_\lambda(z, w)$.

Using the sage mathematical software, I have developed a program which allows us to determine whether or not a Bergman kernel (corresponding to a given weight $\lambda(r)$) has a zero within the disk for its $K$-truncation. The program takes as input a given weight, and a range of values $K$ for which to calculate the minimum modulus $|t_0|$ where $t_0 = z_0\bar{w}_0$ at which $|K|B_\lambda(z_0, w_0) = 0$.

These results are then output as two columns, with the value for $K$ on the left, and the real number modulus of $t_0$ on the right. These findings have allowed me to verify that for very high values of $K$ my bounds on A hold. In addition to this, I have output tables of "minimum moduli" corresponding to a wide range of truncation levels for several "interesting" weights. This program while allowing me to verify my results was initially inspired by the question posed below of what happens to our kernel when the value for $\frac{1}{4}$ is changed.

## 4.1 Determining the Usefulness of Obtained Computational Results

In this section we set out to prove one key point which tells us that the above computations allow us to answer questions related to actual Bergman kernels, and not just their approximations.

**Lemma 1.** *Given a Bergman kernel $B_\lambda(z, w)$ which doesn't have zeroes inside the disk, there exists a natural number $K_0$ such that for all $K > K_0$ the function $|K|B_\lambda(z, w)$ will have zeroes only when $|t| > 1$ where $t = z\bar{w}$.*

*Proof.* Assume there exists a sequence $\{a_n\}$ where $a_n \in \mathbb{D}$ which converges to $a \in \mathbb{D}$. Also, assume that for each $a_n$ there is a $k_n-$truncation of $B_\lambda(z, w)$ which vanishes at $t = a_n$. In [4] it is shown that these truncations converge uniformly on compact subsets using the Bergman Inequality. In every small compact neighborhood $E$ of $a$, with $E \subseteq \mathbb{D}$, the truncations are almost 0 as they vanish at nearby points (the $a_n$'s) within the neighborhood. By the uniform convergence of the truncations to $a$ in $E$ we can make $B_\lambda(z, w)$ as small as we wish in $E$ and thus conclude that $B_\lambda(z, w)$ must vanish somewhere inside of $\mathbb{D} \times \mathbb{D}$. □

# 5 Direction for Future Work

One of the main goals of this theory is determining kernel functions which vanish inside of the disk. In this case, I have shown that for a specific class of kernel functions (depending on $A$) we have vanishing points inside of the disk. This attempt can be furthered in two directions:
First, find out what values of $q$ where $q$ was $\frac{1}{4}$ in the original theorem allow us to still have vanishing points inside of $\mathbb{D}$, and find a relationship between this value of $q$ and what we called $A$ above.
Secondly, one could pursue the route of adding more "levels" to our weight. That is, instead of having our weight be defined in two pieces, determine what happens when we have a weight consisting of more than just two piecewise defined sections.

The last ongoing attempts at answering this question of "which weights $\lambda(r)$ provide us with kernel functions having zeroes inside of the disk" consist of outputting large tables using the computational methodology described and use these results as evidence that one weight function or another vanishes inside of the disk.

# References

[1] Yunus E. Zeytuncu *Weighted Bergman Projections And Kernels: $L^p$ Regularity And Zeroes*

[2] James Ward Brown, Ruel V. Churchill *Complex Variables and Applications*

[3] Kehe Zhu, Haakan Hedenmalm, Boris Korenblum *Theory of Bergman Spaces*

[4] Peter Duren, Alexander Schuster *Bergman Spaces*