

Limit process of stationary TASEP near the characteristic line

Jinho Baik*, Patrik L. Ferrari†, Sandrine Péché‡

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Abstract

The totally asymmetric simple exclusion process (TASEP) on \mathbb{Z} with the Bernoulli- ρ measure as initial conditions, $0 < \rho < 1$, is stationary. It is known that along the characteristic line, the current fluctuates as of order $t^{1/3}$. The limiting distribution has also been obtained explicitly. In this paper we determine the limiting multi-point distribution of the current fluctuations moving away from the characteristics by the order $t^{2/3}$. The main tool is the analysis of a related directed last percolation model. We also discuss the process limit in tandem queues in equilibrium.

1 Introduction and result

Continuous time TASEP. The totally asymmetric simple exclusion process (TASEP) is the simplest non-reversible interacting stochastic particle system. In TASEP, particles are on the lattice of integers, \mathbb{Z} , with at most one particle at each site (exclusion principle). The dynamics is defined as follows. Particles jump to the neighboring right site with rate 1 provided that

*Department of Mathematics, University of Michigan, Ann Arbor, MI, 48109, USA
email: baik@umich.edu

†University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany
e-mail: ferrari@uni-bonn.de

‡Institut Fourier, 100 Rue des maths, 38402 Saint Martin d'Herès, France
e-mail: Sandrine.Peche@ujf-grenoble.fr

the site is empty. Jumps are independent of each other and take place after an exponential waiting time with mean 1, which is counted from the time instant when the right neighbor site is empty.

It is known that the only translation invariant stationary measures are Bernoulli product measures with a given density $\rho \in [0, 1]$ (see [24]). In the sequel we fix a $\rho \in (0, 1)$ to avoid the trivial cases $\rho = 0$ (no particles) and $\rho = 1$ (all sites occupied). One quantity of interest is the fluctuations of the currents of particles during a large time t . Consider the so-called *characteristic line* given by $x = (1 - 2\rho)t$. Then, the current fluctuation seen from the characteristic are $\mathcal{O}(t^{1/3})$ (in agreement with the scaling for the two-point function established in [5]), and the limit distribution function has been determined (conjectured in [28] and proved in [17], see also [1]). The limiting distribution was first discovered in the context of a directed last passage percolation model [4].

On the other hand, if one looks at the current along a line different from the characteristic line, say $x = ct$ with $c \neq 1 - 2\rho$, then for large time t one just sees the Gaussian fluctuations from the initial condition on a $t^{1/2}$ scale, since the dynamics generates fluctuations only $\mathcal{O}(t^{1/3})$, thus irrelevant [14].

A non-trivial interplay between the dynamically generated fluctuations and the one in the initial condition occurs in a region of order $t^{2/3}$ around the characteristic line. One of the main results of this paper is the determination of multi-point limits of the current fluctuations around the characteristics $x = (1 - 2\rho)t + \mathcal{O}(t^{2/3})$ (see Theorem 6 and Theorem 7).

There has been a great deal of work pertaining to the limiting fluctuation of TASEP over the last ten years since the well-known work of Johansson [19]. See for example, the review paper [16] for the limit processes which arise from deterministic initial conditions. A recent paper [9], building on the earlier work [10], is concerned on the situation where random and deterministic initial condition are both present, but not stationary.

Directed percolation model. It is well-known that the currents of TASEP (of arbitrary initial condition) can be expressible in terms of the last passage time of an associated directed last passage percolation model (see for example, [19, 28]). We first discuss the asymptotic result of the following directed percolation model associated to stationary TASEP [28]: see the paragraphs preceding Theorem 6 below for the exact relation to stationary TASEP. Let $w_{i,j}$, $i, j \geq 0$, $i, j \in \mathbb{Z}$, be independent random variables with the following

distributions

$$\begin{aligned}
w_{0,0} &= 0, \\
w_{i,0} &\sim \text{Exp}(1/(1-\rho)), & i \geq 1, \\
w_{0,j} &\sim \text{Exp}(1/\rho), & j \geq 1, \\
w_{i,j} &\sim \text{Exp}(1), & i, j \geq 1.
\end{aligned} \tag{1.1}$$

Here the notation $X \sim \text{Exp}(r)$ means that X is a random variable exponentially distributed with expectation r . An *up-right* path π from $(0,0)$ to $(x,y) \in \mathbb{N}^2$ is a sequence of points $(\pi_\ell \in \mathbb{Z}^2, \ell = 0, \dots, x+y)$, starting from the origin, $\pi_0 = (0,0)$, ending at (x,y) , $\pi_{x+y} = (x,y)$, and satisfying $\pi_{\ell+1} - \pi_\ell \in \{(1,0), (0,1)\}$. Denote by $L(\pi) = \sum_{(i,j) \in \pi} w_{i,j}$. Then, the last passage time is defined by

$$G(x,y) = \max_{\pi: (0,0) \rightarrow (x,y)} L(\pi). \tag{1.2}$$

We are interested in the limit distribution of the properly rescaled last passage time of $G([xN], [yN])$ in the $N \rightarrow \infty$ limit.

When there are no ‘borders’ i.e. $w_{i,j} = 0$ when $i = 0$ or $j = 0$, the limiting distribution of $G(x,y)$ was first obtained in [19]. The case of ‘single-border’ when $w_{i,0} = 0$ but $w_{0,j} = \text{Exp}(1/\rho)$ was studied in [2]. The above ‘double-border’ case was considered in [17,28]. (The Poisson and geometric variations were studied earlier in [4].) One can also consider a more general model when $w_{i,0} \sim \text{Exp}(1/(1/2+a))$ for $i \geq 1$ and $w_{0,j} \sim \text{Exp}(1/(1/2+b))$ for $j \geq 1$, where $a, b > -1/2$. Such model was considered in [28] and a conjecture on the limiting distribution left in that paper was recently confirmed in [1]. For the Poisson and geometric variations, this generalization was considered in [4] earlier when $x = y$.

An interesting characteristic behavior in such bordered models is the transition phenomenon. One can imagine that the last passage time comes from the competition between the contributions from the ‘bulk’ $i, j > 0$, and the edges $i = 0$ or $j = 0$. See for example, Section 6 of [2] for an illustration of such heuristic ideas. For the model (1.1), a crucial role is played by the critical direction (which corresponds to the characteristic line of TASEP),

$$\frac{y}{x} = \frac{\rho^2}{(1-\rho)^2}. \tag{1.3}$$

It is easy to check that along a direction other than the critical direction, the fluctuations of $G([xN], [yN])$ is given by Gaussian distribution on the $N^{1/2}$

scale, see Appendix D. This situation corresponds to the Gaussian fluctuations along a non-characteristic line in a stationary TASEP [14] mentioned earlier.

Along the critical direction and also in a $N^{2/3}$ neighborhood of the critical direction, the fluctuations of $G([xN], [yN])$ are on the $N^{1/3}$ scale. Precisely, the following, among other things, is proven in [17]. Let us set

$$\chi := \rho(1 - \rho). \quad (1.4)$$

Then set the parameter $N = \lfloor (1 - 2\chi)T \rfloor$, where T is considered to be large. Consider the scaling

$$\begin{aligned} x(\tau) &= \left\lfloor (1 - \rho)^2 T + \tau \frac{2\chi^{4/3}}{1 - 2\chi} T^{2/3} \right\rfloor, \\ y(\tau) &= \left\lfloor \rho^2 T - \tau \frac{2\chi^{4/3}}{1 - 2\chi} T^{2/3} \right\rfloor, \\ \ell(\tau, s) &= T - \tau \frac{2(1 - 2\rho)\chi^{1/3}}{1 - 2\chi} T^{2/3} + s \frac{T^{1/3}}{\chi^{1/3}}. \end{aligned} \quad (1.5)$$

The parameter τ measures the displacement of the focus with respect to the critical line (on a $T^{2/3}$ scale). In particular, for $\tau = 0$ one looks exactly along the critical direction. $\ell(\tau, s = 0)$ is the macroscopic value of the last passage time for large T , while the parameter s in $\ell(\tau, s)$ measures the amount of the fluctuations (on a $T^{1/3}$ scale) of the last passage time. Then for any fixed $\tau \in \mathbb{R}$,

$$\lim_{T \rightarrow \infty} \mathbb{P}(G(x(\tau), y(\tau)) \leq \ell(\tau, s)) = F_\tau(s) \quad (1.6)$$

for an explicit distribution function F_τ , which satisfies $F_\tau(s) = F_{-\tau}(s)$. An analogous result was first obtained in [4] for a Poissonized version of the problem in which F_τ was obtained in terms of a solution to the Painlevé II equation (see (3.22) of [4]: $F_\tau(x) = H(s + \tau^2; \tau, -\tau)$). Based on this result, Prähofer and Spohn conjectured in [28] that (1.6) holds. This conjecture was proven in [17] whose analysis was somewhat different from [4]. This results in a different formula of F_τ , expressed in terms of the Airy function (see (1.20) in [17]). It can be checked that these two formulas do agree.

One of the main objects of study in this paper is the limit of multi-point distribution $G(x_j, y_j)$, $j = 1, \dots, k$. The limit of the process for the no-border case was first obtained in [21] following the earlier work of [29] on the

Poissonized version of the model. The limit of the process for the double-border case was considered in [11] for the model when $w_{0,0} \sim \text{Exp}(1/(a+b))$ and $w_{i,0} \sim \text{Exp}(1/(1/2+a))$ for $i \geq 1$, $w_{0,j} \sim \text{Exp}(1/(1/2+b))$ for $j \geq 1$ when $(a,b) \in (-1/2, 1/2)$ and $a+b > 0$. Note that when $a = -b = 1/2 - \rho$, the random variable $w_{0,0}$ becomes singular in this model: the restriction $w_{0,0} = 0$ is significant in connection to stationary TASEP.

The geometric counterpart of the model (1.1) was studied earlier by Imamura and Sasamoto [18]. Denoting by $\text{Geom}(q)$ a random variable with the probability mass function $(1-q)q^k$, $k = 0, 1, 2, \dots$, the authors of [18] considered the model (1.1) where $w_{0,0} = 0$, $w_{i,0} \sim \text{Geom}(\gamma_+\alpha)$, $w_{0,j} \sim \text{Geom}(\gamma_-\alpha)$ and $w_{i,j} \sim \text{Geom}(\alpha^2)$ for $i, j \geq 1$. Since the exponential model (1.1) can be obtained as a limiting case of $\alpha \rightarrow 1$, the analysis of [18] can in principle be used to yield the corresponding result for the model (1.1). Nevertheless, this paper differs from [18] on the following aspects. (a) The authors of [18] obtained explicit limiting distribution functions for the case that the situation $a+b > 0$ (and $a+b < 0$) in (1.1). However, they left the case corresponding to $a+b = 0$ as the limit of the $a+b > 0$ case and did not compute the limiting distribution explicitly (see remarks after Theorem 5.1 in [18]). This critical case is the most interesting (and the most difficult) case for our situation and we give an explicit formula in Theorem 2 below. (b) The justification of the limits of Fredholm determinant and other quantities appearing in the analysis requires proper conjugations of corresponding operators in order to make sense of the Fredholm determinant and trace class limit. These issues were not discussed in [18] (the main issue was to determine the possible limit regimes and not specifically the $a+b = 0$ case). (c) In addition to the multi-point distribution on the line $x+y = \text{constant}$ considered in [18], we also obtain limit of the process result for points (x_k, y_k) not necessarily on the same line (see Theorem 5). (d) The limit of the process for points at more general positions than on a line mentioned in (c) is used to prove the limit of the process (in the sense of finite distribution) of stationary TASEP (see Theorem 6 and Theorem 7 below).

We first state the result extending (1.6) to the joint distributions at points on the line

$$\mathcal{L}_N := \{(x, y) \geq 0 \mid x + y = N\} \tag{1.7}$$

at and near the critical direction. We first need some definitions.

Definition 1. Fix $m \in \mathbb{N}$. For real numbers $\tau_1 < \tau_2 < \dots < \tau_m$ and s_1, \dots, s_m , set

$$\begin{aligned}\mathcal{R} &= s_1 + e^{-\frac{2}{3}\tau_1^3} \int_{s_1}^{\infty} dx \int_0^{\infty} dy \operatorname{Ai}(x+y+\tau_1^2) e^{-\tau_1(x+y)}, \\ \Psi_j(y) &= e^{\frac{2}{3}\tau_j^3 + \tau_j y} - \int_0^{\infty} dx \operatorname{Ai}(x+y+\tau_j^2) e^{-\tau_j x}, \\ \Phi_i(x) &= e^{-\frac{2}{3}\tau_1^3} \int_0^{\infty} d\lambda \int_{s_1}^{\infty} dy e^{-\lambda(\tau_1 - \tau_i)} e^{-\tau_1 y} \operatorname{Ai}(x+\tau_i^2 + \lambda) \operatorname{Ai}(y+\tau_1^2 + \lambda) \\ &\quad + \mathbb{1}_{[i \geq 2]} \frac{e^{-\frac{2}{3}\tau_i^3 - \tau_i x}}{\sqrt{4\pi(\tau_i - \tau_1)}} \int_{-\infty}^{s_1 - x} dy e^{-\frac{y^2}{4(\tau_i - \tau_1)}} - \int_0^{\infty} dy \operatorname{Ai}(y+x+\tau_i^2) e^{\tau_i y}.\end{aligned}\tag{1.8}$$

for $i, j = 1, 2, \dots, m$, where Ai denotes the Airy function.

The first result is the following.

Theorem 2. Fix $m \in \mathbb{N}$. For real numbers $\tau_1 < \tau_2 < \dots < \tau_m$ and s_1, \dots, s_m , with the scaling given in (1.5),

$$\begin{aligned}& \lim_{T \rightarrow \infty} \mathbb{P} \left(\bigcap_{k=1}^m \{G(x(\tau_k), y(\tau_k)) \leq \ell(\tau_k, s_k)\} \right) \\ &= \sum_{k=1}^m \frac{\partial}{\partial s_k} \left(g_m(\tau, s) \det \left(\mathbb{1} - P_s \widehat{K}_{\operatorname{Ai}} P_s \right)_{L^2(\{1, \dots, m\} \times \mathbb{R})} \right).\end{aligned}\tag{1.9}$$

Here $L^2(\{1, \dots, m\} \times \mathbb{R})$ is equipped with the standard measure $\nu \otimes dx$ where ν is the counting measure on $\{1, \dots, m\}$. P_s denotes the projection operator $P_s(k, x) = \mathbb{1}_{[x > s_k]}$, and $\widehat{K}_{\operatorname{Ai}}$ is the so-called extended Airy kernel [29] with shifted entries defined by the kernel

$$\begin{aligned}\widehat{K}_{\operatorname{Ai}}((i, x), (j, y)) &:= [\widehat{K}_{\operatorname{Ai}}]_{i,j}(x, y) \\ &= \begin{cases} \int_0^{\infty} d\lambda \operatorname{Ai}(x+\lambda+\tau_i^2) \operatorname{Ai}(y+\lambda+\tau_j^2) e^{-\lambda(\tau_j - \tau_i)}, & \text{if } \tau_i \leq \tau_j, \\ - \int_{-\infty}^0 d\lambda \operatorname{Ai}(x+\lambda+\tau_i^2) \operatorname{Ai}(y+\lambda+\tau_j^2) e^{-\lambda(\tau_j - \tau_i)}, & \text{if } \tau_i > \tau_j. \end{cases}\end{aligned}\tag{1.10}$$

The function $g_m(\tau, s)$ is defined by

$$\begin{aligned}g_m(\tau, s) &= \mathcal{R} + \langle \rho P_s \Phi, P_s \Psi \rangle \\ &= \mathcal{R} + \sum_{i=1}^m \sum_{j=1}^m \int_{s_i}^{\infty} dx \int_{s_j}^{\infty} dy \Psi_j(y) \rho_{j,i}(y, x) \Phi_i(x),\end{aligned}\tag{1.11}$$

where

$$\rho := (\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s)^{-1}, \quad \rho_{j,i}(y, x) := \rho((j, y), (i, x)), \quad (1.12)$$

and $\Phi((i, x)) := \Phi_i(x)$, $\Psi((j, y)) = \Psi_j(y)$. Finally the functions \mathcal{R} , Φ , and Ψ are defined in Definition 1.

Observe that $\text{dist}((x(\tau_k), y(\tau_k)), \mathcal{L}_N) \leq 2$.

Remark 3. When $m = 1$, (1.9) agrees with the limiting function in (1.20) of [17], as one may expect.

Remark 4. The fact that $\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s$ is invertible follows from the fact that $P_s \widehat{K}_{\text{Ai}} P_s$ is trace class (see [21]) and that $\det(\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s) > 0$ for all given $s \in \mathbb{R}$. See Lemma 33 in Appendix B.

The shift in the integrand by τ_i^2 is due to the fact that the last passage time is (macroscopically) a linear function along the line \mathcal{L}_N , in contrast with the non-border case (i.e., $w_{i,0} = w_{0,j} = 0$) where the last passage time has a non-zero curvature. Therefore, when $|\tau_k| \gg 1$, the contribution from $\det(\mathbb{1} - P K_{\text{Ai}} P)$ will very close to one and the main contribution comes from $g_m(\tau, s)$.

The second theorem is a generalization of Theorem 2 to the case when the points (x_k, y_k) are not necessarily on the same line \mathcal{L}_N , $x + y = N$. We show that the fluctuation is unchanged even if some of the points are away from the line to the order smaller than $\mathcal{O}(T)$. This is due to the slow de-correlation phenomena obtained in [15]: along the critical direction the fluctuations decorrelate to order $\mathcal{O}(T^{1/3})$ over a time scale $\mathcal{O}(T)$ (instead of $\mathcal{O}(T^{2/3})$). More precisely, if we compare the last passage time G at two points (x, y) and (x', y') with $(x' - x, y' - y) = r \cdot ((1 - \rho)^2, \rho^2)$, their fluctuation will be $r + \mathcal{O}(r^{1/3})$ (see Lemma 31 below for details). Consider a $\nu \in (0, 1)$ and any fixed number θ and consider the scaling (a generalization of (1.5)) given by

$$\begin{aligned} x(\tau, \theta) &= \left[(1 - \rho)^2 (T + \theta T^\nu) + \tau \frac{2\chi^{4/3}}{1 - 2\chi} T^{2/3} \right], \\ y(\tau, \theta) &= \left[\rho^2 (T + \theta T^\nu) - \tau \frac{2\chi^{4/3}}{1 - 2\chi} T^{2/3} \right], \\ \ell(\tau, \theta, s) &= T + \theta T^\nu - \tau \frac{2(1 - 2\rho)\chi^{1/3}}{1 - 2\chi} T^{2/3} + s \frac{T^{1/3}}{\chi^{1/3}}. \end{aligned} \quad (1.13)$$

See Figure 1 for an illustration.

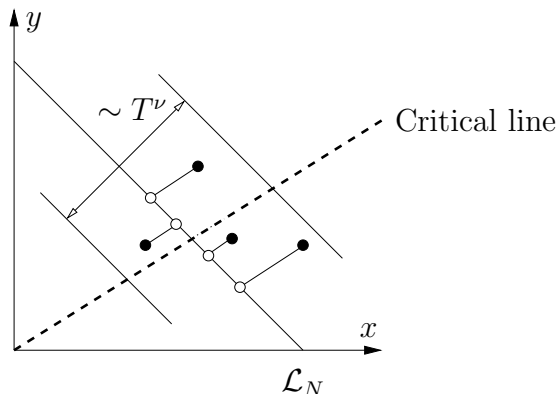


Figure 1: Assume that the black dots are $\mathcal{O}(T^\nu)$ for some $\nu < 1$ away from the line \mathcal{L}_N . Then, the fluctuations of the passage time at the locations of the black dots are, on the $T^{1/3}$ scale, the same as the one of their projection along the critical direction to the line \mathcal{L}_N , the white dots.

Theorem 5. Fix $m \in \mathbb{N}$ and $\nu \in (0, 1)$. For real numbers $\tau_1 < \tau_2 < \dots < \tau_m$, $\theta_1, \dots, \theta_m$, and s_1, \dots, s_m , with the scaling given in (1.13),

$$\begin{aligned} & \lim_{T \rightarrow \infty} \mathbb{P} \left(\bigcap_{k=1}^m \{G(x(\tau_k, \theta_k), y(\tau_k, \theta_k)) \leq \ell(\tau_k, \theta_k, s_k)\} \right) \\ &= \sum_{k=1}^m \frac{\partial}{\partial s_k} \left(g_m(\tau, s) \det \left(\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s \right)_{L^2(\{1, \dots, m\} \times \mathbb{R})} \right). \end{aligned} \quad (1.14)$$

This generalization of Theorem 2 is proven in Section 5.

Stationary TASEP and directed percolation. We now discuss the result in terms of stationary TASEP. The mapping between TASEP and last passage percolation model is as follows. We assign label 0 to the particle sitting at the smallest positive integer site initially. For the rest we use the right-to-left ordering so that $\dots < \mathbf{x}_2(0) < \mathbf{x}_1(0) < 0 \leq \mathbf{x}_0(0) < \mathbf{x}_{-1}(0) < \dots$. Then $\mathbf{x}_k(t) > \mathbf{x}_{k+1}(t)$ for all $t \geq 0$, since the TASEP preserves the ordering of the particles.

For $i, j \in \mathbb{Z}$ such that $i - j > x_j(0)$, let $\Omega(i, j)$ be the waiting time of particle with label j to jump from site $i - j - 1$ to site $i - j$ (the waiting time is counted from the instant where particle can jump, i.e., particle is at $i - j - 1$ and site $i - j$ is empty). The $\Omega(i, j)$ are iid $\text{Exp}(1)$ random variables. Let

$L(x, y)$ be the last passage time to $(x, y) \in \mathbb{N}^2$ along a directed path in the domain $D := \{(i, j) \in \mathbb{Z}^2 | i \leq x, j \leq y, i - j > x_j(0)\}$, starting any point in the domain. This is a curve-to-point optimization problem. An example of the non-trivial part of the boundary of this domain is illustrated in Figure 2 below as the curve at $t = 0$. Then a straightforward generalization of the step-initial case of [19] shows that

$$\mathbb{P}(\cap_{k=1}^m \{L(x_k, y_k) \leq t_k\}) = \mathbb{P}(\cap_{k=1}^m \{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}). \quad (1.15)$$

When the initial condition is random, the domain of directed percolation is also random. Note that (see Figure 2), the part of the domain D in the first quadrant is the rectangle $\{1 \leq i \leq x, 1 \leq j \leq y\}$, but the parts in the second quadrant $i \leq 0, j \geq 1$ and the forth quadrant $i \geq 1, j \leq 0$ are of random shape. Observe that D does not intersect with the third quadrant $i, j \leq 0$.

Define $-(\zeta_- + 1)$ to be the right-most empty site in $\{\dots, -2, -1\}$ in the initial particles' configuration and ζ_+ to be the position of the left-most particle in $\{0, 1, \dots\}$. First let us focus on the case $\zeta_- = 0, \zeta_+ = 0$ (as in Figure 2). Then, since the initial condition is stationary Bernoulli, an interpretation of Burke's theorem [12], see also [13], shows that the $\{L(0, j) | 1 \leq j \leq y\}$ is distributed as $\{X_1, X_1 + X_2, \dots, X_1 + \dots + X_y\}$ where X_j 's are iid $\text{Exp}(1/\rho)$ distributed. Similarly, by considering holes instead of particles one finds that $\{L(i, 0) | 1 \leq i \leq x\}$ is distributed as $\{Y_1, Y_1 + Y_2, \dots, Y_1 + \dots + Y_x\}$ where Y_i 's are iid $\text{Exp}(1/(1 - \rho))$ random variables. Hence $L(x, y)$ has the same distribution as $G(x, y)$ defined from (1.1). This argument was sketched in Section 2 of [28]. Consequently, we have for $x_k, y_k \geq 1, t_k > 0$.

$$\mathbb{P}(\cap_{k=1}^m \{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}) = \mathbb{P}(\cap_{k=1}^m \{G(x_k, y_k) \leq t_k\}), \quad (1.16)$$

for stationary TASEP when initially the position 0 is empty and the position 1 is occupied.

For generic $\zeta_- \geq 0$ and $\zeta_+ \geq 0$, $L(x, y)$ is distributed as $G(x, y)$ where now we assume in (1.1) that $w_{1,0} = \dots = w_{\zeta_+,0} = w_{0,1} = \dots = w_{0,\zeta_-} = 0$ where $\zeta_+ \sim \text{Geom}(1 - \rho)$ and $\zeta_- \sim \text{Geom}(\rho)$. But as shown in Proposition 2.2 in [17], this change does not affect the asymptotics.

Geometrically, the TASEP and directed percolation can be thought of as two different cuts of the three-dimensional object

$$\{x, y, G(x, y) | x, y \geq 1\} : \quad (1.17)$$

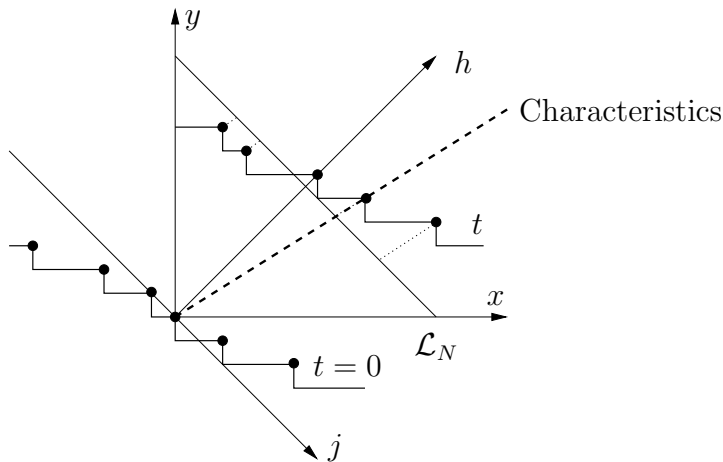


Figure 2: The dots are (random) particle configurations at time $t = 0$ and some later time t . The position of particle with label $n = y$ is its projection on the J -axis. Interpolating between particles as in this example, one gets a line configuration, which is interpreted as height function $h_t(j)$ above position j .

- (a) the directed percolation problem we analyze is the cut at $\{x + y = N\}$,
- (b) the particles' configuration of the TASEP at time t is the cut at $\{G = t\}$.

We have shown in Theorem 5 that the limit process does not depend on the cut chosen for the analysis as long as we avoid the cut along the characteristic, $\{x/y = (1 - \rho)^2/\rho^2\}$. Thus, to get the fluctuations around the characteristic line it is enough to project on $\{x + y = (1 - 2\chi)t\}$ (see Figure 2 for an illustration), for which the limit theorem was proven in Theorem 2.

The following two theorems are proven in Section 5. Define the functions

$$\begin{aligned} n(\tau) &= \lfloor \rho^2 T - 2\tau\rho\chi^{1/3}T^{2/3} \rfloor, \\ q(\tau) &= \lfloor (1 - \rho)^2 T + 2\tau\chi^{1/3}T^{2/3} - (1 - \rho)sT^{1/3}/\chi^{1/3} \rfloor. \end{aligned} \tag{1.18}$$

Theorem 6 (Particles' position representation). *Fix $m \in \mathbb{N}$. For real numbers $\tau_1 < \tau_2 < \dots < \tau_m$ and s_1, \dots, s_m ,*

$$\lim_{T \rightarrow \infty} \mathbb{P} \left(\bigcap_{k=1}^m \{ \mathbf{x}_{n(\tau_k)}(T) \geq q(\tau_k) \} \right) = \sum_{k=1}^m \frac{\partial}{\partial s_k} \left(g_m(\tau, s) \det \left(\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s \right) \right). \tag{1.19}$$

An equivalent but geometrically slightly different way of representing the TASEP is via a height function (see e.g. [17,28]). Let us define the occupation variable, $\eta_i(t) = 1$ if there is a particle at site i at time t and $\eta_i(t) = 0$ otherwise. Then, define the height function

$$h_t(j) = \begin{cases} 2N_t + \sum_{i=1}^j (1 - 2\eta_i(t)), & \text{for } j \geq 1, \\ 2N_t, & \text{for } j = 0, \\ 2N_t - \sum_{i=j+1}^0 (1 - 2\eta_i(t)), & \text{for } j \leq -1, \end{cases} \quad (1.20)$$

where N_t is the number of particles which jumped from site 0 to site 1 during the time-span $[0, t]$. Then the link between the height functions and the locations of particles is

$$\mathbb{P}(\cap_{k=1}^m \{h_{t_k}(x_k - y_k) \geq x_k + y_k\}) = \mathbb{P}(\cap_{k=1}^m \{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}). \quad (1.21)$$

Let us consider the scaling

$$\begin{aligned} J(\tau) &= \lfloor (1 - 2\rho)T + 2\tau\chi^{1/3}T^{2/3} \rfloor, \\ H(\tau) &= \lfloor (1 - 2\chi)T + 2\tau(1 - 2\rho)\chi^{1/3}T^{2/3} - 2s\chi^{2/3}T^{1/3} \rfloor. \end{aligned} \quad (1.22)$$

Theorem 7 (Height function representation). *Fix $m \in \mathbb{N}$. For real numbers $\tau_1 < \tau_2 < \dots < \tau_m$ and s_1, \dots, s_m . Then,*

$$\lim_{T \rightarrow \infty} \mathbb{P} \left(\bigcap_{k=1}^m \{h_{T}(J(\tau_k)) \geq H(\tau_k)\} \right) = \sum_{k=1}^m \frac{\partial}{\partial s_k} \left(g_m(\tau, s) \det \left(\mathbb{1} - P_s \widehat{K}_{A_i} P_s \right) \right). \quad (1.23)$$

Remark 8. For simplicity, in Theorems 6 and 7 we stated the result only for fixed time. However, the statements can be extended to different times in a similar manner of the extension from Theorem 2 to Theorem 5.

Queues in tandem. There is a direct relation between queues in tandem and TASEP. Suppose that there are infinitely many servers, and we assume that the service time of a customer at each server is independent and distributed as $\text{Exp}(1)$. Once a customer is served at the server i , then the customer joins at the $(i+1)$ th queue, and so on. In other words, the departure process from the i th queue is the arrival process at the $(i+1)$ th queue. Suppose that the system is in equilibrium with parameter ρ : the arrival process

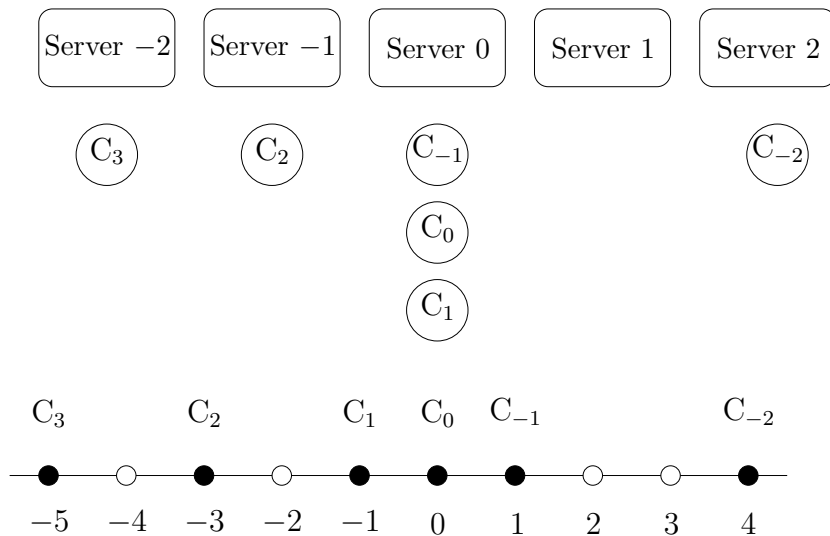


Figure 3: Queues in tandem in equilibrium. The black dots represents the costumers and at every white dots one changes to the next counter.

at each queue is independent Poisson process of rate ρ . Then the departure process at each queue is also independent Poisson process of rate ρ due to Burke's theorem (see for example, [12, 23]; see also [25]). This also implies that at each time, the number of customers in each queue is distributed as $\text{Geom}^*(1 - \rho)$. (Here $X \sim \text{Geom}^*(1 - \rho)$ means that $\mathbb{P}(X = k) = \rho(1 - \rho)^k$, $k = 0, 1, 2, \dots$.) Now consider a fixed time $t = 0$ and arbitrary select one customer and assign label 0 to that customer. For convenience, we call the queue in which that customer is in at time 0 as the 0th queue. We assign labels to the other customers so that the labels decreases for the customers ahead in the queues (see Figure 3). Let $Q_j(t)$ denote the label of the queue in which the j th customer is in at time t . The mapping from the queueing model to the TASEP is obtained by setting $\mathbf{x}_j(t) = Q_j(t) - j$ (see Figure 3). The equilibrium condition implies that the initial condition for the corresponding TASEP is stationary. Hence if we define $E_j(i)$ be the time the j th customer exits the queue i , then we find that

$$\begin{aligned} \mathbb{P}(\cap_{k=1}^m \{E_{y_k}(x_k - 1) \leq t_k\}) &= \mathbb{P}(\cap_{k=1}^m \{Q_{y_k}(t_k) \geq x_k\}) \\ &= \mathbb{P}(\cap_{k=1}^m \{\mathbf{x}_{y_k}(t_k) \geq x_k - y_k\}). \end{aligned} \tag{1.24}$$

Hence using (1.16) and Theorem 5, we immediately obtain the following

result.

Theorem 9. Fix $m \in \mathbb{N}$ and $\nu \in (0, 1)$. For real numbers $\tau_1 < \tau_2 < \dots < \tau_m$, $\theta_1, \dots, \theta_m$, and s_1, \dots, s_m , with the scaling given in (1.13),

$$\begin{aligned} & \lim_{T \rightarrow \infty} \mathbb{P} \left(\bigcap_{k=1}^m \{E_{y(\tau_k, \theta_k)}(x(\tau_k, \theta_k)) \leq \ell(\tau_k, \theta_k, s_k)\} \right) \\ &= \sum_{k=1}^m \frac{\partial}{\partial s_k} \left(g_m(\tau, s) \det \left(\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s \right)_{L^2(\{1, \dots, m\} \times \mathbb{R})} \right). \end{aligned} \tag{1.25}$$

Outline

To prove Theorem 2, which is the basis for Theorems 6 and 7, we first consider a slightly different directed percolation model which is known to be determinantal. This model is then related to (1.1) by a shift argument (Section 2) followed by an analytic continuation (Section 3), resulting in an explicit formula of the joint distribution of the last passage times of a more general version of (1.1) when $w_{i,0} \sim \text{Exp}(1/(1/2 + a))$ for $i \geq 1$ and $w_{0,j} \sim \text{Exp}(1/(1/2 + b))$ for $j \geq 1$, for $a, b \in (-1/2, 1/2)$ (see Theorem 16). The formula for the model (1.1) is obtained by setting $a = -b = \rho - 1/2$. These arguments are a generalization of the arguments given in [4] and [17] for one-point distribution. In Section 4 we carry out the asymptotic analysis of the formula obtained in Theorem 16. The proofs of Theorems 2, 6 and 7 are given in Section 5. Some technical computations are given in Appendices: various expressions of integrals involving Airy functions are given in Appendix A, the invertibility of an operator appearing in Theorem 2 is discussed in Appendix B, certain operator are shown to be trace-class in Appendix C, and finally we explain the Gaussian fluctuation along non-critical directions in Appendix D.

Remark 10. As mentioned above, Theorem 16 below contains a formula for the model with $w_{i,0} \sim \text{Exp}(1/(1/2 + a))$ for $i \geq 1$ and $w_{0,j} \sim \text{Exp}(1/(1/2 + b))$ for $j \geq 1$, for $a, b \in (-1/2, 1/2)$, which is more general than (1.1). This would correspond to other random initial data which possibly allows a shock. Essentially all the ingredients for the asymptotic analysis for this more general case are in this paper, and one can obtain the limit laws for such models. However, for the sake of simplicity, we do not pursue this direction here.

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2 The shift argument

We first consider a slightly different directed percolation model. Let

$$\begin{aligned}\tilde{w}_{0,0} &\sim \text{Exp}(1/(a+b)), \\ \tilde{w}_{i,0} &\sim \text{Exp}(1/(1/2+b)), & i \geq 1, \\ \tilde{w}_{0,j} &\sim \text{Exp}(1/(1/2+a)), & j \geq 1, \\ \tilde{w}_{i,j} &\sim \text{Exp}(1), & i, j \geq 1,\end{aligned}\tag{2.1}$$

where the parameters a and b satisfy

$$a, b \in (-1/2, 1/2), \quad a + b > 0.\tag{2.2}$$

Denote by $G_{a,b}^+(x, y)$ to be the last passage time from $(0, 0)$ to (x, y) for this modified model. This model is well-studied and has nice mathematical structure. In particular, the correlation functions and joint distribution functions on \mathcal{L}_N are determinantal¹, which is well-suited for an asymptotic analysis. Note that the original model given in (1.1) corresponds to the case when $a + b = 0$ and $\tilde{w}_{0,0} = 0$. We will show in Sections 2 and 3 how to obtain a joint distribution formula for the original model (1.1) from this modified model (2.1). Let $G_{a,b}(x, y)$ be the last passage time for the model (2.1) with $\tilde{w}_{0,0}$ replaced by 0. We proceed as follows:

(1) Shift argument: for $a, b \in (-1/2, 1/2)$ with $a + b > 0$, we relate the distribution of $G_{a,b}$ with the one of $G_{a,b}^+$.

(2) Analytic continuation: we determine an expression for $G_{a,b}$ which can be analytically continued in all $a, b \in (-1/2, 1/2)$.

(3) Choice of parameter: finally we set $a = \rho - 1/2$ and $b = 1/2 - \rho$.

¹It is a limit of geometric random variables studied in [20, 21] or of the Schur measure [26]. Exponential random variables are studied directly in [11] on a more general case than the present one.

The step (1) is done in Section 2 and step (2) is presented in Section 3.

Proposition 11 (Shift argument). *Let $a, b \in (-1/2, 1/2)$ with $a + b > 0$. Let $(x_1, y_1), \dots, (x_m, y_m)$ be the a set distinct of points in \mathbb{Z}_+^2 and define*

$$\begin{aligned} P(u_1, \dots, u_m) &:= \mathbb{P} \left(\bigcap_{k=1}^m \{G_{a,b}(x_k, y_k) \leq u_k\} \right), \\ P^+(u_1, \dots, u_m) &:= \mathbb{P} \left(\bigcap_{k=1}^m \{G_{a,b}^+(x_k, y_k) \leq u_k\} \right). \end{aligned} \quad (2.3)$$

Then,

$$P(u_1, \dots, u_m) = \left(1 + \frac{1}{a+b} \sum_{k=1}^m \frac{\partial}{\partial u_k} \right) P^+(u_1, \dots, u_m). \quad (2.4)$$

Proof of Proposition 11. Let us only consider $m = 2$: the proof for general $m \geq 2$ is a straightforward generalization and we leave it to the reader. Set $r = a + b$. Then

$$\begin{aligned} P^+(u_1, u_2) &= \int_0^\infty dy \mathbb{P}(G_{a,b}^+(x_1, y_1) \leq u_1, G_{a,b}^+(x_2, y_2) \leq u_2 | \tilde{w}_{0,0} = y) \mathbb{P}(\tilde{w}_{0,0} = y) \\ &= \int_0^\infty dy P(u_1 - y, u_2 - y) r e^{-ry}. \end{aligned} \quad (2.5)$$

Consider the Laplace transform

$$\begin{aligned} &\int_0^\infty du_1 \int_0^\infty du_2 P^+(u_1, u_2) e^{-t_1 u_1 - t_2 u_2} \\ &= r \int_0^\infty dy \int_0^\infty du_1 \int_0^\infty du_2 P(u_1 - y, u_2 - y) e^{-ry - t_1 u_1 - t_2 u_2} \\ &= r \int_0^\infty dy \int_{-y}^\infty dz_1 \int_{-y}^\infty dz_2 P(z_1, z_2) e^{-ry - t_1(z_1+y) - t_2(z_2+y)}. \end{aligned} \quad (2.6)$$

Since $P(z_1, z_2) = 0$ when either $z_1 \leq 0$ or $z_2 \leq 0$, we can restrict the integral from $-y < z_i < \infty$ to $0 \leq z_i < \infty$. Then the above equals

$$\begin{aligned} &r \int_0^\infty dy \int_0^\infty dz_1 \int_0^\infty dz_2 P(z_1, z_2) e^{-(r+t_1+t_2)y - t_1 z_1 - t_2 z_2} \\ &= \frac{r}{r+t_1+t_2} \int_0^\infty dz_1 \int_0^\infty dz_2 P(z_1, z_2) e^{-t_1 z_1 - t_2 z_2}. \end{aligned} \quad (2.7)$$

Multiplying by $\frac{r+t_1+t_2}{r}$ and integrating by parts leads to

$$\begin{aligned}
& \int_0^\infty du_1 \int_0^\infty du_2 P(u_1, u_2) e^{-t_1 u_1 - t_2 u_2} \\
&= \frac{r+t_1+t_2}{r} \int_0^\infty du_1 \int_0^\infty du_2 P^+(u_1, u_2) e^{-t_1 u_1 - t_2 u_2} \\
&= \int_0^\infty du_1 \int_0^\infty du_2 \left(1 + \frac{t_1+t_2}{r}\right) P^+(u_1, u_2) e^{-t_1 u_1 - t_2 u_2} \\
&= \int_0^\infty du_1 \int_0^\infty du_2 \left(P^+(u_1, u_2) + \frac{1}{r} \left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right) P^+(u_1, u_2)\right) e^{-t_1 u_1 - t_2 u_2}.
\end{aligned} \tag{2.8}$$

Since this holds for all $t_1, t_2 \geq 0$, by inverting the Laplace transform, we obtain

$$P(u_1, u_2) = P^+(u_1, u_2) + \frac{1}{r} \left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right) P^+(u_1, u_2). \tag{2.9}$$

□

As mentioned earlier, the probability $P^+(u_1, \dots, u_m)$ has an explicit determinantal expression. In fact, the joint distribution function at points on \mathcal{L}_N for the directed percolation model with $w_{i,j} \sim \text{Exp}(1/(a_i + b_j))$, $a_i + b_j > 0$, has a determinantal structure (it is a limit of geometric random variables studied in [20, 21] or of the Schur measure [26]; see [7] for a direct approach to exponential random variables).² Specifically, this follows, for example, from Theorem 3.14 and (1.3) of [21] after substituting $a_i \mapsto 1 - \frac{a_i}{L}$ and $b_i \mapsto 1 - \frac{b_i}{L}$ in (1.3) and taking the limit $L \rightarrow \infty$. The kernel in this limit is explicitly derived in Theorem 3 of [11]. For good survey papers on the topic, see [22, 32]. By specializing to the case with weights as in (2.1), we have the following result. Consider a set of distinct points $(x_1, y_1), \dots, (x_m, y_m)$ on the line $\mathcal{L}_{2t} = \{(i, j) \in \mathbb{Z}^2 \mid i + j = 2t\}$, which can be ordered and parametrized by $t_1 < \dots < t_m \in [-t, t]$,

$$x_k = t + t_k, \quad y_k = t - t_k. \tag{2.10}$$

Set

$$\phi_i(w) = \frac{(\frac{1}{2} + w)^{t-t_i}}{(\frac{1}{2} - w)^{t+t_i}}, \tag{2.11}$$

²The result can be extended to any set of points which can be connected by down-right paths, called space-like path, but we do not enter in the details here. See [7, 8] for some examples in similar situations.

and ³

$$f_i(x) = \frac{-1}{2\pi i} \oint_{\Gamma_{a,1/2}} dw \frac{\phi_i(w)e^{-wx}}{a-w}, \quad g_j(y) = \frac{1}{2\pi i} \oint_{\Gamma_{-b,-1/2}} dz \frac{\phi_j(z)^{-1}e^{zy}}{z+b}. \quad (2.12)$$

Define the kernels

$$\overline{K}_{i,j}(x,y) = \widetilde{K}_{i,j}(x,y) - V_{i,j}(x,y) \quad (2.13)$$

where

$$\begin{aligned} V_{i,j}(x,y) &= \frac{\mathbb{1}_{[i>j]}}{2\pi i} \int_{i\mathbb{R}} d\tilde{z} e^{\tilde{z}(y-x)} \frac{\phi_i(\tilde{z})}{\phi_j(\tilde{z})}, \\ \widetilde{K}_{i,j}(x,y) &= \frac{-1}{(2\pi i)^2} \oint_{\Gamma_{-1/2}} dz \oint_{\Gamma_{1/2}} dw e^{zy-wx} \frac{\phi_i(w)}{\phi_j(z)} \frac{1}{w-z}. \end{aligned} \quad (2.14)$$

Then,

$$P^+(u_1, \dots, u_m) = \det(\mathbb{1} - P_u K P_u)_{L^2(\{1, \dots, m\} \times \mathbb{R}_+)} \quad (2.15)$$

where $P_u(k, x) = \mathbb{1}_{[x > u_k]}$ and the operator K is defined by the kernel

$$K((i, x), (j, y)) = K_{i,j}(x, y) := \overline{K}_{i,j}(x, y) + (a+b)f_i(x)g_j(y). \quad (2.16)$$

In (2.15), we have used a slight abuse of notation as $P_u K P_u$ is not a trace-class operator: one can indeed observe that for $i > j$, $P_{u_i} V_{i,j} P_{u_j}(x, x) \not\rightarrow 0$ as $x \rightarrow \infty$. Nevertheless with a suitable multiplication operator M , the conjugate operator $M P_u K P_u M^{-1}$ becomes trace-class so that the Fredholm determinant is well-defined, and the identity becomes valid analytically. More concretely, if we take M as in (2.18) below, then it is shown in Appendix C below that $M P_u K P_u M^{-1}$ is trace-class operator for $a, b \in (0, 1/2)$. Thus, after conjugation, the Fredholm determinant is well-defined, and the identity becomes valid analytically, as proved in the following.

Proposition 12. *Let $a, b \in (0, 1/2)$. Fix constants $\alpha_1, \dots, \alpha_m$ satisfying*

$$-a < \alpha_1 < \alpha_2 < \dots < \alpha_m < b. \quad (2.17)$$

Define the conjugated operator K^{conj} by the kernel

$$K_{i,j}^{\text{conj}}(x, y) = M_i(x) K_{i,j}(x, y) M_j(y)^{-1}, \quad M_i(x) := e^{-\alpha_i x}. \quad (2.18)$$

³For any set of points S , the notation $\oint_{\Gamma_S} dz f(z)$ means that the integration path goes anticlockwise around the points in S but does not include any other poles.

Then, $P_u K^{\text{conj}} P_u$ is a trace-class operator on $L^2(\{1, \dots, m\} \times \mathbb{R})$ and

$$P^+(u_1, \dots, u_m) = \det \left(\mathbb{1} - P_u K^{\text{conj}} P_u \right)_{L^2(\{1, \dots, m\} \times \mathbb{R}_+)} . \quad (2.19)$$

Proposition 12 and (2.15) give the standard extension from one-point kernel to the multi-point (or extended) kernel. All the ingredients are included in Appendix B and Section 3 of [17], and also in [31]. The extended kernel can be found in the more recent work [11]. There the setting is more general allowing several lines/columns to have waiting times different from 1.

Observe \bar{K} is independent of a and b , and the only dependence on a and b in K is through the rank-one term $(a+b)f_i(x)g_j(y)$. Using this, we can find an expression of $P^+(u_1, \dots, u_m)$ in which the condition (2.17) can be relaxed to (2.20). This relaxation is important when we take the limit $a+b \rightarrow 0$ in the next section.

Proposition 13. *Let $a, b \in (0, 1/2)$. Fix constants $\alpha_1, \dots, \alpha_m$ satisfying*

$$-\frac{1}{2} < \alpha_1 < \alpha_2 < \dots < \alpha_m < \frac{1}{2}. \quad (2.20)$$

Define the conjugated operator \bar{K}^{conj} by the kernel

$$\bar{K}_{i,j}^{\text{conj}}(x, y) = M_i(x) \bar{K}_{i,j}(x, y) M_j(y)^{-1}, \quad M_i(x) := e^{-\alpha_i x}. \quad (2.21)$$

Then

$$\begin{aligned} & P^+(u_1, \dots, u_m) \\ &= (1 - (a+b) \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u f, P_u g \rangle) \cdot \det \left(\mathbb{1} - P_u \bar{K}^{\text{conj}} P_u \right). \end{aligned} \quad (2.22)$$

where the notation \langle, \rangle denotes the real inner product in $L^2(\{1, \dots, m\} \times \mathbb{R}_+)$.

Remark 14. One can check that the expression (2.22) can be analytically extended to $a, b \in (-1/2, 1/2)$ with $a+b > 0$. Nevertheless since we will discuss the issue of extending the domain analyticity of a formula of $P(u_1, \dots, u_m)$ to $a, b \in (-1/2, 1/2)$ (with no restriction of $a+b > 0$) in the next section, we do not discuss the detail here.

Proof of Proposition 13. From (2.16),

$$P_u K^{\text{conj}} P_u = P_u \bar{K}^{\text{conj}} P_u + (a+b) (P_u f^{\text{conj}}) \otimes (g^{\text{conj}} P_u) \quad (2.23)$$

where f^{conj} and g^{conj} are multiplication operators by the functions $f^{\text{conj}}(i, x) = M_i(x)f_i(x)$, $g^{\text{conj}}(j, y) = g_j(y)M_j(y)^{-1}$. The functions $f^{\text{conj}}(i, x)$ and $g^{\text{conj}}(j, y)$ are $L^2(\mathbb{R})$: see the end of the proof of Proposition 34. For the convenience of notations, set $[K] = P_u K^{\text{conj}} P_u$, $[\overline{K}] = P_u \overline{K}^{\text{conj}} P_u$, $[f] = P_u f^{\text{conj}}$ and $[g] = g^{\text{conj}} P_u$. Thus

$$\begin{aligned} \det(\mathbb{1} - [K]) &= \det(\mathbb{1} - (a+b)(\mathbb{1} - [\overline{K}])^{-1}[f] \otimes [g]) \cdot \det(\mathbb{1} - [\overline{K}]), \\ &= (1 - (a+b)\langle(\mathbb{1} - [\overline{K}])^{-1}[f], [g]\rangle) \cdot \det(\mathbb{1} - [\overline{K}]). \end{aligned} \tag{2.24}$$

The above step holds assuming that $\mathbb{1} - [\overline{K}] = \mathbb{1} - P_u \overline{K}^{\text{conj}} P_u$ is invertible. The invertibility can be verified as follows. Consider the modification of the directed percolation model where $\widehat{w}_{i,j} = 0$ if i and/or j are equal to 0, and $\widehat{w}_{i,j} \sim \text{Exp}(1)$ for $i, j \geq 1$ (i.e., model without sources). Denote by \widehat{P} the new measure. This is the model with $a = b = 0$ in (2.1), and hence $\widehat{P}(u_1, \dots, u_m) = \det(\mathbb{1} - P_u \overline{K}^{\text{conj}} P_u)$ for any given $u_1, \dots, u_m > 0$. It is easy to obtain a lower bound $\widehat{P}(u_1, \dots, u_m) \geq (1 - e^{-\varepsilon})^{2t^2} > 0$, with $\varepsilon = \min\{u_1, \dots, u_m\}/2t$. Indeed, it is enough to take the configurations with $\widehat{w}_{i,j} \leq \min\{u_1, \dots, u_m\}/2t$ for $i, j \geq 1$ such that $i+j \leq 2t$. This implies that $\det(\mathbb{1} - P_u \overline{K}^{\text{conj}} P_u) \neq 0$, and hence $\mathbb{1} - P_u \overline{K}^{\text{conj}} P_u$ is invertible.

Finally, since $P_u \overline{K} P_u$ is a bounded operator in $L^2(\mathbb{R})$, and $P_u f, P_u g \in L^2(\mathbb{R})$, $\langle(\mathbb{1} - [\overline{K}])^{-1}[f], [g]\rangle$ equals $\langle(\mathbb{1} - P_u \overline{K} P_u)^{-1} P_u f, P_u g\rangle$ by conjugating back M . Now the condition (2.17) for the α_i 's can be relaxed to the condition (2.20) since $\overline{K}^{\text{conj}}$ is trace-class under this assumption: see Proposition 34. \square

3 Analytic continuation

We now find a formula extending (2.4) to the case when $a, b \in (-1/2, 1/2)$ without the restriction that $a + b > 0$. For this purpose, we show that both sides of (2.4) are analytic in the parameter $a, b \in (-1/2, 1/2)$. Analyticity of the left-hand-side of (2.4) i.e., of $P(u_1, \dots, u_m)$ is a straightforward generalization of Proposition 5.1 in [17]. This section is devoted to finding the analytic continuation of the right-hand-side of (2.4).

By using (2.22), the right-hand-side of (2.4) becomes

$$\left((a+b) + \sum_{k=1}^m \frac{\partial}{\partial u_k} \right) \left(\frac{1}{a+b} - \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u f, P_u g \rangle \right) \det (\mathbb{1} - P_u \bar{K}^{\text{conj}} P_u). \quad (3.1)$$

The last determinant is independent of a, b . It is enough show that

$$\frac{1}{a+b} - \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u f, P_u g \rangle \quad (3.2)$$

is analytically continued in $a, b \in (-1/2, 1/2)$. Note that by changing the contour,

$$f_i(x) = f_i^{(a)}(x) + f_i^{(1/2)}(x) \quad (3.3)$$

where

$$f_i^{(a)}(x) = \phi_i(a) e^{-ax}, \quad f_i^{(1/2)}(x) = \frac{-1}{2\pi i} \oint_{\Gamma_{1/2}} dw \frac{\phi_i(w) e^{-wx}}{a-w}. \quad (3.4)$$

Proposition 15 (Analytic continuation). *For $a, b \in (0, 1/2)$,*

$$\frac{1}{a+b} - \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u f, P_u g \rangle = - \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u F, P_u g \rangle + R_{a,b}. \quad (3.5)$$

where

$$R_{a,b} = \frac{1}{2\pi i} \oint_{\Gamma_{a,-b,-1/2}} dw \frac{\phi_1(a)}{\phi_1(w)} \frac{e^{-u_1(a-w)}}{(w-a)(w+b)}, \quad (3.6)$$

and $F((i, x)) = F_i(x)$ with

$$F_i(x) = f_i^{(1/2)}(x) + \int_{u_1}^{\infty} dy \tilde{K}_{i,1}(x, y) f_1^{(a)}(y) + \mathbb{1}_{[i \geq 2]} \int_{-\infty}^{u_1} dy V_{i,1}(x, y) f_1^{(a)}(y). \quad (3.7)$$

The term $R_{a,b}$ is analytic in $a, b \in (-1/2, 1/2)$, and

$$\langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u F, P_u g \rangle := \sum_{i=1}^m \int_{u_i}^{\infty} dx ((\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u F)((i, x)) g_i(x) \quad (3.8)$$

is convergent and is analytic in $a, b \in (-1/2, 1/2)$.

Hence the right-hand-side of (3.5) is an analytic continuation of (3.2) to $a, b \in (-\frac{1}{2}, \frac{1}{2})$. Combining (2.4), (3.1) and (3.5), we finally obtain the following representation of $P(u_1, \dots, u_n)$ defined in (2.3).

Theorem 16. Recall the conditions and definitions from (2.10) through (2.14) above. For $a, b \in (-1/2, 1/2)$,

$$\begin{aligned} & P(u_1, \dots, u_m) \\ &= \left(a + b + \sum_{k=1}^m \frac{\partial}{\partial u_k} \right) \left[(R_{a,b} - \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u F, P_u g \rangle) \det \left(\mathbb{1} - P_u \bar{K}^{\text{conj}} P_u \right) \right], \end{aligned} \quad (3.9)$$

for $u_1, \dots, u_m \in \mathbb{R}_+$, where $R_{a,b}$ and F are defined in Proposition 15.

Proof of Proposition 15. Part I: Decomposition. First we decompose the contribution coming from the pole at $1/2$ and a of f , namely

$$\begin{aligned} (3.2) &= \frac{1}{a+b} - \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u f^{(1/2)}, P_u g \rangle \\ &\quad - \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u f^{(a)}, P_u g \rangle. \end{aligned} \quad (3.10)$$

From Lemma 17 below, $P_u f^{(a)} = (\mathbb{1} + P_u V P_u) P_u F^{(a)} + P_u V (\mathbb{1} - P_u) F^{(a)}$, where $F^{(a)}$ is defined in (3.18). Here, $(V(\mathbb{1} - P_u) F^{(a)})(i, x)$ is well-defined pointwise and is in $L^2(\mathbb{R})$ as we can check as in the proof of Lemma 17. Hence using the identity (recall that $\bar{K} = \tilde{K} - V$ in (2.13))

$$(\mathbb{1} - P_u \bar{K} P_u)^{-1} (\mathbb{1} + P_u V P_u) = \mathbb{1} + (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u \tilde{K} P_u, \quad (3.11)$$

the last term in (3.10) becomes

$$\langle P_u F^{(a)}, P_u g \rangle + \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} (P_u \tilde{K} P_u + P_u V (\mathbb{1} - P_u)) F^{(a)}, P_u g \rangle. \quad (3.12)$$

Observe that the function $(\tilde{K} P_u + V(\mathbb{1} - P_u)) F^{(a)}((i, x))$ is precisely the last two terms in (3.7). Hence from (3.10), we obtain

$$(3.2) = -\langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u F, P_u g \rangle + \frac{1}{a+b} - \langle P_u F^{(a)}, P_u g \rangle. \quad (3.13)$$

Now a direct computation shows that

$$\begin{aligned} \langle P_u F^{(a)}, P_u g \rangle &= \int_{u_1}^{\infty} dx f_1^{(a)}(x) g_1(x) \\ &= \frac{\phi_1(a)}{2\pi i} \oint_{\Gamma_{-1/2, -b}} dw \frac{\phi_1^{-1}(w)}{w+b} \int_{u_1}^{\infty} dx e^{-ax} e^{wx} \\ &= \frac{\phi_1(a)}{2\pi i} \oint_{\Gamma_{-1/2, -b}} dw \frac{\phi_1(w)^{-1} e^{-u_1(a-w)}}{(w+b)(a-w)} \\ &= \frac{1}{2\pi i} \oint_{\Gamma_{-1/2, -b, a}} dw \frac{\phi_1(a)}{\phi_1(w)} \frac{e^{-u_1(a-w)}}{(w+b)(a-w)} + \frac{1}{a+b}. \end{aligned} \quad (3.14)$$

Here in the third equality, we used the fact that the contour $\Gamma_{-1/2, -b}$ can be made to be on the left of the point $w = a$ since $a, b > 0$. Hence (3.13) and (3.14) imply (3.5).

Part II: Analyticity. We now show that the functions on the right-hand-side of (3.5) are analytic in $a, b \in (-\frac{1}{2}, \frac{1}{2})$. Clearly, $R_{a,b}$ is analytic in $a, b \in (-1/2, 1/2)$ since both the poles $w = a$ and $w = -b$ lie inside the integration contour. We need to show that $\langle (\mathbb{1} - P_u \overline{K} P_u)^{-1} P_u F, P_u g \rangle$ is analytic. Note that \overline{K} , \tilde{K} and V are independent of a, b . As $f_1^{(a)}(x) = \phi_i(a) e^{-ax}$ is analytic in $a \in (-\frac{1}{2}, \frac{1}{2})$, $F_i(x)$ is analytic in a in the same domain. Also it is clear from the integral representation (2.12) that $g_i(y)$ is analytic in $b \in (-\frac{1}{2}, \frac{1}{2})$. Hence it is enough to show that $\langle (\mathbb{1} - P_u \overline{K} P_u)^{-1} P_u F, P_u g \rangle$ is well-defined for $a, b \in (-\frac{1}{2}, \frac{1}{2})$. Fix $\delta_0 \in (0, \frac{1}{2})$. Let $a, b \in [-\frac{1}{2} + \delta_0, \frac{1}{2} - \delta_0]$. Using the identity

$$(\mathbb{1} - P_u \overline{K} P_u)^{-1} P_u F = (\mathbb{1} + P_u \overline{K} P_u (\mathbb{1} - P_u \overline{K} P_u)^{-1}) P_u F, \quad (3.15)$$

and the estimates (3.23) and (3.25) in Lemma 18 below, we see that

$$|((\mathbb{1} - P_u \overline{K} P_u)^{-1} P_u F)(i, x)| \leq C e^{-(1/2 - \delta_0/2)x}, \quad x \geq u_i \quad (3.16)$$

for some constant $C > 0$. On the other hand, from (3.24),

$$|g_j(y)| \leq C e^{(1/2 - \delta_0)|y|}, \quad y \in \mathbb{R}, \quad (3.17)$$

for some constant $C > 0$. Therefore, the inner product is convergent, and the Proposition is obtained. \square

Lemma 17. *Let $a \in (0, \frac{1}{2})$. Define the function $F^{(a)}$ in $L^2(\{1, \dots, m\} \times \mathbb{R})$ by*

$$F_i^{(a)}(x) = f_1^{(a)}(x) \delta_{i,1}. \quad (3.18)$$

Then $VF^{(a)}$ defined by

$$(VF^{(a)})(i, x) = \int_{\mathbb{R}} dy V_{i,1}(x, y) f_1^{(a)}(y) \quad (3.19)$$

is well-defined for each $x \in \mathbb{R}$ and is in $L^2(\mathbb{R})$. Moreover,

$$f^{(a)} = F^{(a)} + VF^{(a)}. \quad (3.20)$$

Proof of Lemma 17. Recall from (3.4) that $f_1^{(a)}(x) = e^{-ax}\phi_1(a)$. The estimate (3.22) shows that the integral in (3.19) is well-defined. $(VF^{(a)})(i, x)$ is well-defined pointwise. Note that $V_{i,1}(x, y) = \hat{v}(x - y)$ for some function $v \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ (see (C.9)). Hence the integral in (3.19) equals $(\hat{v} * f_1^{(a)})(x)$. Hence the $L^2(\mathbb{R})$ norm of the integral is bounded by $\|\hat{v}\|_{L^1(\mathbb{R})} \|f_1^{(a)}\|_{L^2(\mathbb{R})}$. But as $|\hat{v}(x)| \leq Ce^{-(1/2-\delta)|x|}$ by (3.22), we find that the integral in (3.19) is in $L^2(\mathbb{R})$.

Now let $\eta_1 \in (0, a)$ and $\eta_2 \in (a, \frac{1}{2})$ be fixed. In the integral formula of $V_{i,1}(x, y)$ in (2.14), we can change the contour $i\mathbb{R}$ to either $i\mathbb{R} + \eta_1$ or $i\mathbb{R} + \eta_2$. We will use the contour $i\mathbb{R} + \eta_1$ when $y \geq 0$ and the contour $i\mathbb{R} + \eta_2$ when $y < 0$. Then for $i \geq 2$,

$$\begin{aligned} (VF^{(a)})(i, x) &= \frac{\phi_1(a)}{2\pi i} \left(\int_{i\mathbb{R} + \eta_1} dz \frac{e^{-zx} \phi_i(z)}{a - z \phi_1(z)} - \int_{i\mathbb{R} + \eta_2} dz \frac{e^{-zx} \phi_i(z)}{a - z \phi_1(z)} \right) \\ &= e^{-ax} \phi_i(a) = f_i^{(a)}(x) \end{aligned} \quad (3.21)$$

where the integral is evaluated using Cauchy's formula. Taking into account the case when $i = 1$, we obtain (3.20). \square

Lemma 18. For any $\delta \in (0, \frac{1}{2}]$, there exists a constant $C > 0$ such that

$$|V_{i,j}(x, y)| \leq Ce^{-(1/2-\delta)|x-y|} \mathbb{1}_{[i>j]}, \quad x, y \in \mathbb{R}, \quad (3.22)$$

and

$$|\overline{K}_{i,j}(x, y)| \leq Ce^{-(1/2-\delta)(x+y)}, \quad x, y > 0. \quad (3.23)$$

For any $b \in (-\frac{1}{2}, \frac{1}{2})$, there is a constant $C > 0$ such that

$$|g_j(y)| \leq Ce^{-by}, \quad y \in \mathbb{R}. \quad (3.24)$$

Finally, suppose that $a \in (-\frac{1}{2}, \frac{1}{2})$ is given. For any $\delta \in (0, (\frac{1}{2} + a)(\frac{1}{2} - a)]$, there is a constant $C > 0$ such that

$$|F_i(x)| \leq Ce^{-(1/2-\delta)x}, \quad x \geq u_i. \quad (3.25)$$

Here the constants are uniform if the parameters a, b, δ are in compact sets.

Proof of Lemma 18. When $y - x \geq 0$, by deforming the contour to $i\mathbb{R} - 1/2 + \delta$ in (2.14), we obtain for $i \geq j$,

$$\begin{aligned} |V_{i,j}(x, y)| &= e^{-(1/2-\delta)(y-x)} \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} ds e^{is(y-x)} \frac{\phi_i(-1/2 + \delta + is)}{\phi_j(-1/2 + \delta + is)} \right| \\ &\leq Ce^{-(1/2-\delta)(y-x)} \end{aligned} \quad (3.26)$$

since the integrand is absolutely convergent (recall that $t_i - t_j \geq 1$ for $i > j$). We obtain the bound (3.22) similarly by using the contour $i\mathbb{R} + 1/2 - \delta$ when $y - x \leq 0$.

The estimate (3.23) is easily obtained by taking the contours $\Gamma_{-1/2}$ and $\Gamma_{1/2}$ as the circles of radii of δ centered at $-1/2$ and $1/2$, respectively, in (2.13).

Recalling $g_j(y)$ (from (2.12)) we obtain the estimate (3.24) by evaluating the residue at $z = -b$ and making the remaining contour $\Gamma_{-1/2}$ small enough.

Finally, in order to estimate $F_i(x)$, first note that $|f_i^{(1/2)}(x)| \leq Ce^{-(1/2-\delta)x}$ by using the integral representation (3.4) with the contour given by the circle of radius δ centered at $1/2$. Also from (3.22) and (3.23), $|\tilde{K}_{i,1}(x, y)| \leq Ce^{-(1/2-\delta)(x+y)}$. Using these estimates and $|f_1^{(a)}(x)| \leq Ce^{-ax}$ in the definition (3.7),

$$F_i(x) = f_i^{(1/2)}(x) + \int_{u_1}^{\infty} dy \tilde{K}_{i,1}(x, y) f_1^{(a)}(y) + \mathbb{1}_{[i \geq 2]} \int_{-\infty}^{u_1} dy V_{i,1}(x, y) f_1^{(a)}(y), \quad (3.27)$$

we obtain (3.25) for x bounded below. In particular, to have the integrals over y bounded we need $0 < \delta < \frac{1}{2} + a$ for the integral with $\tilde{K}_{i,1}$ and $0 < \delta < \frac{1}{2} - a$ for the integral with $V_{i,1}$. Both conditions are satisfied for $0 < \delta < (\frac{1}{2} - a)(\frac{1}{2} + a)$. \square

4 Asymptotic analysis

By setting $a = -b = \rho - 1/2$ in Theorem 16, we obtain

$$\begin{aligned} & P(u_1, \dots, u_n) \\ &= \sum_{k=1}^m \frac{\partial}{\partial u_k} \left[(R_{a,-a} - \langle (\mathbb{1} - P_u \bar{K} P_u)^{-1} P_u F, P_u g \rangle) \det \left(\mathbb{1} - P_u \bar{K}^{\text{conj}} P_u \right) \right]. \end{aligned} \quad (4.1)$$

We now begin asymptotic analysis of this formula.

To obtain our main theorem (Theorem 2) we need to consider the following scaling limit. Fix $\rho \in (0, 1)$. Set $\chi = \rho(1 - \rho)$, $b = 1/2 - \rho$, and $a = \rho - 1/2$. For a large parameter T , according to (1.5) and (2.10), we

consider

$$\begin{aligned} t &= \frac{1-2\chi}{2}T, & t_i &= \frac{1-2\rho}{2}T + \tau_i \frac{2\chi^{4/3}}{1-2\chi}T^{2/3}, \\ u_i &= T - \tau_i \frac{2(1-2\rho)\chi^{1/3}}{1-2\chi}T^{2/3} + s_i \frac{T^{1/3}}{\chi^{1/3}}, \end{aligned} \quad (4.2)$$

where we order $\tau_1 < \tau_2 < \dots < \tau_m$ with $t_i \in [-t, t]$ for all i . The convergence of the Fredholm determinants is ensured only after (yet another) proper conjugation. For this purpose, set

$$A(i) = Z(i) \exp(2\tau_i^3/3 + \tau_i s_i), \quad Z(i) = \phi_i(a) e^{-au_i}. \quad (4.3)$$

Proposition 19. *Consider the scaling (4.2). Then*

$$\lim_{T \rightarrow \infty} \left(\frac{T}{\chi} \right)^{1/3} \frac{A(j)}{A(i)} \overline{K}_{i,j}(u_i, u_j) = [\widehat{K}_{\text{Ai}}]_{i,j}(s_i, s_j), \quad (4.4)$$

uniformly for s_i, s_j in a bounded set. The operator \widehat{K}_{Ai} is defined in (1.10).

Proof of Proposition 19. Recall that $\overline{K}_{i,j} = \widetilde{K}_{i,j} - V_{i,j}$, with $V_{i,j} = 0$ for $\tau_i \leq \tau_j$. The same structure holds for \widehat{K}_{Ai} . Indeed, using the identity (A.7) of Lemma 32, we can rewrite (1.10) as follows,

$$\begin{aligned} [\widehat{K}_{\text{Ai}}]_{i,j}(s_i, s_j) &= \int_0^\infty d\lambda \text{Ai}(s_i + \lambda + \tau_i^2) \text{Ai}(s_j + \lambda + \tau_j^2) e^{-\lambda(\tau_j - \tau_i)} \\ &\quad - \frac{\exp\left(-\frac{(s_i - s_j)^2}{4(\tau_i - \tau_j)} + \frac{2}{3}(\tau_j^3 - \tau_i^3) + (\tau_j s_j - \tau_i s_i)\right)}{\sqrt{4\pi(\tau_i - \tau_j)}} \mathbb{1}(\tau_i > \tau_j) \end{aligned} \quad (4.5)$$

The proof is divided into the convergence for $V_{i,j}$ in Lemma 20 and of $\widetilde{K}_{i,j}$ in Lemma 22 below. \square

Lemma 20. *Consider the scaling (4.2) and $i > j$. Then*

$$\begin{aligned} &\left(\frac{T}{\chi} \right)^{1/3} \frac{A(j)}{A(i)} V_{i,j}(u_i, u_j) \\ &= \frac{\exp\left(-\frac{(s_i - s_j)^2}{4(\tau_i - \tau_j)} + \frac{2}{3}(\tau_j^3 - \tau_i^3) + (\tau_j s_j - \tau_i s_i)\right)}{\sqrt{4\pi(\tau_i - \tau_j)}} + \mathcal{O}(T^{-1/3}) \end{aligned} \quad (4.6)$$

uniform for $s_i - s_j$ in a bounded set.

Proof of Lemma 20. Recall that $\tau_i > \tau_j$. We derive the asymptotics by saddle point analysis. Set

$$\begin{aligned} g_0(\tilde{z}) &:= (\tau_i - \tau_j) \frac{2\chi^{1/3}}{1 - 2\chi} \left[(1 - 2\rho)\tilde{z} - \chi \ln\left(\frac{1}{4} - \tilde{z}^2\right) \right], \\ g_1(\tilde{z}) &:= -\tilde{z}(s_i - s_j)\chi^{-1/3}, \\ g(z) &:= g_0(z) + T^{-1/3}g_1(z). \end{aligned} \tag{4.7}$$

Then by plugging (4.2) into (2.14),

$$\begin{aligned} V_{i,j}(u_i, u_j) &= \frac{1}{2\pi i} \int_{i\mathbb{R}} d\tilde{z} \exp(T^{2/3}g(\tilde{z})) \\ &= \frac{1}{2\pi i} \int_{i\mathbb{R}} d\tilde{z} \exp(T^{2/3}g_0(\tilde{z}) + T^{1/3}g_1(\tilde{z})). \end{aligned} \tag{4.8}$$

There is a unique critical point for g_0 in the interval $(-1/2, 1/2)$, namely

$$\tilde{z}_c = a = \rho - 1/2. \tag{4.9}$$

A straightforward computation gives

$$g_0''(\tilde{z}) = (\tau_i - \tau_j) \frac{2\chi^{4/3}}{1 - 2\chi} \left(\frac{1}{(1/2 + \tilde{z})^2} + \frac{1}{(1/2 - \tilde{z})^2} \right), \tag{4.10}$$

from which $g_0''(\tilde{z}_c) = 2(\tau_i - \tau_j)\chi^{-2/3} > 0$. Also observe that

$$e^{T^{2/3}g(a)} = e^{T^{2/3}g_0(a) + T^{1/3}g_1(a)} = \frac{Z(i)}{Z(j)}. \tag{4.11}$$

For the saddle point analysis we use the contour $\tilde{\gamma} := \{\tilde{z}_c + it \mid t \in \mathbb{R}\}$. First let us show that the contribution coming from $|t| > \delta > 0$ is negligible in the $T \rightarrow \infty$ limit. Denote by $\tilde{\gamma}_\delta^c := \{z \in \tilde{\gamma} \mid |\operatorname{Im}(z)| > \delta\}$. Then we have

$$\begin{aligned} I_\delta^c &:= \left| \frac{Z(j)}{Z(i)} \left(\frac{T}{\chi}\right)^{1/3} \frac{1}{2\pi i} \int_{\tilde{\gamma}_\delta^c} d\tilde{z} \exp(T^{2/3}g(\tilde{z})) \right| \\ &\leq \left(\frac{T}{\chi}\right)^{1/3} \frac{1}{\pi i} \int_\delta^\infty dt \exp(\operatorname{Re}[T^{2/3}g(\tilde{z}_c + it) - T^{2/3}g(a)]) \\ &= \left(\frac{T}{\chi}\right)^{1/3} \frac{1}{\pi} \int_\delta^\infty dt \frac{1}{(|1 - it/(1/2 - \tilde{z}_c)| |1 + it/(1/2 + \tilde{z}_c)|)^\eta} \\ &= \left(\frac{T}{\chi}\right)^{1/3} \frac{1}{\pi} \int_\delta^\infty dt \frac{1}{(1 + t^2/\rho^2)^{\eta/2} (1 + t^2/(1 - \rho)^2)^{\eta/2}} \end{aligned} \tag{4.12}$$

with $\eta := t_i - t_j = (\tau_i - \tau_j) \frac{2\chi^{4/3}}{1-2\chi} T^{2/3} \geq 1$ when T is large enough. For $\rho \geq 1/2$, we have the bound

$$\frac{1}{(1 + t^2/\rho^2)^{\eta/2} (1 + t^2/(1-\rho)^2)^{\eta/2}} \leq \frac{1}{(1 + t^2/\rho^2)^\eta}. \quad (4.13)$$

Hence using the linear lower bound

$$1 + (t/\rho)^2 \geq (1 + \delta^2/\rho^2)(1 + (t-\delta)2\delta/(\delta^2 + \rho^2)) \quad (4.14)$$

for all t, δ , we have

$$\int_\delta^\infty dt \frac{1}{(1 + t^2/\rho^2)^\eta} \leq (1 + \delta^2/\rho^2)^{-\eta} \frac{\delta^2 + \rho^2}{2\delta(\eta - 1)}. \quad (4.15)$$

Therefore there exist a constant $\mu = \mu(\delta) > 0$ and a constant $C = C(\mu) > 0$ such that

$$I_\delta^c \leq C \frac{\exp(-\mu(\delta)T^{2/3})}{T^{1/3}}. \quad (4.16)$$

When $\rho \leq 1/2$, we obtain the same estimate by just replacing in some of the bounds ρ by $1 - \rho$.

Next we determine the contribution from a δ -neighborhood of the critical point. Noting the Taylor series

$$\begin{aligned} g_0(\tilde{z}) &= g_0(\tilde{z}_c) + (\tau_i - \tau_j)\chi^{-2/3}(\tilde{z} - \tilde{z}_c)^2(1 + \mathcal{O}(\tilde{z} - \tilde{z}_c)), \\ g_1(\tilde{z}) &= g_1(\tilde{z}_c) - (s_i - s_j)\chi^{-1/3}(\tilde{z} - \tilde{z}_c), \end{aligned} \quad (4.17)$$

we find that

$$\begin{aligned} & \left| g(z) - g(\tilde{z}_c) - (\tau_i - \tau_j)\chi^{-2/3}(\tilde{z} - \tilde{z}_c)^2 + \frac{(s_i - s_j)(\tilde{z} - \tilde{z}_c)}{\chi^{1/3}T^{1/3}} \right| \\ & \leq \sup_{t \in B(\tilde{z}_c, |\tilde{z} - \tilde{z}_c|)} \left| \frac{g_0^{(3)}(t)}{3!} \right| |\tilde{z} - \tilde{z}_c|^3 \\ & \leq (\tau_i - \tau_j) \frac{4\chi^{4/3}}{3!(1-2\chi)} \left(\frac{1}{(1-\rho - |\tilde{z} - \tilde{z}_c|)^3} + \frac{1}{(\rho - |\tilde{z} - \tilde{z}_c|)^3} \right) |\tilde{z} - \tilde{z}_c|^3. \end{aligned} \quad (4.18)$$

Thus by choosing δ small enough, we find that for $|\tilde{z} - \tilde{z}_c| \leq \delta$,

$$(4.18) \leq \frac{\tau_i - \tau_j}{2\chi^{2/3}} |\tilde{z} - \tilde{z}_c|^2, \quad (4.19)$$

and also

$$(4.18) \leq C|\tilde{z} - \tilde{z}_c|^3, \quad (4.20)$$

for some constant $C > 0$. Using (4.19), (4.20) and the general identity

$$|e^z - e^w| \leq |z - w| \max\{|e^z|, |e^w|\}, \quad (4.21)$$

we deduce that

$$\begin{aligned} & \left| \frac{Z(j)}{Z(i)} \left(\frac{T}{\chi} \right)^{1/3} \frac{1}{2\pi i} \int_{\tilde{\gamma} \setminus \tilde{\gamma}_\delta^c} dz e^{T^{2/3}g(z)} - \frac{1}{2\pi} \int_{-\delta(T/\chi)^{1/3}}^{\delta(T/\chi)^{1/3}} e^{-Y^2(\tau_i - \tau_j) - iY(s_i - s_j)} dY \right| \\ &= \left| \frac{1}{2\pi} \int_{-\delta(T/\chi)^{1/3}}^{\delta(T/\chi)^{1/3}} dY \left(e^{T^{2/3}(g(a+iY(T/\chi)^{-1/3}) - g(a))} - e^{-Y^2(\tau_i - \tau_j) - iY(s_i - s_j)} \right) \right| \\ &\leq \frac{C\chi}{2\pi T^{1/3}} \int_{-\delta(T/\chi)^{1/3}}^{\delta(T/\chi)^{1/3}} dY Y^3 e^{-\frac{1}{2}Y^2(\tau_i - \tau_j)} = \mathcal{O}(T^{-1/3}). \end{aligned} \quad (4.22)$$

Finally, since

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_{-\delta(T/\chi)^{1/3}}^{\delta(T/\chi)^{1/3}} e^{-Y^2(\tau_i - \tau_j) - iY(s_i - s_j)} dY - \frac{1}{\sqrt{4\pi(\tau_i - \tau_j)}} \exp\left(-\frac{(s_i - s_j)^2}{4(\tau_i - \tau_j)}\right) \right| \\ &= \left| \frac{1}{2\pi} \int_{-\delta(T/\chi)^{1/3}}^{\delta(T/\chi)^{1/3}} e^{-Y^2(\tau_i - \tau_j) - iY(s_i - s_j)} dY - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-Y^2(\tau_i - \tau_j) - iY(s_i - s_j)} dY \right| \\ &\leq e^{-cT^{2/3}} \end{aligned} \quad (4.23)$$

for some constant $c > 0$, which is uniform for $s_i - s_j$ in a bounded set, we obtain (4.6) from (4.16) and (4.22). \square

Before discussing the asymptotic of \tilde{K} we obtain also an exponential bound of $V_{i,j}(u_i, u_j)$ when $s_i, s_j \rightarrow +\infty$, which will be used later.

Lemma 21. *Consider the scaling (4.2) and $i > j$ (i.e., $\tau_i > \tau_j$). Then, for any given $\kappa > 0$, there exists a $T_0 > 0$ large enough such that*

$$\left| \left(\frac{T}{\chi} \right)^{1/3} \frac{A(j)}{A(i)} V_{i,j}(u_i, u_j) \right| \leq C e^{-(\tau_i - \tau_j)(s_i + s_j)/2} e^{-\kappa|s_i - s_j|} \quad (4.24)$$

for all $s_i, s_j \in \mathbb{R}$ and $T \geq T_0$. The constant C is uniform in s_i, s_j and in $T \geq T_0$.

Proof of Lemma 21. Set

$$z_c := \begin{cases} a + \frac{(s_i - s_j)\chi^{1/3}}{2(\tau_i - \tau_j)T^{1/3}}, & \text{if } |s_i - s_j| \leq \varepsilon T^{1/3}, \\ a + \frac{\varepsilon\chi^{1/3}}{2(\tau_i - \tau_j)}, & \text{if } s_i - s_j > \varepsilon T^{1/3}, \\ a - \frac{\varepsilon\chi^{1/3}}{2(\tau_i - \tau_j)}, & \text{if } s_i - s_j < -\varepsilon T^{1/3}, \end{cases} \quad (4.25)$$

where $\varepsilon > 0$ is a fixed constant chosen small enough so that z_c lies in a compact subset of $(-1/2, 1/2)$. Taking the integration path as $\gamma := \{z_c + it \mid t \in \mathbb{R}\}$, a computation as in (4.12) yields (the function $g(z)$ is defined in (4.7))

$$\begin{aligned} \left| \frac{Z(j)}{Z(i)} V_{i,j}(u_i, u_j) \right| &= \left| \frac{1}{2\pi i} \int_{\gamma} d\tilde{z} \exp(T^{2/3}g(\tilde{z}) - T^{2/3}g(a)) \right| \\ &\leq e^{T^{2/3}g(z_c) - T^{2/3}g(a)} \frac{1}{2\pi} \int_{\mathbb{R}} dt \frac{1}{(|1 - it/(1/2 - z_c)| |1 + it/(1/2 + z_c)|)^\eta} \end{aligned} \quad (4.26)$$

where $\eta = (\tau_i - \tau_j) \frac{2\chi^{4/3}}{1-2\chi} T^{2/3} \geq 1$ when T is large enough. Hence

$$\begin{aligned} (4.26) &\leq e^{T^{2/3}g(z_c) - T^{2/3}g(a)} \frac{1}{\pi} \int_0^\infty \frac{dt}{(1 + t^2/(1/2 - |z_c|)^2)^\eta} \\ &= e^{T^{2/3}g(z_c) - T^{2/3}g(a)} \frac{1}{\pi} \left(\frac{1}{2} - |z_c| \right) \int_0^\infty \frac{dt}{(1 + t^2)^\eta}. \end{aligned} \quad (4.27)$$

The last integral satisfies

$$\int_0^\infty \frac{dt}{(1 + t^2)^\eta} = \frac{1}{\sqrt{\eta}} \int_0^\infty \left(1 + \frac{s^2}{\eta} \right)^{-\eta} ds \sim \frac{1}{\sqrt{\eta}} \int_0^\infty e^{-s^2} ds \quad (4.28)$$

as $\eta = \mathcal{O}(T^{2/3}) \rightarrow \infty$. Therefore we find that there is a constant $C > 0$ such that for T large enough,

$$\left| \left(\frac{T}{\chi} \right)^{1/3} \frac{Z(j)}{Z(i)} V_{i,j}(u_i, u_j) \right| \leq C \cdot \exp(T^{2/3}(g(z_c) - g(a))). \quad (4.29)$$

Now from (4.18) and (4.19) (with $z = z_c$ and $\tilde{z}_c = a$), if we have taken ε small enough, then

$$g(z_c) - g(a) \leq \frac{3(\tau_i - \tau_j)}{2\chi^{2/3}} (z_c - a)^2 - \frac{s_i - s_j}{\chi^{1/3} T^{1/3}} (z_c - a). \quad (4.30)$$

Now we plugging the value of z_c in (4.25) for three difference cases. When $|s_i - s_j| \leq \varepsilon T^{1/3}$,

$$g(z_c) - g(a) \leq -\frac{(s_i - s_j)^2}{8(\tau_i - \tau_j)T^{2/3}}. \quad (4.31)$$

When $s_i - s_j > \varepsilon T^{1/3}$, then

$$g(z_c) - g(a) \leq -\frac{\varepsilon(4(s_i - s_j) - 3\varepsilon T^{1/3})}{8(\tau_i - \tau_j)T^{1/3}} \leq -\frac{\varepsilon(s_i - s_j)}{8(\tau_i - \tau_j)T^{1/3}}. \quad (4.32)$$

When $s_i - s_j < -\varepsilon T^{1/3}$, then

$$g(z_c) - g(a) \leq -\frac{\varepsilon(4|s_i - s_j| - 3\varepsilon T^{1/3})}{8(\tau_i - \tau_j)T^{1/3}} \leq -\frac{\varepsilon|s_i - s_j|}{8(\tau_i - \tau_j)T^{1/3}}. \quad (4.33)$$

Therefore, we obtain

$$\left| \left(\frac{T}{\chi} \right)^{1/3} \frac{Z(j)}{Z(i)} V_{i,j}(u_i, u_j) \right| \leq \begin{cases} C \cdot \exp\left(-\frac{(s_i - s_j)^2}{8(\tau_i - \tau_j)}\right), & |s_i - s_j| \leq \varepsilon T^{1/3}, \\ C \cdot \exp\left(-\frac{T^{1/3}\varepsilon|s_i - s_j|}{8(\tau_i - \tau_j)}\right), & |s_i - s_j| > \varepsilon T^{1/3}. \end{cases} \quad (4.34)$$

Therefore, for any given $\kappa' > 0$, by taking $\varepsilon > 0$ small enough but fixed and then taking T_0 large enough, there is a constant $C > 0$ such that for all $T \geq T_0$,

$$\left| \left(\frac{T}{\chi} \right)^{1/3} \frac{Z(j)}{Z(i)} V_{i,j}(u_i, u_j) \right| \leq C \cdot \exp(-\kappa'|s_i - s_j|), \quad (4.35)$$

and hence

$$\left| \left(\frac{T}{\chi} \right)^{1/3} \frac{A(j)}{A(i)} V_{i,j}(u_i, u_j) \right| \leq C \exp(-\kappa'|s_i - s_j| + \tau_j s_j - \tau_i s_i). \quad (4.36)$$

Finally, given any $\kappa > 0$, by take $\kappa' \geq \kappa + \max\{|\tau_1|, \dots, |\tau_m|\}$. Then when $s_i - s_j \geq 0$, then

$$\begin{aligned} \tau_j s_j - \tau_i s_i + (\kappa - \kappa')|s_i - s_j| &= -(\tau_i - \tau_j)s_i + \tau_j(s_i - s_j) + (\kappa - \kappa')|s_i - s_j| \\ &\leq -(\tau_i - \tau_j)s_i \leq -\frac{1}{2}(\tau_i - \tau_j)(s_i + s_j) \end{aligned} \quad (4.37)$$

since $\tau_i - \tau_j > 0$. Similarly, when $s_i - s_j < 0$, then

$$\begin{aligned} \tau_j s_j - \tau_i s_i + (\kappa - \kappa')|s_i - s_j| &= -(\tau_i - \tau_j)s_j + \tau_i(s_i - s_j) + (\kappa - \kappa')|s_i - s_j| \\ &\leq -(\tau_i - \tau_j)s_j \leq -\frac{1}{2}(\tau_i - \tau_j)(s_i + s_j). \end{aligned} \tag{4.38}$$

This implies (4.24). \square

We now prove an asymptotic result for \tilde{K} .

Lemma 22. *Consider the scaling (4.2). Then,*

$$\begin{aligned} \left(\frac{T}{\chi}\right)^{1/3} \frac{A(j)}{A(i)} \tilde{K}_{i,j}(u_i, u_j) \\ = \int_0^\infty d\lambda \operatorname{Ai}(\lambda + s_i + \tau_i^2) \operatorname{Ai}(\lambda + s_j + \tau_j^2) e^{-\lambda(\tau_j - \tau_i)} + \mathcal{O}(T^{-1/3}) \end{aligned} \tag{4.39}$$

uniformly for s_i, s_j in a bounded set.

Proof of Lemma 22. By the definition (2.14),

$$\left(\frac{T}{\chi}\right)^{1/3} \tilde{K}_{i,j}(u_i, u_j) = \left(\frac{T}{\chi}\right)^{1/3} \frac{-1}{(2\pi i)^2} \oint_{\Gamma_{-1/2}} dz \oint_{\Gamma_{1/2}} dw \frac{e^{-wu_i} \phi_i(w)}{e^{-zu_j} \phi_j(z)} \frac{1}{w - z}. \tag{4.40}$$

The steepest-descent analysis of integrals very similar to this one with the same scaling (4.2) has been performed repeatedly in various places (see for example, [2, 6, 11, 17, 19]). The only difference here is that we have a double integral and we have to make sure that the two paths do not touch. However this can be easily handled by locally modifying the steep(est)-descent contours near the critical point. Except for this modification, the analysis of our case is similar to those in the literature. Nevertheless we provide the proof for the completeness of the paper, and also since the analysis of this Lemma and Lemma 23 is a prototype of all the other Lemmas in this section below (except for Lemma 27).

The first step is to determine a steep descent path for $\phi_i(w)e^{-wu_i}$ and $\phi_j(w)^{-1}e^{zu_j}$. Let us write

$$\phi_i(w)e^{-wu_i} = \exp(T h_0(w) + T^{2/3} h_{1,i}(w) + T^{1/3} h_{2,i}(w)), \tag{4.41}$$

where

$$\begin{aligned}
h_0(w) &= -w + \rho^2 \ln(1/2 + w) - (1 - \rho)^2 \ln(1/2 - w), \\
h_{1,i}(w) &= \tau_i \frac{2\chi^{1/3}}{1 - 2\chi} [(1 - 2\rho)w - \chi \ln(1/4 - w^2)], \\
h_{2,i}(w) &= -s_i w \chi^{-1/3}.
\end{aligned} \tag{4.42}$$

Note that $h_0(w)$ is independent of i . For the steep descent analysis we need to determine a steep descent path for h_0 . The steep descent path will always be taken symmetric with respect to complex conjugation. Thus we can restrict the discussion below to the part lying in the upper half plane. We have

$$h'_0(w) = -1 + \frac{\rho^2}{1/2 + w} + \frac{(1 - \rho)^2}{1/2 - w}, \quad h''_0(w) = -\frac{\rho^2}{(1/2 + w)^2} + \frac{(1 - \rho)^2}{(1/2 - w)^2}, \tag{4.43}$$

which both vanishes at the critical point $w_c = a = \rho - 1/2$, and $h'''_0(w_c) = 2/\chi$.

Let $\gamma'_\alpha = \{w = \frac{1}{2} - \alpha e^{i\theta}, \theta \in [\pi/2, 3\pi/2]\}$ with $\alpha > 0$. Then, for $\theta \in [\pi/2, \pi)$, $\text{Re}(h_0(w))$ is strictly decreasing in θ . Indeed,

$$\frac{d}{d\theta} \text{Re}(h_0(\frac{1}{2} - \alpha e^{i\theta})) = -\alpha \sin(\theta) \left(1 - \frac{\rho^2}{|w + 1/2|^2}\right) < 0, \tag{4.44}$$

since the last parenthesis is strictly positive: for $\theta \in [\pi/2, \pi)$, $\text{Re}(w) \geq 1/2$ and $|w + 1/2| \geq 1$, while $\rho < 1$. In the neighborhood of the critical point we consider a second path, $\gamma_1 = \{w = \rho - 1/2 + e^{-i\pi/3}t, t \in [0, 2(1 - \rho)]\}$. Along that path, we have

$$\frac{d}{dt} \text{Re}(h_0(w)) = -\frac{t^2(2\rho(1 - \rho) + (1 - 2\rho)t + t^2)}{2|1/2 - w|^2|1/2 + w|^2}. \tag{4.45}$$

The second degree term, $2\rho(1 - \rho) + (1 - 2\rho)t + t^2$, is strictly positive for all $\rho \in (0, 1)$ and $t \in [0, 2(1 - \rho)]$. Therefore also γ_1 is a steep descent path, and close to the critical point will be steepest descent.

Now we can define the steep descent path used in the analysis. Let $\gamma = \gamma_1 \cup \gamma'_{\sqrt{3}(1-\rho)} \cup \bar{\gamma}_1$. In a similar way, one obtains a steep descent path for $-h_0(z)$, namely, Γ is the path obtained by rotation around the origin of the steep descent path γ but with $1 - \rho$ instead of ρ . Finally, as we shall discuss below, any local modification of the contours in a region of order $T^{-1/3}$ around the critical point is allowed, see Figure 4, so to have $|z - w| \geq \varepsilon T^{-1/3}$ for a fixed $\varepsilon > 0$.

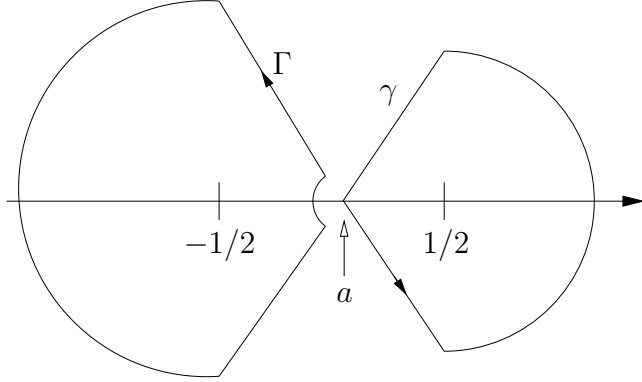


Figure 4: The modified contours γ and Γ are both anticlockwise oriented. The modification of Γ close to the critical point a is of order $T^{-1/3}$.

The path γ and Γ are steep descent path for $h_0(w)$ and $-h_0(z)$. Therefore, for any given small $\delta > 0$, the contribution to the double integral (4.40) coming from $\gamma \times \Gamma \setminus \{|z-a| \leq \delta, |w-a| \leq \delta\}$ is only of order $Z(i)/Z(j)\mathcal{O}(e^{-\mu T})$ for some $\mu = \mu(\delta) > 0$ (with $\mu \sim \delta^3$ for small δ), which is smaller than $Z(i)/Z(j)\mathcal{O}(T^{-1/3})$. Note that $Z(i)/Z(j)$ is the value of the integrand at the critical point.

Next we analyze the contribution coming from a δ -neighborhood of the critical point $w_c = a$. There we can use Taylor series expansion of $h_0, h_{1,i}, h_{2,i}$, which are given by

$$\begin{aligned} h_0(w) &= h_0(w_c) + \frac{1}{3}\chi^{-1}(w - w_c)^3 + \mathcal{O}((w - w_c)^4), \\ h_{1,i}(w) &= h_{1,i}(w_c) + \tau_i\chi^{-2/3}(w - w_c)^2 + \mathcal{O}(\tau_i(w - w_c)^3), \\ h_{2,i}(w) &= h_{2,i}(w_c) - s_i\chi^{-1/3}(w - w_c). \end{aligned} \quad (4.46)$$

Therefore, the main contribution to (4.40) is given by

$$\begin{aligned} \frac{Z(i)}{Z(j)} \left(\frac{T}{\chi}\right)^{1/3} \frac{-1}{(2\pi i)^2} \int_{\Gamma_\delta} dz \int_{\gamma_\delta} dw \frac{e^{T(w-w_c)^3/3\chi + T^{2/3}\tau_i(w-w_c)^2/\chi^{2/3} - T^{1/3}s_i(w-w_c)/\chi^{1/3}}}{e^{T(z-w_c)^3/3\chi + T^{2/3}\tau_j(z-w_c)^2/\chi^{2/3} - T^{1/3}s_j(z-w_c)/\chi^{1/3}}} \\ \times \frac{e^{\mathcal{O}(T(w-w_c)^4, \tau_i T^{2/3}(w-w_c)^3)}}{e^{\mathcal{O}(T(z-w_c)^4, \tau_j T^{2/3}(z-w_c)^3)}} \frac{1}{w - z}. \end{aligned} \quad (4.47)$$

where γ_δ and Γ_δ are the pieces of γ and Γ which lies in a δ -neighborhood of the critical point $w_c = a$. More precisely, setting $H_i := h_0 + T^{-1/3}h_{1,i} + T^{-2/3}h_{2,i}$ one has that

$$\begin{aligned} & \left| TH_i(w_c + t) - TH_i(w_c) - Th_0^{(3)}(w_c)\frac{t^3}{3!} - T^{2/3}h_{1,i}''(w_c)\frac{t^2}{2!} - T^{1/3}\frac{s_i}{\chi^{1/3}}t \right| \\ & \leq T \sup_{B(w_c,|t|)} |h_0^{(4)}(w)| |t^4|/4! + T^{2/3} \sup_{B(w_c,|t|)} |h_{1,i}^{(3)}(w)| |t^3|/3!. \end{aligned} \quad (4.48)$$

Assume that $0 < \eta \leq \frac{1}{2} \min\{\rho, 1 - \rho\}$, then

$$\sup_{B(w_c, \eta)} |h_0^{(4)}(w)| \leq 6.2^4(\rho^{-2} + (1 - \rho)^{-2}) \quad (4.49)$$

and

$$\sup_{B(w_c, \eta)} |h_{1,i}^{(3)}(w)| \leq |\tau_i| \frac{32\chi^{4/3}}{1 - 2\chi} (\rho^{-3} + (1 - \rho)^{-3}). \quad (4.50)$$

Thus, it is an easy computation to check that one can find $\eta > 0$ small enough so that

$$\eta^4 \sup_{B(w_c, \eta)} |h_0^{(4)}(w)| \leq h_0^{(3)}(w_c) \frac{\eta^3}{3!} \quad \text{and} \quad \sup_{B(w_c, \eta)} |h_{1,i}^{(3)}(w)| \eta^3 \leq |h_{1,i}''(w_c)| \frac{\eta^2}{2!}. \quad (4.51)$$

Assume that $0 < \delta < \eta$. Then it is easy to show as in (4.22), Lemma 20 that

$$\begin{aligned} & \left| \frac{Z(i)}{Z(j)} \left(\frac{T}{\chi} \right)^{1/3} \frac{-1}{(2\pi i)^2} \int_{\Gamma_\delta} dz \int_{\gamma_\delta} dw e^{TH_i(w) - TH_j(z)} \frac{1}{w - z} \right. \\ & \quad \left. - \frac{Z(i)}{Z(j)} \left(\frac{T}{\chi} \right)^{1/3} \frac{-1}{(2\pi i)^2} \int_{\Gamma_\delta} dz \int_{\gamma_\delta} dw \frac{e^{\frac{T(w-w_c)^3}{3\chi} + T^{2/3} \frac{\tau_i(w-w_c)^2}{\chi^{2/3}} - T^{1/3} \frac{s_i(w-w_c)}{\chi^{1/3}}}}{e^{\frac{T(z-w_c)^3}{3\chi} + T^{2/3} \frac{\tau_j(z-w_c)^2}{\chi^{2/3}} - T^{1/3} \frac{s_j(z-w_c)}{\chi^{1/3}}}} \frac{1}{w - z} \right| \\ & \leq \mathcal{O}(T^{-1/3}). \end{aligned} \quad (4.52)$$

It thus remains us only to determine the asymptotic of (4.52). We make the change of variables $W = (w - w_c)(T/\chi)^{1/3}$ and $Z = (z - w_c)(T/\chi)^{1/3}$ and obtain

$$\frac{Z(i)}{Z(j)} \frac{-1}{(2\pi i)^2} \int_{e^{-2\pi i/3}\delta(T/\chi)^{1/3}}^{e^{2\pi i/3}\delta(T/\chi)^{1/3}} dZ \int_{e^{\pi i/3}\delta(T/\chi)^{1/3}}^{e^{-\pi i/3}\delta(T/\chi)^{1/3}} dW \frac{e^{W^3/3 + \tau_i W^2 - s_i W}}{e^{Z^3/3 + \tau_j Z^2 - s_j Z}} \frac{1}{W - Z} \quad (4.53)$$

where the integration paths do not touch. We can now extend the integration path to infinity, since

$$\begin{aligned} & \left| \frac{Z(i)}{Z(j)} \frac{-1}{(2\pi i)^2} \int_{e^{-2\pi i/3}\infty}^{e^{2\pi i/3}\infty} dZ \int_{e^{\pi i/3}\infty}^{e^{-\pi i/3}\infty} dW \frac{e^{W^3/3+\tau_i W^2-s_i W}}{e^{Z^3/3+\tau_j Z^2-s_j Z}} \frac{1}{W-Z} \right| \\ & \frac{Z(i)}{Z(j)} \frac{-1}{(2\pi i)^2} \int_{e^{-2\pi i/3\delta(T/\chi)^{1/3}}\infty}^{e^{2\pi i/3\delta(T/\chi)^{1/3}}\infty} dZ \int_{e^{\pi i/3\delta(T/\chi)^{1/3}}\infty}^{e^{-\pi i/3\delta(T/\chi)^{1/3}}\infty} dW \frac{e^{W^3/3+\tau_i W^2-s_i W}}{e^{Z^3/3+\tau_j Z^2-s_j Z}} \frac{1}{W-Z} \Big| \\ & \leq C e^{-\delta^3 T/(6\chi)}. \end{aligned} \tag{4.54}$$

The conjugation by $A(j)/A(i)$ and the identity (A.2) end the proof. \square

Now we give a bound which holds uniformly for s_i, s_j bounded from below. Let $s_0 \in \mathbb{R}$ be given.

Lemma 23. *Consider the scaling (4.2). Then, for any given $\kappa > 0$, there exists a $T_0 > 0$ large enough such that for all $T \geq T_0$ and $s_i, s_j \geq s_0$*

$$\left| \left(\frac{T}{\chi} \right)^{1/3} \frac{A(j)}{A(i)} \tilde{K}_{i,j}(u_i, u_j) \right| \leq C e^{-\kappa(s_i+s_j)}. \tag{4.55}$$

The constant C depends on κ, τ_i, τ_j only.

Proof of Lemma 23. The upper bound of the integrals similar to (4.40) has also been obtained in various places (see for example, [2, 6, 11, 17, 19]). The analysis in our case is similar to those in the literature. However, again we provide the proof for the completeness of the paper; also the analysis in this proof is going to be used and adapted in all the Lemmas below in this section (except for Lemma 27).

First of all, we can rewrite

$$\begin{aligned} & \tilde{K}_{i,j}(u_i, u_j) \\ & = \left(\frac{T}{\chi} \right)^{1/3} \int_0^\infty d\lambda \left(\frac{-1}{2\pi i} \oint_{\Gamma_{1/2}} dw \phi_i(w) e^{-w(u_i + \frac{T^{1/3}}{\chi^{1/3}}\lambda)} \right) \left(\frac{1}{2\pi i} \oint_{\Gamma_{-1/2}} dz \frac{e^{z(u_j + \frac{T^{1/3}}{\chi^{1/3}}\lambda)}}{\phi_j(z)} \right). \end{aligned} \tag{4.56}$$

Set $u_0 = T - \tau_i \frac{2(1-2\rho)\chi^{1/3}}{1-2\chi} T^{2/3}$ (which corresponds to $s_i = 0$). Then

$$\left(\frac{T}{\chi} \right)^{1/3} \frac{A(j)}{A(i)} \tilde{K}_{i,j}(u_i, u_j) = \int_0^\infty d\lambda E_1(s_i + \lambda) E_2(s_j + \lambda), \tag{4.57}$$

where

$$\begin{aligned} E_1(x) &:= \left(\frac{T}{\chi}\right)^{1/3} \left(\frac{-1}{2\pi i} \oint_{\Gamma_{1/2}} dw \phi_i(w) e^{-w(u_0+x(T/\chi)^{1/3})}\right) A(i)^{-1} \\ E_2(x) &:= \left(\frac{T}{\chi}\right)^{1/3} \left(\frac{1}{2\pi i} \oint_{\Gamma_{-1/2}} dz \frac{e^{z(u_0+x(T/\chi)^{1/3})}}{\phi_j(z)}\right) A(j). \end{aligned} \quad (4.58)$$

We now show the exponential decay of $E_1(s_i)$ for large positive s_i . Lemma 22 indeed implies the result when $s_0 \leq s_i \leq 0$. It is thus enough to consider the case where $s_i > 0$. The analysis of $E_2(s_j)$ is made in exactly the same way, up to a rotation around the origin and exchange of ρ with $1 - \rho$.

To this aim we modify the contour given on Figure 4 as follows, defining a new contour γ' . Call γ'^+ the part of the contour lying in the upper half-plane $\{\text{Im}z \geq 0\}$. Let $\kappa > 0$ be a fixed constant and

$$\varepsilon(i) := 2|\tau_i| + 2\kappa. \quad (4.59)$$

Then we set

$$\gamma'^+ = \{a + te^{i\pi/3}, t \geq \varepsilon(i)(T/\chi)^{-1/3}\} \cup \{a + \varepsilon(i)(T/\chi)^{-1/3}e^{i\theta}, 0 \leq \theta \leq \pi/3\}. \quad (4.60)$$

The contour γ' is then completed by adjoining the conjugate of γ'^+ . This modification of the contour γ has no impact on the saddle point analysis made in the proof of Lemma 22. Indeed, for T large enough

$$\begin{aligned} |Th_0(a) + T^{2/3}h_{1,i}(a) - Th_0(a + \varepsilon(i)(T/\chi)^{-1/3}e^{i\theta}) \\ - T^{2/3}h_{1,i}(a + \varepsilon(i)(T/\chi)^{-1/3}e^{i\theta})| \leq C(i), \end{aligned} \quad (4.61)$$

where $C(i) = 2\varepsilon(i)^2(|\tau_i| + \varepsilon(i))$ is a uniformly bounded constant as τ_i is chosen in a compact interval of \mathbb{R} .

Along γ' it holds,

$$\text{Re}(w - a) \geq \frac{\varepsilon(i)}{2}(T/\chi)^{-1/3} \geq (\kappa + |\tau_i|)(T/\chi)^{-1/3}. \quad (4.62)$$

Thus, along γ'

$$\left| e^{-(w-a)s_i(T/\chi)^{1/3} + \tau_i s_i} \right| \leq e^{-\kappa s_i}. \quad (4.63)$$

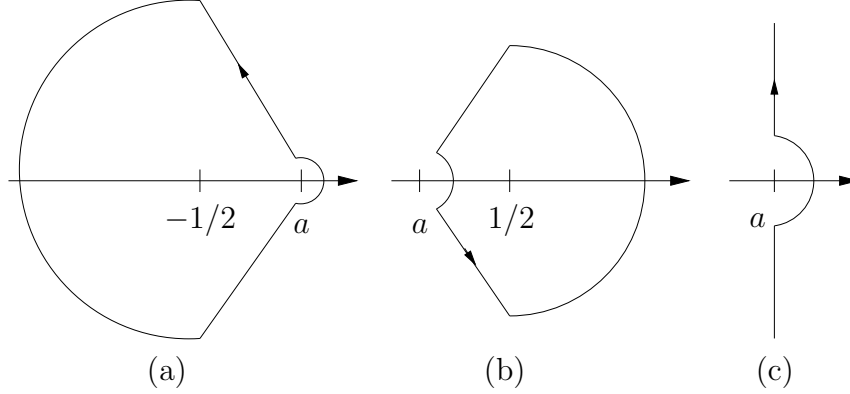


Figure 5: Contours used in the following Lemmas. The paths have local modifications close to the critical point a that are only of order $T^{-1/3}$.

Using the saddle point argument used in the proof of Lemma 22 and (4.61), we easily deduce that there exists $C > 0$ independent of s_i such that for T large enough

$$|E_1(s_i)| \leq C e^{-\kappa s_i}. \quad (4.64)$$

Thus,

$$|(4.57)| \leq \int_0^\infty d\lambda |E_1(s_i + \lambda)| |E_2(s_j + \lambda)| \leq C e^{-\kappa(s_i + s_j)} \quad (4.65)$$

for some other constant C . □

The saddle point analysis made for \tilde{K} in the preceding Lemmas will now be used, up to minor modifications, to consider the asymptotics of the remaining terms.

Lemma 24. *Uniformly for s_1 in a bounded interval one has that*

$$\lim_{T \rightarrow \infty} \left(\frac{T}{\chi}\right)^{1/3} R_{a,-a} = s_1 + e^{-\frac{2}{3}\tau_1^3 - \tau_1 s_1} \int_0^\infty dx \int_0^\infty dy \text{Ai}(\tau_1^2 + s_1 + x + y) e^{-\tau_1(x+y)}. \quad (4.66)$$

Moreover, for any given $\kappa > 0$, there exists a $T_0 > 0$ large enough, such that for all $T \geq T_0$ it holds

$$\left| \left(\frac{T}{\chi}\right)^{1/3} R_{a,-a} - s_1 \right| \leq C e^{-\kappa s_1}, \quad (4.67)$$

for any $s \geq s_0$. The constant C does not depend on T or s_1 .

Proof of Lemma 24. We have

$$\left(\frac{T}{\chi}\right)^{-1/3} R_{a,-a} = \left(\frac{T}{\chi}\right)^{-1/3} \frac{1}{2\pi i} \oint_{\Gamma_{-1/2,a}} dz \frac{\phi_1(a)e^{-u_1 a}}{\phi_1(z)e^{-zu_1}} \frac{1}{(z-a)^2} \quad (4.68)$$

and also

$$\frac{1}{\phi_1(z)e^{-za}} = \exp \left[-(Th_0(z) + T^{2/3}h_{1,1}(z) + T^{1/3}h_{2,1}(z)) \right] \quad (4.69)$$

with h_k 's defined in (4.46). As steep descent path we use almost the same contour Γ as in Figure 4, up to $T^{-1/3}$ -local deformation so that it passes to the right of a , see Figure 5 (a). Then, using the same asymptotic argument as in the proof of Lemma 22, we deduce that

$$\lim_{T \rightarrow \infty} \left(\frac{T}{\chi}\right)^{1/3} R_{a,-a} = \frac{1}{2\pi i} \int_{\Gamma_0} dZ \frac{e^{-Z^3/3 - \tau_1 Z^2 + s_1 Z}}{Z^2}, \quad (4.70)$$

where with Γ_0 we mean the path from $e^{-2\pi i/3}\infty$ to $e^{2\pi i/3}\infty$ and passing on the right of 0. We can take it to its left up to adding a residue term, which is just s_1 . Then the identity (A.4) gives the first formula of the Lemma. For the exponential decay, one first computes the residue at $z = a$, yielding that

$$\left(\frac{T}{\chi}\right)^{-1/3} R_{a,-a} = s_1 + \left(\frac{T}{\chi}\right)^{-1/3} \frac{1}{2\pi i} \oint_{\Gamma_{-1/2}} dz \frac{\phi_1(a)e^{-u_1 a}}{\phi_1(z)e^{-zu_1}} \frac{1}{(z-a)^2}. \quad (4.71)$$

The contour now lies to the left of a , which ensures using the same argument as in Lemma 23, the exponential decay for large positive s_1 . □

Now we we turn to the asymptotics of the function g defined in (2.12) and the different terms of the function F defined in (3.7).

Lemma 25. *Consider the scaling (4.2). Then*

$$\lim_{T \rightarrow \infty} \frac{1}{A(i)} f_i^{(1/2)}(u_i) = - \int_0^\infty dy \operatorname{Ai}(\tau_i^2 + s_i + y) e^{\tau_i y} \quad (4.72)$$

uniformly for s_i in a bounded set. Moreover, for any given $\kappa > 0$, there exists a $T_0 > 0$ large enough, such that for all $T \geq T_0$ it holds

$$\left| \frac{1}{A(i)} f_i^{(1/2)}(u_i) \right| \leq C e^{-\kappa s_i} \quad (4.73)$$

for some constant C independent of T and uniform for s_i bounded from below.

Proof of Lemma 25. We have

$$f_i^{(1/2)}(u_i) = \frac{1}{2\pi i} \oint_{\Gamma_{1/2}} dw \frac{e^{Th_0(w)+T^{2/3}h_{1,i}(w)+T^{1/3}h_{2,i}(w)}}{w-a} \quad (4.74)$$

with the h_k 's defined in (4.46). As a steep descent path we use γ in Figure 4 with a local modification on the $T^{-1/3}$ scale so that it passes on the right of a , see Figure 5 (b). Then, the same asymptotic argument of the proofs of Lemma 22 and Lemma 23 still holds and gives us the claimed result. We also use the identity (A.5) to express the result in terms of Airy functions. \square

Lemma 26. Consider the scaling (4.2). Then

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{A(i)} \int_{u_1}^{\infty} dy \tilde{K}_{i,1}(u_i, y) f_1^{(a)}(y) \\ &= e^{-\frac{2}{3}\tau_1^3 - \tau_1 s_1} \int_0^{\infty} d\lambda \int_0^{\infty} dy e^{-\lambda(\tau_1 - \tau_i)} e^{-\tau_1 y} \text{Ai}(\tau_i^2 + s_i + \lambda) \text{Ai}(\tau_1^2 + s_1 + \lambda + y) \end{aligned} \quad (4.75)$$

uniformly for s_1, s_i in a bounded set. Moreover, for any given $\kappa > 0$, there exists a $T_0 > 0$ large enough, such that for all $T \geq T_0$ it holds

$$\left| \frac{1}{A(i)} \int_{u_1}^{\infty} dy \tilde{K}_{i,1}(u_i, y) f_1^{(a)}(y) \right| \leq C e^{-\kappa(s_i + s_1)} \quad (4.76)$$

for some constant C independent of T and uniform for s_i, s_1 bounded from below.

Proof of Lemma 26. Notice that $f_1^{(a)}(y) = \phi_i(a) e^{-ay}$. Then, choosing the integration path for z in (2.14) so that $\text{Re}(z) < a$, we can explicitly integrate the variable y and get

$$\int_{u_1}^{\infty} dy \tilde{K}_{i,1}(u_i, y) f_1^{(a)}(y) = \frac{-1}{(2\pi i)^2} \oint_{\Gamma_{-1/2}} dz \oint_{\Gamma_{1/2}} dw \frac{e^{zu_1 - wu_i} \phi_i(w) \phi_1(a) e^{-au_1}}{(w-z)\phi_1(z) a-z}, \quad (4.77)$$

where the integration path for z does not include a and does not cross the integration path for w . The rest follows along the same lines of Lemma 22 and Lemma 23. Finally one uses (A.6) to rewrite the double integral expression in terms of Airy functions. \square

Lemma 27. *Consider the scaling (4.2) and let $i \geq 2$. Then*

$$\lim_{T \rightarrow \infty} \frac{1}{A(i)} \int_{-\infty}^{u_1} dy V_{i,1}(u_i, y) f_1^{(a)}(y) = \frac{e^{-\frac{2}{3}\tau_i^3 - \tau_i s_i}}{\sqrt{4\pi(\tau_i - \tau_1)}} \int_{-\infty}^{s_1 - s_i} dy e^{-\frac{y^2}{4(\tau_i - \tau_1)}}. \quad (4.78)$$

uniformly for $s_i - s_1$ in a bounded set. Moreover, for any given $\kappa > 0$, there exists a $T_0 > 0$ large enough, such that for all $T \geq T_0$ it holds

$$\left| \frac{1}{A(i)} \int_{-\infty}^{u_1} dy V_{i,1}(u_i, y) f_1^{(a)}(y) \right| \leq C \left(e^{-\kappa|s_i - s_1| - \tau_i s_i + \tau_1 s_1} + \mathbb{1}_{[s_1 \geq s_i - 1]} e^{-2\tau_i^3/3 - \tau_i s_i} \right) \quad (4.79)$$

for some constant C independent of T and uniform in s_i, s_1 .

Proof of Lemma 27. Using $f_1^{(a)}(y) = \phi_i(a)e^{-ay}$ and modifying the integration path in (2.14) to $i\mathbb{R} + \alpha$ for any $\alpha \in (a, 1/2)$, we can integrate out the y variable with the result

$$\begin{aligned} & \frac{1}{Z(i)} \int_{-\infty}^{u_1} dy V_{i,1}(u_i, y) f_1^{(a)}(y) \\ &= \frac{1}{Z(i)} \frac{1}{2\pi i} \int_{i\mathbb{R} + \alpha} dz e^{z(u_1 - u_i)} \frac{\phi_i(z)}{\phi_1(z)} \frac{\phi_1(a)e^{-au_1}}{z - a} \\ &= \frac{Z(1)}{Z(i)} \frac{1}{2\pi i} \int_{i\mathbb{R} + \alpha} dz e^{z(u_1 - u_i)} \frac{\phi_i(z)}{\phi_1(z)} \frac{1}{z - a} \end{aligned} \quad (4.80)$$

For the saddle point analysis we then deform back our integration path to $a + i\mathbb{R}$ with only a local deviation order $T^{-1/3}$ around $z = a$ to make it passing on the right of a . Specifically, for a given small $\varepsilon > 0$, we use the integration path

$$\gamma = \{a + ix, x \in \mathbb{R} \setminus (-\varepsilon T^{-1/3}, \varepsilon T^{-1/3})\} \cup \{a + \varepsilon T^{-1/3} e^{i\theta}, \theta \in [-\pi/2, \pi/2]\}, \quad (4.81)$$

see Figure 5 (c). The asymptotic analysis for $s_i - s_1$ in a bounded set is the same as in Lemma 20 except that the integral has an extra $1/Y$ in the

denominator. Explicitly, we get

$$\lim_{T \rightarrow \infty} \frac{1}{Z(i)} \int_{-\infty}^{u_1} dy V_{i,1}(u_i, y) f_1^{(a)}(y) = \frac{1}{2\pi} \int_{\mathbb{R}-i\varepsilon} dY \frac{e^{-Y^2(\tau_i - \tau_1) - iY(s_i - s_1)}}{iY}, \quad (4.82)$$

for any $\varepsilon > 0$.

The function

$$v(s) := \frac{1}{2\pi} \int_{\mathbb{R}-i\varepsilon} dY \frac{e^{-Y^2(\tau_i - \tau_1) + iYs}}{iY} \quad (4.83)$$

satisfies

$$\frac{d}{ds} v(s) = \frac{1}{\sqrt{4\pi(\tau_i - \tau_1)}} \exp\left(-\frac{s^2}{4(\tau_i - \tau_1)}\right) \quad (4.84)$$

and the boundary condition

$$\lim_{s \rightarrow -\infty} v(s) = 0. \quad (4.85)$$

Therefore, $v(s)$ is a Gaussian distribution function, so that our limiting function is

$$\lim_{T \rightarrow \infty} \frac{1}{Z(i)} \int_{-\infty}^{u_1} dy V_{i,1}(u_i, y) f_1^{(a)}(y) = \frac{1}{\sqrt{4\pi(\tau_i - \tau_1)}} \int_{-\infty}^{s_1 - s_i} dx e^{-\frac{x^2}{4(\tau_i - \tau_1)}}. \quad (4.86)$$

In order to prove (4.79), we adapt the proof of Lemma 21. First, suppose that $s_i - s_1 \geq 1$. For this case, we take the contour $\gamma = \{z_c + it | t \in \mathbb{R}\}$ as in the proof of Lemma 21. Note that $\text{dist}(a, \gamma) = \text{dist}(a, z_c) \geq \frac{\chi^{1/3}}{2(\tau_i - \tau_1)T^{1/3}}$. Hence and the term $\frac{1}{z-a}$ in (4.80) is bounded by $\mathcal{O}(T^{1/3})$ and an analysis as in the proof of Lemma 21 as in the proof (noting the presence of $T^{1/3}$ in (4.24)) implies (4.79).

Secondly, suppose that $s_i - s_1 \leq -1$. We take the same contour $\gamma = \{z_c + it | t \in \mathbb{R}\}$. In this case, the residue at the simple pole a should be taken into account. Except for this term, the analysis on the contour γ is the same and we obtain (4.79).

Finally, when $-1 \leq s_i - s_1 \leq 1$, we take the contour $\gamma = \gamma_1 \cup \gamma_2$ where γ_1 is the part of the contour $\{z_c + it | t \in \mathbb{R}\}$ which lies outside the circle of radius $r := \frac{1}{(\tau_i - \tau_1)T^{1/3}}$ centered at a , and γ_2 is the part of the circle of radius r centered at a whose real part is at least z_c . This contour is same as the

straight line $\{z_c + it | t \in \mathbb{R}\}$, except that it goes around a on the right in a $\mathcal{O}(T^{-1/3})$ -neighborhood of a . On γ_1 , as $\text{dist}(a, \gamma_1) \geq r = \mathcal{O}(T^{-1/3})$, and hence we obtain, as in the proof of Lemma 21,

$$\left| \frac{Z(1)}{Z(i)} \frac{1}{2\pi i} \int_{\gamma_1} dz e^{z(u_1 - u_i)} \frac{\phi_i(z)}{\phi_1(z)} \frac{1}{z - a} \right| \leq e^{-\kappa |s_i - s_1|}. \quad (4.87)$$

On the other hand, on γ_2 , observe that $\text{dist}(a, \gamma_2) = \mathcal{O}(T^{-1/3})$ and the arc length of γ_2 is also $\mathcal{O}(T^{-1/3})$. By using the estimate (4.18) and (4.19), which holds on γ_2 if T is large enough, we obtain that $T^{2/3}g(z)$ is bounded for $z \in \gamma_2$ (where $g(z)$ is in (4.7)). Hence (recall the first equation of (4.26))

$$\left| \frac{Z(1)}{Z(i)} \frac{1}{2\pi i} \int_{\gamma_2} dz e^{z(u_1 - u_i)} \frac{\phi_i(z)}{\phi_1(z)} \frac{1}{z - a} \right| \leq \frac{1}{2\pi} \int_{\gamma_2} |dz| \frac{e^{T^{2/3}|g(z) - g(a)|}}{|z - a|} = \mathcal{O}(1). \quad (4.88)$$

This, together with (4.87) implies (4.79). \square

The last term to be computed is the asymptotic of g .

Lemma 28. *Consider the scaling (4.2). Then*

$$\lim_{T \rightarrow \infty} A(j)g_j(u_j) = e^{\frac{2}{3}\tau_j^3 + \tau_j s_j} - \int_0^\infty dx \text{Ai}(\tau_j^2 + s_j + x) e^{-\tau_j x} \quad (4.89)$$

uniformly for s_j in a bounded set. Moreover, for any given $\kappa > 0$, there exists a $T_0 > 0$ large enough, such that for all $T \geq T_0$ it holds

$$|A(j)g_j(u_j)| \leq e^{\frac{2}{3}\tau_j^3 + \tau_j s_j} + C e^{-\kappa s_j} \quad (4.90)$$

for some constant C independent of T and uniform for s_j bounded from below.

Proof of Lemma 28. We have

$$g_j(u_j) = \frac{1}{2\pi i} \oint_{\Gamma_{-1/2, a}} dz \frac{e^{-(Th_0(z) + T^{2/3}h_{1,j}(z) + T^{1/3}h_{2,j}(z))}}{z - a} \quad (4.91)$$

with the h_k 's defined in (4.46). As step descent path we use Γ in Figure 4 with a local modification on the $T^{-1/3}$ scale so that it passes on the right of a , see Figure 5 (a). Then, the asymptotic argument of the proofs of Lemma 22 and Lemma 23 still holds and gives us the claimed result. The only small

difference is that for the bound the integration path for large s_j 's has to be chosen to pass on the left of a , which is fine up to adding the residue at a . The factor $e^{2\tau_j^3/3+\tau_j s_j}$ comes from the multiplication by $A(j)$. To rewrite the result in terms of Airy functions we used (A.3). \square

5 Proof of Main Theorems

In this section we prove Theorem 2. Recall (4.1)

$$\mathbb{P} \left(\bigcap_{k=1}^m \{G(x(\tau_k), y(\tau_k)) \leq \ell(\tau_k, s_k)\} \right) = \sum_{k=1}^m \frac{\partial}{\partial s_k} \left[\mathcal{G}_\ell \cdot \det \left(\mathbb{1} - P_\ell \overline{K}^{\text{conj}} P_\ell \right) \right]. \quad (5.1)$$

where

$$\mathcal{G}_\ell := \left(\frac{\chi}{T} \right)^{1/3} (R_{a,-a} - \langle (\mathbb{1} - P_\ell \overline{K} P_\ell)^{-1} P_\ell F, P_\ell g \rangle). \quad (5.2)$$

and $a = \rho - 1/2$. We take the limit $T \rightarrow \infty$ with the scaling (1.5).

Note that the scaling of (1.5) is related to the scaling (4.2) by the identities $|x(\tau_i) + y(\tau_i) - 2t| \leq 2$ and $|x(\tau_i) - y(\tau_i) - 2t_i| \leq 2$. The difference (at most) ± 2 does not contribute to the asymptotics and only makes the notations complicated. For this reason, we will ignore this difference ± 2 in the following presentation.

Proposition 29. *Fix $m \in \mathbb{N}$, and real numbers $\tau_1 < \tau_2 < \dots < \tau_m$, $s_1, \dots, s_m \in \mathbb{R}$. Let $x(\tau_k)$, $y(\tau_k)$ and $\ell(\tau_k, s_k)$ be defined as in (1.5). Then it holds*

$$\lim_{T \rightarrow \infty} \det \left(\mathbb{1} - P_\ell \overline{K}^{\text{conj}} P_\ell \right)_{L^2(\{1, \dots, m\} \times \mathbb{R})} = \det \left(\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s \right)_{L^2(\{1, \dots, m\} \times \mathbb{R})}, \quad (5.3)$$

where $P_\ell(k, x) = \mathbb{1}_{[x > \ell(\tau_k, s_k)]}$ and $P_s(k, x) = \mathbb{1}_{[x > s_k]}$.

Proof of Proposition 29. Proposition 19 implies that for s_i, s_j in a bounded interval I

$$\lim_{T \rightarrow \infty} \left(\frac{T}{\chi} \right)^{1/3} \frac{A(j)}{A(i)} \overline{K}_{i,j}^{\text{conj}}(u_i, u_j) = [\widehat{K}_{\text{Ai}}]_{i,j}(s_i, s_j), \quad (5.4)$$

and that the convergence is uniform on I . Now Lemmas 21 and 23 then imply the convergence in trace class norm of the rescaled kernels

$(T/\chi)^{1/3} \frac{A(j)}{A(i)} \widetilde{K}_{i,j}(u_i, u_j)$ and $(T/\chi)^{1/3} \frac{A(j)}{A(i)} V_{i,j}(u_i, u_j)$. This completes the proof. \square

Proposition 30. Fix $m \in \mathbb{N}$ real numbers $\tau_1 < \tau_2 < \dots < \tau_m$, $s_1, \dots, s_m \in \mathbb{R}$. Let $x(\tau_k)$, $y(\tau_k)$ and $\ell(\tau_k, s_k)$ be defined as in (1.5). Then

$$\lim_{T \rightarrow \infty} \mathcal{G}_\ell = g_m(\tau, s) \quad (5.5)$$

where $g_m(\tau, s)$ is defined in (1.11).

Proof of Proposition 30. Consider the term $\langle (\mathbb{1} - P_\ell \overline{K} P_\ell)^{-1} P_\ell F, P_\ell g \rangle$. We first scale and also conjugate by some operators. Define the function

$$u_i(x) := T - \tau_i \frac{2(1-2\rho)\chi^{1/3}}{1-2\chi} T^{2/3} + x \frac{T^{1/3}}{\chi^{1/3}}. \quad (5.6)$$

Set $\beta = 1 + \max_{i=1, \dots, m} |\tau_i|$, and define the multiplication operator

$$W(i, x) := A(i) e^{-\beta x}. \quad (5.7)$$

Set

$$L_{i,j}^T(x, x') := \left(\frac{T}{\chi} \right)^{1/3} W(i, x) \overline{K}_{i,j}(u_i(x), u_j(x')) W^{-1}(j, x'), \quad (5.8)$$

and

$$\Psi_i^T(x) := W(i, x) g_i(u_i(x)), \quad \Phi_i^T(x) := W^{-1}(i, x) F_i(u_i(x)), \quad (5.9)$$

where one recalls the definitions of F_i in (3.7) and g_i in (2.12). Here the superscript T indicates the dependence on T .

By Lemmas 25, 26, 27 and 28, for any given $\kappa > 0$, there exists a $T_0 > 0$ large enough, such that for all $T \geq T_0$, Ψ_i^T and Φ_i^T are $L^2([s_i, \infty))$ for any given s_i . Then for T large enough,

$$\left(\frac{\chi}{T} \right)^{1/3} \langle (\mathbb{1} - P_\ell \overline{K} P_\ell)^{-1} P_\ell F, P_\ell g \rangle = \langle (\mathbb{1} - P_s L^T P_s)^{-1} P_s \Phi^T, P_s \Psi^T \rangle. \quad (5.10)$$

Let Q be the multiplication operator $Q(i, x) = e^{-\beta x}$ and set

$$\tilde{\rho} = (\mathbb{1} - P_s Q \widehat{K}_{\text{Ai}} Q^{-1} P_s)^{-1}, \quad \tilde{\Psi} = Q \Psi, \quad \tilde{\Phi} = Q^{-1} \Phi \quad (5.11)$$

where the functions Ψ, Φ are defined in (1.8). Observe that the functions $\tilde{\Psi}_i, \tilde{\Phi}_i \in L^2((s_i, \infty))$. Since

$$\langle (\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s)^{-1} P_s \Phi, P_s \Psi \rangle = \langle \tilde{\rho} P_s \tilde{\Phi}, P_s \tilde{\Psi} \rangle, \quad (5.12)$$

it is enough to prove $\lim_{T \rightarrow \infty} \langle \rho^T P_s \Phi^T, P_s \Psi^T \rangle = \langle \tilde{\rho} P_s \tilde{\Phi}, P_s \tilde{\Psi} \rangle$ where $\rho^T = (\mathbb{1} - P_s L^T P_s)^{-1}$.

Clearly,

$$\begin{aligned} & \left| \langle \rho^T P_s \Phi^T, P_s \Psi^T \rangle - \langle \tilde{\rho} P_s \tilde{\Phi}, P_s \tilde{\Psi} \rangle \right| \leq \|\rho^T - \tilde{\rho}\| \|P_s \Phi^T\| \|P_s \Psi^T\| \\ & \quad + \|\rho\| \|P_s(\Psi^T - \tilde{\Psi})\| \|P_s \Phi^T\| + \|\rho\| \|P_s \tilde{\Psi}\| \|P_s(\Phi^T - \tilde{\Phi})\|. \end{aligned} \quad (5.13)$$

First consider $\|\rho^T - \tilde{\rho}\|$. As the operator-norm is bounded by the Hilbert-Schmidt norm $\|\cdot\|_2$,

$$\begin{aligned} & \|P_s L^T P_s - P_s Q \widehat{K}_{\text{Ai}} Q^{-1} P_s\|_2^2 \leq \|P_s L^T P_s - P_s Q \widehat{K}_{\text{Ai}} Q^{-1} P_s\|_2^2 \\ & = \sum_{i,j=1}^m \int_{s_i}^{\infty} dx \int_{s_j}^{\infty} dx' |L_{i,j}^T(x, x') - e^{-\beta(x-x')} [\widehat{K}_{\text{Ai}}]_{i,j}(x, x')|^2. \end{aligned} \quad (5.14)$$

By taking $\kappa = \beta + 1$ in Lemmas 23 and 21, and using the dominated convergence theorem and Proposition 19, the limit as $T \rightarrow \infty$ of (5.14) becomes 0. By Lemma 33, the operator $\mathbb{1} - P_s \widehat{K}_{\text{Ai}} P_s$ is invertible and it follows that $\|\rho^T - \tilde{\rho}\| \rightarrow 0$ as $T \rightarrow \infty$.

Now consider the term $P_s \Psi^T$. From (4.90) of Lemma 28, we have $|\Psi_j^T(x)| \leq C_1 e^{-(\beta-|\tau_i|)x} + C_2 e^{-(\beta+\kappa)x}$ for $x \geq s_j$ for a fixed s_j . The conjugation $e^{-\beta x}$ is introduced to make this function decay exponentially as $x \rightarrow \infty$. Hence $\|P_s \Psi^T\|$ is bounded uniformly for large enough T . Also noting that the right-hand-side of (4.89) is precisely $\Psi_j(s_j)$, it follows from dominated convergence theorem that $\|P_s(\Psi^T - \tilde{\Psi})\| \rightarrow 0$ as $T \rightarrow \infty$.

It follows from Lemmas 25, 26 and 27 that $|\Phi_j^T(x)| \leq C e^{-(\kappa-\beta)x}$ for $x \geq s_j$ for a fixed s_j . Hence $\|P_s \Phi^T\|$ is uniformly bounded for large enough T if we set for example $\kappa = \beta + 1 > 0$. Also it follows that $\|P_s(\Phi^T - \tilde{\Phi})\| \rightarrow 0$ as $T \rightarrow \infty$.

Hence we have shown that (5.10) converges to (5.12). The remaining term in \mathcal{G}_ℓ is $(\chi/T)^{1/3} R_{a,-a}$. But Lemma 24 shows that this converges to \mathcal{R} (after the changes of variables $x \mapsto x - s_1$). This completes the proof. \square

The above two Propositions proves that $\mathcal{G}_\ell \cdot \det \left(\mathbb{1} - P_\ell \overline{K}^{\text{conj}} P_\ell \right)$ converges to $g_m(\tau, s) \det \left(\mathbb{1} - P_s \widehat{K}_{\Lambda_i} P_s \right)$ as $T \rightarrow \infty$. It remains to show that the derivatives with respect to s_k 's also converge. Let us first consider $(\chi/T)^{1/3} \langle (\mathbb{1} - P_\ell \overline{K} P_\ell)^{-1} P_\ell F, P_\ell g \rangle$ of \mathcal{G}_ℓ . As in (5.10), this equals $\langle (\mathbb{1} - P_s L^T P_s)^{-1} P_s \Phi^T, P_s \Psi^T \rangle$. Note that the dependence on s_k is only through the projection operator P_s . Hence by simple translations,

$$\langle (\mathbb{1} - P_s L^T P_s)^{-1} P_s \Phi^T, P_s \Psi^T \rangle = \langle (\mathbb{1} - \mathbf{L}^T)^{-1} \Phi^T, \Psi^T \rangle \quad (5.15)$$

where $\mathbf{L}^T((i, x), (j, y)) := L^T((i, x+s_i), (j, y+s_j))$, $\Phi^T((i, x)) = \Phi^T((i, x+s_i))$ and $\Psi^T((i, x)) = \Psi^T((i, x+s_i))$, and all the (real) inner product, the operator and the functions are all defined on $L^2(\{1, \dots, m\} \times \mathbb{R}_+)$. Then

$$\begin{aligned} \frac{\partial}{\partial s_k} \langle (\mathbb{1} - \mathbf{L}^T)^{-1} \Phi^T, \Psi^T \rangle &= \langle (\mathbb{1} - \mathbf{L}^T)^{-1} \left(\frac{\partial}{\partial s_k} \mathbf{L}^T \right) (\mathbb{1} - \mathbf{L}^T)^{-1} \Phi^T, \Psi^T \rangle \\ &+ \langle (\mathbb{1} - \mathbf{L}^T)^{-1} \left(\frac{\partial}{\partial s_k} \Phi^T \right), \Psi^T \rangle + \langle (\mathbb{1} - \mathbf{L}^T)^{-1} \Phi^T, \frac{\partial}{\partial s_k} \Psi^T \rangle. \end{aligned} \quad (5.16)$$

Hence we need to control the asymptotics of the derivatives of the kernels. But since all the asymptotic bounds for large x in the Lemmas in Section 4 are all exponential, while the derivatives of the kernels yields only polynomial terms, we can check that the dominated convergence theorem still applies and obtain the convergence of (5.16) to the corresponding derivatives of the limit. We omit the detail. The other terms are also similar. This completes the proof of Theorem 2.

To prove Theorems 5-7 we use the following slow-decorrelation result, which is an extension of Proposition 8 of [15].

Lemma 31. *Let $A = (c_1 T, c_2 T)$ for some $c_1, c_2 > 0$. Let then $B = A + r((1 - \rho)^2, \rho^2)$ with $r \sim T^\nu$ with $0 < \nu < 1$. Then, for any $\beta \in (\nu/3, 1/3)$, it holds*

$$\lim_{T \rightarrow \infty} \mathbb{P} \left(|G(B) - G(A) - r| \leq T^\beta \right) = 1. \quad (5.17)$$

Proof of Lemma 31. First of all, it is easy to check by Proposition 2.2 of [17] that the difference between the model (1.1) and the last passage percolation model corresponding exactly to the stationary TASEP becomes irrelevant in the order as $T \rightarrow \infty$. Denote \mathbb{P}_{TA} the measure for the last passage percolation

corresponding to the stationary TASEP. Due to the stationarity in space and time of TASEP, we observe that $G(B) - G(A)$ and $G(B - A)$ have the same distribution. Therefore,

$$\begin{aligned}\mathbb{P}_{\text{TA}}(|G(B) - G(A) - r| \leq T^\beta) &= \mathbb{P}_{\text{TA}}(|G(B - A) - r| \leq T^\beta) \\ &= \mathbb{P}_{\text{TA}}(|G(r(1 - \rho)^2, r\rho^2) - r| \leq T^\beta).\end{aligned}\tag{5.18}$$

But since the distribution of $(G(r(1 - \rho)^2, r\rho^2) - r)/(r/\chi)^{1/3}$ converges to F_0 [17], and $r^{1/3}/T^\beta = \mathcal{O}(T^{\nu/3-\beta}) \rightarrow 0$ as $T \rightarrow \infty$, we find that (5.18) converges to 1 as $T \rightarrow \infty$. This completes the proof. \square

Proof of Theorem 5. The projection of $(x(\tau, \theta), y(\tau, \theta))$ on the line $\{x + y = (1 - 2\chi)T\}$ along the critical direction $((1 - \rho)^2, \rho^2)$ is $(x(\tau), y(\tau))$ with x, y defined in (1.5). We have

$$x(\tau) = x(\tau, \theta) - r(1 - \rho)^2, \quad y(\tau) = y(\tau, \theta) - r\rho^2\tag{5.19}$$

with $r = r(\theta) = \theta T^\nu$. In particular $\ell(\tau, 0, s) = \ell(\tau, \theta, s) - r(\theta)$. Then,

$$\begin{aligned}&\mathbb{P}\left(\bigcap_{k=1}^m \{G(x(\tau_k, \theta_k), y(\tau_k, \theta_k)) \leq \ell(\tau_k, \theta_k, s_k)\}\right) \\ &= \mathbb{P}\left(\bigcap_{k=1}^m \{G(x(\tau_k, 0), y(\tau_k, 0)) \leq \ell(\tau_k, 0, s_k) + \Xi_k\}\right)\end{aligned}\tag{5.20}$$

with

$$\Xi_k := G(x(\tau_k, 0), y(\tau_k, 0)) + r(\theta_k) - G(x(\tau_k, \theta_k), y(\tau_k, \theta_k)).\tag{5.21}$$

By Lemma 31 we have $\Xi_k = o(T^{1/3})$ so that

$$\lim_{T \rightarrow \infty} (5.20) = \lim_{T \rightarrow \infty} \mathbb{P}\left(\bigcap_{k=1}^m \{G(x(\tau_k, 0), y(\tau_k, 0)) \leq \ell(\tau_k, 0, s_k)\}\right),\tag{5.22}$$

see the proof of Theorem 1 of [15] for detailed steps. This is by Theorem 2 the desired result. \square

Proof of Theorem 6. The goal is to express our setting into the one of Theorem 5. Denote $p(\tau) := q(\tau) + n(\tau)$, which is given by

$$p(\tau) = (1 - \rho)^2 T + \tau 2(1 - \rho)\chi^{1/3} T^{2/3} - (1 - \rho)s T^{1/3}/\chi^{1/3}\tag{5.23}$$

and recall

$$n(\tau) = \rho^2 T - \tau 2\rho\chi^{1/3} T^{2/3}. \quad (5.24)$$

(Since $n(\tau)$ and $q(\tau)$ are integers, the above formulas are exact up to the error of size 2 at most: since this difference does not affect the asymptotics but only complicates the formulas, we drop this difference in the following presentation.) By (1.16) we have

$$\mathbb{P} \left(\bigcap_{k=1}^m \{ \mathbf{x}_{n(\tau_k)}(T) \geq q(\tau_k) \} \right) = \mathbb{P} \left(\bigcap_{k=1}^m \{ G(p(\tau_k), n(\tau_k)) \leq T \} \right). \quad (5.25)$$

Denote by $(X(\tau), Y(\tau))$ the projection of $(p(\tau), n(\tau))$ on the line $\{x + y = (1 - 2\chi)T\}$ along the critical direction $((1 - \rho)^2, \rho^2)$. We get

$$X(\tau) = p(\tau) - r(1 - \rho)^2, \quad Y(\tau) = n(\tau) - r\rho^2 \quad (5.26)$$

where

$$r = r(\tau, s) = \tau \frac{2(1 - 2\rho)\chi^{1/3} T^{2/3}}{1 - 2\chi} - s \frac{1 - \rho}{1 - 2\chi} (T/\chi)^{1/3}. \quad (5.27)$$

Let us further denote

$$\tau^s = \tau - \frac{s\rho}{2\chi^{2/3}} T^{-1/3}. \quad (5.28)$$

Then,

$$T = \ell(\tau^s, s) + r(\tau, s) \quad (5.29)$$

where $\ell(\tau, s)$ is defined in (1.5). Then, replacing (5.27) into (5.26) we get

$$\begin{aligned} X(\tau) &= (1 - \rho)^2 T + \tau \frac{2\chi^{4/3} T^{2/3}}{1 - 2\chi} - s \frac{\rho\chi^{2/3}}{1 - 2\chi} T^{1/3} = x(\tau^s), \\ Y(\tau) &= \rho^2 T - \tau \frac{2\chi^{4/3} T^{2/3}}{1 - 2\chi} + s \frac{\rho\chi^{2/3}}{1 - 2\chi} T^{1/3} = y(\tau^s), \end{aligned} \quad (5.30)$$

where $x(\tau), y(\tau)$ are defined in (1.5).

With these notations we can rewrite (5.25) as follows,

$$(5.25) = \mathbb{P} \left(\bigcap_{k=1}^m \{ G(x(\tau_k^s) + r_k(1 - \rho)^2, y(\tau_k^s) + r_k\rho^2) \leq \ell(\tau_k^s, s_k) + r_k \} \right) \quad (5.31)$$

where $r_k := r(\tau_k, s_k) \sim \mathcal{O}(T^{2/3})$. Then, by Theorem 5 the result follows. \square

Proof of Theorem 7. The proof of Theorem 7 is very similar to the one of Theorem 6. The scaling (1.22) corresponds, in terms of x, y , to the scaling

$$\begin{aligned} x &= (1 - \rho)^2 T + 2\tau(1 - \rho)\chi^{1/3} T^{2/3} - s\chi^{2/3} T^{1/3}, \\ y &= \rho^2 T - 2\tau\rho\chi^{1/3} T^{2/3} - s\chi^{2/3} T^{1/3}. \end{aligned} \quad (5.32)$$

The projections $(X(\tau), Y(\tau))$ of (x, y) on the line $\{x + y = (1 - 2\chi)T\}$ along the critical direction $((1 - \rho)^2, \rho^2)$ are given by $X(\tau) = x - r(1 - \rho)^2$, $Y(\tau) = y - r\rho^2$. Explicitly we find

$$\begin{aligned} r(\tau, s) &= \tau \frac{2(1 - 2\rho)\chi^{1/3} T^{2/3}}{1 - 2\chi} - s \frac{2\chi^{2/3} T^{1/3}}{1 - 2\chi}, \\ X(\tau) &= \rho^2 T + \tau \frac{2\chi^{4/3} T^{2/3}}{1 - 2\chi} + s \frac{(1 - 2\rho)\chi^{2/3} T^{1/3}}{1 - 2\chi}, \\ Y(\tau) &= \rho^2 T - \tau \frac{2\chi^{4/3} T^{2/3}}{1 - 2\chi} - s \frac{(1 - 2\rho)\chi^{2/3} T^{1/3}}{1 - 2\chi}. \end{aligned} \quad (5.33)$$

Setting

$$\tau^s = \tau + s \frac{(1 - 2\rho)T^{-1/3}}{2\chi^{2/3}} \quad (5.34)$$

we get

$$T = \ell(\tau^s, s) + r(\tau, s), \quad X(\tau) = x(\tau^s), \quad Y(\tau) = y(\tau^s). \quad (5.35)$$

Then, setting $r_k := r(\tau_k, s_k)$ we get (5.31) and then by Theorem 5 the result follows. \square

A Some Airy function identities

In the asymptotic analysis we get two basic integral expressions which can be rewritten in terms of Airy functions and exponential, namely

$$\begin{aligned} \frac{-1}{2\pi i} \int_{e^{\pi i/3}\infty}^{e^{-\pi i/3}\infty} dW e^{W^3/3 + bW^2 - cW} &= \text{Ai}(b^2 + c) e^{\frac{2}{3}b^3 + bc}, \\ \frac{1}{2\pi i} \int_{e^{-2\pi i/3}\infty}^{e^{2\pi i/3}\infty} dZ e^{-Z^3/3 - bZ^2 + cZ} &= \text{Ai}(b^2 + c) e^{-\frac{2}{3}b^3 - bc}. \end{aligned} \quad (A.1)$$

Starting from these two formulas we state some identities.

Lemma 32.

$$\begin{aligned}
\text{(A)} \quad & \frac{-1}{(2\pi i)^2} \int_{e^{-2\pi i/3}\infty}^{e^{2\pi i/3}\infty} dZ \int_{e^{\pi i/3}\infty}^{e^{-\pi i/3}\infty} dW \frac{e^{W^3/3+b_1W^2-c_1W}}{e^{Z^3/3+b_2Z^2-c_2Z}} \frac{1}{W-Z} \\
& = \frac{e^{\frac{2}{3}b_1^3+b_1c_1}}{e^{\frac{2}{3}b_2^3+b_2c_2}} \int_0^\infty d\lambda e^{-\lambda(b_2-b_1)} \text{Ai}(b_1^2+c_1+\lambda) \text{Ai}(b_2^2+c_2+\lambda), \quad (\text{A.2})
\end{aligned}$$

$$\begin{aligned}
\text{(B)} \quad & \frac{-1}{2\pi i} \int_{e^{-2\pi i/3}\infty, \text{ left of } 0}^{e^{2\pi i/3}\infty} dZ e^{-Z^3/3-bZ^2+cZ} \frac{1}{Z} \\
& = e^{-\frac{2}{3}b^3-bc} \int_0^\infty dx \text{Ai}(b^2+c+x)e^{-bx}, \quad (\text{A.3})
\end{aligned}$$

$$\begin{aligned}
\text{(C)} \quad & \frac{1}{2\pi i} \int_{e^{-2\pi i/3}\infty, \text{ left of } 0}^{e^{2\pi i/3}\infty} dZ e^{-Z^3/3-bZ^2+cZ} \frac{1}{Z^2} \\
& = e^{-\frac{2}{3}b^3-bc} \int_0^\infty dx \int_0^\infty dy \text{Ai}(b^2+c+x+y)e^{-b(x+y)}, \quad (\text{A.4})
\end{aligned}$$

$$\begin{aligned}
\text{(D)} \quad & \frac{-1}{2\pi i} \int_{e^{\pi i/3}\infty, \text{ right of } 0}^{e^{-\pi i/3}\infty} dW e^{Z^3/3+bW^2-cW} \frac{1}{W} \\
& = e^{\frac{2}{3}b^3+bc} \int_0^\infty dx \text{Ai}(b^2+c+x)e^{bx}, \quad (\text{A.5})
\end{aligned}$$

$$\begin{aligned}
\text{(E)} \quad & \frac{1}{(2\pi i)^2} \int_{e^{-2\pi i/3}\infty, \text{ left of } 0}^{e^{2\pi i/3}\infty} dZ \int_{e^{\pi i/3}\infty}^{e^{-\pi i/3}\infty} dW \frac{e^{W^3/3+b_1W^2-c_1W}}{e^{Z^3/3+b_2Z^2-c_2Z}} \frac{1}{(W-Z)Z} \\
& = \frac{e^{\frac{2}{3}b_1^3+b_1c_1}}{e^{\frac{2}{3}b_2^3+b_2c_2}} \int_0^\infty d\lambda \int_0^\infty dx e^{-\lambda(b_2-b_1)} \text{Ai}(b_1^2+c_1+\lambda) \text{Ai}(b_2^2+c_2+\lambda) e^{-b_2x}, \\
& \hspace{20em} (\text{A.6})
\end{aligned}$$

$$\begin{aligned}
\text{(F)} \quad & \text{For } b_2 < b_1, \quad \int_{\mathbb{R}} d\lambda e^{-\lambda(b_2-b_1)} \text{Ai}(b_1^2+c_1+\lambda) \text{Ai}(b_2^2+c_2+\lambda) \\
& = \frac{1}{\sqrt{4\pi(b_1-b_2)}} \exp\left(-\frac{(c_2-c_1)^2}{4(b_1-b_2)} + \frac{2}{3}(b_2^3-b_1^3) + b_2c_2 - b_1c_1\right). \quad (\text{A.7})
\end{aligned}$$

Proof of Lemma 32. (A) Since we can choose the paths for W and Z such that $\operatorname{Re}(W - Z) > 0$, using

$$\frac{1}{W - Z} = \int_0^\infty d\lambda e^{-\lambda(W - Z)} \quad (\text{A.8})$$

and (A.1) we get (A.2).

(B) Let $f(c) := \text{l.h.s.}(A.3)$. Then differentiating and using (A.1) we get $f'(c) = -\operatorname{Ai}(b^2 + c)e^{-\frac{2}{3}b^3 - bc}$. Together with the boundary condition $\lim_{c \rightarrow \infty} f'(c) = 0$, we get (A.3).

(C) Let $f(c) := \text{l.h.s.}(A.4)$. Differentiating twice and applying (A.1) we obtain $f''(c) = \operatorname{Ai}(b^2 + c)e^{-\frac{2}{3}b^3 - bc}$. The boundary conditions are $\lim_{c \rightarrow \infty} f'(c) = 0 = \lim_{c \rightarrow \infty} f''(c)$. Integrating twice and shifting the integration bounds to zero, we get (A.4).

(D) This can be derived by (A.3) by change of variable $W := -Z$.

(E) This identity can be found by first using (A.8) and then the two complex integrals are decoupled. For the integral over W one uses (A.1), while for the integral over Z one uses (A.3).

(F) One applies the identity (2.20) of [21], which holds for $b_2 < b_1$,

$$\begin{aligned} & \int_{\mathbb{R}} d\lambda e^{-\lambda(b_2 - b_1)} \operatorname{Ai}(d_1 + \lambda) \operatorname{Ai}(d_2 + \lambda) \\ &= \frac{1}{\sqrt{4\pi(b_1 - b_2)}} \exp\left(-\frac{(d_2 - d_1)^2}{4(b_1 - b_2)} - (b_1 - b_2)\frac{d_1 + d_2}{2} + \frac{(b_1 - b_2)^3}{12}\right) \end{aligned} \quad (\text{A.9})$$

and then we set $d_1 = b_1^2 + c_1$, $d_2 = b_2^2 + c_2$. \square

B Invertibility of $\mathbb{1} - P_s \widehat{K}_{\operatorname{Ai}} P_s$

Lemma 33. *For any fixed real numbers s_1, \dots, s_m ,*

$$(\mathbb{1} - P_s \widehat{K}_{\operatorname{Ai}} P_s)^{-1} \quad (\text{B.1})$$

exists.

Proof of Lemma 33. First of all, notice that

$$\det(\mathbb{1} - P_s \widehat{K}_{\operatorname{Ai}} P_s) = \mathbb{P}\left(\bigcap_{k=1}^m \{\mathcal{A}(\tau_k) - \tau_k^2 \leq s_k\}\right). \quad (\text{B.2})$$

Now, let $S := \min\{s_1, \dots, s_m\}$. Then,

$$\begin{aligned}
(B.2) &\geq \mathbb{P}\left(\bigcap_{k=1}^m \{\mathcal{A}(\tau_k) - \tau_k^2 \leq S\}\right) \\
&\geq \mathbb{P}\left(\sup_{\tau \in \mathbb{R}} (\mathcal{A}(\tau) - \tau^2) \leq S\right) = F_{\text{GOE}}(S) > 0.
\end{aligned}
\tag{B.3}$$

The equality with the GOE Tracy-Widom distribution, F_{GOE} , is proven in Corollary 1.3 of [21], while the strict inequality follows from the monotonicity of the distribution function and the large $-S$ asymptotics (see for example, [3]).

Moreover, as shown in Section 2.2 of [21], $P_s \widehat{K}_{A_i} P_s$ is trace-class (the shift by τ_k^2 is irrelevant for that property, since it holds for any s_k 's). The proof ends by applying a known result on Fredholm determinant, see e.g. Theorem XIII.105 (b) in [30]: let A be a trace-class operator, then

$$\det(\mathbb{1} + A) \neq 0 \iff \mathbb{1} + A \text{ is invertible.} \tag{B.4}$$

□

C Trace-class properties of the kernel

In this section, we discuss the basic properties of the operators \overline{K} and K in Section 2.

Proposition 34.

- (i) The operators $P_u \overline{K} P_u$ and $P_u K P_u$ are bounded in $L^2(\{1, \dots, m\} \times \mathbb{R})$.
- (ii) Let $a, b \in (0, \frac{1}{2})$. Fix constants $\alpha_1, \dots, \alpha_m$ such that

$$-\frac{1}{2} < \alpha_1 < \alpha_2 < \dots < \alpha_m < \frac{1}{2}. \tag{C.1}$$

Then $P_u \overline{K}^{\text{conj}} P_u$ defined in (2.21) is a trace-class operator on $L^2(\{1, \dots, m\} \times \mathbb{R})$. On the other hand, $P_u K^{\text{conj}} P_u$ defined in (2.18) is a trace-class in the same space if the constants satisfy more restricted condition

$$-a < \alpha_1 < \alpha_2 < \dots < \alpha_m < b. \tag{C.2}$$

Proof of Proposition 34. We only prove (ii). The proof of (i) follows easily by suitably modifying the analysis for part (ii).

We start with $P_u \overline{K}^{\text{conj}} P_u$. Since the set of trace-class operators is a linear space, it is enough to prove that $P_{u_i} M_i \overline{K}_{i,j} M_j^{-1} P_{u_j}$ is a trace-class operator in $L^2(\mathbb{R})$ for each i, j . As $\overline{K}_{i,j} = \tilde{K}_{i,j} - V_{i,j}$ (see (2.13)), we prove that each of the two operators is trace-class.

(a) *Operator $\tilde{K}_{i,j}$:* It is easy to check that the operator can be re-expressed as (see (4.56))

$$P_{u_i}(x) e^{-\alpha_i x} \tilde{K}_{i,j}(x, y) e^{\alpha_j y} P_{u_j}(y) = \int_{\mathbb{R}} dz L(x, z) R(z, y) \quad (\text{C.3})$$

where

$$\begin{aligned} L(x, z) &= P_{u_i}(x) e^{-\alpha_i x} \frac{-1}{2\pi i} \oint_{\Gamma_{1/2}} dw \phi_i(w) e^{-w(x+z)} P_0(z), \\ R(z, y) &= e^{\alpha_j y} P_{u_j}(y) \frac{1}{2\pi i} \oint_{\Gamma_{-1/2}} dz \frac{e^{z(y+z)}}{\phi_j(z)} P_0(z). \end{aligned} \quad (\text{C.4})$$

By choosing the integration paths close enough to $1/2$, resp. $-1/2$, a simple estimate shows that there exists $0 < \delta < \min\{\alpha_1 + 1/2, 1/2 - \alpha_m\}$ so that

$$\begin{aligned} |L(x, z)| &\leq C e^{-(1/2-\delta+\alpha_i)x} e^{-(1/2-\delta)z} \mathbb{1}_{[x, z \geq 0]}, \\ |R(z, y)| &\leq C e^{-(1/2-\delta-\alpha_j)x} e^{-(1/2-\delta)z} \mathbb{1}_{[z, y \geq 0]}. \end{aligned} \quad (\text{C.5})$$

From this it follows immediately that L and R are Hilbert-Schmidt operators on $L^2(\mathbb{R})$. So is the conjugated kernel of $\tilde{K}_{i,j}$ trace-class.

(b) *Operator $V_{i,j}$:* As $V_{i,j} = 0$ for $i \leq j$, assume that $i > j$.

In this case, we have

$$\Delta t := t_i - t_j \geq 1. \quad (\text{C.6})$$

From (2.14), by plugging in ϕ_i and ϕ_j and by changing the contour from $i\mathbb{R}$ to \mathbb{R} , we obtain

$$V_{i,j}(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iz(x-y)}}{(\frac{1}{2} + iz)^{\Delta t} (\frac{1}{2} - iz)^{\Delta t}} dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iz(x-y)} f(z) dz \quad (\text{C.7})$$

where

$$f(z) = \frac{1}{(\frac{1}{2} + iz)^{\Delta t} (\frac{1}{2} - iz)^{\Delta t}} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}). \quad (\text{C.8})$$

Hence

$$V_{i,j}(x, y) = \hat{f}(x - y), \quad (\text{C.9})$$

the Fourier transform of f .

Note that $f(z) = g(z)h(z)$ where

$$g(z) = \frac{1}{(\frac{1}{2} + iz)^{\Delta t}}, \quad h(z) = \frac{1}{(\frac{1}{2} - iz)^{\Delta t}}. \quad (\text{C.10})$$

As $g \in L^2(\mathbb{R})$ so then is $\hat{g} \in L^2(\mathbb{R})$. Moreover, as

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-izs}}{(\frac{1}{2} + iz)^{\Delta t}} dz \quad (\text{C.11})$$

is well-defined pointwise except at $s = 0$ (since it is conditionally convergent), $\hat{g}(s)$ equals (C.11) almost everywhere. Furthermore, we can compute (C.11) explicitly and obtain

$$\hat{g}(s) = \frac{(-s)^{\Delta t - 1}}{(\Delta t - 1)!} e^{s/2} \mathbb{1}_{[s < 0]}. \quad (\text{C.12})$$

Similarly,

$$\hat{h}(s) = \frac{s^{\Delta t - 1}}{(\Delta t - 1)!} e^{-s/2} \mathbb{1}_{[s > 0]}. \quad (\text{C.13})$$

We have

$$(\hat{g} * \hat{h})^\vee(s) = g(s)h(s) = f(s). \quad (\text{C.14})$$

Hence

$$\begin{aligned} V_{i,j}(x, y) &= \hat{f}(x - y) = (\hat{g} * \hat{h})(x - y) \\ &= \int_{\mathbb{R}} \hat{g}(x - y - s) \hat{h}(s) ds = \int_{\mathbb{R}} \hat{g}(x - s) \hat{h}(s - y) ds \end{aligned} \quad (\text{C.15})$$

Here all the steps make sense since $\hat{g}, \hat{h} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.

Note $\alpha_i > \alpha_j$ for $i > j$, and $1 \pm (\alpha_i + \alpha_j) > 0$. By using (C.15), we see that

$$P_{u_i} M_i V_{i,j} M_j^{-1} P_{u_j} = (P_{u_i} M_i \mathcal{G} N^{-1})(N \mathcal{H} M_j^{-1} P_{u_j}) \quad (\text{C.16})$$

where \mathcal{G} and \mathcal{H} are the operators with kernel

$$\begin{aligned} \mathcal{G}(x, y) &= \frac{(y - x)^{t_i - t_j - 1}}{(t_i - t_j - 1)!} e^{\frac{(x-y)}{2}} \mathbb{1}_{[x-y < 0]}, \\ \mathcal{H}(x, y) &= \frac{(x - y)^{t_i - t_j - 1}}{(t_i - t_j - 1)!} e^{-\frac{(x-y)}{2}} \mathbb{1}_{[x-y > 0]}, \end{aligned} \quad (\text{C.17})$$

and N is the multiplication operator

$$N(x) = e^{-\frac{1}{2}(\alpha_i + \alpha_j)x}. \quad (\text{C.18})$$

Then

$$\begin{aligned} & \iint_{\mathbb{R}^2} dx dy |(P_{u_i} M_i \mathcal{G} N^{-1})(x, y)|^2 dy dx \\ &= \iint_{y > x \geq u_i} dx dy \left| e^{-\alpha_i x} \frac{(y-x)^{t_i - t_j - 1}}{(t_i - t_j - 1)!} e^{\frac{(x-y)}{2}} e^{\frac{1}{2}(\alpha_i + \alpha_j)y} \right|^2 \end{aligned} \quad (\text{C.19})$$

Changing the variable y by $s := y - x$, the above equals

$$\frac{1}{((t_i - t_j - 1)!)^2} \int_{u_i}^{\infty} dx e^{-(\alpha_i - \alpha_j)x} \int_0^{\infty} ds s^{2(t_i - t_j - 1)} e^{-(1 - \alpha_i - \alpha_j)y} \quad (\text{C.20})$$

which is finite. Hence $P_{u_i} M_i \mathcal{G} N^{-1}$ is a Hilbert-Schmidt operator. Similarly,

$$\begin{aligned} & \iint_{\mathbb{R}^2} dx dy |(N \mathcal{H} M_j^{-1} P_{u_j})(x, y)|^2 \\ &= \iint_{x > y \geq u_j} dx dy \left| e^{-\frac{1}{2}(\alpha_i + \alpha_j)x} \frac{(x-y)^{t_i - t_j - 1}}{(t_i - t_j - 1)!} e^{-\frac{(x-y)}{2}} e^{\alpha_j y} \right|^2 \\ &= \frac{1}{((t_i - t_j - 1)!)^2} \int_{u_j}^{\infty} dx e^{-(\alpha_i - \alpha_j)x} \int_0^{\infty} ds s^{2(t_i - t_j - 1)} e^{-(1 + \alpha_i + \alpha_j)y} \end{aligned} \quad (\text{C.21})$$

is finite, and hence $N \mathcal{H} M_j^{-1} P_{u_j}$ is a Hilbert-Schmidt operator. Therefore, $M_i P_{u_i} V_{i,j} P_{u_j} M_j^{-1}$ is a trace-class operator.

Now we consider K^{conj} . From (2.16),

$$P_{u_i} K_{i,j}^{\text{conj}} P_{u_j} = P_{u_i} K_{i,j}^{\text{conj}} P_{u_j} + (a + b) P_{u_i} M_i f_i \otimes g_j M_j^{-1} P_{u_j}. \quad (\text{C.22})$$

But by changing the contour in (2.12),

$$f_i(x) = \phi_i(a) e^{-ax} + \frac{-1}{2\pi i} \oint_{\Gamma_{1/2}} dw \frac{\phi_i(w) e^{-wx}}{a - w}. \quad (\text{C.23})$$

Hence there is a constant $C > 0$ such that $|f_i(x)| \leq C e^{-ax}$ for some constant C for $x \geq u_i$. Therefore, $M_i(x) f_i(x) \in L^2((u_i, \infty))$ if $\alpha_i > -a$. Similarly, $g_j(y) M_j(y) \in L^2((u_j, \infty))$ if $\alpha_j < b$. Being a product of two Hilbert-Schmidt operator, $P_{u_i} M_i f_i \otimes g_j M_j^{-1} P_{u_j}$ is trace-class if (C.2) holds. This completes the proof. \square

D Gaussian fluctuations

Consider the directed percolation model defined in (1.1), but look away from the characteristic line. We set

$$x = \frac{\gamma}{1+\gamma}N, \quad y = \frac{1}{1+\gamma}N. \quad (\text{D.1})$$

Denote $Q(a, b) = G((a, b), (x, y))$ the passage time from (a, b) to (x, y) . Then

$$G(x, y) = \max\{Q(0, 1), Q(1, 0)\}. \quad (\text{D.2})$$

In Section 6 of [2] (see also [27]) a last passage percolation with one source only is considered. Nevertheless, the arguments given therein can readily be used to prove the following: if $\gamma > \gamma_c = \rho^2/(1-\rho)^2$

$$\lim_{N \rightarrow \infty} \mathbb{P}(Q(1, 0) \leq c_1 N + c_2 s N^{1/2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s dx e^{-x^2/2} \equiv \Phi(s), \quad (\text{D.3})$$

where $c_1 = \frac{\gamma}{1+\gamma}(\frac{1}{\rho} + \frac{1}{\gamma(1-\rho)})$ and $c_2 = \sqrt{\frac{\gamma}{1+\gamma}} \sqrt{\frac{1}{\rho^2} - \frac{1}{\gamma(1-\rho^2)}}$. If $\gamma < \gamma_c$ then there exists a constant c'_2 (see [2]) such that

$$\lim_{N \rightarrow \infty} \mathbb{P}(Q(1, 0) \leq c'_1 N + c'_2 s N^{1/3}) = F_{\text{GUE}}(s), \quad (\text{D.4})$$

where $c'_1 = \frac{\gamma}{1+\gamma}(1 + \frac{1}{\sqrt{\gamma}})^2$ and F_{GUE} is the GUE Tracy-Widom distribution. By symmetry, if $\gamma < \gamma_c$, setting $b_1 = \frac{\gamma}{1+\gamma}(\frac{1}{1-\rho} + \frac{1}{\gamma\rho})$ and $b_2 = \sqrt{\frac{\gamma}{1+\gamma}} \sqrt{\frac{1}{(1-\rho)^2} - \frac{1}{\gamma\rho^2}}$, it holds

$$\lim_{N \rightarrow \infty} \mathbb{P}(Q(0, 1) \leq b_1 N + b_2 s N^{1/2}) = \Phi(s) \quad (\text{D.5})$$

and for $\gamma > \gamma_c$

$$\lim_{N \rightarrow \infty} \mathbb{P}(Q(0, 1) \leq c'_1 N + c'_2 s N^{2/3}) = F_{\text{GUE}}(s). \quad (\text{D.6})$$

Consider now the case $\gamma > \gamma_c$. Then since $c'_1 < c_1$ we have

$$\lim_{N \rightarrow \infty} \mathbb{P}(Q(0, 1) \leq c_1 N + c_2 s N^{1/2}) = 1. \quad (\text{D.7})$$

With the notation $d := c_1 N + c_2 s N^{1/2}$ and (D.2) we obtain

$$\begin{aligned} \mathbb{P}(Q(1, 0) \leq d) &\geq \mathbb{P}(G(x, y) \leq d) \\ &= \mathbb{P}(Q(0, 1) \leq d \cap Q(1, 0) \leq d) \\ &= \mathbb{P}(Q(0, 1) \leq d) - \mathbb{P}(Q(0, 1) \leq d \cap Q(1, 0) > d) \\ &\geq \mathbb{P}(Q(0, 1) \leq d) - 1 + \mathbb{P}(Q(1, 0) \leq d). \end{aligned} \quad (\text{D.8})$$

We take the limit $N \rightarrow \infty$ on both sides and by (D.3) and (D.7) we get

$$\lim_{N \rightarrow \infty} \mathbb{P}(G(x, y) \leq c_1 N + c_2 s N^{1/2}) = \Phi(s). \quad (\text{D.9})$$

Similarly, for the case $\gamma < \gamma_c$, one shows in the same way that

$$\lim_{N \rightarrow \infty} \mathbb{P}(G(x, y) \leq b_1 N + b_2 s N^{1/2}) = \Phi(s). \quad (\text{D.10})$$

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