Pinched-up periodic KPZ fixed point

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Abstract

The periodic KPZ fixed point is the conjectural universal limit of the KPZ universality class models on a ring when both the period and time critically tend to infinity. For the case of the periodic narrow wedge initial condition, we consider the conditional distribution when the periodic KPZ fixed point is unusually large at a particular position and time. We prove a conditional limit theorem up to the "pinch-up" time. When the period is large enough, the result is the same as that for the KPZ fixed point on the line obtained by Liu and Wang in 2022. We identify the regimes in which the result changes and find probabilistic descriptions of the limits.

1 Introduction and main results

The KPZ fixed point is a universal two-dimensional random field [17] to which the height functions of many random growth models on the line are expected to converge in the large-time limit. Among various properties found for the KPZ fixed point (see, for example, [7, 11, 6, 14, 15, 19, 5] and references therein) is the recent study on conditional distributions when the field is uncharacteristically large at a specific position and time [16]. This paper aims to study similar conditional distributions for the periodic KPZ fixed point, which arises as the universal limit for random growth models on a ring. The size of the ring affects the field, and the interest is to determine the effect of domain size on the conditional distribution. We first review a result for the "pinched-up" KPZ fixed point and then introduce the periodic KPZ fixed point.

1.1 KPZ fixed point when it is pinched-up

Let $\mathsf{H}(x,t)$ denote the KPZ fixed point with the narrow wedge initial condition. Consider the event that $\mathsf{H}(0,1) = L$ is large. It was showed in [18] that conditional on this event, $\mathsf{H}(x,t) - L$ is distributed as a properly scaled Tracy-Widom distribution for every fixed $(x,t) \in \mathbb{R} \times (1,\infty)$ as $L \to \infty$. This is consistent with the intuition that the conditioning makes the shifted height $\mathsf{H}(x,1) - L$ close to the narrow wedge, and thus, from the Markovian property, the pinched-up process after time $t = 1$ should look again like the KPZ fixed point with the narrow wedge initial condition, starting at $t = 1$. On the other hand, for $t \in (0,1)$, the following result was proven.

Theorem 1.1 ([16])**.** *Let $\mathsf{H}(x,t)$ be the KPZ fixed point with the narrow wedge initial condition. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be independent Brownian bridges. Then, conditional on the event that $\mathsf{H}(0,1) = L$,*

$$\frac{\mathsf{H}(\frac{x}{\sqrt{2}L^{1/4}}, t) - tL}{\sqrt{2}L^{1/4}} \to (\mathbb{B}_1(t) - x) \wedge (\mathbb{B}_2(t) + x) \tag{1}$$

for $(x,t) \in \mathbb{R} \times (0,1)$ in the finite-dimensional distributions sense as $L \to \infty$.

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The KPZ fixed point with the narrow wedge initial condition satisfies the invariance properties

$$\alpha \mathsf{H}(\alpha^{-2}x, \alpha^{-3}t) \stackrel{d}{=} \mathsf{H}(x,t) \quad \text{and} \quad \mathsf{H}(x,t) \stackrel{d}{=} \mathsf{H}(x+\beta t, t) + \frac{1}{t}\left((x+\beta t)^2 - x^2\right) \tag{2}$$

for every $\alpha > 0$ and $\beta \in \mathbb{R}$. Thus, (1) also implies a result when the conditioning is given at a general point (X, T) instead of $(0, 1)$ (see [16, Remark 1.5]): conditioned that $\mathsf{H}(X, T) = L$,

$$\frac{\mathsf{H}(tX + \frac{xT^{3/4}}{\sqrt{2}L^{1/4}}, tT) - tL}{\sqrt{2}T^{1/4}L^{1/4}} \to (\mathbb{B}_1(t) - x) \wedge (\mathbb{B}_2(t) + x) \tag{3}$$

as $L \to \infty$ for $(x, t) \in \mathbb{R} \times (0, 1)$.

The papers [8, Theorem 1.9] and [10] also considered conditional limit theorems and obtained the first-order term and concentration results. The result (1) has an implication on the geodesics as well. Based on the above result, the authors of [16] conjectured that under the same conditional event, the geodesic converges to the Brownian bridge. This conjecture was recently proved by [9] using geometric and probabilistic methods. These results show that the geodesic between $(0, 0)$ to $(0, 1)$ typically stays within the distance of order $L^{-1/4}$ from the straight line.

1.2 Periodic KPZ fixed point

Let $\mathbf{h}(n, t)$ be the height function of the totally asymmetric simple exclusion process (TASEP) on the discrete ring of size $2a$. We identify the ring as the set $\{-a + 1, \cdots, a\}$ and extend the TASEP periodically on the integers $\mathbb{Z}$ by setting $\mathbf{h}(n \pm 2a, t) = \mathbf{h}(n, t)$. We may call the extended TASEP a periodic TASEP of period $2a$. Suppose that initially, $\mathbf{h}(n, 0) = |n|$ for $-a + 1 \le n \le a$ and is extended periodically. This initial condition is called the periodic step initial condition.

An interesting large time limit arises when the period $2a$ is proportional to $t^{2/3}$, which is called a relaxation time scale. In this limit, the ring size affects the fluctuations of the height function critically. It was shown[1] in [2] that for every positive integer m, for every m distinct points $(\gamma_i, \tau_i) \in \mathbb{R} \times \mathbb{R}_+$ and every m real numbers β_i, $i = 1, \cdots, m$,

$$\lim_{T=(2a)^{3/2}\to\infty} \mathbb{P}\left(\bigcap_{i=1}^{m}\left\{\frac{\mathbf{h}(\gamma_i T^{2/3}, 2\tau_i T) - \tau_i T}{-T^{1/3}} \le \beta_i\right\}\right) = \mathbf{F}_m(\beta; \gamma, \tau) \tag{4}$$

converges, where $\beta = (\beta_1, \cdots, \beta_m)$, $\tau = (\tau_1, \cdot, \tau_m)$, and $\gamma = (\gamma_1, \cdots, \gamma_m)$. The function $\mathbf{F}_m(\beta; \gamma, \tau)$ are periodic with the shift $\gamma_i \mapsto \gamma_i + 1$ for any i, and can be extend to continuously for $(\gamma, \tau, \beta) \in \mathbb{R}^m \times \mathbb{R}_+^m \times \mathbb{R}^m$. The functions $\mathbf{F}_m$, $m = 1, 2, \cdots$, form a consistent collection of multivariate cumulative distribution functions See Section 2.2 and Appendix A for the formula and properties of these functions.

Let $\mathcal{H}^{(per)}(\gamma, \tau)$, $(\gamma, \tau) \in \mathbb{R} \times \mathbb{R}_+$, be a process defined by the collection $\mathbf{F}_m$. It satisfies the spatial periodicity

$$\mathcal{H}^{(per)}(\gamma + 1, \tau) = \mathcal{H}^{(per)}(\gamma, \tau).$$

Since we will only discuss finite-dimensional properties in this paper, we will simply call it the periodic KPZ fixed point with the periodic narrow wedge initial condition. The limit (4) was also proved for the discrete-time TASEP [13] and the PushASEP [12] on a ring. The distribution functions $\mathbf{F}_m$ are expected to be the universal limits for the multi-time, multi-position distributions of the KPZ universality class in the periodic domain at the relaxation time scale. The convergence (4) was also extended to other initial conditions satisfying some technical assumptions [3]. Since we will only consider the periodic narrow wedge initial condition case in this paper and leave other initial conditions for future consideration, we will simply call $\mathcal{H}^{(per)}(\gamma, \tau)$ the periodic KPZ fixed point without mentioning the initial condition.

[1] In [2], the case when $\gamma_i \neq \gamma_{i'}$, $\tau_i = \tau_{i'}$ and $\beta_i = \beta_{i'}$ for some $i \neq i'$ was not analyzed. See Appendix A how we can obtain the result in this case.

Unlike the KPZ fixed point, the periodic KPZ fixed point does not satisfy the invariance properties (2). Instead, it was conjectured in [2] that

$$\epsilon^{-1/3}\mathcal{H}^{(per)}(2\epsilon^{2/3}\gamma,\epsilon\tau)\to \mathsf{H}(\gamma,\tau) \qquad \text{as } \epsilon\to 0 \tag{5}$$

and

$$\frac{\sqrt{2}}{\pi^{1/4}T^{1/2}}\left(\mathcal{H}^{(per)}(\gamma,\tau T)+\tau T\right)\to \mathsf{B}(\tau) \qquad \text{as } T\to\infty \tag{6}$$

where B is a Brownian motion. The limit in (6) does not depend on γ. The one-point distribution case of (5) with $\gamma=0$ was verified in [4, Theorem 1.6] and the one-point distribution case of (6) was proved in [4, Theorem 1.5].

The process $\mathcal{H}^{(per)}$ has period 1. It is illuminating to consider the general periods. For $p>0$, let

$$\mathcal{H}_p(\gamma,\tau):=p^{1/2}\mathcal{H}^{(per)}(p^{-1}\gamma,p^{-3/2}\tau). \tag{7}$$

Then, it satisfies

$$\mathcal{H}_p(\gamma+p,\tau)=\mathcal{H}_p(\gamma,\tau) \tag{8}$$

We call it the p-periodic KPZ fixed point (with the periodic narrow wedge initial condition). The processes with different parameters are related by the formula

$$p^{-1/2}\mathcal{H}_p(p\gamma,p^{3/2}\tau)\stackrel{d}{=}q^{-1/2}\mathcal{H}_q(q\gamma,q^{3/2}\tau) \tag{9}$$

for all $p,q>0$. The conjectures (5) and (6) are translated to the conjecture on the large period limit

$$\mathcal{H}_p(2\gamma,\tau)\to\mathsf{H}(\gamma,\tau) \qquad \text{as } p\to\infty \tag{10}$$

and the conjecture on the small period limit

$$\frac{\sqrt{2}p^{1/4}}{\pi^{1/4}}\left(\mathcal{H}_p(\gamma,\tau)+p^{-1}\tau\right)\to\mathsf{B}(\tau) \qquad \text{as } p\to 0. \tag{11}$$

1.3 Results

The goal of this paper is to study the periodic KPZ fixed point when it is unusually large at the origin at a specific time. We obtain the following results up to the time before the “pinch-up”. There are three theorems depending on the period.

Theorem 1.2. *Conditional on the event that $\mathcal{H}_p(0,1)=\ell$,*

$$\frac{\mathcal{H}_p(\frac{\sqrt{2}x}{\ell^{1/4}},t)-t\ell}{\sqrt{2}\ell^{1/4}}\to(\mathbb{B}_1(t)-x)\wedge(\mathbb{B}_2(t)+x)$$

for $(x,t)\in\mathbb{R}\times(0,1)$ in the sense of convergence of finite-dimensional distributions as $\ell\to\infty$ if [2]

$$\ell^{-1/4}\ll p \qquad \textit{and} \qquad \log p\ll\ell^{3/2}.$$

Here, $\mathbb{B}_1$ and $\mathbb{B}_2$ are independent Brownian bridges.

[2]The notations mean that $p\ell^{1/4}\to\infty$ and $\log p/\ell^{3/2}\to 0$ as $\ell\to\infty$.

Theorem 1.3. *Conditional on the event that* $\mathcal{H}_p(0,1)=\ell$,

$$\frac{\mathcal{H}_p(\frac{\sqrt{2}x}{\ell^{1/4}},t)-t\ell}{\sqrt{2}\ell^{1/4}} \to \mathsf{M}_{\mathsf{r}}(x,t)$$

for $(x,t)\in\mathbb{R}\times(0,1)$ *in the sense of convergence of finite-dimensional distributions as* $\ell\to\infty$ *if*

$$p=\mathsf{r}\sqrt{2}\ell^{-1/4} \qquad \textit{with } \mathsf{r}>0.$$

Here, the random field $\mathsf{M}_{\mathsf{r}}(x,t)$ *is defined in Subsection 1.4.*

Theorem 1.4. *Conditional on the event that* $\mathcal{H}_p(0,1)=\ell$,

$$\frac{\mathcal{H}_p(\frac{\sqrt{2}x}{\ell^{1/4}},t)-t\ell}{\sqrt{2}\ell^{1/4}} \to \frac{1}{\sqrt{2}}\mathbb{B}(t)$$

for $(x,t)\in\mathbb{R}\times(0,1)$ *in the sense of convergence of finite-dimensional distributions as* $\ell\to\infty$ *if*

$$\ell^{-1}\log\ell \ll p \ll \ell^{-1/4}.$$

Here, $\mathbb{B}$ *is a Brownian bridge.*

We have several remarks.

- Let $\widetilde{\mathcal{H}}_p(x,t):=\frac{\mathcal{H}_p(\frac{\sqrt{2}x}{\ell^{1/4}},t)-t\ell}{\sqrt{2}\ell^{1/4}}$. The above results should be understood as, for every positive integer m and real numbers $h_1,\cdots,h_{m-1}$, the limit

$$\lim_{\ell\to\infty}\mathbb{P}\left(\widetilde{\mathcal{H}}_p(x_1,t_1)\ge h_1,\cdots,\widetilde{\mathcal{H}}_p(x_{m-1},t_{m-1})\ge h_{m-1}\,\middle|\,\mathcal{H}_p(0,1)=\ell\right) \tag{12}$$

  exists in each case and is given by the probabilities described in the theorems. The conditioning on $\mathcal{H}_p(0,1)=\ell$ means

$$\lim_{\epsilon\to0}\frac{\mathbb{P}\left(\widetilde{\mathcal{H}}_p(x_1,t_1)\ge h_1,\cdots,\widetilde{\mathcal{H}}_p(x_{m-1},t_{m-1})\ge h_{m-1},\mathcal{H}_p(0,1)\in(\ell-\epsilon,\ell+\epsilon)\right)}{\mathbb{P}(\mathcal{H}_p(0,1)\in(\ell-\epsilon,\ell+\epsilon))}$$

- In all three cases, the position and the height are scaled the same way as in the KPZ fixed point case (1), except for multiplicative 2 in the spatial variable (see (10)).

- As mentioned at the end of Section 1.1, if the KPZ fixed point is conditioned on the event $\mathsf{H}(0,1)=\ell\to\infty$, then its geodesic stays within the order $\ell^{-1/4}$ from the straight line from $(0,0)$ to $(0,1)$. Since $\mathcal{H}_p$ has the period p, it is natural to conjecture that limit theorems for the periodic KPZ fixed point take different forms depending on $p\gg\ell^{-1/4}$ or $p\ll\ell^{-1/4}$. The results above show that the critical regime is indeed when p is same order as $\ell^{-1/4}$.

- In Theorem 1.2, p is allowed to tend to zero, stay $O(1)$, or tend to infinity as long as it satisfies $p\gg\ell^{-1/4}$ and $\log p\ll\ell^{3/2}$. In this case, the limit is exactly same that of the KPZ fixed point (1) as the KPZ fixed point (except for the factor 2 in the spatial scale). The condition $\log p\ll\ell^{3/2}$ is technical one. We expect that Theorem 1.2 holds true as long as $p\gg\ell^{-1/4}$ but it is not clear how to remove this condition from our proof.

- In Theorem 1.3 and 1.4, the period p necessarily tends to zero. Theorem 1.3 corresponds to the case when the period and the geodesic interact critically, while Theorem 1.4 is about the situation when the geodesic is overwhelmed by the period. The condition $p \gg \ell^{-1}\log\ell$ in theorem 1.4 is also a technical one, which may be weakened, but we do not expected that it can be completely removed.
- From the scaling property (9), the theorems imply similar results when we condition at time τ instead of time 1: conditional on the event that $\mathcal{H}_p(0,\tau)=\ell$, the limit of

$$\frac{\mathcal{H}_p(\frac{\sqrt{2}x\tau^{3/4}}{\ell^{1/4}}, t\tau) - t\ell}{\sqrt{2}\tau^{1/4}\ell^{1/4}}$$

  as $\ell \to \infty$ is equal, in the sense of finite dimensional distributions, to one of the above three theorems.
- For the case when $p = O(1)$ or $p \to \infty$ in Theorem 1.2, it is also interesting to consider the case when the conditioning is given at a general position instead of the origin. We expect a result similar to (3). However, unlike the KPZ fixed point situation, the periodic KPZ fixed point does not satisfy the invariance properties (2), and the result does not directly follow from Theorem 1.2. This situation will be studied in a separate paper.

1.4 The limiting distribution for Theorem 1.3

We now describe the random field M_r appearing in Theorem 1.3. It depends on a positive parameter r.

Definition 1.5. *For* $\mathsf{r} > 0$*, define the function*

$$\mathsf{w}_\mathsf{r}(x,y) = \max\left\{ s\in\mathbb{R} : \left[\frac{x-s}{\mathsf{r}}\right] + \left[\frac{y-s}{\mathsf{r}}\right] \ge 0 \right\}, \qquad x,y\in\mathbb{R}, \tag{13}$$

where $[a]$ *denotes the largest integer less than or equal to* a*.*

The maximum is achieved since the function $s \mapsto \left[\frac{x-s}{\mathsf{r}}\right] + \left[\frac{y-s}{\mathsf{r}}\right]$ is non-increasing and left-continuous. The function $\mathsf{w}_\mathsf{r}(x,y)$ satisfies the following properties.

Lemma 1.6. *(a) If* $(x,y)-(x',y') = k(\mathsf{r},-\mathsf{r})$ *for some* $k\in\mathbb{Z}$*, then* $\mathsf{w}_\mathsf{r}(x,y) = \mathsf{w}_\mathsf{r}(x',y')$*.*

(b) For every $x,y\in\mathbb{R}$*,*

$$\mathsf{w}_\mathsf{r}(x,y) = \max\left\{ x+\mathsf{r}\left[\frac{y-x}{2\mathsf{r}}\right], y-\mathsf{r}\left[\frac{y-x}{2\mathsf{r}}\right]-\mathsf{r}\right\} = \max\left\{ y+\mathsf{r}\left[\frac{x-y}{2\mathsf{r}}\right], x-\mathsf{r}\left[\frac{x-y}{2\mathsf{r}}\right]-\mathsf{r}\right\}. \tag{14}$$

(c) For every $x,y\in\mathbb{R}$*,*

$$\lim_{\mathsf{r}\to\infty}\mathsf{w}_\mathsf{r}(x,y) = x\wedge y \quad \text{and} \quad \lim_{\mathsf{r}\to 0}\mathsf{w}_\mathsf{r}(x,y) = \frac{x+y}{2}. \tag{15}$$

(d) For every $\mathsf{r}>0$*, the function* $\mathsf{w}_\mathsf{r}(x,y)$ *is a non-decreasing continuous function in both* x *and* y *variables.*

(e) For every x,y*, the function* $\mathsf{r}\mapsto\mathsf{w}_\mathsf{r}(x,y)$ *is continuous.*

Proof. (a) is straightforward.

(b) Both formulas (13) and (14) are invariant if (x,y) is changed to $(x+\mathsf{r}, y-\mathsf{r})$. Hence it is sufficient to show that they are the same when $0\le y-x<2\mathsf{r}$. In this case, the first formula of (14) is equal to $\mathsf{w}_\mathsf{r}(x,y)=\max\{x,y-r\}$. On the other hand, the formula (13) is equal to x if $x\le y<x+\mathsf{r}$ and is equal to $y-\mathsf{r}$ if $x+\mathsf{r}\le y<x+2\mathsf{r}$. Thus, (13) and the first formula of (14) are the same. The formula (13) implies that $\mathsf{w}_\mathsf{r}(x,y)=\mathsf{w}_\mathsf{r}(y,x)$ and hence the second formula of (14) follows from the first formula.

(c) As $\mathsf{r} \to \infty$, $[\frac{\alpha}{2\mathsf{r}}] \to -1$ if $\alpha < 0$ and $[\frac{\alpha}{2\mathsf{r}}] \to 0$ if $\alpha > 0$. On the other hand, as $\mathsf{r} \to 0$, $\mathsf{r}[\frac{\alpha}{2\mathsf{r}}] \to \frac{\alpha}{2}$ for every real number α. Thus, (c) follows from (b).

(d) The non-decreasing property is easy to check from (13). On the other hand, if $\mathsf{w}_{\mathsf{r}}(x,y) = s$, $\mathsf{w}_{\mathsf{r}}(x',y') = s'$, and $(x,y) = (x',y') + (\epsilon_1, \epsilon_2)$, then we find from (13) that $s - \max\{\epsilon_1, \epsilon_2\} \le s' \le s + \max\{-\epsilon_1, -\epsilon_2\}$. Thus, $\mathsf{w}_{\mathsf{r}}(x,y)$ is jointly continuous in (x,y).

(e) The property clearly holds if $x = y$ since $\mathsf{w}_{\mathsf{r}}(x,x) = x$. For $x \neq y$, since $\mathsf{w}_{\mathsf{r}}(x,y) = \mathsf{w}_{\mathsf{r}}(y,x)$, it is enough to consider the case when $y > x$. Let $f(\mathsf{r}) = x + \mathsf{r}\left[\frac{y-x}{2\mathsf{r}}\right]$ and $g(\mathsf{r}) = y - \mathsf{r}\left[\frac{y-x}{2\mathsf{r}}\right] - \mathsf{r}$. They are continuous functions of $\mathsf{r} \in (0,\infty) \setminus \{\frac{y-x}{2n} : n = 1, 2, \cdots\}$, and so is $\mathsf{r} \mapsto \mathsf{w}_{\mathsf{r}}(x,y) = \max\{f(\mathsf{r}), g(\mathsf{r})\}$. At $\mathsf{r}_0 = \frac{y-x}{2n}$, we find that $f(\mathsf{r}) \to \frac{x+y}{2}$ as $\mathsf{r} \uparrow \mathsf{r}_0$ and $f(\mathsf{r}) \to \frac{x+y}{2} - \frac{y-x}{2n}$ as $\mathsf{r} \downarrow \mathsf{r}_0$ while $g(\mathsf{r}) \to \frac{x+y}{2} - \frac{y-x}{2n}$ as $\mathsf{r} \uparrow \mathsf{r}_0$ and $g(\mathsf{r}) \to \frac{x+y}{2}$ as $\mathsf{r} \downarrow \mathsf{r}_0$. Thus, $\mathsf{w}_{\mathsf{r}}(x,y) \to \mathsf{w}_{\mathsf{r}_0}(x,y)$ as $\mathsf{r} \to \mathsf{r}_0$ in either direction. This completes the proof. □

We now define $\mathsf{M}_{\mathsf{r}}(x,t)$.

Definition 1.7. *For every* $\mathsf{r} > 0$*, define*

$$\mathsf{M}_{\mathsf{r}}(x,t) = \mathsf{w}_{\mathsf{r}}\left(\frac{1}{\sqrt{2}}\big(\mathbb{B}_1^{(\mathsf{r})}(t) + \mathbb{B}_2(t)\big) - x, \frac{1}{\sqrt{2}}\big(-\mathbb{B}_1^{(\mathsf{r})}(t) + \mathbb{B}_2(t)\big) + x\right), \qquad (x,t) \in \mathbb{R} \times (0,1),$$

where $\mathbb{B}_1^{(\mathsf{r})}(t)$ *is a Brownian bridge on the circle* $S^1(\mathsf{r}) = \mathbb{R}/\sqrt{2}\mathsf{r}\mathbb{Z}$ *and* $\mathbb{B}_2(t)$ *is an independent standard Brownian bridge.*

Here we identify $S^1(\mathsf{r})$ with the interval $(-\frac{\mathsf{r}}{\sqrt{2}}, \frac{\mathsf{r}}{\sqrt{2}}]$. Lemma 1.6 (a) shows that $\mathsf{M}_{\mathsf{r}}(x,t)$ is unchanged even if we change the value of $\mathbb{B}_1^{(\mathsf{r})}(t)$ by an addition of $\sqrt{2}\mathsf{r}k$ for some integer k. Thus, $\mathsf{M}_{\mathsf{r}}(x,t)$ is well-defined.

The limits $\mathsf{M}_{\mathsf{r}}(x,t)$ as $\mathsf{r} \to \infty$ and $\mathsf{r} \to 0$ are the ones appearing Theorems 1.2 and 1.4, respectively. As $\mathsf{r} \to \infty$, $S^1(\mathsf{r})$ becomes $\mathbb{R}$ and $\mathbb{B}_1^{(\mathsf{r})}(t)$ converges to a Brownian bridge $\mathbb{B}_1(t)$. Hence, by (15), $\mathsf{M}_{\mathsf{r}}(x,t)$ converges to $(\frac{\mathbb{B}_1(t)+\mathbb{B}_2(t)}{\sqrt{2}} - x) \wedge (\frac{-\mathbb{B}_1(t)+\mathbb{B}_2(t)}{\sqrt{2}} + x)$. Since $\frac{\mathbb{B}_1(t)+\mathbb{B}_2(t)}{\sqrt{2}}$ and $\frac{-\mathbb{B}_1(t)+\mathbb{B}_2(t)}{\sqrt{2}}$ have the same distribution as a pair of independent Brownian bridges, we see that $\mathsf{M}_{\mathsf{r}}(x,t) \to (\mathbb{B}_1(t) - x) \wedge (\mathbb{B}_2(t) + x)$ in distribution as $\mathsf{r} \to \infty$. On the other hand, the second equation of (15) implies that $\mathsf{M}_{\mathsf{r}}(x,t) \to \frac{1}{\sqrt{2}}\mathbb{B}_2(t)$ as $\mathsf{r} \to 0$.

1.5 Right tail of one-point density

The analysis of the paper also yields the following right tail asymptotics.

Theorem 1.8. *Let*

$$f_p(\beta;\gamma,\tau) = \frac{\mathrm{d}}{\mathrm{d}\beta}\mathbb{P}\left(\mathcal{H}_p(\gamma,\tau) \le \beta\right) \tag{16}$$

be the one-point density function of the periodic KPZ fixed point. Then, as $\ell \to \infty$*,*

$$f_p(\ell;0,1) = \begin{cases} \dfrac{1}{8\pi\ell}e^{-\frac{4}{3}\ell^{3/2}}(1+o(1)) & \text{if } \ell^{-1/4} \ll p \text{ and } \log p \ll \ell^{3/2}, \\ \dfrac{c(\mathsf{r})}{8\pi\ell}e^{-\frac{4}{3}\ell^{3/2}}(1+o(1)) & \text{if } p = \mathsf{r}\sqrt{2}\ell^{-1/4}, \\ \dfrac{1}{4\sqrt{2\pi}\ell^{5/4}p}e^{-\frac{4}{3}\ell^{3/2}}(1+o(1)) & \text{if } \ell^{-1}\log\ell \ll p \ll \ell^{-1/4}, \end{cases} \tag{17}$$

where

$$c(\mathsf{r}) = \sum_{k\in\mathbb{Z}} e^{-\mathsf{r}^2k^2} = \frac{\sqrt{\pi}}{\mathsf{r}}\sum_{k\in\mathbb{Z}} e^{-\frac{\pi^2}{\mathsf{r}^2}k^2}. \tag{18}$$

Thus, if $\ell^{-1/4} \ll p$ and $\log p \ll \ell^{3/2}$, the right-tail is same as that of the density function of the GUE Tracy-Widom distribution. For the KPZ fixed point on the line, which is the $p = \infty$ case of the periodic KPZ fixed point, the one-point distribution is given by the GUE Tracy-Widom distribution. Hence, we expect that the right-tail does not change as long as $p \gg \ell^{-1/4}$.

The right tail when $p = 1$ was previously obtained for the the one-point distribution function. The result [4, Theorem 1.7] shows that[3]

$$\mathbb{P}(\mathcal{H}_1(0,1) > \ell) = \frac{1}{16\pi\ell^{3/2}} e^{-\frac{4}{3}\ell^{3/2}}(1 + O(\ell^{-3/2})) \qquad \text{as } \ell \to \infty.$$

The above theorem when $p = 1$ is consistent with the formal derivative of this result.

1.6 Structure of the paper

The proofs of Theorems 1.2–1.4 are based on an analysis of an explicit formula of the multi-time, multi-position distributions of the periodic KPZ fixed point obtained in [2]. The method is similar to that of [16] for the KPZ fixed point, but the formulas for the periodic KPZ fixed point are more complicated, and hence analysis becomes more involved. The other difficulty is to find probabilistic descriptions of the limits of the formulas, especially for Theorem 1.3, which we first obtain in terms of complicated contour integrals. We guess the probabilistic interpretations of the formulas and check that they are correct by direct computations.

The explicit formula of the multi-time, multi-position distributions of the periodic KPZ fixed point involves an integral of a Fredholm determinant. In Section 2, we introduce this formula and show that upon the conditioning, the integral of some terms of the series expansion of the Fredholm determinant vanishes. We then state four propositions, Proposition 2.8–2.11, and prove Theorems 1.2–1.4 assuming these propositions. We also prove Theorem 1.8 in this section. Section 3 is a preparatory section where we consider a function appearing in the distribution formula to compute its limit and obtain several bounds. Section 4 is the main analytic part of the paper. We perform asymptotic analysis and prove Proposition 2.8–2.10. Proposition 2.11, is proved in Section 5. There are two sections in the Appendix. In Appendix A, we prove (4) for the exceptional values of parameters that were not treated in [2] and also prove the continuity and consistency of the distribution functions $\mathbf{F}_m$. Finally, we show in Section B that the series formula of the Fredholm determinant in Section 2 is the same as that of [2].

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2 Proof of theorems

2.1 Set-up

The conditions on p and ℓ in Theorem 1.2–1.4 are

- (Case 1) $p \gg \ell^{-1/4}$, $\log p \ll \ell^{3/2}$, and $\ell \to \infty$,
- (Case 2) $p = \mathsf{r}\sqrt{2}\ell^{-1/4}$ and $\ell \to \infty$,
- (Case 3) $\ell^{-1} \log \ell \ll p \ll \ell^{-1/4}$ and $\ell \to \infty$,

[3] Asymptotic result was also obtained for $\mathbb{P}(\mathcal{H}_1(\gamma, \tau) > \ell)$ for all γ, τ.

respectively. In the rest of the paper, we will refer these limits as "for Case 1", and so on. In each case, we evaluate the limit of

$$\mathbb{P}\left(\bigcap_{i=1}^{m-1}\left\{\widetilde{\mathcal{H}}_p(x_i,t_i)\ge h_i\right\}\,\middle|\,\mathcal{H}_p(0,1)=\ell\right). \tag{19}$$

in (12). We will also often state that a result holds "eventually" to mean that it holds when the appropriate parameters are large enough. For example, for Case 3, it means that there are positive constants $c_1,c_2,c_3>0$ such that the result holds for all ℓ and p satisfying $\ell\ge c_1$, $p^{-1}\ell^{-1/4}\ge c_2$, and $\frac{p\ell}{\log\ell}\ge c_3$.

The following result from [16] shows that when we consider the limit of (19) it is enough to consider the case when $t_1,\cdots,t_{m-1}$ are all distinct.

Lemma 2.1. *[16, Lemma 3.6] Let Y be a random field on $\mathbb{R}\times(0,T)$ with the property that for every positive integer d and $x_1,\cdots,x_d\in\mathbb{R}$, the cumulative distribution function $\mathbb{P}\left(\cap_{i=1}^d\{Y(x_i,t_i)\le\beta_i\}\right)$ is continuous in the variables β_i and t_i for every $1\le i\le d$. If a sequence of random fields Y_n on $\mathbb{R}\times(0,T)$ satisfies $(Y_n(x_i,t_i))_{i=1,\cdots,d}\to(Y(x_i,t_i))_{i=1,\cdots,d}$ in distribution as $n\to\infty$ for every d and for every $(x_i,t_i)\in\mathbb{R}\times(0,T)$, $i=1,\cdots,d$, where $t_1,\cdots,t_d$ are distinct numbers, then $Y_n(x,t)\to Y(x,t)$ in the sense of convergence of finite-dimensional distributions as $n\to\infty$.*

Thus, the convergence for distinct times imply the convergence for arbitrary times if the limit distributions satisfy a continuity property. The limit fields in Theorems 1.2-1.3 clearly satisfy the continuity properties, and thus, it is enough to prove the convergence in distribution for distinct times only.

2.2 Formula of the distribution functions

Recall the relation (7) between $\mathcal{H}_p$ and $\mathcal{H}^{(per)}$. When the times are distinct,[4] there is an exact formula for the multi-point distributions of $\mathcal{H}^{(per)}$. We state the formula here. We analyze this formula to prove the theorems.

Let $\beta_1,\cdots,\beta_m$ be real numbers. Set $I_i^+=[\beta_i,\infty)$ and $I_i^-=(-\infty,\beta_i]$. Consider m points (γ_1,τ_1), $\cdots$, $(\gamma_m,\tau_m)\in\mathbb{R}\times(0,\infty)$ satisfying $0<\tau_1<\cdots<\tau_m$. The paper [2] obtained formulas for the joint probabilities

$$\mathbb{P}\left(\mathcal{H}^{(per)}(\gamma_1,\tau_1)\in I_1^{\pm},\cdots,\mathcal{H}^{(per)}(\gamma_{m-1},\tau_{m-1})\in I_{m-1}^{\pm},\mathcal{H}^{(per)}(\gamma_m,\tau_m)\in I_m^{-}\right) \tag{20}$$

for an arbitrary choices of $+$ and $-$ in each place. When all but the last signs are positive, we have (see [2, eq. (7.17)] and note the relation (7) between $\mathcal{H}^{(per)}$ and $\mathcal{H}_p$)

$$\mathbb{P}\left(\bigcap_{i=1}^{m-1}\{\mathcal{H}_p(\gamma_i,\tau_i)\ge\beta_i\}\cap\{\mathcal{H}_p(\gamma_m,\tau_m)\le\beta_m\}\right)=\frac{(-1)^{m-1}}{(2\pi\mathrm{i})^m}\oint\cdots\oint C(\mathbf{z})D(\mathbf{z})\prod_{i=1}^m\frac{\mathrm{d}z_i}{z_i} \tag{21}$$

where the contours are circles centered at the origin with the radii satisfying $0<|z_1|<\cdots<|z_m|<1$. With $\mathbf{z}=(z_1,\cdots,z_m)$, the functions $C(\mathbf{z})$ and $D(\mathbf{z})$ are defined in (22) and (24) below. In Appendix A, we will use the case when all signs are negative.

To introduce the function $C(\mathbf{z})$, let $\mathrm{Li}_s(z)$ denote the polylogarithm function of order s. Then,

$$C(\mathbf{z})=\prod_{i=1}^{m-1}\frac{z_i}{z_i-z_{i+1}}\prod_{i=1}^m\frac{e^{\frac{\beta_i}{p^{1/2}}A_1(z_i)+\frac{\tau_i}{p^{3/2}}A_2(z_i)}}{e^{\frac{\beta_i}{p^{1/2}}A_1(z_{i+1})+\frac{\tau_i}{p^{3/2}}A_2(z_{i+1})}}e^{2B(z_i,z_i)-2B(z_{i+1},z_i)} \tag{22}$$

[4]The result is also obtained for equal-time case when $\tau_i=\tau_{i+1}$ for some i as long as $\beta_i<\beta_{i+1}$.

where

$$A_1(z) = -\frac{1}{\sqrt{2\pi}}\mathrm{Li}_{3/2}(z), \qquad A_2(z) = -\frac{1}{\sqrt{2\pi}}\mathrm{Li}_{5/2}(z), \qquad B(z,z') = \frac{1}{4\pi}\sum_{k,k'=1}^{\infty}\frac{z^k(z')^{k'}}{(k+k')\sqrt{kk'}}. \tag{23}$$

Here, we set $z_{m+1} = 0$ in the expressions.

The function $D(\mathbf{z})$ is a Fredholm determinant. The series formula of it is

$$D(\mathbf{z}) = \sum_{\mathbf{n}\in\{0,1,\cdots\}^m}\frac{1}{(\mathbf{n}!)^2}D_{\mathbf{n}}(\mathbf{z}) \tag{24}$$

where $\mathbf{n}! = n_1!n_2!\cdots n_m!$ for $\mathbf{n} = (n_1,\cdots,n_m)$ and $D_{\mathbf{n}}(\mathbf{z})$ is given below. The formula of $D_{\mathbf{n}}(\mathbf{z})$ below is slightly different from that of [2, Lemma 2.10] and we explain in Appendix B how to obtain the formula.[5]

For $|z| < 1$, define the discrete set

$$\mathrm{L}_z = \{w : e^{-w^2/2} = z,\, \mathrm{Re}(w) < 0\}. \tag{25}$$

For $\mathbf{n} = (n_1,\cdots,n_m)$ and distinct complex numbers $z_1,\cdots,z_m$ in the punctured unit disk, let

$$D_{\mathbf{n}}(\mathbf{z}) = \prod_{i=2}^{m}\left(1-\frac{z_{i-1}}{z_i}\right)^{n_i}\left(1-\frac{z_i}{z_{i-1}}\right)^{n_{i-1}}\sum_{U,\hat{U}\in\mathrm{L}_{z_1}^{n_1}\times\cdots\times\mathrm{L}_{z_m}^{n_m}} H_{\mathbf{n}}(U,\hat{U})R_{\mathbf{n}}(U,\hat{U})E_{\mathbf{n}}(U,\hat{U}) \tag{26}$$

with the functions defined as follows. Let

$$h(w,z) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{w}\mathrm{Li}_{1/2}(ze^{(w^2-y^2)/2})\mathrm{d}y \qquad \text{for } \mathrm{Re}(w) < 0. \tag{27}$$

For $U = (U^{(1)},\cdots,U^{(m)})$ and $\hat{U} = (\hat{U}^{(1)},\cdots,\hat{U}^{(m)})$ with $U^{(i)},\hat{U}^{(i)} \in \mathrm{L}_{z_i}^{n_i}$, we write the components $U^{(i)} = (u_i^{(i)},\cdots,u_{n_i}^{(i)})$ and $\hat{U}^{(i)} = (\hat{u}_i^{(i)},\cdots,\hat{u}_{n_i}^{(i)})$. Then,

$$H_{\mathbf{n}}(U,\hat{U}) = \prod_{i=1}^{m}\prod_{j=1}^{n_i} e^{2h_i(u_j^{(i)})-h_{i+1}(u_j^{(i)})-h_{i-1}(u_j^{(i)})+2h_i(\hat{u}_j^{(i)})-h_{i+1}(\hat{u}_j^{(i)})-h_{i-1}(\hat{u}_j^{(i)})} \tag{28}$$

where $h_i(w) := h(w;z_i)$ and $h_0(w) = h_{m+1}(w) = 0$. Next, for $X = (x_1,\cdots,x_a)$ and $Y = (y_1,\cdots,y_a)$, let

$$\mathsf{K}(X;Y) = \det\left(\frac{1}{x_i+y_j}\right)_{i,j=1}^{a} = \frac{\prod_{1\le i<j\le a}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^{a}(x_i+y_j)} \tag{29}$$

denote the Cauchy determinant. We have

$$R_{\mathbf{n}}(U,\hat{U}) = \left[\prod_{i=1}^{m}\prod_{j_i=1}^{n_i}\frac{1}{u_{j_i}^{(i)}\hat{u}_{j_i}^{(i)}}\right]\prod_{i=0}^{m}\mathsf{K}(U^{(i)},-\hat{U}^{(i+1)};\hat{U}^{(i)},-U^{(i+1)}) \tag{30}$$

with the convention that $U^{(0)} = \hat{U}^{(0)} = U^{(m+1)} = \hat{U}^{(m+1)} = \emptyset$. Finally,

$$E_{\mathbf{n}}(U,\hat{U}) = \prod_{i=1}^{m}\prod_{j_i=1}^{n_i}E^{i,+}(u_{j_i}^{(i)})E^{i,-}(\hat{u}_{j_i}^{(i)}), \qquad E^{i,\pm}(s) := e^{-\frac{\tau_i-\tau_{i-1}}{3p^{3/2}}s^3 \pm \frac{\gamma_i-\gamma_{i-1}}{2p}s^2 + \frac{\beta_i-\beta_{i-1}}{p^{1/2}}s}. \tag{31}$$

[5]The paper [5] also discusses another Fredholm determinant formula.

2.3 Derivative of the distribution function

The conditional probability is interpreted as (see (19))

$$\begin{aligned}
&\mathbb{P}\Big(\bigcap_{k=1}^{m-1} \{\mathcal{H}_p(\gamma_k,\tau_k)\ge\beta_k\} \,\Big|\, \mathcal{H}_p(\gamma_m,\tau_m)=\beta_m \Big) \\
&= \lim_{\epsilon\to 0} \frac{\mathbb{P}\left(\cap_{k=1}^{m-1}\{\mathcal{H}_p(\gamma_k,\tau_k)\ge\beta_k\}\cap\{\mathcal{H}_p(\gamma_m,\tau_m)\in(\beta_m-\epsilon,\beta_m+\epsilon)\}\right)}{\mathbb{P}\left(\mathcal{H}_p(\gamma_m,\tau_m)\in(\beta_m-\epsilon,\beta_m+\epsilon)\right)} \\
&= \frac{\frac{\partial}{\partial\beta_m}\mathbb{P}\left(\cap_{k=1}^{m-1}\{\mathcal{H}_p(\gamma_k,\tau_k)\ge\beta_k\}\cap\{\mathcal{H}_p(\gamma_m,\tau_m)\le\beta_m\}\right)}{\frac{\partial}{\partial\beta_m}\mathbb{P}\left(\mathcal{H}_p(\gamma_m,\tau_m)\le\beta_m\right)}.
\end{aligned} \tag{32}$$

We now take a derivative of (21) to find a formula for (32). We have the following result for the numerator. The denominator is given by the same formula with $m=1$. In the result below, compared with (24), the sums are only over $\mathbf{n}\in\{1,2,\cdots\}^m$, instead of being over $\mathbf{n}\in\{0,1,2,\cdots\}^m$. Also $\hat{D}_{\mathbf{n}}(\mathbf{z})$ is the same as $D_{\mathbf{n}}(\mathbf{z})$, except for the extra factor $\sum_{j=1}^{n_m}(u_j^{(m)}+\hat{u}_j^{(m)})$ in the summand. This proof is modeled on a computation from [16].

Proposition 2.2. *Let $\mathbb{N}=\{1,2,\cdots\}$, the set of positive integers. Then,*

$$\begin{aligned}
&\frac{\partial}{\partial\beta_m}\mathbb{P}\left(\cap_{k=1}^{m-1}\{\mathcal{H}_p(\gamma_k,\tau_k)\ge\beta_k\}\cap\{\mathcal{H}_p(\gamma_m,\tau_m)\le\beta_m\}\right) \\
&= \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m p^{1/2}} \oint\cdots\oint \left(A_1(z_m)C(\mathbf{z})\sum_{\mathbf{n}\in\mathbb{N}^m}\frac{D_{\mathbf{n}}(\mathbf{z})}{(\mathbf{n}!)^2} + C(\mathbf{z})\sum_{\mathbf{n}\in\mathbb{N}^m}\frac{\hat{D}_{\mathbf{n}}(\mathbf{z})}{(\mathbf{n}!)^2}\right)\prod_{i=1}^m\frac{\mathrm{d}z_i}{z_i}
\end{aligned} \tag{33}$$

where the contours are the circles centered at the origin with radii satisfying $0<|z_1|<\cdots<|z_m|<1$. The terms $A_1(z)$, $C(\mathbf{z})$, and $D_{\mathbf{n}}(\mathbf{z})$ are defined in (23), (22), and (24), and for $\mathbf{n}=(n_1,\cdots,n_m)$,

$$\hat{D}_{\mathbf{n}}(\mathbf{z}) = \prod_{i=2}^m\left(1-\frac{z_{i-1}}{z_i}\right)^{n_i}\left(1-\frac{z_i}{z_{i-1}}\right)^{n_{i-1}} \sum_{U,\hat{U}\in \mathrm{L}_{z_1}^{n_1}\times\cdots\times\mathrm{L}_{z_m}^{n_m}} H_{\mathbf{n}}(U,\hat{U})\hat{R}_{\mathbf{n}}(U,\hat{U})E_{\mathbf{n}}(U,\hat{U}) \tag{34}$$

where

$$\hat{R}_{\mathbf{n}}(U,\hat{U}) = R_{\mathbf{n}}(U,\hat{U})\sum_{j=1}^{n_m}(u_j^{(m)}+\hat{u}_j^{(m)}). \tag{35}$$

Proof. In the formula (21), β_m appears in two places. Since

$$\frac{\mathrm{d}C(\mathbf{z})}{\mathrm{d}\beta_m} = \frac{1}{p^{1/2}}A_1(z_m)C(\mathbf{z}) \quad\text{and}\quad \frac{\mathrm{d}E_{\mathbf{n}}(U,\hat{U})}{\mathrm{d}\beta_m} = \frac{1}{p^{1/2}}E_{\mathbf{n}}(U,\hat{U})\sum_{j=1}^{n_m}(u_j^{(m)}+\hat{u}_j^{(m)}),$$

we find that

$$\begin{aligned}
&\frac{\partial}{\partial\beta_m}\mathbb{P}\left(\cap_{k=1}^{m-1}\{\mathcal{H}_p(\gamma_k,\tau_k)\ge\beta_k\}\cap\{\mathcal{H}_p(\gamma_m,\tau_m)\le\beta_m\}\right) \\
&= \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m p^{1/2}} \oint\cdots\oint \left(A_1(z_m)C(\mathbf{z})\sum_{\mathbf{n}\in\{0,1,\cdots\}^m}\frac{D_{\mathbf{n}}(\mathbf{z})}{(\mathbf{n}!)^2} + C(\mathbf{z})\sum_{\mathbf{n}\in\{0,1,\cdots\}^m}\frac{\hat{D}_{\mathbf{n}}(\mathbf{z})}{(\mathbf{n}!)^2}\right)\prod_{i=1}^m\frac{\mathrm{d}z_i}{z_i}
\end{aligned} \tag{36}$$

where the sums are over $\mathbf{n}\in\{0,1,\cdots\}^m$. Note the fact that $E(U,\hat{U})$ decays super-exponentially fast as a variable tends to ∞ in the sets L_z where the rate of decay depends only on $|z|\in(0,1)$. Hence the summation of $D_{\mathbf{n}}$ and $\hat{D}_{\mathbf{n}}$ are uniformly convergent. Thus, we can change the order of the integral with respect to $\mathbf{z}$ and summation over $\mathbf{n}$. Now Lemma 2.3 below shows that the integral is zero if one of the components of $\mathbf{n}$ is zero. Thus, we obtain the result. □

Recall that the contours for the integral are circles satisfying $0 < |z_1| < \cdots < |z_m| < 1$.

Lemma 2.3. *If one of the components of* $\mathbf{n} = (n_1, \cdots, n_m)$ *is zero, then*

$$\oint \cdots \oint A_1(z_m) C(\mathbf{z}) D_{\mathbf{n}}(\mathbf{z}) \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{z_i} = 0 \tag{37}$$

and

$$\oint \cdots \oint C(\mathbf{z}) \hat{D}_{\mathbf{n}}(\mathbf{z}) \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{z_i} = 0. \tag{38}$$

Proof. The case when $m = 1$ can be checked directly. Note that in this case $D_{\mathbf{n}}(\mathbf{z}) = 1$, $\hat{D}_{\mathbf{n}}(\mathbf{z}) = 0$. And the functions $C(\mathbf{z})$, $A_1(z_1)/z_1$ are both analytic at $z_1 = 0$. These implies the two identities (37) and (38). Below we assume that $m \geq 2$.

Let $\mathbf{n} = (n_1, \cdots, n_m)$ be given and one of the components is zero. Let k be the smallest integer such that $n_k = 0$. We will show the integrands of both integrals are analytic as a function of z_k in the integration domain, and thus the integrals are zero.

We first focus on the integral in (37). Since $n_k = 0$, the set $\mathrm{L}_{z_k}^{n_k}$ is empty. Thus, the integrand does not contain any factor involving $U^{(k)}$ and $\hat{U}^{(k)}$, which depend on z_k. If $k = 1$, the only term that depends on z_1 in $A_1(z_m) C(\mathbf{z}) D_{\mathbf{n}}(\mathbf{z}) \prod_{i=1}^{m} \frac{1}{z_i}$ is the factor

$$\frac{1}{z_1 - z_2} e^{-\sum_{j=1}^{n_2} (h_1(u_j^{(2)}) + h_1(\hat{u}_j^{(2)}))} \left(1 - \frac{z_1}{z_2}\right)^{n_2} e^{\beta_1 A_1(z_1) + \tau_1 A_2(z_1) + 2B(z_1) - 2B(z_2, z_1)}.$$

Since $|z_1| < |z_2|$, this function is analytic at $z_1 = 0$. This implies the integrand in (37) is analytic in z_1 around the origin. Hence (37) holds when $k = 1$.

If $1 < k < m$, the only term that depends on z_k in $A_1(z_m) C(\mathbf{z}) D_{\mathbf{n}}(\mathbf{z}) \prod_{i=1}^{m} \frac{1}{z_i}$ is

$$\begin{aligned} &\frac{1}{(z_{k-1} - z_k)(z_k - z_{k+1})} e^{-\sum_{i=1}^{n_{k-1}} (h_k(u_i^{(k-1)}) + h_k(\hat{u}_i^{(k-1)})) - \sum_{j=1}^{n_{k+1}} (h_k(u_j^{(k+1)}) + h_k(\hat{u}_j^{(k+1)}))} \\ &\times \left(1 - \frac{z_k}{z_{k-1}}\right)^{n_{k-1}} \left(1 - \frac{z_k}{z_{k+1}}\right)^{n_{k+1}} e^{(\beta_k - \beta_{k-1}) A_1(z_k) + (\tau_k - \tau_{k-1}) A_2(z_k) + 2B(z_k) - 2B(z_k, z_{k-1}) - 2B(z_{k+1}, z_k)}. \end{aligned}$$

As a function of z_k, it is of the form

$$(z_k - z_{k-1})^{n_{k-1} - 1} (z_k - z_{k+1})^{n_{k+1} - 1} \times (\text{a term analytic in } |z_k| < 1)$$

Since $n_{k-1} \geq 1$, the first factor is analytic in z_k. On the other hand, due to the contour conditions, the second factor is analytic in $|z_k| < |z_{k+1}|$. Thus, the whole term is analytic at $z_k = 0$, and we obtain (37) when $1 < k < m$.

If $k = m$, the only term that depends on z_m in $A_1(z_m) C(\mathbf{z}) D_{\mathbf{n}}(\mathbf{z}) \prod_{i=1}^{m} \frac{1}{z_i}$ is

$$\begin{aligned} &\frac{A_1(z_m)}{z_m(z_{m-1} - z_m)} e^{-\sum_{j=1}^{n_{m-1}} (h_m(u_j^{(m-1)}) + h_m(\hat{u}_j^{(m-1)}))} \\ &\times \left(1 - \frac{z_m}{z_{m-1}}\right)^{n_{m-1}} e^{(\beta_m - \beta_{m-1}) A_1(z_m) + (\tau_m - \tau_{m-1}) A_2(z_m) + 2B(z_m) - 2B(z_m, z_{m-1})}. \end{aligned}$$

As a function of z_m, it is of the form

$$(z_m - z_{m-1})^{n_{m-1} - 1} \frac{A_1(z_m)}{z_m} \times (\text{a term analytic in } |z_m| < 1)$$

Since $n_{m-1} \geq 1$, the above is analytic at $z_m = z_{m-1}$. On the other hand, since $A_1(0) = 0$, the term $\frac{A_1(z_m)}{z_m}$ is analytic at $z_m = 0$. Thus, the integrand in (37) is analytic in z_m within the integration contour. We obtain (37).

The proof of (38) is exactly the same as that of (37) when $k < m$ since $\hat{D}_{\mathbf{n}}(\mathbf{z})$ is the same as $D_{\mathbf{n}}(\mathbf{z})$ except an extra factor $\sum_{j=1}^{n_m}(u_j^{(m)} + \hat{u}_j^{(m)})$ which does not depend on z_k. When $k = m$, we have $n_m = 0$. This factor $\sum_{j=1}^{n_m}(u_j^{(m)} + \hat{u}_j^{(m)}) = 0$ hence the integrand is zero. We still have (38). □

From the above results, we can write the probability in (19) as (43) below.

Definition 2.4. *For $\mathbf{z} = (z_1, \cdots, z_m)$ with $0 < |z_1| < \cdots < |z_m| < 1$, define*

$$C^\bullet(\mathbf{z}) = C(\mathbf{z}) \prod_{i=1}^{m-1} \frac{z_i - z_{i+1}}{z_i} = \prod_{i=1}^{m} \frac{e^{\frac{\beta_i}{p^{1/2}} A_1(z_i) + \frac{\tau_i}{p^{3/2}} A_2(z_i)}}{e^{\frac{\beta_i}{p^{1/2}} A_1(z_{i+1}) + \frac{\tau_i}{p^{3/2}} A_2(z_{i+1})}} e^{2B(z_i,z_i) - 2B(z_{i+1},z_i)} \tag{39}$$

where A_1, A_2, B are given in (23)*, and we set $z_{m+1} = 0$. Define*

$$\begin{aligned} D_{\mathbf{n}}^\bullet(\mathbf{z}) &= \sum_{U, \hat{U} \in \mathrm{L}_{z_1}^{n_1} \times \cdots \times \mathrm{L}_{z_m}^{n_m}} H_{\mathbf{n}}(U, \hat{U}) R_{\mathbf{n}}(U, \hat{U}) E_{\mathbf{n}}(U, \hat{U}), \\ \hat{D}_{\mathbf{n}}^\bullet(\mathbf{z}) &= \sum_{U, \hat{U} \in \mathrm{L}_{z_1}^{n_1} \times \cdots \times \mathrm{L}_{z_m}^{n_m}} H_{\mathbf{n}}(U, \hat{U}) \hat{R}_{\mathbf{n}}(U, \hat{U}) E_{\mathbf{n}}(U, \hat{U}) \end{aligned} \tag{40}$$

where the functions $H_{\mathbf{n}}(U, \hat{U})$, $R_{\mathbf{n}}(U, \hat{U})$, and $E_{\mathbf{n}}(U, \hat{U})$ are defined in (28), (30)*, and* (31)*, while the function $\hat{R}_{\mathbf{n}}(U, \hat{U})$ is defined in* (34)*. Also define*

$$T_{\mathbf{n}}^\bullet(\mathbf{z}) = \prod_{i=2}^{m} \left(1 - \frac{z_{i-1}}{z_i}\right)^{n_i} \left(1 - \frac{z_i}{z_{i-1}}\right)^{n_{i-1}-1}. \tag{41}$$

Corollary 2.5. *Let $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbf{1} = (\underbrace{1, \cdots, 1}_{m})$. Let $C^\bullet(\mathbf{z})$, $D_{\mathbf{n}}^\bullet(\mathbf{z})$, and $\hat{D}_{\mathbf{n}}^\bullet(\mathbf{z})$ be given above with the parameters*

$$\tau_i = t_i, \qquad \gamma_i = \frac{\sqrt{2} x_i}{\ell^{1/4}}, \qquad \beta_i = t_i \ell + h_i \sqrt{2} \ell^{1/4} \qquad \text{for } i = 1, \cdots, m, \tag{42}$$

where $0 < t_1 < \cdots < t_{m-1} < 1$, $x_1, \cdots, x_{m-1} \in \mathbb{R}$, $h_1, \cdots, h_{m-1} \in \mathbb{R}$, and $t_m = 1$, $x_m = 0$, $h_m = 0$. Then,

$$\mathbb{P}\left(\bigcap_{i=1}^{m-1} \left\{ \frac{\mathcal{H}(\frac{\sqrt{2}x_i}{\ell^{1/4}}, t_i) - t_i \ell}{\sqrt{2}\ell^{1/4}} \geq h_i \right\} \middle| \mathcal{H}_p(0,1) = \ell \right) = \frac{P_{m,1} + P_{m,2} + \hat{P}_{m,1} + \hat{P}_{m,2}}{P_{1,1} + P_{1,2} + \hat{P}_{1,1} + \hat{P}_{1,2}} \tag{43}$$

where

$$\begin{aligned} P_{m,1} &= \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \oint \cdots \oint A_1(z_m) C^\bullet(\mathbf{z}) D_{\mathbf{1}}^\bullet(\mathbf{z}) T_{\mathbf{1}}^\bullet(\mathbf{z}) \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{z_i}, \\ P_{m,2} &= \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \sum_{\mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}} \frac{1}{(\mathbf{n}!)^2} \oint \cdots \oint A_1(z_m) C^\bullet(\mathbf{z}) D_{\mathbf{n}}^\bullet(\mathbf{z}) T_{\mathbf{n}}^\bullet(\mathbf{z}) \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{z_i}, \\ \hat{P}_{m,1} &= \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \oint \cdots \oint C^\bullet(\mathbf{z}) \hat{D}_{\mathbf{1}}^\bullet(\mathbf{z}) T_{\mathbf{1}}^\bullet(\mathbf{z}) \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{z_i}, \\ \hat{P}_{m,2} &= \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \sum_{\mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}} \frac{1}{(\mathbf{n}!)^2} \oint \cdots \oint C^\bullet(\mathbf{z}) \hat{D}_{\mathbf{n}}^\bullet(\mathbf{z}) T_{\mathbf{n}}^\bullet(\mathbf{z}) \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{z_i}. \end{aligned} \tag{44}$$

2.4 Four propositions

We analyze the equation (43) to prove Theorems 1.2–1.4. We will see that the main contributions to the limit comes from $\frac{\hat{P}_{m,1}}{\hat{P}_{1,1}}$ for all three Cases. There are four propositions in this subsection. Proposition 2.8 computes the limit of $\hat{P}_{m,1}$. Proposition 2.9 shows that $P_{m,1}$ is of a smaller order. Similarly, Proposition 2.10 shows that $P_{m,2}$ and $\hat{P}_{m,2}$ are also of smaller orders. Probabilistic interpretations of the limits from Proposition 2.8 are obtained in Proposition 2.11. In the next subsection, we prove the main theorems assuming these four propositions. The proofs of these propositions are the main analysis of this paper and they given in Section 4 and 5.

All results in this subsection hold uniformly for the parameters in compact subsets of $0 < t_1 < \cdots < t_{m-1} < 1$, $(x_1, \cdots, x_{m-1}) \in \mathbb{R}^{m-1} \in \mathbb{R}$, $(h_1, \cdots, h_{m-1}) \in \mathbb{R}^{m-1}$ although we do not state this fact explicitly.

We first need some definitions.

Definition 2.6. *For every vector* $\mathbf{a} = (a_1, \cdots, a_m)$ *of real numbers, we denote*

$$\Delta a_i = \begin{cases} a_1, & i = 1, \\ a_i - a_{i-1}, & 2 \le i \le m. \end{cases} \tag{45}$$

Definition 2.7. *For* $\mathbf{a} = (a_1, \cdots, a_m) \in \mathbb{R}^m$ *satisfying* $0 < a_1 < \cdots < a_m$ *and* $\mathbf{b} = (b_1, \cdots, b_m) \in \mathbb{R}^m$, *define*

$$S_\infty(\mathbf{a}, \mathbf{b}) = \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \int \cdots \int \prod_{i=2}^m \frac{1}{\xi_i - \xi_{i-1}} \prod_{i=1}^m e^{\frac{\Delta a_i}{2}\xi_i^2 - \Delta b_i \xi_i} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_m \tag{46}$$

where the contours are vertical lines, oriented upward, satisfying $\mathrm{Re}(\xi_1) > \cdots > \mathrm{Re}(\xi_m)$. *For* $\mathbf{w} = (w_1, \cdots, w_m) \in \mathbb{C}^m$ *satisfying* $0 < |w_1| < \cdots < |w_m|$, *define*

$$S_{\mathsf{r}}(\mathbf{a}, \mathbf{b}; \mathbf{w}) = \frac{(-1)^{m-1}}{\mathsf{r}^m} \sum_{\xi_1, \cdots, \xi_m} \prod_{i=2}^m \frac{1}{\xi_i - \xi_{i-1}} \prod_{i=1}^m e^{\frac{\Delta a_i}{2}\xi_i^2 - \Delta b_i \xi_i} \qquad \text{for } \mathsf{r} > 0 \tag{47}$$

where the sum is over the roots of the equations

$$e^{-\mathsf{r}\xi_i} = w_i \qquad \text{for } i = 1, \cdots, m. \tag{48}$$

Let $\mathbf{t} = (t_1, \cdots, t_m) = (t_1, \cdots, t_{m-1}, 1)$, $\mathbf{x} = (x_1, \cdots, x_m) = (x_1, \cdots, x_{m-1}, 0)$, and $\mathbf{h} = (h_1, \cdots, h_m) = (h_1, \cdots, h_{m-1}, 0)$. The first proposition is about $\hat{P}_{m,1}$.

Proposition 2.8. *We have*

$$\frac{4\ell}{p^{1/2}} e^{\frac{4}{3}\ell^{\frac{3}{2}}} \hat{P}_{m,1} \to \begin{cases} S_\infty(\mathbf{t}, \mathbf{h} - \mathbf{x}) S_\infty(\mathbf{t}, \mathbf{h} + \mathbf{x}) & \text{for Case 1,} \\ \oint \cdots \oint S_{\mathsf{r}}(\mathbf{t}, \mathbf{h} - \mathbf{x}; \mathbf{w}) S_{\mathsf{r}}(\mathbf{t}, \mathbf{h} + \mathbf{x}; \mathbf{w}) \prod_{i=2}^m (1 - \frac{w_{i-1}}{w_i}) \prod_{i=1}^m \frac{\mathrm{d}w_i}{2\pi \mathrm{i} w_i} & \text{for Case 2,} \end{cases} \tag{49}$$

and

$$2^{3/2} \ell^{5/4} p^{1/2} e^{\frac{4}{3}\ell^{\frac{3}{2}}} \hat{P}_{m,1} \to S_\infty(2\mathbf{t}, 2\mathbf{h}) \qquad \text{for Case 3.} \tag{50}$$

The integral contours for Case 2 are counterclockwise circles satisfying $0 < |w_1| < \cdots < |w_m|$.

The formula of $\hat{P}_{m,1}$ in (44) contains $\hat{D}_{\mathbf{1}}(\mathbf{z})$, which, from (34), is a series. The above result is obtained by showing that after scaling $\mathbf{z}$ appropriately, the series converges to an integral for Case 1 and to a series for Case 2. Note that S_∞ is an integral while S_{r} is a series. For Case 3, only one term dominates the series $\hat{D}_{\mathbf{1}}(\mathbf{z})$.

The second proposition shows that $P_{m,1}$ is smaller than $\hat{P}_{m,1}$. Note from our assumptions in section 2.1, $p\ell \to \infty$ for all three Cases.

Proposition 2.9. *There is a constant $C > 0$ such that*

$$\left|\frac{\ell}{p^{1/2}} e^{\frac{4}{3}\ell^{\frac{3}{2}}} P_{m,1}\right| \le \frac{C}{\sqrt{p\ell}} e^{-\frac{p\ell}{2}} \qquad \textit{for Case 1 and 2} \tag{51}$$

and

$$\left|\ell^{5/4} p^{1/2} e^{\frac{4}{3}\ell^{\frac{3}{2}}} P_{m,1}\right| \le \frac{C}{\sqrt{p\ell}} e^{-\frac{p\ell}{2}} \qquad \textit{for Case 3} \tag{52}$$

eventually.

The third proposition shows that $P_{m,2}$ and $\hat{P}_{m,2}$ are small.

Proposition 2.10. *There are positive constants δ and C such that*

$$\left|e^{\frac{4}{3}\ell^{\frac{3}{2}}} P_{m,2}\right| \le Ce^{-\delta\ell^{3/2}} \quad \textit{and} \quad \left|e^{\frac{4}{3}\ell^{\frac{3}{2}}} \hat{P}_{m,2}\right| \le Ce^{-\delta\ell^{3/2}}. \tag{53}$$

eventually for all three Cases.

The fourth and final proposition is a probabilistic interpretation of the limits in Proposition 2.8. The result (54) was obtained[6] in [16].

Proposition 2.11. *Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^m$ satisfying $0 < a_1 < \cdots < a_{m-1} < a_m$.*

(a) ([16, Lemma 3.4]) We have[7]

$$S_\infty(\mathbf{a}, \mathbf{b}) = \mathbb{P}\left(\mathsf{B}(a_1) \ge b_1, \cdots, \mathsf{B}(a_{m-1}) \ge b_{m-1}, \mathsf{B}(a_m) = b_m\right) \tag{54}$$

where B is a Brownian motion.

(b) For every $\mathsf{r} \in (0, \infty)$,

$$\begin{aligned} &\oint \cdots \oint S_{\mathsf{r}}(\mathbf{a}, \mathbf{b}; \mathbf{w}) S_{\mathsf{r}}(\mathbf{a}, \mathbf{c}; \mathbf{w}) \prod_{i=2}^{m} \left(1 - \frac{w_{i-1}}{w_i}\right) \prod_{i=1}^{m} \frac{\mathrm{d}w_i}{2\pi \mathrm{i} w_i} \\ &= \mathbb{P}\left(\bigcap_{i=1}^{m-1} \left\{\mathsf{w}_{\mathsf{r}}\big(\mathsf{B}_1(a_i) - \frac{b_i - c_i}{2}, \mathsf{B}_2(a_i) + \frac{b_i - c_i}{2}\big) \ge \frac{b_i + c_i}{2}\right\} \bigcap \left\{\frac{\mathsf{B}_1(a_m) - b_m}{\mathsf{r}} = -\frac{\mathsf{B}_2(a_m) - c_m}{\mathsf{r}} \in \mathbb{Z}\right\}\right) \end{aligned} \tag{55}$$

where the contours are circles satisfying $0 < |w_1| < \cdots < |w_m| < 1$, B_1 and B_2 are independent Brownian motions, and w_{r} is the function defined in (13).

2.5 Proof of Theorems 1.2, 1.3, 1.4, and 1.8

We now prove the main theorems assuming Proposition 2.8–2.11. In (43), denote $P_{m,1} + P_{m,2} + \hat{P}_{m,1} + \hat{P}_{m,2} = P_m$.

Proof of Theorems 1.2, 1.3, and 1.4. For Case 1, Proposition 2.8, 2.9, and 2.10 imply that

$$\frac{4\ell}{p^{1/2}} e^{\frac{4}{3}\ell^{\frac{3}{2}}} P_m \to S_\infty(\mathbf{t}, \mathbf{h} + \mathbf{x}) S_\infty(\mathbf{t}, \mathbf{h} - \mathbf{x}).$$

[6] We need to set $\xi_i = -u_i$ in (46) to find the formula (3.6) of [16].

[7] The notation $\mathbb{P}(\mathsf{B}(a_1) \in I_1, \cdots, \mathsf{B}(a_{m-1}) \in I_{m-1}, \mathsf{B}(a_m) = b_m)$ means $\int_{I_1} \mathrm{d}y_1 \cdots \int_{I_{m-1}} \mathrm{d}y_{m-1}\, f(y_1, \cdots, y_{m-1}, b_m)$ where f is the joint density function of $\mathsf{B}(a_1), \cdots, \mathsf{B}(a_m)$.

By Proposition 2.11 (a), recalling that $t_m = 1$ and $x_m = h_m = 0$, we find that

$$\frac{P_m}{P_1} \to \frac{\mathbb{P}\left(\bigcap_{i=1}^{m-1} \{\mathsf{B}_1(t_i) - x_i \geq h_i,\, \mathsf{B}_2(t_i) + x_i \geq h_i\} \bigcap \{\mathsf{B}_1(1) = \mathsf{B}_2(1) = 0\}\right)}{\mathbb{P}\left(\mathsf{B}_1(1) = \mathsf{B}_2(1) = 0\right)}$$

for independent Brownian motions B_1 and B_2. The limit is equal to

$$\mathbb{P}\Big(\bigcap_{i=1}^{m-1} \big\{(\mathbb{B}_1(t_i) - x_i) \wedge (\mathbb{B}_2(t_i) + x_i) \geq h_i\big\}\Big)$$

for independent Brownian bridges $\mathbb{B}_1$ and $\mathbb{B}_2$. Theorem 1.2 then follows from (19), (43), and Lemma 2.1.

Similarly, for Case 2, Proposition 2.8, 2.9, 2.10 and Proposition 2.11 (b) imply that $\frac{P_m}{P_1}$ converges to

$$\mathbb{P}\Big(\bigcap_{i=1}^{m-1} \big\{\mathsf{w}_{\mathsf{r}}\big(\mathsf{B}_1'(t_i) - x_i, \mathsf{B}_2'(t_i) + x_i\big) \geq h_i\big\} \Big| \mathsf{B}_1'(1) = -\mathsf{B}_2'(1) \in \mathsf{r}\mathbb{Z}\Big)$$

for independent Brownian motions B_1' and B_2'. Let $\mathsf{B}_1(t) = \frac{\mathsf{B}_1'(t) - \mathsf{B}_2'(t)}{\sqrt{2}}$ and $\mathsf{B}_2(t) = \frac{\mathsf{B}_1'(t) + \mathsf{B}_2'(t)}{\sqrt{2}}$. Then, B_1 and B_2 are again independent Brownian motions and the above probability is equal to

$$\mathbb{P}\Big(\bigcap_{i=1}^{m-1} \big\{\mathsf{w}_{\mathsf{r}}\big(\frac{\mathsf{B}_1(t) + \mathsf{B}_2(t)}{\sqrt{2}} - x_i, \frac{-\mathsf{B}_1(t) + \mathsf{B}_2(t)}{\sqrt{2}} + x_i\big) \geq h_i\big\} \Big| \mathsf{B}_1(1) \in \sqrt{2}\mathsf{r}\mathbb{Z},\, \mathsf{B}_2(1) = 0\Big).$$

Since w_{r} in the formula is unchanged if $\mathsf{B}_1(t)$ is replaced by $\mathsf{B}_1(t) + \sqrt{2}\mathsf{r}k$ for any integer k (see Lemma 1.6 (a)), this is equal to

$$\mathbb{P}\Big(\bigcap_{i=1}^{m-1} \big\{\mathsf{w}_{\mathsf{r}}\big(\frac{\mathbb{B}_1^{(\mathsf{r})}(t) + \mathbb{B}_2(t)}{\sqrt{2}} - x_i, \frac{-\mathbb{B}_1^{(\mathsf{r})}(t) + \mathbb{B}_2(t)}{\sqrt{2}} + x_i\big) \geq h_i\big\}\Big)$$

for a Brownian bridge $\mathbb{B}_1^{(\mathsf{r})}$ on $S^1(\mathsf{r}) = \mathbb{R}/\sqrt{2}\mathsf{r}\mathbb{Z}$ and independent Brownian bridge $\mathbb{B}_2$. Since $\mathsf{w}_{\mathsf{r}}\big(\frac{\mathbb{B}_1^{(\mathsf{r})}(t) + \mathbb{B}_2(t)}{\sqrt{2}} - x, \frac{-\mathbb{B}_1^{(\mathsf{r})}(t) + \mathbb{B}_2(t)}{\sqrt{2}} + x\big) = \mathsf{M}_{\mathsf{r}}(x,t)$ by Definition 1.7, Theorem 1.3 follows.

Finally, for Case 3, Proposition 2.8, 2.9, 2.10 and Proposition 2.11 (a) again show that $\frac{P_m}{P_1}$ converges to

$$\mathbb{P}\left(\bigcap_{i=1}^{m-1} \{\mathsf{B}(2t_i) \geq 2h_i\} \Big| \mathsf{B}(2) = 0\right) = \mathbb{P}\left(\bigcap_{i=1}^{m-1} \Big\{\mathbb{B}(t_i) \geq \sqrt{2}h_i\Big\}\right)$$

where B is a Brownian motion and $\mathbb{B}$ is a Brownian bridge on $[0,1]$. Thus, we obtain Theorem 1.4. □

Proof of Theorem 1.8. Proposition 2.2, Lemma 2.3, and Corollary 2.5 show that

$$f_p(\ell; 0, 1) = \frac{1}{p^{1/2}}(P_{1,1} + P_{1,2} + \hat{P}_{1,1} + \hat{P}_{1,2})$$

with $t_1 = 1$, $x_1 = 0$, and $\ell_1 = \ell$. Propositions 2.8–2.10 thus imply the result. The equality of the two formula of $c(\mathsf{r})$ is due to the Poisson summation formula, $\sum_{k\in\mathbb{Z}} g(k) = \sum_{k\in\mathbb{Z}} \hat{g}(k)$ with $\hat{g}(t) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i t x}\mathrm{d}x$, for suitable functions g. □

We prove Proposition 2.8, 2.9, 2.10 and 2.11 in Section 4 and 5. In the next section, we prove a limit and estimates for a key function that appear in the proofs.

3 Preparations

Let $a > 0$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, and $d \geq 0$. For $\ell > 0$, consider the function from $\mathsf{G}_\ell : \mathbb{R} \to \mathbb{C}$ defined by

$$\mathsf{G}_\ell(x) = 3a\xi(x)^2 + (c-2b)\xi(x) + \frac{1}{\ell^{3/4}}\left(b\xi(x)^2 - a\xi(x)^3\right) \quad \text{where} \quad \xi(x) = -\frac{2(d+\mathrm{i}x)}{1+\sqrt{1+\frac{2(d+\mathrm{i}x)}{\ell^{3/4}}}}. \tag{56}$$

While proving Proposition 2.8, 2.9, and 2.10, we need to analyze the functions $E^{i,\pm}(s)$ in (31). In the appropriate choice of the variable s, $E^{i,\pm}(s)$ are related to the function G_ℓ with particular values of a, b, c, and d: see (85). We compute a pointwise limit and uniform bounds of $\mathsf{G}_\ell(x)$ in this section.

3.1 Pointwise limit

Lemma 3.1 (Pointwise limit)**.** *For every $x \in \mathbb{R}$,*

$$\mathsf{G}_\ell(x_\ell) \to 3a(d+\mathrm{i}x)^2 - (c-2b)(d+\mathrm{i}x) \tag{57}$$

if $\ell \to \infty$ and $x_\ell \to x$. The convergence is uniform for x in a compact subset of $\mathbb{R}$ and for (a,b,c,d) in a compact subset of $(0,\infty) \times \mathbb{R} \times \mathbb{R} \times [0,\infty)$.

Proof. It is clear since $\xi(x_\ell) \to -(d+\mathrm{i}x)$. □

3.2 A lemma

The following simple lemma will be used in the next subsection.

Lemma 3.2. *Let $A \geq 0$. Let r be a solution to the equation $r^2 - \frac{A}{r^2} = 1$. Then, $|r| \geq 1 + \frac{2A}{3(\sqrt{13}+1)}$ if $0 \leq A \leq 3$ and $|r| \geq 1 + \frac{A^{1/4}}{3\sqrt{2}}$ if $A \geq 1$.*

Proof. Solving a quadratic equation, all solutions satisfy

$$|r| = \left(\frac{1+\sqrt{1+4A}}{2}\right)^{1/2} = \left(1 + \frac{2A}{\sqrt{1+4A}+1}\right)^{1/2}.$$

Note that

$$\sqrt{1+y} \geq \begin{cases} 1 + \dfrac{y}{3} & \text{for } 0 \leq y \leq 3, \\ 1 + \dfrac{\sqrt{y}}{3} & \text{for } y \geq 9/16. \end{cases}$$

If $0 \leq A \leq 3$, then $\frac{2A}{\sqrt{1+4A}+1} \leq A \leq 3$, while if $A \geq 1$, then $\frac{\sqrt{1+4A}-1}{2} \geq \frac{\sqrt{5}-1}{2} > \frac{9}{16}$. Hence,

$$|r| \geq 1 + \frac{2A}{3(\sqrt{1+4A}+1)} \qquad \text{if } 0 \leq A \leq 3$$

and

$$|r| \geq 1 + \frac{1}{3}\left(\frac{2A}{\sqrt{1+4A}+1}\right)^{1/2} \qquad \text{if } A \geq 1.$$

The result follows by noting that

$$\sqrt{1+4A}+1 \leq \begin{cases} \sqrt{13}+1 & \text{for } 0 \leq A \leq 3, \\ 1+\sqrt{4A}+1 \leq 4\sqrt{A} & \text{for } A \geq 1. \end{cases}$$

□

3.3 Uniform estimates

We find a uniform upper bound of $|e^{\mathsf{G}_\ell(x)}| = e^{\mathrm{Re}\mathsf{G}_\ell(x)}$. From its definition, $\xi(x)$ satisfies the equation

$$\frac{\xi(x)^2}{\ell^{3/4}} = 2\xi(x) + 2(d + \mathrm{i}x). \tag{58}$$

Thus,

$$\mathsf{G}_\ell(x) = \mathsf{G}_1(x) + 2b(d + \mathrm{i}x) \qquad \text{where} \quad \mathsf{G}_1(x) = 3a\xi(x)^2 + c\xi(x) - \frac{a}{\ell^{3/4}}\xi(x)^3. \tag{59}$$

Consider $\mathsf{G}_1(x)$. Note that $\mathsf{G}_1(x) = -\frac{a}{\ell^{3/4}}(\xi(x) - \ell^{3/4})^3 + 3a\xi(x)\ell^{3/4} - a\ell^{3/2} + c\xi(x)$. Write

$$\frac{\xi(x)}{\ell^{3/4}} - 1 = -v_x + \mathrm{i}w_x.$$

where $v_x > 0$ and $w_x \in \mathbb{R}$. The quadratic equation (58) for $\xi(x)$ implies that v_x and w_x satisfy

$$v_x^2 - w_x^2 = 1 + 2d\ell^{-3/4} \quad \text{and} \quad v_x w_x = -x\ell^{-3/4}. \tag{60}$$

Using the first equation, we see that $\mathrm{Re}((\xi(x) - \ell^{3/4})^3) = (-v_x^3 + 3v_x w_x^2)\ell^{9/4} = (2v_x^3 - 3v_x)\ell^{9/4} - 6dv_x\ell^{3/2}$. Hence,

$$\mathrm{Re}\,(\mathsf{G}_1(x)) = -2a(v_x^3 - 1)\ell^{3/2} - (c - 6ad)v_x\ell^{3/4} + c\ell^{3/4}. \tag{61}$$

In order to obtain an upper bound of $\mathrm{Re}\,(\mathsf{G}_1(x))$, we need an estimate of v_x.

Lemma 3.3. *Define*

$$\delta_x = \frac{v_x}{(1 + 2d\ell^{-3/4})^{1/2}} - 1. \tag{62}$$

Then, there is a constant $c_0 > 0$ such that if $\ell \geq c_0$,

$$\delta_x \geq \frac{x^2}{10\ell^{3/2}} \qquad \textit{for } |x| \leq \sqrt{3}\ell^{3/4}$$

and

$$\delta_x \geq \frac{|x|^{1/2}}{5\ell^{3/8}} \qquad \textit{for } |x| \geq \frac{6}{5}\ell^{3/4}.$$

Proof. From (60), v_x^2 satisfies the equation

$$v_x^2 - \frac{B}{v_x^2} = C \qquad \text{where } B = x^2\ell^{-3/2} \text{ and } C = 1 + 2d\ell^{-3/4}.$$

Let $r = C^{-1/2}v_x$ and apply Lemma 3.2 with $A = \frac{B}{C^2}$. Note that since $v_x > 0$, we have $r > 0$. Also note that $r = 1 + \delta_x$. Thus, Lemma 3.2 implies that

$$\delta_x \geq \frac{2x^2\ell^{-3/2}}{3(\sqrt{13} + 1)(1 + 2d\ell^{-3/4})^2} \qquad \text{for } |x| \leq \sqrt{3}\ell^{3/4}(1 + 2d\ell^{-3/4})$$

and

$$\delta_x \geq \frac{|x|^{1/2}\ell^{-3/8}}{3\sqrt{2}(1 + 2d\ell^{-3/4})^{1/2}} \qquad \text{for } |x| \geq \ell^{3/4}(1 + 2d\ell^{-3/4}).$$

We take ℓ large enough so that $2d\ell^{-3/4} \leq \frac{1}{5}$. The result follows by noting that $\frac{2}{3(\sqrt{13}+1)(6/5)^2} > \frac{1}{10}$ and $\frac{1}{3\sqrt{2}\sqrt{6/5}} > \frac{1}{5}$. ∎

From the definition (62),

$$v_x = (1+2d\ell^{-3/4})^{1/2}(1+\delta_x). \tag{63}$$

In (61), $\mathrm{Re}\,(\mathsf{G}_1(x))$ is a cubic function of v_x. We write the linear term of v_x in terms of a linear term δ_x using (63). For the cubic term of v_x, we note that since $(1+x)^c \ge 1+cx$ for all $x>0$ and $c\ge 1$,

$$v_x^3 =(1+2d\ell^{-3/4})^{3/2}(1+\delta_x)^3 \ge (1+3d\ell^{-3/4})(1+3\delta_x).$$

Thus, since $a>0$ and $d>0$, we find that

$$\mathrm{Re}\,(\mathsf{G}_1(x)) \le -6a(1+3d\ell^{-3/4})\delta_x\ell^{3/2} - (c-6ad)(v_x-1)\ell^{3/4} \le -6a\delta_x\ell^{3/2} - (c-6ad)(v_x-1)\ell^{3/4} \tag{64}$$

Since $(1+x)^{1/2} \le 1+\frac{1}{2}x$ for all $x>0$, we see from (63) that $v_x \le (1+d\ell^{-3/4})(1+\delta_x)$. Therefore, we find that

$$\mathrm{Re}\,(\mathsf{G}_1(x)) \le \left(-6a\delta_x\ell^{3/2} + |c-6ad|\ell^{3/4} + |c-6ad|d\right)\delta_x + |c-6ad|d. \tag{65}$$

Thus, since $a>0$, there is a constant $c_0\ge 1$ such that if $\ell \ge c_0$, then

$$\mathrm{Re}\,(\mathsf{G}_1(x)) \le -5a\ell^{3/2}\delta_x + |c-6ad|d. \tag{66}$$

Since $\mathrm{Re}(\mathsf{G}_\ell(x)) = \mathrm{Re}(\mathsf{G}_1(x)) + 2bd$ from (59), (66), and Lemma 3.3 imply the following bound.

Lemma 3.4. *Uniformly for (a,b,c,d) in a compact subset of $(0,\infty)\times\mathbb{R}\times\mathbb{R}\times[0,\infty)$, there are constants $C>0$ and $c_0>0$ such that if $\ell\ge c_0$, then*

$$|e^{\mathsf{G}_\ell(x)}| \le Ce^{-\frac{a}{2}x^2} \qquad \text{for } |x| \le \sqrt{3}\ell^{3/4} \tag{67}$$

and

$$|e^{\mathsf{G}_\ell(x)}| \le Ce^{-a\ell^{9/8}\sqrt{|x|}} \qquad \text{for } |x| \ge \frac{6}{5}\ell^{3/4}. \tag{68}$$

Corollary 3.5. *Let $\mathsf{G}_\ell(x)$ be the function defined in (56). Uniformly for (a,b,c,d) in a compact subset of $(0,\infty)\times\mathbb{R}\times\mathbb{R}\times[0,\infty)$, there are constants $c_0\ge 1$, $c_1>0$, and $c_2>0$ such that*

$$|e^{\mathsf{G}_\ell(x)}| \le c_1e^{-c_2\sqrt{|x|}} \qquad \text{for all } x\in\mathbb{R} \tag{69}$$

and for all $\ell \ge c_0$.

Proof. The result follows from Lemma 3.4 by noting $\ell^{9/8}\ge 1$ and $x^2+1 \ge \sqrt{|x|}$ for all $x\in\mathbb{R}$. □

4 Asymptotic analysis

We prove Proposition 2.8, 2.9, and 2.10 in this section. The proofs are almost uniform for all three cases except that we need to add the restriction $p \ll \ell^{5/4}$ in the proof of Proposition 2.10 for Case 1. The remaining situation for Case 1 is handled separately at the end of this section.

4.1 Choice of contours

It is convenient to introduce the notation

$$\mathsf{r} = \frac{p\ell^{1/4}}{\sqrt{2}}. \tag{70}$$

Note that $\mathsf{r}\to\infty$ for Case 1, r is a constant for Case 2, and $\mathsf{r}\to 0$ for Case 3.

The contours for the integrals of (44) are circles around the origin satisfying $0 < |z_1| < \cdots < |z_m| < 1$. We make the following specific choice of the radii. The choice is the same for all three Cases except in the last subsection which we change the analysis slightly. Let

$$\rho_1 > \cdots > \rho_m > 0$$

be real numbers which we keep fixed. We choose the contours as

$$z_i = e^{-\frac{\ell p}{2} - \mathrm{r}\rho_i + \mathrm{i}\theta_i}, \qquad \theta_i \in (-\pi, \pi], \tag{71}$$

for each $i = 1, \cdots, m$. Throughout this section except for the last subsection 4.10, we assume that z_i are given by the above equation. We write $\mathbf{z} = (z_1, \cdots, z_m)$.

4.2 Bound of $C^\bullet$

The function $C^\bullet(\mathbf{z})$ is given by the formula (39). For every $a > 0$, polylogarithm functions satisfy

$$|\mathrm{Li}_a(z)| = \left| \sum_{n=1}^{\infty} \frac{z^n}{n^a} \right| \le \sum_{n=1}^{\infty} |z|^n \le 2|z| \qquad \text{for } |z| \le 1/2. \tag{72}$$

Thus, if $|z| \le 1/2$, then (see (23))

$$|A_1(z)| = \left| -\frac{1}{\sqrt{2\pi}} \mathrm{Li}_{3/2}(z) \right| \le |z| \quad \text{and} \quad |A_2(z)| = \left| -\frac{1}{\sqrt{2\pi}} \mathrm{Li}_{5/2}(z) \right| \le |z|. \tag{73}$$

Similarly, for $|z|, |z'| \le 1/2$,

$$|B(z, z')| = \left| \frac{1}{4\pi} \sum_{k,k'=1}^{\infty} \frac{z^k (z')^{k'}}{(k + k')\sqrt{kk'}} \right| \le \frac{1}{4\pi} \sum_{k,k'=1}^{\infty} |z|^k |z'|^{k'} \le |z||z'|. \tag{74}$$

We find the following bound.

Lemma 4.1. *For* $\mathbf{z}$ *given in* (71)*, there is a constant* $c > 0$ *such that*

$$|C^\bullet(\mathbf{z}) - 1| \le c\ell p^{-1/2} e^{-\frac{\ell p}{2}} e^{c\ell p^{-1/2} e^{-\frac{\ell p}{2}}}$$

for all $\theta \in (-\pi, \pi]^m$ *and* $\ell, p > 0$ *satisfying* $\ell p \ge 2$*. Furthermore,* $|C^\bullet(\mathbf{z})| \le 2$ *and* $C^\bullet(\mathbf{z}) \to 1$ *uniformly in* $\theta \in (-\pi, \pi]^m$ *eventually for all three Cases.*

Proof. From (71), $|z_i| \le e^{-\frac{\ell p}{2}}$. If $\ell p \ge 2$, then $|z_i| \le e^{-1} \le 1/2$. From the formula (39) of $C^\bullet(\mathbf{z})$, the bounds (73) and (74), and the choice of the parameters (42), we find, using the inequality $|e^w - 1| \le |w| e^{|w|}$ for all complex number w, that there is a constant $c > 0$ so that

$$|C^\bullet(\mathbf{z}) - 1| \le c(\ell p^{-1/2} + p^{-3/2}) \Big(\sum_{i=1}^{m} |z_i| \Big) e^{c(\ell p^{-1/2} + p^{-3/2}) \sum_{i=1}^{m} |z_i|}.$$

Since $\ell p \ge 2$, we see $\ell p^{-1/2} + p^{-3/2} \le \frac{3}{2} \ell p^{-1/2}$. Using $|z_i| \le e^{-\frac{\ell p}{2}}$, we obtain the bound after replacing the constant c by $\frac{2c}{3m}$.

Note that $\ell p^{-1/2} e^{-\frac{\ell p}{2}} \le \frac{1}{(\ell p^4)^{1/2}} (\ell p)^{3/2} e^{-\frac{\ell p}{2}}$. In all Cases, $\ell p \ge \log \ell \to \infty$. Hence, the term $(\ell p)^{3/2} e^{-\frac{\ell p}{2}} \to 0$. For Case 1 and 2, The term $(\ell p^4)^{1/2}$ is bounded below, and thus, $\ell p^{-1/2} e^{-\frac{\ell p}{2}} \to 0$. For Case 3, we have $\ell^{-1} \log \ell \ll p$. Thus, $\ell p^{-1/4} \ll \frac{\ell^{5/4}}{(\log \ell)^{1/4}}$ and $\ell p \ge 4 \log \ell$ eventually. Thus, $\ell p^{-1/2} e^{-\frac{\ell p}{2}} \le \frac{\ell^{5/4}}{(\log \ell)^4} e^{-2 \log \ell} \to 0$. Hence, $C^\bullet(\mathbf{z}) \to 1$ for all three Cases, which also implies that $|C^\bullet(\mathbf{z})| \le 2$ eventually. □

4.3 The functions $\mathsf{u}_i(k)$

For $|z| < 1$, a complex number u is in the discrete set $\mathrm{L}_z = \{u : e^{-u^2/2} = z,\ \mathrm{Re}(u) < 0\}$ if and only if it is of the form $u = -\sqrt{-2\log z + 4\pi \mathrm{i} k}$ for some $k \in \mathbb{Z}$. With (71) in mind, define the function

$$\mathsf{u}_i(k) = \mathsf{u}_i(k;\theta_i) = -\sqrt{\ell p + 2\mathsf{r}\rho_i - 2\mathrm{i}\theta_i + 4\pi\mathrm{i}k}, \qquad k \in \mathbb{Z} \tag{75}$$

for $i = 1, \cdots, m$, where the branch of the square root is chosen so that $\mathrm{Re}(\mathsf{u}_i(k)) < 0$. We also define

$$\mathsf{u}_i(\mathbf{k}^{(i)}) = (\mathsf{u}_i(k_1^{(i)}), \cdots, \mathsf{u}_i(k_{n_i}^{(i)})) \quad \text{for } \mathbf{k}^{(i)} = (k_1^{(i)}, \cdots, k_{n_i}^{(i)}) \in \mathbb{Z}^{n_i}. \tag{76}$$

Furthermore, we write

$$\mathsf{U}(\mathbf{k}) = (\mathsf{u}_1(\mathbf{k}^{(1)}), \cdots, \mathsf{u}_m(\mathbf{k}^{(m)})) \quad \text{for } \mathbf{k} = (\mathbf{k}^{(1)}, \cdots, \mathbf{k}^{(m)}) \in \mathbb{Z}^{\mathbf{n}} = \mathbb{Z}^{n_1} \times \cdots \times \mathbb{Z}^{n_m} \tag{77}$$

for $\mathbf{n} = (n_1, \cdots, n_m)$.

Using these notations, the functions in (40) become

$$D_{\mathbf{n}}^{\bullet}(\mathbf{z}) = \sum_{\mathbf{k},\hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}} s_{\mathbf{n}}^{\bullet}(\mathbf{k}, \hat{\mathbf{k}}) \qquad \text{and} \qquad \hat{D}_{\mathbf{n}}^{\bullet}(\mathbf{z}) = \sum_{\mathbf{k},\hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}} \hat{s}_{\mathbf{n}}^{\bullet}(\mathbf{k}, \hat{\mathbf{k}}) \tag{78}$$

with

$$\begin{aligned} s_{\mathbf{n}}^{\bullet}(\mathbf{k}, \hat{\mathbf{k}}) &:= H_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) R_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) E_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})), \\ \hat{s}_{\mathbf{n}}^{\bullet}(\mathbf{k}, \hat{\mathbf{k}}) &:= H_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) \hat{R}_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) E_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})), \end{aligned} \tag{79}$$

where by $\mathbf{k} \in \mathbb{Z}^{\mathbf{n}}$, we mean that $\mathbf{k} = (\mathbf{k}^{(1)}, \cdots, \mathbf{k}^{(m)}) \in \mathbb{Z}^{n_1} \times \cdots \times \mathbb{Z}^{n_m}$.

4.4 Bound of $H_{\mathbf{n}}$

The function $H_{\mathbf{n}}(U, \hat{U})$ in (28) involves the function

$$h(w, z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{w} \mathrm{Li}_{1/2}(z e^{(w^2 - y^2)/2}) \mathrm{d}y = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\mathrm{Re}(w)} \mathrm{Li}_{1/2}(z e^{(w^2 - (x + \mathrm{i}\mathrm{Im}(w))^2)/2}) \mathrm{d}x$$

for $\mathrm{Re}(w) < 0$. From (72), we see that for $\mathrm{Re}(w) < 0$ and $|z| \le 1/2$,

$$|h(w, z)| \le \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\mathrm{Re}(w)} |z| e^{(\mathrm{Re}(w)^2 - x^2)/2} \mathrm{d}x \le |z|. \tag{80}$$

Lemma 4.2. *For* $\mathbf{z}$ *given in* (71)*, we have*

$$|H_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) - 1| \le 8|\mathbf{n}| e^{-\frac{\ell p}{2}} e^{8|\mathbf{n}| e^{-\frac{\ell p}{2}}} \le 4|\mathbf{n}| e^{4|\mathbf{n}|}$$

for all $\mathbf{n} \in \mathbb{N}^m$*,* $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}$*,* $\theta \in (-\pi, \pi]^m$*, and* $\ell, p > 0$ *satisfying* $\ell p \ge 2$*. As a consequence,*

$$|H_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))| \le 5|\mathbf{n}| e^{4|\mathbf{n}|}.$$

Proof. Using the inequality $|e^w - 1| \le |w| e^{|w|}$ and the estimate (80),

$$|H_{\mathbf{n}}(U, \hat{U}) - 1| \le 8|\mathbf{n}| (\max_{i=1}^{m} |z_i|) e^{8|\mathbf{n}| (\max_{i=1}^{m} |z_i|)} \le 4|\mathbf{n}| e^{4|\mathbf{n}|}$$

for all $U, \hat{U} \in \mathrm{L}_{z_1} \times \cdots \times \mathrm{L}_{z_m}$ if $|z_1|, \cdots, |z_m| \le 1/2$. For $\mathbf{z}$ given in (71), $|z_i| \le e^{-\frac{\ell p}{2}} \le 1/2$ for all i if $\ell p \ge 2$. Inserting $U = \mathsf{U}(\mathbf{k})$ and $\hat{U} = \mathsf{U}(\hat{\mathbf{k}})$, we obtain the result. □

4.5 Bound and limits of $E_{\mathbf{n}}$

Recall from (31) and (42) that for $\mathbf{n} = (n_1, \cdots, n_m)$,

$$E_{\mathbf{n}}(U, \hat{U}) = \prod_{i=1}^{m} \prod_{j_i=1}^{n_i} E^{i,+}(u_{j_i}^{(i)}) E^{i,-}(\hat{u}_{j_i}^{(i)}) \quad \text{where} \quad E^{i,\pm}(s) = e^{-\frac{\Delta t_i}{3p^{3/2}} s^3 \pm \frac{\Delta x_i}{\sqrt{2} p \ell^{1/4}} s^2 + \frac{\ell \Delta t_i + \sqrt{2} \ell^{1/4} \Delta h_i}{p^{1/2}} s}.$$

We compute the limit and bounds of $E_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$ where $\mathsf{U}(\mathbf{k})$ is the function from (77). For the limit, we only need the case when $\mathbf{n} = \mathbf{1}$, and thus we do not state the results when $\mathbf{n} \neq \mathbf{1}$.

Define

$$\widetilde{E}^{i,\pm}(s) = E^{i,\pm}(s) e^{\frac{2\Delta t_i}{3} \ell^{3/2} + \sqrt{2}(\Delta h_i \mp \frac{\Delta x_i}{2}) \ell^{3/4}}. \tag{81}$$

Then,

$$E_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) = e^{-\frac{4}{3} \ell^{3/2} (\sum_{i=1}^m n_i \Delta t_i) - 2\sqrt{2} \ell^{3/4} (\sum_{i=1}^m n_i \Delta h_i)} \prod_{i=1}^{m} \prod_{j=1}^{n_j} \widetilde{E}^{i,+}(\mathsf{u}_i(k_j^{(i)})) \widetilde{E}^{i,-}(\mathsf{u}_i(\hat{k}_j^{(i)})) \tag{82}$$

where $\mathsf{u}_i(k)$ is the function from (75). When $\mathbf{n} = \mathbf{1}$, since $\sum_{i=1}^m \Delta h_i = 0$ and $\sum_{i=1}^m \Delta t_i = 1$, this formula becomes

$$E_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) = e^{-\frac{4}{3} \ell^{3/2}} \prod_{i=1}^{m} \widetilde{E}^{i,+}(\mathsf{u}_i(k_1^{(i)})) \widetilde{E}^{i,-}(\mathsf{u}_i(\hat{k}_1^{(i)})). \tag{83}$$

The function $\widetilde{E}^{i,\pm}(\mathsf{u}_i(k))$ is expressible in terms of the function G_ℓ from (56). Recall the function $\xi(x)$ in (56), which contains the parameter d. Comparing with the formula (75) of u_i, we find that

$$\mathsf{u}_i(k) = -\sqrt{\ell p} \big(1 - \frac{1}{\ell^{3/4}} \xi\big(\frac{2\pi k - \theta_i}{\sqrt{2}\mathsf{r}}\big)\big) \qquad \text{with } d = \frac{\rho_i}{2}. \tag{84}$$

A direct computation shows that

$$\widetilde{E}^{i,\pm}(\mathsf{u}_i(k)) = e^{\mathsf{G}_\ell \left(\frac{2\pi k - \theta_i}{\sqrt{2}\mathsf{r}}\right)} \tag{85}$$

with the parameters

$$a = \frac{\Delta t_i}{3}, \qquad b = \pm \frac{\Delta x_i}{\sqrt{2}}, \qquad c = \sqrt{2} \Delta h_i, \qquad \text{and} \qquad d = \frac{\rho_i}{\sqrt{2}}. \tag{86}$$

Thus, the results from Subsection 3 are applicable.

We first find a limit of $E_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$. For $\mathbf{y} = (y_1, \cdots, y_m) \in \mathbb{R}^m$, we use the notation $[\mathbf{y}] = ([y_1], \cdots, [y_m])$.

Lemma 4.3 (Limit of $E_{\mathbf{n}}$ when $\mathbf{n} = \mathbf{1}$). *For Case 1, for every $\mathbf{y}, \hat{\mathbf{y}} \in \mathbb{R}^m$,*

$$\begin{aligned} &e^{\frac{4}{3} \ell^{3/2}} E_{\mathbf{1}}\big(\mathsf{U}([\mathsf{r}\mathbf{y}]), \mathsf{U}([\mathsf{r}\hat{\mathbf{y}}])\big) \\ &\quad \to \prod_{i=1}^{m} e^{\frac{\Delta t_i}{2} (\rho_i + 2\pi \mathrm{i} y_i)^2 - (\Delta h_i - \Delta x_i)(\rho_i + 2\pi \mathrm{i} y_i)) + \frac{\Delta t_i}{2} (\rho_i + 2\pi \mathrm{i} \hat{y}_i))^2 - (\Delta h_i + \Delta x_i)(\rho_i + 2\pi \mathrm{i} \hat{y}_i))} \end{aligned} \tag{87}$$

uniformly in $\theta \in (-\pi, \pi]^m$. For Case 2, for every $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}^m$,

$$e^{\frac{4}{3} \ell^{3/2}} E_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) \to \prod_{i=1}^{m} e^{\frac{\Delta t_i}{2} \xi_i(k_i)^2 - (\Delta h_i - \Delta x_i) \xi_i(k_i) + \frac{\Delta t_i}{2} \xi_i(\hat{k}_i)^2 - (\Delta h_i + \Delta x_i) \xi_i(\hat{k}_i)} \tag{88}$$

uniformly in $\theta \in (-\pi, \pi]^m$*, where*

$$\xi_i(k) = \rho_i + \frac{1}{\mathsf{r}}(2\pi \mathrm{i} k - \mathrm{i}\theta_i). \tag{89}$$

For Case 3, if

$$\theta_i = \mathsf{r}\varphi_i$$

for $i = 1, \cdots, m$*, then*

$$e^{\frac{4}{3}\ell^{3/2}} E_{\mathbf{1}}(\mathsf{U}(0), \mathsf{U}(0)) \to \prod_{i=1}^{m} e^{\Delta t_i(\rho_i - \mathrm{i}\varphi_i)^2 - 2\Delta h_i(\rho_i - \mathrm{i}\varphi_i)} \tag{90}$$

uniformly for $\varphi = (\varphi_1, \cdots, \varphi_m)$ *in a compact subset of* $\mathbb{R}^m$*.*

Proof. From (83), it is enough to compute the limits of $\widetilde{E}^{i,\pm}(\mathsf{u}_i(k)) = e^{\mathsf{G}_\ell\left(\frac{2\pi k - \theta_i}{\sqrt{2}\mathsf{r}}\right)}$. We use Lemma 3.1. From (86), $d = \frac{\rho_i}{\sqrt{2}}$. Recall that $\mathsf{r} \to \infty$ for Case 1, r is a constant for Case 2, and $\mathsf{r} \to 0$ for Case 3. Thus, for Case 1,

$$d + \mathrm{i}\frac{2\pi[\mathsf{r}y] - \theta_i}{\sqrt{2}\mathsf{r}} = \frac{\rho_i}{\sqrt{2}} + \mathrm{i}\frac{2\pi[\mathsf{r}y] - \theta_i}{\sqrt{2}\mathsf{r}} \to \frac{\rho_i + 2\pi \mathrm{i} y}{\sqrt{2}},$$

and hence Lemma 3.1 yields the result (87). For Case 2, $d + \mathrm{i}\frac{2\pi k - \theta_i}{\sqrt{2}\mathsf{r}} = \frac{\rho_i}{\sqrt{2}} + \mathrm{i}\frac{2\pi k - \theta_i}{\sqrt{2}\mathsf{r}} = \frac{\xi_i(k)}{\sqrt{2}}$ and we obtain (88). For Case 3, with $\theta_i = \mathsf{r}\varphi_i$,

$$d - \mathrm{i}\frac{\theta_i}{\sqrt{2}\mathsf{r}} = \frac{\rho_i - \mathrm{i}\varphi_i}{\sqrt{2}}.$$

Thus, we obtain (90). □

We now find a uniform bound for $E_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$. We start with the following result.

Lemma 4.4. *There are constants* $c_0 \geq 1$*,* $c_1 > 0$*, and* $c_* > 0$ *such that*

$$\left| E_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) \right| \prod_{i=1}^{m} e^{\frac{4n_i\Delta t_i}{3}\ell^{3/2} + 2\sqrt{2}n_i\Delta h_i \ell^{3/4}} \leq c_1^{|\mathbf{n}|} \prod_{i=1}^{m}\prod_{j=1}^{n_i} e^{-2c_*\sqrt{\frac{1}{\mathsf{r}}|k_j^{(i)} - \frac{\theta_i}{2\pi}|} - 2c_*\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j^{(i)} - \frac{\theta_i}{2\pi}|}} \tag{91}$$

for all $\mathbf{n} \in \mathbb{N}^m$*,* $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}$*,* $\theta \in (-\pi, \pi]^m$*, and* $\ell \geq c_0$*. As a consequence,*

$$\left| E_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) \right| \prod_{i=1}^{m} e^{\frac{4n_i\Delta t_i}{3}\ell^{3/2} + 2\sqrt{2}n_i\Delta h_i \ell^{3/4}} \leq c_1^{|\mathbf{n}|} \prod_{i=1}^{m}\prod_{j=1}^{n_i} e^{-c_*\sqrt{\frac{1}{\mathsf{r}}|k_j^{(i)}|} - c_*\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j^{(i)}|}}. \tag{92}$$

Proof. Since $\widetilde{E}^{i,\pm}(\mathsf{u}_i(k)) = e^{\mathsf{G}_\ell(\frac{2\pi k - \theta_i}{\sqrt{2}\mathsf{r}})}$ with the parameters (86), Corollary 3.5 gives a bound: there are constants $c_0 \geq 1$, $c_1 > 0$, and $c_2 > 0$ such that

$$|\widetilde{E}^{i,\pm}(\mathsf{u}_i(k))| \leq c_1 e^{-c_2\sqrt{\left|\frac{2\pi k - \theta_i}{\sqrt{2}\mathsf{r}}\right|}} \tag{93}$$

for all $k \in \mathbb{Z}$, $\theta \in (-\pi, \pi]$, and $\ell \geq c_0$. Thus, from (82), we obtain the bound (91) where we replaced c_1^2 by c_1 and $c_2\sqrt{\sqrt{2}\pi}$ by $2c_*$. The bound (92) follows from (91) since

$$\left| k - \frac{\theta}{2\pi} \right| \geq \frac{|k|}{2} \geq \frac{|k|}{4} \tag{94}$$

for every $k \in \mathbb{Z}$ and $\theta \in (-\pi, \pi]$. □

When $\mathbf{n} = \mathbf{1}$, due to (83), the above result implies the next bound.

Corollary 4.5 (Bound of $E_{\mathbf{n}}$ for $\mathbf{n} = \mathbf{1}$)**.** *Suppose $\mathbf{n} = \mathbf{1}$. With the same constants $c_0 \geq 1$, $c_1 > 0$, and $c_* > 0$ in Lemma 4.4,*

$$e^{\frac{4}{3}\ell^{3/2}} \left| E_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) \right| \leq c_1^m \prod_{i=1}^m e^{-2c_* \sqrt{\frac{1}{r}|k_i - \frac{\theta_i}{2\pi}|} - 2c_* \sqrt{\frac{1}{r}|\hat{k}_i - \frac{\theta_i}{2\pi}|}} \tag{95}$$

and

$$e^{\frac{4}{3}\ell^{3/2}} \left| E_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) \right| \leq c_1^m \prod_{i=1}^m e^{-c_* \sqrt{\frac{|k_i|}{r}} - c_* \sqrt{\frac{|\hat{k}_i|}{r}}} \tag{96}$$

for all $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^m$, $\theta \in (-\pi, \pi]^m$, and $\ell \geq c_0$.

For the case when $\mathbf{n} \neq \mathbf{1}$, we have the following estimate. We use the fact that $t_1, \cdots, t_{m-1}$ are distinct.

Corollary 4.6 (Bound of $E_{\mathbf{n}}$ for $\mathbf{n} \neq \mathbf{1}$)**.** *Let $c_* > 0$ be the constant from Lemma 4.4. There are positive constants c_0, δ, and c_2 such that*

$$e^{\frac{4}{3}\ell^{3/2}} \left| E_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}})) \right| \leq e^{-\frac{4\delta}{3}\ell^{3/2} - c_2|\mathbf{n}|\ell^{3/2}} \prod_{i=1}^m \prod_{j=1}^{n_i} e^{-c_* \sqrt{\frac{1}{r}|k_j^{(i)}|} - c_* \sqrt{\frac{1}{r}|\hat{k}_j^{(i)}|}} \tag{97}$$

for all $\mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}$, $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}$, $\theta \in (-\pi, \pi]^m$, and $\ell \geq c_0$.

Proof. Since Δt_i are positive constants (recall that $t_1, \cdots, t_{m-1}$ are distinct) that add up to 1, we find that

$$\sum_{i=1}^m n_i \Delta t_i = 1 + \sum_{i=1}^m (n_i - 1)\Delta t_i \geq 1 + \min_{i=1}^m \{\Delta t_i\} \qquad \text{for every } \mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}.$$

Let c and δ be any positive constants satisfying $1 + \min_{i=1}^m\{\Delta t_i\} = \frac{1+\delta}{1-c}$. Then,

$$(1-c)\sum_{i=1}^m n_i \Delta t_i \geq 1 + \delta \qquad \text{for all } \mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}. \tag{98}$$

This inequality implies that

$$\sum_{i=1}^m n_i \Delta t_i \geq 1 + \delta + c\sum_{i=1}^m n_i \Delta t_i \geq 1 + \delta + c|\mathbf{n}| \min_{i=1}^m \{\Delta t_i\}$$

for all $\mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}$. Thus,

$$\begin{aligned} &\log\left(c_1^{|\mathbf{n}|} \prod_{i=1}^m e^{-\frac{4n_i\Delta t_i}{3}\ell^{3/2} - 2\sqrt{2}n_i \Delta h_i \ell^{3/4}} \right) \\ &\leq -\frac{4(1+\delta)}{3}\ell^{3/2} - |\mathbf{n}|\ell^{3/2}\left(\frac{4c}{3}\min_{i=1}^m\{\Delta t_i\} - 2\sqrt{2}\ell^{-3/4}\max_{i=1}^m |\Delta h_i| - \ell^{-3/2}\log|c_1| \right). \end{aligned}$$

The last parenthesis term is larger than or equal to a positive constant c_2 if ℓ is large enough. Thus, we obtain the result from Lemma 4.4 after adjusting the constant c_0. □

4.6 Bounds and limits of $R_{\mathbf{n}}$ and $\hat{R}_{\mathbf{n}}$

From (30),

$$|R_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))| = \prod_{i=1}^m \prod_{j_i=1}^{n_i} \frac{1}{|\mathsf{u}_i(k_{j_i}^{(i)})\mathsf{u}_i(\hat{k}_{j_i}^{(i)})|} \prod_{i=0}^m \left| \mathsf{K}(U^{(i)}, -\hat{U}^{(i+1)}; \hat{U}^{(i)}, -U^{(i+1)}) \right| \tag{99}$$

with $U^{(i)} = \mathsf{u}_i(\mathbf{k}^{(i)})$ and $\hat{U}^{(i)} = \mathsf{u}_i(\hat{\mathbf{k}}^{(i)})$ and the convention that $U^{(0)} = \hat{U}^{(0)} = U^{(m+1)} = \hat{U}^{(m+1)} = \emptyset$. Recall from (29) that

$$\mathsf{K}(X;Y) = \det\left(\frac{1}{x_i+y_j}\right)_{i,j=1}^{a} = \frac{\prod_{1\le i<j\le a}(x_j-x_i)(y_j-y_i)}{\prod_{i,j=1}^{a}(x_i+y_j)} \tag{100}$$

for $X = (x_1,\cdots,x_a)$ and $Y = (y_1,\cdots,y_a)$.

From the definition (75), we have a trivial bound

$$|\mathsf{u}_i(k)| \ge \sqrt{\ell p} \tag{101}$$

for all $i = 1,\cdots,m$ and $k \in \mathbb{Z}$.

To estimate (99), we need the following lemmas.

Lemma 4.7. *For every $i, i' = 1,\cdots,m$ and $k, k' \in \mathbb{Z}$,*

$$|\mathsf{u}_i(k) + \mathsf{u}_{i'}(k')| \ge \sqrt{\ell p}. \tag{102}$$

Proof. From the definition (75), $\mathrm{Re}(\mathsf{u}_i(k)^2) > 0$ and $\mathrm{Re}(\mathsf{u}_i(k)) < 0$. Thus, $\arg(-\mathsf{u}_i(k)) \in (-\pi/4,\pi/4)$. Using polar forms $-\mathsf{u}_i(k) = ce^{\mathrm{i}\varphi}$ and $-\mathsf{u}_{i'}(k') = c'e^{\mathrm{i}\varphi'}$ for some $c, c' > 0$ and $\varphi, \varphi' \in (-\pi/4,\pi/4)$, we find that

$$|\mathsf{u}_i(k) + \mathsf{u}_{i'}(k')| = |c + c'e^{\mathrm{i}(\varphi'-\varphi)}| \ge |c + c'\cos(\varphi-\varphi')| \ge c = |\mathsf{u}_i(k)| \ge \sqrt{\ell p}$$

for all i, i' and k, k'. The last inequality is due to (101). □

Lemma 4.8. *We have*

$$|\mathsf{u}_i(k)| \le 5\sqrt{\ell p} + 5\sqrt{|k|} \tag{103}$$

for all $i = 1,\ldots,m$, $k \in \mathbb{Z}$, and $\ell, p > 0$ satisfying $\ell^3 \ge 4\rho_1^4$ and $\ell p \ge 1$.

Proof. If $\ell^3 \ge 4\rho_1^4$, then $2\mathsf{r}\rho_i \le 2\mathsf{r}\rho_1 \le \ell p$ (recall (70)). Thus, (see (75))

$$|\mathsf{u}_i(k)|^4 = (\ell p + 2\mathsf{r}\rho_i)^2 + (4\pi k - 2\theta_i)^2 \le 4(\ell p)^2 + (4\pi|k| + 2\pi)^2 \le 4(\ell p)^2 + 32\pi^2|k|^2 + 8\pi^2.$$

for all $\theta_i \in (-\pi,\pi]$. Hence, for $\ell p \ge 1$,

$$|\mathsf{u}_i(k)| \le \left((4+8\pi^2)(\ell p)^2 + 32\pi^2|k|^2\right)^{1/4} \le (4+8\pi^2)^{1/4}\sqrt{\ell p} + (32\pi^2)^{1/4}\sqrt{|k|}.$$

Since $(4+8\pi^2)^{1/4} \approx 3.01$ and $(32\pi^2)^{1/4} \approx 4.21$, we obtain the result. □

Lemma 4.9. *(a) For every $r > 0$, $a \ge 0$ and $\epsilon > 0$,*

$$\frac{1}{r^a}\sum_{k=1}^{\infty}|k|^a e^{-\epsilon\sqrt{\frac{k}{r}}} \le r\int_{\frac{1}{r}}^{\infty} y^a e^{-\epsilon\sqrt{\frac{y}{2}}}\mathrm{d}y. \tag{104}$$

(b) Recall that $\mathsf{r} = \frac{\ell^{1/4}p}{\sqrt{2}}$. For every $\epsilon > 0$ and $a \ge 0$, there is a positive constant $C = C(a,\epsilon)$ such that

$$\sum_{k=-\infty}^{\infty}|\mathsf{u}_i(k)|^a e^{-\epsilon\sqrt{\frac{|k|}{\mathsf{r}}}} \le C(\ell p)^{\frac{a}{2}+1} \tag{105}$$

for all $i = 1,\ldots,m$ and $\ell, p > 0$ satisfying $\ell^3 \ge 4\rho_1^4$ and $\ell p \ge 1$.

Proof. (a) For $k \geq 1$, we have $k \geq \frac{k+1}{2} \geq \frac{x}{2}$ for all $x \in [k, k+1]$. Thus,

$$\frac{1}{r^a}\sum_{k=1}^{\infty} k^a e^{-\epsilon\sqrt{\frac{k}{r}}} \leq \frac{1}{r^a}\sum_{k=1}^{\infty}\int_k^{k+1} x^a e^{-\epsilon\sqrt{\frac{x}{2r}}}\mathrm{d}x = \frac{1}{r^a}\int_1^{\infty} x^a e^{-\epsilon\sqrt{\frac{x}{2r}}}\mathrm{d}x = r\int_{\frac{1}{r}}^{\infty} y^a e^{-\epsilon\sqrt{\frac{y}{2}}}\mathrm{d}y.$$

(b) The result (a) implies that

$$\frac{1}{\mathsf{r}^a}\sum_{k=-\infty}^{\infty}|k|^a e^{-\epsilon\sqrt{\frac{|k|}{\mathsf{r}}}} \leq \delta_{a=0} + 2\mathsf{r}B_a \quad \text{where } B_a = \int_0^{\infty} y^a e^{-\epsilon\sqrt{\frac{y}{2}}}\mathrm{d}y.$$

Hence, if $\ell^3 \geq 4\rho_1^4$ and $\ell p \geq 1$, then (103) implies that

$$\sum_{k=-\infty}^{\infty}|\mathsf{u}_i(k)|^a e^{-\epsilon\sqrt{\frac{|k|}{\mathsf{r}}}} \leq 5^a\sum_{k=-\infty}^{\infty}(2^a(\ell p)^{a/2} + 2^a|k|^{a/2})e^{-\epsilon\sqrt{\frac{|k|}{\mathsf{r}}}} \leq 10^a\left((\ell p)^{a/2}(1+2\mathsf{r}B_0) + 2\mathsf{r}^{a/2+1}B_{a/2}\right).$$

Since $\ell^3 \geq 4\rho_1^4$ implies that $\mathsf{r} \leq \frac{\ell p}{2\rho_1}$, the above is bounded by a constant times $(\ell p)^{a/2+1}$ if $\ell^3 \geq 4\rho_1^4$ and $\ell p \geq 1$. We thus obtain the result. □

Lemma 4.10. *For every $\epsilon > 0$, there is a positive constant C_0 such that*

$$\frac{\left|\mathsf{K}(\mathsf{u}_i(\mathbf{k}), -\mathsf{u}_{i'}(\hat{\mathbf{k}}'); \mathsf{u}_i(\hat{\mathbf{k}}), -\mathsf{u}_{i'}(\mathbf{k}'))\right|}{\prod_{j=1}^n|u_i(\hat{k}_j)|\prod_{j=1}^{n'}|u_{i'}(k'_j)|}\prod_{j=1}^n e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}|k_j|}-\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j|}}\prod_{j=1}^{n'} e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}|k'_j|}-\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}'_j|}} \leq \left(\frac{C_0(\ell p)^2}{\mathsf{r}^2}\right)^{\frac{n+n'}{2}} \tag{106}$$

for all two distinct integers i and i' from $\{0, \cdots, m+1\}$, $n, n' \in \mathbb{N}$, $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^n$, $\mathbf{k}', \hat{\mathbf{k}}' \in \mathbb{Z}^{n'}$, and $\ell, p > 0$ satisfying $\ell^3 \geq 4\rho_1^4$ and $\ell p \geq 1$.

Proof. Since K is a Cauchy determinant (100), the left-hand side of (106) is zero if two components of any of $\mathbf{k}, \mathbf{k}', \hat{\mathbf{k}}$, or $\hat{\mathbf{k}}'$ are equal. Thus, it is enough to consider the case that $\mathbf{k}, \mathbf{k}', \hat{\mathbf{k}}$, or $\hat{\mathbf{k}}'$ all have distinct components. Let $\epsilon > 0$ be an arbitrary constant.

For vectors $X = (x_1, \cdots, x_a)$ and $Y = (y_1, \cdots, y_a)$, and scalars $f_1, \cdots, f_a$, the Hadamard's inequality implies that

$$\left|\mathsf{K}(X;Y)\prod_{p=1}^a f_p\right| = \left|\det\left(\frac{f_p}{x_q + y_p}\right)_{q,p=1}^a\right| \leq \prod_{q=1}^a\sqrt{\sum_{p=1}^a\frac{f_p^2}{(x_q+y_p)^2}}.$$

Thus,

$$\begin{aligned}
&\frac{\left|\mathsf{K}(\mathsf{u}_i(\mathbf{k}), -\mathsf{u}_{i'}(\hat{\mathbf{k}}'); \mathsf{u}_i(\hat{\mathbf{k}}), -\mathsf{u}_{i'}(\mathbf{k}'))\right|}{\prod_{j=1}^n|u_i(\hat{k}_j)|\prod_{j=1}^{n'}|u_{i'}(k'_j)|}\prod_{j=1}^n e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j|}}\prod_{j'=1}^{n'} e^{-\epsilon\sum_{j=1}^{n'}\sqrt{\frac{1}{\mathsf{r}}|k'_j|}} \\
&\leq \prod_{q=1}^n\sqrt{\sum_{p=1}^n\frac{e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_p|}}}{|u_i(k_q)+u_i(\hat{k}_p)|^2|u_i(\hat{k}_p)|^2} + \sum_{p=1}^{n'}\frac{e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k'_p|}}}{|u_i(k_q)-u_{i'}(k'_p)|^2|u_{i'}(k'_p)|^2}} \\
&\quad\times\prod_{q=1}^{n'}\sqrt{\sum_{p=1}^n\frac{e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_p|}}}{|u_{i'}(\hat{k}'_q)-u_i(\hat{k}_p)|^2|u_i(\hat{k}_p)|^2} + \sum_{p=1}^{n'}\frac{e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k'_p|}}}{|u_{i'}(\hat{k}'_q)+u_{i'}(k'_p)|^2|u_{i'}(k'_p)|^2}}.
\end{aligned} \tag{107}$$

Consider the first sum. From (101) and (102), $|u_i(k)| \geq \sqrt{\ell p}$ and $|u_i(k) + u_{i'}(k')| \geq \sqrt{\ell p}$. Since we assume that the components of $\hat{\mathbf{k}}$ are distinct, the case $a = 0$ of (105) implies that there is a constant $C_1 > 0$ so that

$$\sum_{p=1}^n\frac{e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_p|}}}{|u_i(k_q)+u_i(\hat{k}_p)|^2|u_i(\hat{k}_p)|^2} \leq \frac{1}{(\ell p)^2}\sum_{j=1}^n e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j|}} \leq \frac{1}{(\ell p)^2}\sum_{k=-\infty}^{\infty}e^{-2\epsilon\sqrt{\frac{|k|}{\mathsf{r}}}} \leq \frac{C_1}{\ell p} \tag{108}$$

$\ell^3 \geq 4\rho_1^4$ and $\ell p \geq 1$. The same bound holds for the fourth sum. For the second sum, note that $\frac{1}{|a-b|^2} \leq \frac{2(1+|a|^2)(1+|b|^2)}{|a^2-b^2|^2}$ for all complex a, b. Since

$$|u_i(k_q)^2 - u_{i'}(k'_p)^2| = |2\mathsf{r}\rho_i + \mathrm{i}(-2\theta_i + 4\pi k_q) - 2\mathsf{r}\rho_{i'} - \mathrm{i}(-2\theta_{i'} + 4\pi k'_p)| \geq 2\mathsf{r}|\rho_i - \rho_{i'}|, \tag{109}$$

the $a = 0$ and $a = 2$ cases of (105) show that there is a constant $C_2 > 0$ such that

$$\begin{aligned}\sum_{p=1}^{n} \frac{e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k'_p|}}}{|u_i(k_q) - u_{i'}(k'_p)|^2 |u_{i'}(k'_p)|^2} &\leq \frac{(1 + |u_i(k_q)|^2)}{2\mathsf{r}^2(\rho_i - \rho_{i'})^2 \ell p} \sum_{j=1}^{n} (1 + |u_{i'}(k'_j)|^2) e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k'_j|}} \\ &\leq \frac{(1 + |u_i(k_q)|^2)}{2\mathsf{r}^2(\rho_i - \rho_{i'})^2 \ell p} \sum_{k=-\infty}^{\infty} (1 + |u_{i'}(k)|^2) e^{-2\epsilon\sqrt{\frac{|k|}{\mathsf{r}}}} \leq \frac{C_2 \ell p}{\mathsf{r}^2}(1 + |u_i(k_q)|^2)\end{aligned} \tag{110}$$

if $\ell^3 \geq 4\rho_1^4$ and $\ell p \geq 1$. The third sum is similar. Hence, the left-hand side of (106) is bounded by

$$\prod_{j=1}^{n} e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}|k_j|}} \prod_{j=1}^{n'} e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}'_j|}} \left[\prod_{q=1}^{n} \sqrt{\frac{C_1}{\ell p} + \frac{C_2 \ell p}{\mathsf{r}^2}(1 + |u_i(k_q)|^2)}\right] \left[\prod_{q=1}^{n'} \sqrt{\frac{C_1}{\ell p} + \frac{C_2 \ell p}{\mathsf{r}^2}(1 + |u_{i'}(\hat{k}'_q)|^2)}\right]. \tag{111}$$

From (103), since $e^{-2\epsilon\sqrt{\frac{|k|}{\mathsf{r}}}} \leq 1$ and $e^{-2\epsilon\sqrt{\frac{|k|}{\mathsf{r}}}} \frac{|k|}{\mathsf{r}} \leq \max\{x e^{-2\epsilon\sqrt{x}} : x \geq 0\} < \infty$, with additional constants C_3 and C_4,

$$e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k_q|}} \left(\frac{C_1}{\ell p} + \frac{C_2 \ell p}{\mathsf{r}^2}(1 + |u_i(k_q)|^2)\right) \leq \frac{C_1}{\ell p} + \frac{C_2 \ell p}{\mathsf{r}^2} + \frac{C_3 (\ell p)^2}{\mathsf{r}^2} + \frac{C_4 \ell p}{\mathsf{r}}. \tag{112}$$

Since $\ell^3 \geq 4\rho_1^4$ implies that $\frac{\ell p}{\mathsf{r}} \geq 2\rho_1$, there is a positive constant C_0 so that the right-side of (112) is bounded by $\frac{C_0(\ell p)^2}{\mathsf{r}^2}$ if $\ell^3 \geq 4\rho_1^4$ and $\ell p \geq 1$. Hence, (111) is bounded by $(\frac{C_0(\ell p)^2}{\mathsf{r}^2})^{(n+n')/2}$. ☐

Corollary 4.11 (Bound of $R_{\mathbf{n}}$ and $\hat{R}_{\mathbf{n}}$)**.** *For every $\epsilon > 0$, there is a positive constant C_0 such that*

$$|R_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))| \leq \left(\frac{C_0(\ell p)^2}{\mathsf{r}^2}\right)^{|\mathbf{n}|} \prod_{i=1}^{m}\prod_{j=1}^{n_i} e^{2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k_j^{(i)}|} + 2\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j^{(i)}|}} \tag{113}$$

and

$$|\hat{R}_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))| \leq |\mathbf{n}|(\ell p)^{1/2} \left(\frac{C_0(\ell p)^2}{\mathsf{r}^2}\right)^{|\mathbf{n}|} \prod_{i=1}^{m}\prod_{j=1}^{n_i} e^{2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k_j^{(i)}|} + 2\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j^{(i)}|}} \tag{114}$$

for all $\mathbf{n} \in \mathbb{N}^m$, $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}$, and $\ell, p > 0$ satisfying $\ell^3 \geq 4\rho_1^4$ and $\ell p \geq 1$.

Proof. The bound (113) follows by inserting the estimate (106) in the formula (99). For the bound (114), we need to modify the argument a little bit. Recall that $\hat{R}_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$ is equal to $R_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$ times the sum $\sum_{j=1}^{n_m}(u_m(k_j^{(m)}) + u_m(\hat{k}_j^{(m)}))$. We may assume that $k_j^{(m)}$ are distinct for $1 \leq j \leq n_m$, and $\hat{k}_j^{(m)}$ are also distinct since otherwise the left-hand side is zero. Using the lower bound $|u_i(k)| \leq 5\sqrt{\ell p} + 5\sqrt{k}$ in (103),

$$\begin{aligned}&\left|\sum_{j=1}^{n_m}(u_m(k_j^{(m)}) + u_m(\hat{k}_j^{(m)}))\right| \\ &\leq \left(10 n_m\sqrt{\ell p} + 5\sum_{j=1}^{n_m}\sqrt{k_j^{(m)}} e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}k_j^{(m)}}} + 5\sum_{j=1}^{n_m}\sqrt{\hat{k}_j^{(m)}} e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}\hat{k}_j^{(m)}}}\right) \prod_{j=1}^{n_m} e^{\epsilon\sqrt{\frac{1}{\mathsf{r}}|k_j^{(m)}|} + \epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j^{(m)}|}}\end{aligned} \tag{115}$$

Note that the maximum of the function $\sqrt{x}e^{-\epsilon\sqrt{x/\mathsf{r}}}$ over $x \in [0, \infty]$ is $C\sqrt{\mathsf{r}}$. Here C is a positive contant. Also note that $2\rho_1\mathsf{r} < \ell p$ by our assumption $\ell^3 \geq 4\rho_1^4$. Hence the left hand side of (115) is bounded by a constant times $|\mathbf{n}||\ell p|^{1/2}\prod_{i=1}^m\prod_{j=1}^{n_i}e^{\epsilon\sqrt{\frac{1}{\mathsf{r}}|k_j^{(i)}|}+\epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j^{(i)}|}}$. Combining with the estimate (113) and adjusting the ϵ value accordingly we obtain (114). □

The exponential bounds of Corollary 4.11 are enough for $\mathbf{n}\neq\mathbf{1}$. However, for $\mathbf{n}=\mathbf{1}$, we need a stronger estimate. In the next lemma, we obtain a polynomial bound in this case. Note that when $\mathbf{n}=\mathbf{1}$, the product formula of the Cauchy determinant implies that

$$R_{\mathbf{1}}(U,\hat{U})=(-1)^m\prod_{i=1}^m\frac{1}{(u_i+\hat{u}_i)^2u_i\hat{u}_i}\prod_{i=2}^m\frac{(u_i+\hat{u}_{i-1})(\hat{u}_i+u_{i-1})}{(u_i-u_{i-1})(\hat{u}_i-\hat{u}_{i-1})} \tag{116}$$

and $\hat{R}_{\mathbf{1}}(U,\hat{U})=(u_m+\hat{u}_m)R_{\mathbf{1}}(U,\hat{U})$. We insert $U=\mathsf{U}(\mathbf{k})$ and $\hat{U}=\mathsf{U}(\hat{\mathbf{k}})$ where $\mathbf{k}=(k_1,\cdots,k_m)\in\mathbb{N}^m$ and $\hat{\mathbf{k}}=(\hat{k}_1,\cdots,\hat{k}_m)\in\mathbb{N}^m$.

Lemma 4.12 (Bound of $R_{\mathbf{n}}$ and $\hat{R}_{\mathbf{n}}$ for $\mathbf{n}=\mathbf{1}$)**.** *There is a polynomial P of $2m$ variables such that*

$$(\ell p)^2\mathsf{r}^{2m-2}|R_{\mathbf{1}}(\mathsf{U}(\mathbf{k}),\mathsf{U}(\hat{\mathbf{k}}))|\leq|P(\frac{\mathbf{k}}{\mathsf{r}},\frac{\hat{\mathbf{k}}}{\mathsf{r}})|\quad\text{and}\quad(\ell p)^{3/2}\mathsf{r}^{2m-2}|\hat{R}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}),\mathsf{U}(\hat{\mathbf{k}}))|\leq|P(\frac{\mathbf{k}}{\mathsf{r}},\frac{\hat{\mathbf{k}}}{\mathsf{r}})|$$

for all $\mathbf{k},\hat{\mathbf{k}}\in\mathbb{Z}^m$, $\theta\in(-\pi,\pi]^m$, and ℓ,p satisfying $\ell^3\geq4\rho_1^4$ and $\ell p\geq1$.

Proof. Recall the trivial bound $|\mathsf{u}_i(k)|\geq\sqrt{\ell p}$ from (101) for all i and k and the bound from (102) that $|\mathsf{u}_i(k)+\mathsf{u}_{i'}(k')|\geq\sqrt{\ell p}$ for all i,i' and k,k'. The bound (103) implies that

$$\left|\mathsf{u}_i(k)\right|\leq5\sqrt{\ell p}\left(1+\sqrt{\frac{|k|}{\ell p}}\right)\leq5\sqrt{\ell p}\left(1+\sqrt{\frac{|k|}{2\rho_1\mathsf{r}}}\right)$$

if $\ell\geq4\rho_1^4$ and $\ell p\geq1$. On the other hand, since $|\mathsf{u}_i(k)^2-\mathsf{u}_{i'}(k')^2|=|2\mathsf{r}\rho_i+\mathrm{i}(-2\theta_i+2\pi k)-2\mathsf{r}\rho_{i'}-\mathrm{i}(-2\theta_{i'}+2\pi k')|\geq2\mathsf{r}|\rho_i-\rho_{i'}|$, we have

$$\left|\frac{1}{\mathsf{u}_i(k)-\mathsf{u}_{i'}(k')}\right|=\left|\frac{\mathsf{u}_i(k)+\mathsf{u}_{i'}(k')}{\mathsf{u}_i(k)^2-\mathsf{u}_{i'}(k')^2}\right|\leq\frac{|\mathsf{u}_i(k)|+|\mathsf{u}_{i'}(k')|}{2\mathsf{r}|\rho_i-\rho_{i'}|} \tag{117}$$

for all $k,k'\in\mathbb{N}$, and $i\neq i'$. Inserting these estimates into (116), we obtain the desired inequalities. □

The proof shows that we also have the bound given by $P(\frac{\mathbf{k}}{\ell p},\frac{\mathbf{k}}{\ell p})$. For a later convenience, we replaced it by a less precise bound $P(\frac{\mathbf{k}}{\mathsf{r}},\frac{\hat{\mathbf{k}}}{\mathsf{r}})$.

We also need pointwise limits of $\hat{R}_{\mathbf{n}}$ when $\mathbf{n}=\mathbf{1}$.

Lemma 4.13 (Limit of $\hat{R}_{\mathbf{n}}$ for $\mathbf{n}=\mathbf{1}$)**.** *For Case 1, for every $\mathbf{y},\hat{\mathbf{y}}\in\mathbb{R}^m$,*

$$(-1)^{m-1}2(\ell p)^{3/2}\mathsf{r}^{2m-2}\hat{R}_{\mathbf{1}}\big(\mathsf{U}([\mathsf{r}\mathbf{y}]),\mathsf{U}([\mathsf{r}\hat{\mathbf{y}}])\big)\to\prod_{i=2}^m\frac{1}{(\rho_i+2\pi\mathrm{i}y_i-\rho_{i-1}-2\pi\mathrm{i}y_{i-1})(\rho_i+2\pi\mathrm{i}\hat{y}_i-\rho_{i-1}-2\pi\mathrm{i}\hat{y}_{i-1})} \tag{118}$$

uniformly for $\theta\in(-\pi,\pi]^m$. For Case 2, for every $\mathbf{k},\hat{\mathbf{k}}\in\mathbb{Z}^n$,

$$(-1)^{m-1}2(\ell p)^{3/2}\mathsf{r}^{2m-2}\hat{R}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}),\mathsf{U}(\hat{\mathbf{k}}'))\to\prod_{i=2}^m\frac{1}{(\xi_i(k_i)-\xi_{i-1}(k_{i-1}))(\xi_i(\hat{k}_i)-\xi_{i-1}(\hat{k}_{i-1}))} \tag{119}$$

uniformly for $\theta\in(-\pi,\pi]^m$, where $\xi_i(k)=\rho_i+\frac{1}{\mathsf{r}}(2\pi\mathrm{i}k-\mathrm{i}\theta_i)$ as in (89). For Case 3, if

$$\theta_i=\mathsf{r}\varphi_i,\qquad i=1,\cdots,m,$$

then

$$(-1)^{m-1}2(\ell p)^{3/2}\mathsf{r}^{2m-2}\hat{R}_{\mathbf{1}}(\mathsf{U}(\mathbf{0}),\mathsf{U}(\mathbf{0}))\to\prod_{i=2}^{m}\frac{1}{(\rho_i-\mathrm{i}\varphi_i-\rho_{i-1}+\mathrm{i}\varphi_{i-1})^2}\tag{120}$$

uniformly for φ is a compact subset of $\mathbb{R}^m$.

Proof. From the definition (75) of $u_i(k)$,

$$u_i(k)=-\sqrt{\ell p+2\mathsf{r}\rho_i-2\mathrm{i}\theta_i+4\pi\mathrm{i}k}=-\sqrt{\ell p+2\mathsf{r}\xi_i(k)}=-\sqrt{\ell p}\left(1+\frac{2\mathsf{r}}{\ell p}\xi_i(k)\right)^{1/2}$$

using $\xi(k)=\rho_i+\frac{1}{\mathsf{r}}(2\pi\mathrm{i}k-\mathrm{i}\theta_i)$. Hence, for every k,

$$u_i(k)=-\sqrt{\ell p}\left(1+O\left(\frac{1}{\ell^{3/4}}\right)\right)$$

uniformly in $\theta_i\in(-\pi,\pi]$. Also for every $i\neq i'$ and k,k',

$$\frac{1}{u_i(k)-u_{i'}(k')}=\frac{u_i(k)+u_{i'}(k')}{u_i(k)^2-u_{i'}(k')^2}=\frac{-2\sqrt{\ell p}\left(1+O\left(\frac{1}{\ell^{3/4}}\right)\right)}{2\mathsf{r}(\xi_i(k)-\xi_{i'}(k'))}$$

uniformly in $\theta_i,\theta_{i'}\in(-\pi,\pi]$. Inserting them into $\hat{R}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}),\mathsf{U}(\hat{\mathbf{k}}'))$ using the formula (116), we find that for every $\mathbf{k},\mathbf{k}'\in\mathbb{Z}$,

$$(-1)^{m-1}2(\ell p)^{3/2}\mathsf{r}^{2m-2}\hat{R}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}),\mathsf{U}(\hat{\mathbf{k}}'))=\left[\prod_{i=2}^{m}\frac{1}{(\xi_i(k_i)-\xi_{i-1}(k_{i-1}))(\xi_i(k_i')-\xi_{i-1}(k_{i-1}'))}\right]\left(1+O\left(\frac{1}{\ell^{3/4}}\right)\right)\tag{121}$$

uniformly for $\theta\in(-\pi,\pi]^m$. The result (121) implies (119) for Case 2. For Case 1, we have $\mathsf{r}\to\infty$, and thus,

$$\xi_i([\mathsf{r}y])=\rho_i+\frac{-\mathrm{i}\theta_i+2\pi\mathrm{i}[\mathsf{r}y]}{\mathsf{r}}\to\rho_i+2\pi\mathrm{i}y$$

for every $y\in\mathbb{R}$. Hence, (121) implies (118). If $\theta_i=\mathsf{r}\varphi_i$, then

$$\xi_i(0)=\rho_i+\frac{-\mathrm{i}\theta_i}{\mathsf{r}}=\rho_i-\mathrm{i}\varphi_i.$$

Thus, (120) follows from (121) after inserting $\mathbf{k}=\hat{\mathbf{k}}=\mathbf{0}$. □

4.7 Proof of Proposition 2.8

We analyze

$$\hat{P}_{m,1}=(-1)^{m-1}\int_{(-\pi,\pi]^m}C^{\bullet}(\mathbf{z})\hat{D}^{\bullet}_{\mathbf{1}}(\mathbf{z})T^{\bullet}_{\mathbf{1}}(\mathbf{z})\prod_{i=1}^{m}\frac{\mathrm{d}\theta_i}{2\pi}\tag{122}$$

where $z_i=e^{-\frac{\ell p}{2}-\mathsf{r}\rho_i+\mathrm{i}\theta_i}$, $\theta_i\in(-\pi,\pi]$, as given in recall (71). From (41) when $\mathbf{n}=\mathbf{1}$,

$$T^{\bullet}_{\mathbf{1}}(\mathbf{z})=\prod_{i=2}^{m}\left(1-\frac{z_{i-1}}{z_i}\right)=\prod_{i=2}^{m}\left(1-\frac{e^{-\mathsf{r}\rho_{i-1}+\mathrm{i}\theta_{i-1}}}{e^{-\mathsf{r}\rho_i+\mathrm{i}\theta_i}}\right).\tag{123}$$

Since $\rho_1>\cdots>\rho_m>0$, we find

$$|T^{\bullet}_{\mathbf{1}}(\mathbf{z})|\leq 2^m.\tag{124}$$

Recall from (78) that

$$\hat{D}^{\bullet}_{\mathbf{1}}(\mathbf{z}) = \sum_{\mathbf{k},\hat{\mathbf{k}}\in\mathbb{Z}^m} \hat{s}^{\bullet}_{\mathbf{1}}(\mathbf{k},\hat{\mathbf{k}}) \tag{125}$$

where $\hat{s}^{\bullet}_{\mathbf{1}}(\mathbf{k},\mathbf{k}') = H_{\mathbf{1}}(\mathsf{U}(\mathbf{k}),\mathsf{U}(\hat{\mathbf{k}}))\hat{R}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}),\mathsf{U}(\hat{\mathbf{k}}))E_{\mathbf{1}}(\mathsf{U}(\mathbf{k}),\mathsf{U}(\hat{\mathbf{k}}))$. Lemma 4.2, Corollary 4.5, and Lemma 4.12 imply that there are constants $c_0 \geq 1, c_* > 0$ and a polynomial P of $2m$ variables such that

$$(\ell p)^{3/2}\mathsf{r}^{2m-2}e^{\frac{4L^{3/2}}{3T^{1/2}}}\left|\hat{s}^{\bullet}_{\mathbf{1}}(\mathbf{k},\hat{\mathbf{k}})\right| \leq |P(\frac{\mathbf{k}}{\mathsf{r}},\frac{\hat{\mathbf{k}}}{\mathsf{r}})|\prod_{i=1}^m e^{-c_*\sqrt{\frac{|k_i|}{\mathsf{r}}}-c_*\sqrt{\frac{|\hat{k}_i|}{\mathsf{r}}}} \tag{126}$$

for all $\mathbf{k},\hat{\mathbf{k}} \in \mathbb{Z}^m$, $\theta \in (-\pi,\pi]^m$, and $L, T > 0$ satisfying $\ell \geq c_0$ and $\ell p \geq 2$.

4.7.1 Case 2

For Case 2, $\mathsf{r} = \frac{\ell^{1/4}p}{\sqrt{2}}$ is a constant. Thus, the right-hand side of (126) gives a uniform upper bound, independent of L and T, that is summable. Therefore, by the dominated convergence theorem, Lemma 4.2 and equations (88) and (119) imply that

$$(-1)^{m-1}\frac{2}{\mathsf{r}^2}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}\hat{D}^{\bullet}_1(\mathbf{z}) \to \frac{1}{\mathsf{r}^{2m}}\sum_{\mathbf{k},\hat{\mathbf{k}}\in\mathbb{Z}^m}\prod_{i=2}^m \frac{1}{(\xi_i(k_i)-\xi_{i-1}(k_{i-1}))(\xi_i(k_i')-\xi_{i-1}(k_{i-1}'))}$$
$$\times\prod_{i=1}^m e^{\frac{\Delta t_i}{2}\xi_i(k_i))^2-(\Delta h_i-\Delta x_i)\xi_i(k_i)}\prod_{i=1}^m e^{\frac{\Delta t_i}{2}\xi_i(\hat{k}_i)^2-(\Delta h_i+\Delta x_i)\xi_i(\hat{k}_i)}$$

uniformly for $\theta \in (-\pi,\pi]^m$. Furthermore, the left-hand side is uniformly bounded in L, T, θ. The limit factorizes to the product of two series, and we find from Definition 2.7 that it is equal to

$$S_{\mathsf{r}}(\mathbf{t},\mathbf{h}-\mathbf{x};\mathbf{w})S_{\mathsf{r}}(\mathbf{t},\mathbf{h}+\mathbf{x};\mathbf{w}) \quad \text{where} \quad w_i = e^{-\mathsf{r}\rho_i+\mathrm{i}\theta_i}. \tag{127}$$

Consider the limit of (122). Lemma 4.1 shows that $C^{\bullet}(\mathbf{z}) \to 1$ is uniformly in θ. Thus, the above limit for $\hat{D}^{\bullet}_1(\mathbf{z})$ implies that

$$\frac{2}{\mathsf{r}^2}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}\hat{P}_{m,1} \to \int_{(-\pi,\pi]^m} S_{\mathsf{r}}(\mathbf{t},\mathbf{h}-\mathbf{x};\mathbf{w})S_{\mathsf{r}}(\mathbf{t},\mathbf{h}+\mathbf{x};\mathbf{w})\prod_{i=2}^m\left(1-\frac{e^{-(r\rho_{i-1}-\mathrm{i}\theta_{i-1})}}{e^{-(r\rho_i-\mathrm{i}\theta_i)}}\right)\prod_{i=1}^m\frac{\mathrm{d}\theta_i}{2\pi}$$

where $w_i = e^{-\mathsf{r}\rho_i+\mathrm{i}\theta_i}$. Changing the variables θ_i to w_i, this proves Proposition 2.8 for Case 2.

4.7.2 Case 1

For Case 1, we write the series (125) as an integral of a piecewise constant function,

$$\hat{D}^{\bullet}_{\mathbf{1}}(\mathbf{z}) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\hat{s}^{\bullet}([\mathbf{y}],[\hat{\mathbf{y}}])\mathrm{d}\mathbf{y}\mathrm{d}\hat{\mathbf{y}} = \mathsf{r}^{2m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\hat{s}^{\bullet}([\mathsf{r}\mathbf{y}],[\mathsf{r}\hat{\mathbf{y}}])\mathrm{d}\mathbf{y}\mathrm{d}\hat{\mathbf{y}}. \tag{128}$$

Consider the bound (126) and insert $[\mathsf{r}y_i]$ for k_i and $[\mathsf{r}y_i']$ for k_i'. Since $\mathsf{r} \to \infty$ for Case 1, we may assume that $\mathsf{r} \geq 1$. Then

$$\frac{|y|}{2} \leq \frac{|[\mathsf{r}y]|}{\mathsf{r}} \leq 2|y| \qquad \text{for } |y| \geq 2.$$

Thus, the estimate (126) implies an r-independent upper bound,

$$(\ell p)^{3/2}\mathsf{r}^{2m-2}e^{\frac{4}{3}\ell^{3/2}}|\hat{s}^{\bullet}([\mathsf{r}\mathbf{y}],[\mathsf{r}\hat{\mathbf{y}}])| \leq |\widetilde{P}(\mathbf{y},\mathbf{y}')|\prod_{i=1}^m e^{-c_*\sqrt{\frac{|y_i|}{2}}-c_*\sqrt{\frac{|y_i'|}{2}}}$$

where $\widetilde{P}(\mathbf{y},\mathbf{y}')$ is a polynomial of $\mathbf{y},\mathbf{y}'\in\mathbb{R}^m$ that does not depend on r and θ. Therefore, the dominated convergence theorem, Lemma 4.2, and equations (118) and (87) imply

$$\begin{aligned}&(-1)^{m-1}\frac{2}{\mathsf{r}^2}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}\hat{D}^{\bullet}_{\mathbf{1}}(\mathbf{z})\\ &\to \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\prod_{i=2}^m \frac{1}{(\rho_i+2\pi\mathrm{i}y_i-\rho_{i-1}-2\pi\mathrm{i}y_{i-1})(\rho_i+2\pi\mathrm{i}y_i'-\rho_{i-1}-2\pi\mathrm{i}y_{i-1}')}\\ &\qquad\times\prod_{i=1}^m e^{\frac{\Delta t_i}{2}(\rho_i+2\pi\mathrm{i}y_i)^2-(\Delta h_i-\Delta x_i)(\rho_i+2\pi\mathrm{i}y_i)}\prod_{i=1}^m e^{\frac{\Delta t_i}{2}(\rho_i+2\pi\mathrm{i}y_i')^2-(\Delta h_i+\Delta x_i)(\rho_i+2\pi\mathrm{i}y_i')}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{y}'.\end{aligned}$$

Note that the $\mathbf{y}$-integrals and the $\mathbf{y}'$-integrals factorize. Changing the variables $\rho_i+2\pi\mathrm{i}y_i=\xi_i$, the $\mathbf{y}$-integral is equal to $(-1)^{m-1}S_\infty(\mathbf{t},\mathbf{h}-\mathbf{x})$ of Definition 2.7. Similarly the $\mathbf{y}'$-integral is equal to $(-1)^{m-1}S_\infty(\mathbf{t},\mathbf{h}+\mathbf{x})$. Note that the ordering of the contours comes from the condition that $\rho_1>\cdots>\rho_m$. Thus,

$$(-1)^{m-1}\frac{2}{\mathsf{r}^2}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}e^{\frac{4L^{3/2}}{3T^{1/2}}}\hat{D}^{\bullet}(\theta)\to S_\infty(\mathbf{t},\mathbf{h}-\mathbf{x})S_\infty(\mathbf{t},\mathbf{h}+\mathbf{x}) \tag{129}$$

uniformly for $\theta\in(-\pi,\pi]^m$, and the limit does not depend on θ.

From Lemma 4.1, $C^{\bullet}(\mathbf{z})\to 1$ is uniformly in θ. On the other hand, since $\mathsf{r}\to\infty$ for Case 1,

$$T^{\bullet}_{\mathbf{1}}(\mathbf{z})=\prod_{i=2}^m\left(1-\frac{z_{i-1}}{z_i}\right)=\prod_{i=2}^m\left(1-\frac{e^{-\mathsf{r}\rho_{i-1}+\mathrm{i}\theta_{i-1}}}{e^{-\mathsf{r}\rho_i+\mathrm{i}\theta_i}}\right)\to 1$$

uniformly in θ as well. Thus,

$$\frac{2}{\mathsf{r}^2}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}\hat{P}_{m,1}\to S_\infty(\mathbf{t},\mathbf{h}-\mathbf{x})S_\infty(\mathbf{t},\mathbf{h}+\mathbf{x}).$$

This proves Proposition 2.8 for Case 1.

4.7.3 Case 3

For Case 3, $\mathsf{r}\to 0$. We change the variables $\theta_i=\mathsf{r}\varphi_i$ so that (122) becomes

$$\hat{P}_{m,1}=(-1)^{m-1}\frac{\mathsf{r}^m}{(2\pi)^m}\int_{\mathbb{R}^m}C^{\bullet}(\mathbf{z})\hat{D}^{\bullet}_{\mathbf{1}}(\mathbf{z})T^{\bullet}_{\mathbf{1}}(\mathbf{z})\prod_{i=1}^m 1_{(-\frac{\pi}{\mathsf{r}},\frac{\pi}{\mathsf{r}}]}(\varphi_i)\mathrm{d}\varphi_i,\qquad z_i=e^{-\frac{L}{2T}-\mathsf{r}\rho_i+\mathsf{r}\mathrm{i}\varphi_i}. \tag{130}$$

Write

$$\frac{2}{\mathsf{r}}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}\hat{P}_{m,1}=\sum_{\mathbf{k},\hat{\mathbf{k}}\in\mathbb{Z}^m}\int_{\mathbb{R}^m}Q_{\mathsf{r}}(\varphi;\mathbf{k},\hat{\mathbf{k}})\prod_{i=1}^m\frac{\mathrm{d}\varphi_i}{2\pi} \tag{131}$$

where

$$Q_{\mathsf{r}}(\varphi;\mathbf{k},\hat{\mathbf{k}})=2\mathsf{r}^{m-1}(-1)^{m-1}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}C^{\bullet}(\mathbf{z})\hat{s}^{\bullet}_{\mathbf{1}}(\mathbf{k},\hat{\mathbf{k}})T^{\bullet}_{\mathbf{1}}(\mathbf{z})\prod_{i=1}^m 1_{(-\frac{\pi}{\mathsf{r}},\frac{\pi}{\mathsf{r}}]}(\varphi_i). \tag{132}$$

By Lemma 4.1, $C^{\bullet}(\mathbf{z})\to 1$ uniformly. Thus, we may assume that $|C^{\bullet}(\mathbf{z})|\le 2$. For the term $T^{\bullet}_{\mathbf{1}}(\mathbf{z})$, the estimate (124) is not enough for Case 3. We need a better estimate. For every $\varphi\in\mathbb{R}^m$,

$$\frac{T^{\bullet}_{\mathbf{1}}(\mathbf{z})}{\mathsf{r}^{m-1}}=\frac{1}{\mathsf{r}^{m-1}}\prod_{i=2}^m\left(1-\frac{e^{-\mathsf{r}\rho_{i-1}+\mathrm{i}\mathsf{r}\varphi_{i-1}}}{e^{-\mathsf{r}\rho_i+\mathrm{i}\mathsf{r}\varphi_i}}\right)\to(-1)^{m-1}\prod_{i=2}^m(\rho_i-\mathrm{i}\varphi_i-\rho_{i-1}+\mathrm{i}\varphi_{i-1}). \tag{133}$$

Since $|1-e^w|\le|w|$ for complex numbers w satisfying $\mathrm{Re}(w)\le 0$, we also see that

$$\frac{|T^{\bullet}_{\mathbf{1}}(\mathbf{z})|}{\mathsf{r}^{m-1}}\le\prod_{i=2}^m|\rho_{i-1}-\rho_i-\mathrm{i}(\varphi_{i-1}-\varphi_i)|\qquad\text{for all }\varphi\in\mathbb{R}^m. \tag{134}$$

Thus,

$$\frac{|T_{\mathbf{1}}^{\bullet}(\mathbf{z})|}{\mathsf{r}^{m-1}}\prod_{i=1}^{m}1_{(-\frac{\pi}{\mathsf{r}},\frac{\pi}{\mathsf{r}}]}(\varphi_i)\le\big(\rho_1-\rho_m+\frac{2\pi}{\mathsf{r}}\big)^{m-1}\tag{135}$$

since $\rho_1>\cdots>\rho_m$. Using the estimate (126) for $\hat{s}_{\mathbf{1}}^{\bullet}(\mathbf{k},\hat{\mathbf{k}})$, we find that

$$|Q_{\mathsf{r}}(\varphi;\mathbf{k},\hat{\mathbf{k}})|\le 4\big(\rho_1-\rho_m+\frac{2\pi}{\mathsf{r}}\big)^{m-1}|P\big(\frac{\mathbf{k}}{\mathsf{r}},\frac{\hat{\mathbf{k}}}{\mathsf{r}}\big)|\prod_{i=1}^{m}e^{-c_*\sqrt{\frac{|k_i|}{\mathsf{r}}}-c_*\sqrt{\frac{|\hat{k}_i|}{\mathsf{r}}}}\prod_{i=1}^{m}1_{(-\frac{\pi}{\mathsf{r}},\frac{\pi}{\mathsf{r}}]}(\varphi_i)\tag{136}$$

for all $\varphi\in\mathbb{R}^m$.

For the sum over $(\mathbf{k},\hat{\mathbf{k}})\ne(\mathbf{0},\mathbf{0})$ in (131), we find, after integrating over φ_is, that

$$\begin{aligned}&\sum_{(\mathbf{k},\hat{\mathbf{k}})\in\mathbb{Z}^{2m}\setminus\{(\mathbf{0},\mathbf{0})\}}\int_{\mathbb{R}^m}|Q_{\mathsf{r}}(\varphi;\mathbf{k},\hat{\mathbf{k}})|\prod_{i=1}^{m}\frac{\mathrm{d}\varphi_i}{2\pi}\\&\le 4\big(\rho_1-\rho_m+\frac{2\pi}{\mathsf{r}}\big)^{m-1}\left(\frac{2\pi}{\mathsf{r}}\right)^{m}\sum_{\mathbf{k},\hat{\mathbf{k}}\in\mathbb{Z}^m\setminus\{(\mathbf{0},\mathbf{0})\}}|P\big(\frac{\mathbf{k}}{\mathsf{r}},\frac{\hat{\mathbf{k}}}{\mathsf{r}}\big)|\prod_{i=1}^{m}e^{-c_*\sqrt{\frac{|k_i|}{\mathsf{r}}}-c_*\sqrt{\frac{|\hat{k}_i|}{\mathsf{r}}}}.\end{aligned}$$

Recall that $\mathsf{r}\to 0$ for Case 3. Lemma 4.9 (a) implies that for any non-negative integer ℓ, there is a constant $C'_\ell>0$ such that

$$\sum_{k\in\mathbb{Z}\setminus\{0\}}\big(\frac{|k|}{\mathsf{r}}\big)^{\ell}e^{-c_*\sqrt{\frac{|k|}{\mathsf{r}}}}\le 2\mathsf{r}\int_{\frac{1}{\mathsf{r}}}^{\infty}y^{\ell}e^{-c_*\sqrt{\frac{y}{2}}}\mathrm{d}y\le\frac{C'_\ell}{\mathsf{r}^{\ell-1/2}}e^{-\frac{c_*}{\sqrt{2\mathsf{r}}}}$$

and

$$\sum_{k\in\mathbb{Z}}\big(\frac{|k|}{\mathsf{r}}\big)^{\ell}e^{-c_*\sqrt{\frac{|k|}{\mathsf{r}}}}\le 1+\frac{C'_\ell}{\mathsf{r}^{\ell-1/2}}e^{-\frac{c_*}{\sqrt{2\mathsf{r}}}}\le 2$$

for all small enough $\mathsf{r}>0$. Therefore, there are a positive constant C and a non-negative integer n so that

$$\sum_{(\mathbf{k},\hat{\mathbf{k}})\in\mathbb{Z}^{2m}\setminus\{(\mathbf{0},\mathbf{0})\}}\int_{\mathbb{R}^m}|Q_{\mathsf{r}}(\varphi;\mathbf{k},\hat{\mathbf{k}})|\prod_{i=1}^{m}\frac{\mathrm{d}\varphi_i}{2\pi}\le\frac{C}{\mathsf{r}^n}e^{-\frac{c_*}{\sqrt{2\mathsf{r}}}}.$$

Thus, the series tends to 0 as $\mathsf{r}\to 0$.

We now consider the term for $\mathbf{k}=\hat{\mathbf{k}}=\mathbf{0}$ in (131), $\int_{\mathbb{R}^m}Q_{\mathsf{r}}(\varphi;\mathbf{0},\mathbf{0})\prod_{i=1}^{m}\frac{\mathrm{d}\varphi_i}{2\pi}$. In the derivation of (126), we used (96). We now use the bound (95) instead to find

$$\mathsf{r}^{2m-2}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}\left|\hat{s}_{\mathbf{1}}^{\bullet}(\mathbf{k},\hat{\mathbf{k}})\right|\le|P\big(\frac{\mathbf{k}}{\mathsf{r}},\frac{\hat{\mathbf{k}}}{\mathsf{r}}\big)|\prod_{i=1}^{m}e^{-2c_*\sqrt{\frac{1}{\mathsf{r}}|k_i-\frac{\mathsf{r}\varphi_i}{2\pi}|}-2c_*\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_i-\frac{\mathsf{r}\varphi_i}{2\pi}|}}.\tag{137}$$

Thus, when $\mathbf{k}=\hat{\mathbf{k}}=\mathbf{0}$, there is a constant $C>0$ such that

$$\mathsf{r}^{2m-2}(\ell p)^{3/2}e^{\frac{4}{3}\ell^{3/2}}\,|\hat{s}_{\mathbf{1}}^{\bullet}(\mathbf{0},\mathbf{0})|\le C\prod_{i=1}^{m}e^{-\frac{4c_*}{\sqrt{2\pi}}\sqrt{|\varphi_i|}}.\tag{138}$$

Using this estimate in (132), and also using (134) and the fact that $|C^{\bullet}(\mathbf{z})|\le 2$,

$$|Q_{\mathsf{r}}(\varphi;\mathbf{0},\mathbf{0})|\le 4C\prod_{i=2}^{m}|\rho_{i-1}-\rho_i-\mathrm{i}(\varphi_{i-1}-\varphi_i)|\prod_{i=1}^{m}e^{-\frac{4c_*}{\sqrt{2\pi}}\sqrt{|\varphi_i|}}.$$

Since the upper bound is absolutely integrable and does not depend on L, T, we can apply the dominated converge theorem to evaluate the integral of $Q_{\mathsf{r}}(\varphi; \mathbf{0}, \mathbf{0})$. Recall $\hat{s}^{\bullet}_{\mathbf{1}}(\mathbf{0}, \mathbf{0}) = H_{\mathbf{1}}(\mathsf{U}(\mathbf{0}), \mathsf{U}(\mathbf{0}))\hat{R}_{\mathbf{1}}(\mathsf{U}(\mathbf{0}), \mathsf{U}(\mathbf{0}))E_{\mathbf{1}}(\mathsf{U}(\mathbf{0}), \mathsf{U}(\mathbf{0}))$ in (79). Lemma 4.2 implies that $H_{\mathbf{1}}(\mathsf{U}(\mathbf{0}), \mathsf{U}(\mathbf{0})) \to 1$. Thus, (90), (120), and (133) imply, also using $C^{\bullet}(\mathbf{z}) \to 1$, that

$$\int_{\mathbb{R}^m} Q_{\mathsf{r}}(\varphi; \mathbf{0}, \mathbf{0}) \prod_{i=1}^{m} \frac{\mathrm{d}\varphi_i}{2\pi} \to \frac{(-1)^{m-1}}{(2\pi)^m} \int_{\mathbb{R}^m} \prod_{i=2}^{m} \frac{1}{\rho_i - \mathrm{i}\varphi_i - \rho_{i-1} + \mathrm{i}\varphi_{i-1}} \prod_{i=1}^{m} e^{\Delta t_i(\rho_i - \mathrm{i}\varphi_i)^2 - 2\Delta h_i(\rho_i - \mathrm{i}\varphi_i))} \prod_{i=1}^{m} \mathrm{d}\varphi_i.$$

The limit is $S_\infty(2\mathbf{t}, 2\mathbf{h})$ in Definition 2.7.

Combining all together we conclude that $\frac{2}{\mathsf{r}}(\ell p)^{3/2} e^{\frac{4}{3}\ell^{3/2}} \hat{P}_{m,1} \to S_\infty(2\mathbf{t}, 2\mathbf{h})$. Thus, we proved Proposition 2.8 for Case 3.

4.8 Proof of Proposition 2.9

The formula of $P_{m,1}$ and $\hat{P}_{m,1}$ are similar:

$$P_{m,1} = \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \oint \cdots \oint A_1(z_m) C^{\bullet}(\mathbf{z}) D^{\bullet}_{\mathbf{1}}(\mathbf{z}) T^{\bullet}_{\mathbf{1}}(\mathbf{z}) \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{z_i}$$

and

$$\hat{P}_{m,1} = \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \oint \cdots \oint C^{\bullet}(\mathbf{z}) \hat{D}^{\bullet}_{\mathbf{1}}(\mathbf{z}) T^{\bullet}_{\mathbf{1}}(\mathbf{z}) \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{z_i}.$$

In the previous section on the analysis of $\hat{P}_{m,1}$, all upper bounds were obtained from absolute value estimates. In $P_{m,1}$, there is an additional decay factor due to (see (73))

$$|A_1(z_m)| \le |z_m| \le e^{-\frac{\ell p}{2}}$$

and the fact that $\ell p \to \infty$ for all three Cases. Furthermore, the term $D_{\mathbf{1}}(\mathbf{z})$ involves $R^{\bullet}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$ while $\hat{D}_{\mathbf{1}}(\mathbf{z})$ contains $\hat{R}^{\bullet}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$. By Lemma 4.12, we find that an estimate of $R^{\bullet}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$ is the $\frac{1}{(\ell p)^{1/2}}$ times the estimate of $\hat{R}^{\bullet}_{\mathbf{1}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))$. Thus, in all estimates obtained in the last sections for $\hat{D}_{\mathbf{1}}(\mathbf{z})$, we can multiply $\frac{1}{(\ell p)^{1/2}}$ to obtain an estimate for $D_{\mathbf{1}}(\mathbf{z})$. Due to these two factors, since $|\hat{P}_{m,1}|$ is uniformly bounded in all three cases, we find that $|P_{m,1}|$ is of order $\frac{e^{-\frac{\ell p}{2}}}{(\ell p)^{1/2}}$ is all three cases. This proves Proposition 2.9.

4.9 Proof of Proposition 2.10 when $p \ll \ell^{5/4}$

We prove Proposition 2.10 for Case 2 and 3 as well as Case 1 under the extra assumption that $p \ll \ell^{5/4}$ in this section and prove remaining part of Case 1 in the next section. The assumption $p \ll \ell^{5/4}$ will be used only when we simplify (142) at the very end of the analysis.

Recall (78) and (79). Lemma 4.2, Corollary 4.6, and Corollary 4.11 imply a bound for $s^{\bullet}_{\mathbf{n}}(\mathbf{k}, \hat{\mathbf{k}})$ and $\hat{s}^{\bullet}_{\mathbf{n}}(\mathbf{k}, \hat{\mathbf{k}})$. Let $c_* > 0$ be the constant from Lemma 4.4 that appears in Corollary 4.6. When applying Corollary 4.11, we use the constant $\epsilon = \frac{c_*}{2}$. Thus, we find that there are positive constants c_0, c_2, c_*, δ and C_0 such that

$$e^{\frac{4}{3}\ell^{3/2}} |s^{\bullet}_{\mathbf{n}}(\mathbf{k}, \hat{\mathbf{k}})| \le 5|\mathbf{n}| e^{4|\mathbf{n}|} \left(\frac{C_0 (\ell p)^2}{\mathsf{r}^2}\right)^{|\mathbf{n}|} e^{-\frac{4\delta}{3}\ell^{3/2} - c_2|\mathbf{n}|\ell^{3/2}} \prod_{i=1}^{m} \prod_{j=1}^{n_i} e^{-\frac{c_*}{2}\sqrt{\frac{1}{\mathsf{r}}|k_j^{(i)}|} - \frac{c_*}{2}\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j^{(i)}|}} \tag{139}$$

for all $\mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}$, $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}$, $\theta \in (-\pi, \pi]^m$, and $L, T > 0$ satisfying $\ell \ge c_0$ and $\ell p \ge 2$. We also have a similar estimate for $\hat{s}^{\bullet}_{\mathbf{n}}(\mathbf{k}, \hat{\mathbf{k}})$ where we need to multiply $|\mathbf{n}|\ell p$ due to the difference between (114) and (113).

Consider the series (78) which are sums over $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}$. Since $s^\bullet_{\mathbf{n}}(\mathbf{k}, \hat{\mathbf{k}}) = \hat{s}^\bullet_{\mathbf{n}}(\mathbf{k}, \hat{\mathbf{k}}) = 0$ if two components of any one of $k_1, \cdots, k_m, \hat{k}_1, \cdots, \hat{k}_m$ are equal (due to the Cauchy determinants in $R_{\mathbf{n}}(\mathbf{k}, \hat{\mathbf{k}})$), it is enough to take sums over indices of distinct components. Thus, noting $\sum_{i=1}^m n_i = |\mathbf{n}|$,

$$e^{\frac{4}{3}\ell^{3/2}} D^\bullet_{\mathbf{n}}(\mathbf{z}) \le 5|\mathbf{n}| e^{4|\mathbf{n}|} \left(\frac{C_0(\ell p)^2}{\mathsf{r}^2} \right)^{|\mathbf{n}|} e^{-\frac{4\delta}{3}\ell^{3/2} - c_2|\mathbf{n}|\ell^{3/2}} \left(\sum_{k=-\infty}^{\infty} e^{-\frac{c_*}{2}\sqrt{\frac{|k|}{\mathsf{r}}}} \right)^{2|\mathbf{n}|}.$$

The sum can be estimated using the $a = 0$ case of (105), and we find that

$$e^{\frac{4}{3}\ell^{3/2}} D^\bullet_{\mathbf{n}}(\mathbf{z}) \le 5|\mathbf{n}| \left(\frac{C_0(\ell p)^4}{\mathsf{r}^2} \right)^{|\mathbf{n}|} e^{-\frac{4\delta}{3}\ell^{3/2} - c_2|\mathbf{n}|\ell^{3/2}} \tag{140}$$

where the constant C_0 is modified from the last equation. We also have a similar estimate for $\hat{D}^\bullet_{\mathbf{n}}(\mathbf{z})$ where we need to multiply $|\mathbf{n}|\ell p$.

From (44),

$$|e^{\frac{4}{3}\ell^{3/2}} P_{m,2}| \le \sum_{\mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}} \frac{1}{(\mathbf{n}!)^2} \int_{(-\pi,\pi]^m} |A_1(z_m) C^\bullet(\mathbf{z}) D^\bullet_{\mathbf{n}}(\mathbf{z}) T^\bullet_{\mathbf{n}}(\mathbf{z})| \prod_{i=1}^m \frac{\mathrm{d}\theta_i}{2\pi},$$

$$|e^{\frac{4}{3}\ell^{3/2}} \hat{P}_{m,2}| \le \sum_{\mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}} \frac{1}{(\mathbf{n}!)^2} \int_{(-\pi,\pi]^m} |C^\bullet(\mathbf{z}) \hat{D}^\bullet_{\mathbf{n}}(\mathbf{z}) T^\bullet_{\mathbf{n}}(\mathbf{z})| \prod_{i=1}^m \frac{\mathrm{d}\theta_i}{2\pi},$$

where $z_i = e^{-\frac{\ell p}{2} - \mathsf{r}\rho_i + \mathrm{i}\theta_i}$. By (73), $|A_1(z_m)| \le |z_m| \le 1$. By Lemma 4.1, $|C^\bullet(\mathbf{z})| \le 2$ for all three Cases eventually. Using the formula of z_i, since $\rho_1 > \cdots > \rho_m$, we see that (41) satisfies

$$|T^\bullet_{\mathbf{n}}(\mathbf{z})| = \left| \prod_{i=2}^m \left(1 - \frac{z_{i-1}}{z_i}\right)^{n_i} \left(1 - \frac{z_i}{z_{i-1}}\right)^{n_{i-1}-1} \right| \le \prod_{i=2}^m 2^{n_i}(1 + e^{\mathsf{r}(\rho_{i-1}-\rho_i)})^{n_{i-1}} \le 2^{2|\mathbf{n}|} e^{c'\mathsf{r}|\mathbf{n}|} \tag{141}$$

where $c' = \max\{\rho_{i-1} - \rho_i : 2 \le i \le m\} > 0$. Note that this estimate contains an exponential function and is very loose but it is sufficient when we assume that $p \ll \ell^{5/4}$.

Thus, with a new positive constant C_0,

$$|e^{\frac{4}{3}\ell^{3/2}} P_{m,2}| \le e^{-\frac{4\delta}{3}\ell^{3/2}} \sum_{\mathbf{n} \in \mathbb{N}^m \setminus \{\mathbf{1}\}} \frac{|\mathbf{n}|}{(\mathbf{n}!)^2} \left(\frac{C_0(\ell p)^4}{\mathsf{r}^2} \right)^{|\mathbf{n}|} e^{c'\mathsf{r}|\mathbf{n}|} e^{-c_2|\mathbf{n}|\ell^{3/2}}. \tag{142}$$

Since we assume that $p \ll \ell^{5/4}$, we have $\mathsf{r} = p\ell^{1/4}/\sqrt{2} \ll \ell^{3/2}$ and $(\ell p)^4/\mathsf{r}^2 = 2p^2\ell^{7/2} \ll \ell^6$. Recall that $\ell \to \infty$ for all three Cases. Thus the sum on the right hand side of (142) is convergent and uniformly bounded for all three cases. Note that $\frac{\ell}{p^{1/2}} \ll \ell^{3/2}$ since $\ell p \to \infty$. This proves first result of Proposition 2.10. An estimate of $\hat{P}_{m,2}$ is similar; the summand in (142) is multiplied by $|\mathbf{n}|\ell p$. This change does not affect the proof much and we obtain the second result of Proposition 2.10.

4.10 Proof of Proposition 2.10 when $p \gg \ell$

Case 1 is when $\ell^{-1/4} \ll p$ and $\log p \ll \ell^{3/2}$. We prove Proposition 2.10 for Case 1 when $p \ll \ell^{5/4}$ does not hold. The proof given here applies to the situation when p and ℓ satisfy $p \gg \ell$ and $\log p \ll \ell^{3/2}$. Note that we have ℓ and r both tend to infinity in this case.

The main reason that we added the assumption $p \ll \ell^{5/4}$ in the last section is the factor $e^{c'\mathsf{r}|\mathbf{n}|}$ in (142) which comes from the estimate (141) of $|T^\bullet_{\mathbf{n}}(\mathbf{z})| \le 2^{2|\mathbf{n}|}e^{c'\mathsf{r}|\mathbf{n}|}$. In order to improve this estimate, we modify the integral contours. In (71), the contours were chosen as

$$z_i = e^{-\frac{\ell p}{2} - \mathsf{r}\rho_i + \mathrm{i}\theta_i}, \qquad \theta_i \in (-\pi, \pi],$$

where $\rho_1 > \cdots > \rho_m > 0$ were fixed numbers. In this section, we choose these numbers to be dependent on r:

$$\rho_i = \rho_1 - \frac{i-1}{\mathsf{r}} \tag{143}$$

for $1 \le i \le m$, where ρ_1 is a a fixed positive number. With this change, the estimate (141) is changed to

$$|T_{\mathbf{n}}^{\bullet}(\mathbf{z})| \le 2^{2|\mathbf{n}|} e^{c'|\mathbf{n}|}. \tag{144}$$

The difference is that the exponent is changed from $c'\mathsf{r}|\mathbf{n}|$ to $c'|\mathbf{n}|$, which gives a much tighter bound. However, we need to check how other quantities in the estimate (141) change due to the contour changes.

The estimates in Sections 4.2 and 4.4 are still valid without any change. For the estimates in Section 4.5, note that $d = \frac{\rho_i}{\sqrt{2}} = \frac{\rho_1}{\sqrt{2}} - \frac{i-1}{\sqrt{2}\mathsf{r}}$ which depends on r but is close to the constant $\frac{\rho_1}{\sqrt{2}}$. Since Corollary 3.5 holds uniformly on d, Lemma 4.4 and Corollary 4.6 still hold. However, the estimates in Section 4.6 need some changes.

Lemma 4.10 is changed to the following estimate.

Lemma 4.14. *For every $\epsilon > 0$, there is a positive constant C_0 such that*

$$\frac{\left|\mathsf{K}(\mathsf{u}_i(\mathbf{k}), -\mathsf{u}_{i'}(\hat{\mathbf{k}}'); \mathsf{u}_i(\hat{\mathbf{k}}), -\mathsf{u}_{i'}(\mathbf{k}'))\right|}{\prod_{j=1}^{n} |u_i(\hat{k}_j)| \prod_{j=1}^{n'} |u_{i'}(k'_j)|} \prod_{j=1}^{n} e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}|k_j|} - \epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}_j|}} \prod_{j=1}^{n'} e^{-\epsilon\sqrt{\frac{1}{\mathsf{r}}|k'_j|} - \epsilon\sqrt{\frac{1}{\mathsf{r}}|\hat{k}'_j|}} \le C_0^{\frac{n+n'}{2}} \tag{145}$$

for all two distinct integers i and i' from $\{0, \cdots, m+1\}$, $n, n' \in \mathbb{N}$, $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^n$, $\mathbf{k}', \hat{\mathbf{k}}' \in \mathbb{Z}^{n'}$, and $\ell, p > 0$ satisfying $\ell^3 \ge 4\rho_1^4$ and $\ell p \ge 1$.

Proof. Recall that it is enough to consider the case when $\mathbf{k}, \mathbf{k}', \hat{\mathbf{k}}$, or $\hat{\mathbf{k}}'$ all have distinct components. In the proof of Lemma 4.10, the estimates (107) and (108) still hold. However, we need to change the estimate 110. Since the components of k'_p are all distinct, we have

$$\sum_{p=1}^{n} \frac{e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k'_p|}}}{|u_i(k_q) - u_{i'}(k'_p)|^2 |u_{i'}(k'_p)|^2} \le \sum_{p=1}^{n} \frac{1}{|u_i(k_q) - u_{i'}(k'_p)|^2 |u_{i'}(k'_p)|^2} \le \sum_{k' \in \mathbb{Z}} \frac{1}{|u_i(k_q) - u_{i'}(k')|^2 |u_{i'}(k')|^2} \tag{146}$$

We split the last sum into two parts. The first part contains all k' satisfying $|u_i(k_q) - u_{i'}(k')| \ge |u_{i'}(k')|$. This part is bounded by, recalling the definition of $u_{i'}(k')$ in (75),

$$\sum_{k'} \frac{1}{|u_{i'}(k')|^4} = \sum_{k'} \frac{1}{(\ell p + 2\mathsf{r}\rho_{i'})^2 + (4\pi k' - 2\theta_{i'})^2} \le \sum_{k' \in \mathbb{Z}} \frac{1}{1 + (4\pi k' - 2\theta_{i'})^2} \tag{147}$$

which is uniformly bounded by a constant. The second part of the sum contains all k' satisfying $|u_i(k_q) - u_{i'}(k')| < |u_{i'}(k')|$. Noting the fact that $|u_i(k_q) + u_{i'}(k')| \le |u_i(k_q) - u_{i'}(k')| + 2|u_{i'}(k')| \le 3|u_{i'}(k')|$, this part is bounded by

$$\sum_{k'} \frac{|u_i(k_q) + u_{i'}(k')|^2}{|(u_i(k_q))^2 - (u_{i'}(k'))^2|^2 |u_{i'}(k')|^2} \le \sum_{k'} \frac{9}{|(u_i(k_q))^2 - (u_{i'}(k'))^2|^2} \tag{148}$$

which is uniformly bounded by a constant since $i \ne i'$ and

$$\begin{aligned} |(u_i(k_q))^2 - (u_{i'}(k'))^2|^2 &= 4\mathsf{r}^2(\rho_i - \rho_{i'})^2 + (4\pi(k_q - k') + 2(\theta_{i'} - \theta_i))^2 \\ &= 4(i - i')^2 + (4\pi(k_q - k') + 2(\theta_{i'} - \theta_i))^2 \end{aligned} \tag{149}$$

by our choices of ρ_i and $\rho_{i'}$. Combing the above two parts, we obtain

$$\sum_{p=1}^{n} \frac{e^{-2\epsilon\sqrt{\frac{1}{\mathsf{r}}|k'_p|}}}{|u_i(k_q) - u_{i'}(k'_p)|^2 |u_{i'}(k'_p)|^2} \le C_2, \tag{150}$$

which implies that the bound (111) changes to

$$\prod_{j=1}^{n} e^{-\epsilon\sqrt{\frac{1}{r}|k_j|}} \prod_{j=1}^{n'} e^{-\epsilon\sqrt{\frac{1}{r}|\hat{k}_j'|}} \left[\prod_{q=1}^{n}\sqrt{\frac{C_1}{\ell p}+C_2}\right]\left[\prod_{q=1}^{n'}\sqrt{\frac{C_1}{\ell p}+C_2}\right]. \tag{151}$$

We thus obtain (145). □

Using the above bound instead of Lemma 4.14, the same proof shows that Corollary 4.11 changes to the following.

Corollary 4.15. *For every $\epsilon > 0$, there is a positive constant C_0 such that*

$$|R_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))| \le C_0^{|\mathbf{n}|} \prod_{i=1}^{m}\prod_{j=1}^{n_i} e^{2\epsilon\sqrt{\frac{1}{r}|k_j^{(i)}|}+2\epsilon\sqrt{\frac{1}{r}|\hat{k}_j^{(i)}|}} \tag{152}$$

and

$$|\hat{R}_{\mathbf{n}}(\mathsf{U}(\mathbf{k}), \mathsf{U}(\hat{\mathbf{k}}))| \le |\mathbf{n}|(\ell p)^{1/2} C_0^{|\mathbf{n}|} \prod_{i=1}^{m}\prod_{j=1}^{n_i} e^{2\epsilon\sqrt{\frac{1}{r}|k_j^{(i)}|}+2\epsilon\sqrt{\frac{1}{r}|\hat{k}_j^{(i)}|}} \tag{153}$$

for all $\mathbf{n} \in \mathbb{N}^m$, $\mathbf{k}, \hat{\mathbf{k}} \in \mathbb{Z}^{\mathbf{n}}$, and $\ell, p > 0$ satisfying $\ell^3 \ge 4\rho_1^4$ and $\ell p \ge 1$.

We are ready to prove Proposition 2.10 for Case 1 assuming $p \gg \ell$ and $\log p \ll \ell^{3/2}$. We follow the same analysis of subsection 4.9 except that we replace Corollary 4.11 by Corollary 4.15 and the inequality (141) by (144). Then the inequality (139) is replaced by

$$e^{\frac{4}{3}\ell^{3/2}}|s_{\mathbf{n}}^{\bullet}(\mathbf{k},\hat{\mathbf{k}})| \le 5|\mathbf{n}|e^{4|\mathbf{n}|}C_0^{|\mathbf{n}|}e^{-\frac{4\delta}{3}\ell^{3/2}-c_2|\mathbf{n}|\ell^{3/2}} \prod_{i=1}^{m}\prod_{j=1}^{n_i} e^{-\frac{c_*}{2}\sqrt{\frac{1}{r}|k_j^{(i)}|}-\frac{c_*}{2}\sqrt{\frac{1}{r}|\hat{k}_j^{(i)}|}} \tag{154}$$

and the inequality (140) is changed to

$$e^{\frac{4}{3}\ell^{3/2}}D_{\mathbf{n}}^{\bullet}(\mathbf{z}) \le 5|\mathbf{n}|\left(C_0(\ell p)^2\right)^{|\mathbf{n}|}e^{-\frac{4\delta}{3}\ell^{3/2}-c_2|\mathbf{n}|\ell^{3/2}}. \tag{155}$$

For the bounds of $|\hat{s}_{\mathbf{n}}^{\bullet}(\mathbf{k},\hat{\mathbf{k}})|$ and $\hat{D}_{\mathbf{n}}^{\bullet}(\mathbf{z})$, we only need to multiply the bounds of $|s_{\mathbf{n}}^{\bullet}(\mathbf{k},\hat{\mathbf{k}})|$ and $D_{\mathbf{n}}^{\bullet}(\mathbf{z})$ by a factor $|\mathbf{n}|(\ell p)^{1/2}$ due to Corollary 4.15. Finally, using (144), the inequality (142) changes to

$$|e^{\frac{4}{3}\ell^{3/2}}P_{m,2}| \le e^{-\frac{4\delta}{3}\ell^{3/2}} \sum_{\mathbf{n}\in\mathbb{N}^m\setminus\{\mathbf{1}\}} \frac{|\mathbf{n}|}{(\mathbf{n}!)^2}\left(C_0(\ell p)^2\right)^{|\mathbf{n}|}e^{c'|\mathbf{n}|}e^{-c_2|\mathbf{n}|\ell^{3/2}}. \tag{156}$$

The sum is uniformly bounded provided $p \ll e^{c_2\ell^{3/2}/2}$, which holds since $\log p \ll \ell^{3/2}$. This proves the first part of Proposition 2.10. The proof of the second part on $\hat{P}_{m,2}$ is similar.

5 Proof of Proposition 2.11

We start with a lemma.

Lemma 5.1. *Let $\mathbf{a} \in \mathbb{R}^m$ satisfy $0 < a_1 < \cdots < a_{m-1} < a_m$ and let $\mathbf{b} \in \mathbb{R}^m$. Then, for every $r > 0$,*

$$\begin{aligned}&\frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m}\int\cdots\int \prod_{i=2}^{m}\frac{1-e^{r(\xi_i-\xi_{i-1})}}{\xi_i-\xi_{i-1}}\prod_{i=1}^{m}e^{\frac{\Delta a_i}{2}\xi_i^2-\Delta b_i\xi_i}\mathrm{d}\xi_i\\ &= \mathbb{P}\left(\mathsf{B}(a_1)-b_1\in[0,r),\cdots,\mathsf{B}(a_{m-1})-b_{m-1}\in[0,r),\mathsf{B}(a_m)=b_m\right)\end{aligned} \tag{157}$$

where the contours are distinct vertical lines oriented upwards and B is a standard Brownian motion.

Proof. From Gaussian integrals,

$$\frac{1}{(2\pi \mathrm{i})^m}\int\cdots\int\prod_{i=1}^m e^{\frac{\Delta a_i}{2}\xi_i^2-\Delta y_i\xi_i}\mathrm{d}\xi_i=\prod_{i=1}^m\frac{e^{-\frac{(\Delta y_i)^2}{2\Delta a_i}}}{\sqrt{2\pi\Delta a_i}}$$

for every $\mathbf{y}\in\mathbb{R}^m$. The result follows by integrating y_i from b_i to b_i+r for $i=1,\cdots,m-1$ and taking $y_m=b_m$. □

If the contours are ordered as $\mathrm{Re}(\xi_1)>\cdots>\mathrm{Re}(\xi_m)$, then taking $r\to+\infty$, (157) yields

$$\frac{1}{(2\pi \mathrm{i})^m}\int\cdots\int\prod_{i=2}^m\frac{1}{\xi_i-\xi_{i-1}}\prod_{i=1}^m e^{\frac{\Delta a_i}{2}\xi_i^2-\Delta b_i\xi_i}\mathrm{d}\xi_i=\mathbb{P}\left(\mathsf{B}(a_1)\ge b_1,\cdots,\mathsf{B}(a_{m-1})\ge b_{m-1},\mathsf{B}(a_m)=b_m\right),$$

which is Proposition 2.11 (a). This computation is due to [16, Lemma 3.4].

We now prove Proposition 2.11 (b).

Proof of Proposition 2.11 (b). Denote the left-side of (55) by

$$A:=\frac{1}{(2\pi \mathrm{i})^m}\oint\cdots\oint S_{\mathsf{r}}(\mathbf{a},\mathbf{b};\mathbf{w})S_{\mathsf{r}}(\mathbf{a},\mathbf{c};\mathbf{w})\prod_{i=2}^m\left(1-\frac{w_{i-1}}{w_i}\right)\prod_{i=1}^m\frac{\mathrm{d}w_i}{w_i}\tag{158}$$

where the contours are circles satisfying $0<|w_1|<\cdots<|w_m|<1$ and (recall (47))

$$S_{\mathsf{r}}(\mathbf{a},\mathbf{b};\mathbf{w})=\frac{(-1)^{m-1}}{\mathsf{r}^m}\sum_{\xi_1,\cdots,\xi_m}\prod_{i=2}^m\frac{1}{\xi_i-\xi_{i-1}}\prod_{i=1}^m e^{\frac{\Delta a_i}{2}\xi_i^2-\Delta b_i\xi_i}.$$

Since the sum is over the points ξ_i satisfying $e^{-\mathsf{r}\xi_i}=w_i$, we see that

$$\prod_{i=2}^m\left(1-\frac{w_{i-1}}{w_i}\right)=\prod_{i=2}^m\left(1-e^{\mathsf{r}(\xi_i-\xi_{i-1})}\right).$$

Thus, we can write A as

$$A=\frac{1}{(2\pi \mathrm{i})^m}\oint\cdots\oint S_{\mathsf{r}}(\mathbf{a},\mathbf{b};\mathbf{w})T_{\mathsf{r}}(\mathbf{a},\mathbf{c};\mathbf{w})\prod_{i=1}^m\frac{\mathrm{d}w_i}{w_i}\tag{159}$$

where

$$T_{\mathsf{r}}(\mathbf{a},\mathbf{c};\mathbf{w})=\frac{(-1)^{m-1}}{\mathsf{r}^m}\sum_{\xi_1,\cdots,\xi_m}\prod_{i=2}^m\frac{1-e^{\mathsf{r}(\xi_i-\xi_{i-1})}}{\xi_i-\xi_{i-1}}\prod_{i=1}^m e^{\frac{\Delta a_i}{2}\xi_i^2-\Delta c_i\xi_i}.\tag{160}$$

Let $0<|w|<1$. Let $f(\xi)$ be a function that is analytic in a vertical strip $p-2\delta<\mathrm{Re}(\xi)<p+2\delta$ for some $\delta>0$, where $p=-\frac{\log|w|}{r}$, and decays fast as $\mathrm{Im}(\xi)\to\pm\infty$ in the strip. By the Cauchy residue theorem,

$$\sum_{\xi:e^{-\mathsf{r}\xi}=w}f(\xi)=\frac{1}{2\pi \mathrm{i}}\int_{p+\delta-\mathrm{i}\infty}^{p+\delta+\mathrm{i}\infty}\frac{-\mathsf{r}wf(\xi)}{e^{-\mathsf{r}\xi}-w}\mathrm{d}\xi-\frac{1}{2\pi \mathrm{i}}\int_{p-\delta-\mathrm{i}\infty}^{p-\delta+\mathrm{i}\infty}\frac{-\mathsf{r}wf(\xi)}{e^{-\mathsf{r}\xi}-w}\mathrm{d}\xi.$$

Using the geometric series and moving the contours, we find that

$$\sum_{\xi:e^{-\mathsf{r}\xi}=w}f(\xi)=\frac{\mathsf{r}}{2\pi \mathrm{i}}\sum_{k=-\infty}^{\infty}\frac{1}{w^k}\int_{p+\mathrm{i}\mathbb{R}}f(\xi)e^{-k\mathsf{r}\xi}\mathrm{d}\xi$$

where the contour is oriented upwards. Extending the formula in a natural way, we have

$$S_{\mathsf{r}}(\mathbf{a},\mathbf{b};\mathbf{w}) = \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \sum_{\mathbf{n}\in\mathbb{Z}^m} \frac{1}{w_1^{n_1}\cdots w_m^{n_m}} \int\cdots\int \prod_{i=2}^m \frac{(-1)^{m-1}}{\xi_i-\xi_{i-1}} \prod_{i=1}^m e^{\frac{\Delta a_i}{2}\xi_i^2-\Delta b_i\xi_i} e^{-n_i\mathsf{r}\xi_i}\mathrm{d}\xi_i \tag{161}$$

and

$$T_{\mathsf{r}}(\mathbf{a},\mathbf{c};\mathbf{w}) = \frac{(-1)^{m-1}}{(2\pi \mathrm{i})^m} \sum_{\mathbf{n}\in\mathbb{Z}^m} \frac{1}{w_1^{n_1}\cdots w_m^{n_m}} \int\cdots\int \prod_{i=2}^m \frac{1-e^{r(\xi_i-\xi_{i-1})}}{\xi_i-\xi_{i-1}} \prod_{i=1}^m e^{\frac{\Delta a_i}{2}\xi_i^2-\Delta c_i\xi_i} e^{-n_i\mathsf{r}\xi_i}\mathrm{d}\xi_i \tag{162}$$

where the contours are vertical lines, oriented upwards, satisfying $\mathrm{Re}(\xi_1) > \cdots > \mathrm{Re}(\xi_m)$. The ordering of the contours follows from $|w_1| < \cdots < |w_m|$.

Change the summation index $\mathbf{n}$ to $\mathbf{k}$ by setting $n_i = k_1 + \cdots + k_i$ for $i = 1, \cdots, m$ so that $k_i = \Delta n_i$ (where $n_0 := 0$.) Using Lemma 5.1 with b_i replaced by $b_i + \mathsf{r}k_i$ and c_i replaced by $c_i + \mathsf{r}k_i$, we find that

$$S_{\mathsf{r}}(\mathbf{a},\mathbf{b};\mathbf{w}) = \sum_{\mathbf{k}\in\mathbb{Z}^m} \frac{\mathbb{P}\left(\bigcap_{i=1}^{m-1}\{\mathsf{B}_1(a_i)-b_i\geq \mathsf{r}k_i\}\cap\{\mathsf{B}_1(a_m)-b_m=\mathsf{r}k_m\}\right)}{w_1^{\Delta k_1}\cdots w_m^{\Delta k_m}} \tag{163}$$

and

$$T_{\mathsf{r}}(\mathbf{a},\mathbf{c};\mathbf{w}) = \sum_{\mathbf{k}\in\mathbb{Z}^m} \frac{\mathbb{P}\left(\bigcap_{i=1}^{m-1}\{\mathsf{B}_2(a_i)-c_i\in[\mathsf{r}k_i,\mathsf{r}(k_i+1))\}\cap\{\mathsf{B}_2(a_m)-c_m=\mathsf{r}k_m\}\right)}{w_1^{\Delta k_1}\cdots w_m^{\Delta k_m}} \tag{164}$$

where B_1 and B_2 are independent Brownian motions.

Inserting the above formulas into (159) and computing the integrals, we obtain

$$\begin{aligned} A = \sum_{\mathbf{k}\in\mathbb{Z}^m} \mathbb{P}&\left(\bigcap_{i=1}^{m-1}\{\mathsf{B}_1(a_i)-b_i\geq -\mathsf{r}k_i\}\cap\{\mathsf{B}_1(a_m)-b_m=-\mathsf{r}k_m\}\right) \\ &\times \mathbb{P}\left(\bigcap_{i=1}^{m-1}\{\mathsf{B}_2(a_i)-c_i\in[\mathsf{r}k_i,\mathsf{r}(k_i+1))\}\cap\{\mathsf{B}_2(a_m)-c_m=\mathsf{r}k_m\}\right). \end{aligned} \tag{165}$$

Note that for two random variables X and Y,

$$\bigcup_{k=-\infty}^{\infty}\{X\geq -k,\, Y\in[k,k+1)\} = \bigcup_{k=-\infty}^{\infty}\{X\geq -k,\, [Y]=k\} = \{X+[Y]\geq 0\} = \{[X]+[Y]\geq 0\}$$

where we used the simple fact that $x+[y]\geq 0$ if and only if $[x]+[y]\geq 0$. Thus,

$$A = \mathbb{P}\left(\bigcap_{i=1}^{m-1}\left\{\left[\frac{\mathsf{B}_1(a_i)-b_i}{\mathsf{r}}\right]+\left[\frac{\mathsf{B}_2(a_i)-c_i}{\mathsf{r}}\right]\geq 0\right\}\bigcap\left\{\frac{\mathsf{B}_1(a_m)-b_m}{\mathsf{r}} = -\frac{\mathsf{B}_2(a_m)-c_m}{\mathsf{r}}\in\mathbb{Z}\right\}\right). \tag{166}$$

From the definition (13) of $\mathsf{w}_{\mathsf{r}}(x,y)$,

$$\left[\frac{x-s}{\mathsf{r}}\right]+\left[\frac{y-s}{\mathsf{r}}\right]\geq 0 \quad \text{if and only if} \quad \mathsf{w}_{\mathsf{r}}(x,y)\geq s. \tag{167}$$

Hence, (166) implies (55).

□

A Extension and continuity of the distribution functions $\mathbf{F}_m$

The limit result (4) was proved in [2] for most but not all parameters. In this section, we first show that the convergence holds for all parameters. We then show that the limit functions are a consistent collection of multivariate cumulative distribution functions. We further show that they are continuous in all variables.

Let $\mathbf{h}(n,t)$ be the height function for the TASEP on the discrete ring of size $2a$ as in Section 1.2. For $T > 0$, let

$$\widetilde{\mathbf{h}}_T(\gamma,\tau) := \frac{\mathbf{h}(\gamma T^{2/3}, 2\tau T) - \tau T}{-T^{1/3}}, \qquad (\gamma,\tau) \in \mathbb{R}\times\mathbb{R}_+$$

where the ring size is set as $(2a)^{3/2} = T$.[8] Let

$$\mathbb{R}^m_{+,\le} = \{\tau = (\tau_1,\cdots,\tau_m) \in (0,\infty)^m : 0 < \tau_1 \le \cdots \le \tau_m\}.$$

For $\tau \in \mathbb{R}^m_{+,\le}$, define

$$\Omega^m_+(\tau) = \{\beta = (\beta_1,\cdots,\beta_m) \in \mathbb{R}^m : \beta_i < \beta_{i+1} \text{ if } \tau_i = \tau_{i+1}\}.$$

It was shown in [2] that for every $\gamma \in \mathbb{R}^m$ and $\tau \in \mathbb{R}^m_{+,\le}$, the limit

$$\lim_{T\to\infty} \mathbb{P}\left(\bigcap_{i=1}^m \left\{\widetilde{\mathbf{h}}_T(\gamma_i,\tau_i) \le \beta_i\right\}\right) = \mathbf{F}_m(\beta;\gamma,\tau) \text{ converges if } \beta \in \Omega^m_+(\tau), \tag{168}$$

as mentioned in (20).

When $m = 1$, it was already shown in [1] that the one-point distribution $\mathbf{F}_1$ is a distribution function and is continuous. We do not need the explicit form of $\mathbf{F}_m$ for the first two results below.

We first show that the limit (168) convergences for every $\beta \in \mathbb{R}^m$. For $\tau \in \mathbb{R}^m_{+,\le}$, define the set

$$\Omega^m(\tau) := \overline{\Omega^m_+(\tau)} = \{\beta = (\beta_1,\cdots,\beta_m) \in \mathbb{R}^m : \beta_i \le \beta_{i+1} \text{ if } \tau_i = \tau_{i+1}\}.$$

Lemma A.1. *Let $\gamma \in \mathbb{R}^m$ and $\tau \in \mathbb{R}^m_{+,\le}$. For every $\hat\beta \in \Omega^m(\tau)$, the limit*

$$\lim_{\Omega^m(\tau)\ni\beta\to\hat\beta} \mathbf{F}_m(\beta;\gamma,\tau) \tag{169}$$

exists. Furthermore, if we denote the limit as $\mathbf{F}_m(\hat\beta;\gamma,\tau)$, then

$$\lim_{T\to\infty} \mathbb{P}\left(\bigcap_{i=1}^m \left\{\widetilde{\mathbf{h}}_T(\gamma_i,\tau_i) \le \hat\beta_i\right\}\right) = \mathbf{F}_m(\hat\beta,\gamma,\tau). \tag{170}$$

Proof. For $\epsilon > 0$, let

$$\beta'_i = \hat\beta_i - (2 - \frac{i}{m+1})\epsilon \quad \text{and} \quad \beta''_i = \hat\beta_i + (1 + \frac{i}{m+1})\epsilon$$

for $i = 1,\cdots,m$. Then, $\beta',\beta'' \in \Omega^m_+(\tau)$. Furthermore, $(\beta'_1,\cdots,\beta'_k,\beta''_{k+1},\cdots,\beta_m) \in \Omega^m_+(\tau)$ for every k.

Let $\beta \in \Omega^m_+(\tau)$ be an arbitrary number satisfying $\sum_{i=1}^m |\beta_i - \hat\beta_i| < \epsilon$. Note that $\beta'_i \le \beta_i \le \beta''_i$. Thus,

$$\mathbf{F}_m(\beta';\gamma,\tau) \le \mathbf{F}_m(\beta;\gamma,\tau) \le \mathbf{F}_m(\beta'';\gamma,\tau)$$

where we used the fact that being a limit of a distribution function, $\mathbf{F}_m(\beta;\gamma,\tau)$ is a weakly increasing function of $\beta \in \Omega^m_+(\tau)$. From the monotonicity property again, as $\epsilon \downarrow 0$, $\mathbf{F}_m(\beta';\gamma,\tau)$ increases weakly and $\mathbf{F}_m(\beta'';\gamma,\tau)$ decreases weakly. Therefore, the limit (169) converges if we show that

$$\lim_{\epsilon\downarrow 0} \mathbf{F}_m(\beta'';\gamma,\tau) - \mathbf{F}_m(\beta';\gamma,\tau) = 0. \tag{171}$$

[8] To be precise, we set $a = [T^{2/3}]/2$ since a is a half-integer.

For every j, from (168),

$$\begin{aligned}&\mathbf{F}_m(\beta_1',\cdots,\beta_{j-1}',\beta_j'',\beta_{j+1}'',\cdots,\beta_m'';\gamma,\tau)-\mathbf{F}_m(\beta_1',\cdots,\beta_{j-1}',\beta_j',\beta_{j+1}'',\cdots,\beta_m'';\gamma,\tau)\\&=\lim_{T\to\infty}\mathbb{P}\Big(\big\{\beta_j'<\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j)\le\beta_j''\big\}\bigcap_{i=1}^{j-1}\big\{\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j)\le\beta_i'\big\}\bigcap_{i=j+1}^{m}\big\{\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j)\le\beta_i''\big\}\Big)\\&\le\lim_{T\to\infty}\mathbb{P}\big(\beta_j'<\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j)\le\beta_j''\big)=\mathbf{F}_1(\beta_j'';\gamma_j,\tau_j)-\mathbf{F}_1(\beta_j';\gamma_j,\tau_j).\end{aligned}$$

Summing over j, we obtain

$$\mathbf{F}_m(\beta_1'',\cdots,\beta_m'';\gamma,\tau)-\mathbf{F}_m(\beta_1',\cdots,\beta_m';\gamma,\tau)\le\sum_{j=1}^{m}\left(\mathbf{F}_1(\beta_j'';\gamma_j,\tau_j)-\mathbf{F}_1(\beta_j'\gamma_j,\tau_j)\right).$$

Since the one-point distribution $\mathbf{F}_1$ is continuous (see [1]), the right side converges to zero as $\epsilon\to 0$. The left-hand side is also nonnegative due to the monotonicity property of $\mathbf{F}_m$. Thus we obtain (171), which implies the convergence of (169).

With the same notations as above, from the monotonicity of probabilities which holds for the parameters without any restrictions,

$$\mathbb{P}\Big(\bigcap_{i=1}^{m}\big\{\widetilde{\mathbf{h}}_T(\gamma_i,\tau_i)\le\beta_i'\big\}\Big)\le\mathbb{P}\Big(\bigcap_{i=1}^{m}\big\{\widetilde{\mathbf{h}}_T(\gamma_i,\tau_i)\le\hat{\beta}_i\big\}\Big)\le\mathbb{P}\Big(\bigcap_{i=1}^{m}\big\{\widetilde{\mathbf{h}}_T(\gamma_i,\tau_i)\le\beta_i''\big\}\Big)$$

As $T\to\infty$, the lower bound tends to $\mathbf{F}_m(\beta';\gamma,\tau)$ and the upper bound tends to $\mathbf{F}_m(\beta'';\gamma,\tau)$. If we let $\epsilon\downarrow 0$, then both of them converge to $\mathbf{F}_m(\hat{\beta};\gamma,\tau)$. This shows (170). ☐

Corollary A.2. *For every $\gamma\in\mathbb{R}^m$, $\tau\in\mathbb{R}_+^m$, and $\beta\in\mathbb{R}^m$, the limit*

$$\mathbf{F}_m(\beta;\gamma,\tau):=\lim_{T\to\infty}\mathbb{P}\Big(\bigcap_{i=1}^{m}\big\{\widetilde{\mathbf{h}}_T(\gamma_i,\tau_i)\le\beta_i\big\}\Big)\tag{172}$$

converges. The function $\mathbf{F}_m(\beta;\gamma,\tau)$ is invariant under the permutations of the triples $(\beta_i,\gamma_i,\tau_i)$, $i=1,\cdots,m$.

Proof. The probability $\mathbb{P}(\cap_{i=1}^m\{\widetilde{\mathbf{h}}_T(\gamma_i,\tau_i)\le\beta_i\})$ is defined for every $(\gamma,\tau,\beta)\in\mathbb{R}^m\times\mathbb{R}_+^m\times\mathbb{R}^m$, and is invariant under the permutations of the triples $(\beta_i,\gamma_i,\tau_i)$. For a permutation $\sigma\in S_m$, let $\gamma^\sigma,\tau^\sigma,\beta^\sigma$ be the parameters obtained from γ,τ,β applying the permutation σ to the index. Let σ be a permutation so that $\tau^\sigma\in\mathbb{R}_{+,\le}^m$ and $\beta^\sigma\in\Omega_0^m(\tau^\sigma)$. There is at least one such permutation. By the last lemma, $\mathbb{P}(\cap_{i=1}^m\{\widetilde{\mathbf{h}}_T(\gamma_i,\tau_i)\le\beta_i\})=\mathbb{P}(\cap_{i=1}^m\{\widetilde{\mathbf{h}}_T(\gamma_i^\sigma,\tau_i^\sigma)\le\beta_i^\sigma\})$ converges. If there are more than one permutation with the same property, then it is easy to check that they results in the same limit. The invariance under permutations follow easily. ☐

Therefore, the convergence (4) holds for all parameters, and we use the same notation $\mathbf{F}_m(\beta;\gamma,\tau)$ for the limit. We now show that $\mathbf{F}_m(\beta;\gamma,\tau)$ are a collection of consistent multivariate cumulative distribution functions. For the restricted parameters, this fact was proved in [2, Section 7]. Here, we prove it for all parameters.

Proposition A.3. *(a) For every m and $(\gamma,\tau)\in\mathbb{R}^m\times\mathbb{R}_+^m$, $\beta\mapsto\mathbf{F}_m(\beta;\gamma,\tau)$ is a multivariate cumulative distribution function.*

(b) Let $(\gamma,\tau,\beta)\in\mathbb{R}^m\times\mathbb{R}_+^m\times\mathbb{R}^m$. For each $j=1,\cdots,m$, let $\gamma^{(j)},\tau^{(j)},\beta^{(j)}$ be the points in $\mathbb{R}^{m-1}$ obtained from γ,τ,β by removing γ_j,τ_j,β_j, respectively. Then,

$$\lim_{\beta_j\to\infty}\mathbf{F}_m(\beta;\gamma,\tau)=\mathbf{F}_{m-1}(\beta^{(j)};\gamma^{(j)},\tau^{(j)}).$$

Proof. (a) The equation (172) implies the monotone non-decreasing property. It also implies that $\mathbf{F}_m(\beta;\gamma,\tau) \le \mathbf{F}_1(\beta_j;\gamma_j,\tau_j)$ and $1-\mathbf{F}_m(\beta;\gamma,\tau) \le \sum_{j=1}^m (1-\mathbf{F}_1(\beta_j;\gamma_j,\tau_j))$. We thus find the correct limit properties as β becomes small or large.

(b) From (172) again,

$$0 \le \mathbf{F}_{m-1}(\beta^{(j)};\gamma^{(j)},\tau^{(j)}) - \mathbf{F}_m(\beta;\gamma,\tau) = \lim_{T\to\infty} \mathbb{P}\Big(\big\{\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j) > \beta_j\big\} \bigcap_{\substack{1\le i\le m\\ i\ne j}} \big\{\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j) \le \beta_i\big\}\Big)$$
$$\le \lim_{T\to\infty} \mathbb{P}\big(\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j) > \beta_j\big) = 1 - \mathbf{F}_1(\beta_j;\gamma_j,\tau_j).$$

The upper bound tends to 0 as $\beta_j \to +\infty$ since $\mathbf{F}_1$ is a distribution function [1]. □

The final result of this Section is the continuity of $\mathbf{F}_m(\beta;\gamma,\tau)$. When $\beta \in \Omega_+^m(\tau)$, the limit $\mathbf{F}_m(\beta;\gamma,\tau)$ for (168) is given by the formula

$$\mathbf{F}_m(\beta;\gamma,\tau) = \frac{1}{(2\pi \mathrm{i})^m} \oint \cdots \oint C(\mathbf{z}) D(\mathbf{z}) \prod_{i=1}^m \frac{\mathrm{d}z_i}{z_i} \tag{173}$$

where the integrand is same as that of the formula (21) (with $p=1$) but the radii of the contour circles satisfy the reverse inequalities $0 < |z_m| < \cdots < |z_1| < 1$. From the formula of $C(\mathbf{z})$ and $D(\mathbf{z})$ in Section 2.2 (with $p=1$) and Lemma A.1, $\mathbf{F}_m(\beta;\gamma,\tau)$ is jointly continuous for $\gamma \in \mathbb{R}^m$, $\tau \in \mathbb{R}^m_{+,\le}$, and $\beta \in \Omega^m(\tau)$. Due to the invariance under permutations of the triples of the parameters, it is continuous on the set

$$U_+^m := \mathbb{R}^m \times \mathbb{R}_+^m \times \mathbb{R}^m \setminus \{(\gamma,\tau,\beta) : \beta_i = \beta_j \text{ and } \tau_i = \tau_j \text{ for some } 1 \le i < j \le m\}.$$

The next result shows that it is continuous in all of $\mathbb{R}^m \times \mathbb{R}_+^m \times \mathbb{R}^m$.

Proposition A.4. *The function $\mathbf{F}_m(\beta;\gamma,\tau)$ is jointly continuous in $(\gamma,\tau,\beta) \in \mathbb{R}^m \times \mathbb{R}_+^m \times \mathbb{R}^m$.*

Proof. Let (γ,τ,β) be a point in $\mathbb{R}^m \times \mathbb{R}_+^m \times \mathbb{R}^m$. For $(\gamma',\tau',\beta') \in \mathbb{R}^m \times \mathbb{R}_+^m \times \mathbb{R}^m$ and $1 \le j \le m$, let

$$\gamma'_{(j)} = (\cdots, \gamma'_{j-1}, \tau'_j, \gamma_{j+1}, \cdots), \qquad \tau'_{(j)} = (\cdots, \tau'_{j-1}, \tau'_j, \tau_{j+1}, \cdots), \qquad \beta'_{(j)} = (\cdots, \beta'_{j-1}, \beta'_j, \beta_{j+1}, \cdots).$$

Fom (172), for every j,

$$\begin{aligned}
&|\mathbf{F}_m(\beta'_{(j)};\gamma'_{(j)},\tau'_{(j)}) - \mathbf{F}_m(\beta'_{(j-1)};\gamma'_{(j-1)},\tau'_{(j-1)})| \\
&\le \lim_{T\to\infty} \mathbb{P}\big(\widetilde{\mathbf{h}}_T(\gamma'_j,\tau'_j) \le \beta'_j,\, \widetilde{\mathbf{h}}_T(\gamma_j,\tau_j) > \beta_j\big) + \mathbb{P}\big(\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j) \le \beta_j,\, \widetilde{\mathbf{h}}_T(\gamma'_j,\tau'_j) > \beta'_j\big) \\
&= \lim_{T\to\infty} \mathbb{P}\big(\widetilde{\mathbf{h}}_T(\gamma'_j,\tau'_j) \le \beta'_j\big) + \mathbb{P}\big(\widetilde{\mathbf{h}}_T(\gamma_j,\tau_j) \le \beta_j\big) - 2\mathbb{P}\big(\widetilde{\mathbf{h}}_T(\gamma'_j,\tau'_j) \le \beta'_j,\, \widetilde{\mathbf{h}}_T(\gamma_j,\tau_j) \le \beta_j\big).
\end{aligned}$$

Thus,

$$\begin{aligned}
|\mathbf{F}_m(\beta'_{(j)};\gamma'_{(j)},\tau'_{(j)}) - \mathbf{F}_m(\beta'_{(j-1)};\gamma'_{(j-1)},\tau'_{(j-1)})| &= \mathbf{F}_1(\beta'_j;\gamma'_j,\tau'_j) + \mathbf{F}_1(\beta_j;\gamma_j,\tau_j) - 2\mathbf{F}_2(\beta'_j,\beta_j;\gamma'_j,\gamma_j,\tau'_j,\tau_j) \\
&\le \mathbf{F}_1(\beta'_j;\gamma'_j,\tau'_j) + \mathbf{F}_1(\beta_j;\gamma_j,\tau_j) - 2\mathbf{F}_2(\beta'_j,\beta_j-\epsilon;\gamma'_j,\gamma_j,\tau'_j,\tau_j)
\end{aligned}$$

for every $\epsilon > 0$. If $(\gamma'_j,\tau'_j,\beta'_j)$ is close enough to $(\gamma_j,\tau_j,\beta_j)$, then $(\gamma'_j,\gamma_j,\tau'_j,\tau_j,\beta'_j,\beta_j-\epsilon) \in U_+^2$. Thus, the continuity of $\mathbf{F}_2$ on U_+^2 implies that

$$\begin{aligned}
&\limsup_{(\gamma',\tau',\beta')\to(\gamma,\tau,\beta)} |\mathbf{F}_m(\beta'_{(j)};\gamma'_{(j)},\tau'_{(j)}) - \mathbf{F}_m(\beta'_{(j-1)};\gamma'_{(j-1)},\tau'_{(j-1)})| \\
&\le 2\mathbf{F}_1(\beta_j;\gamma_j,\tau_j) - 2\mathbf{F}_2(\beta_j,\beta_j-\epsilon;\gamma_j,\gamma_j,\tau_j,\tau_j) = 2\mathbf{F}_1(\beta_j;\gamma_j,\tau_j) - 2\mathbf{F}_1(\beta_j-\epsilon;\gamma_j,\tau_j)
\end{aligned}$$

where we used the fact that $\mathbf{F}_2(a,b;\gamma,\gamma,\tau,\tau)=\mathbf{F}_1(a;\gamma,\tau)$ if $a<b$, which follows from the definition of (172). Since the inequality holds for every $\epsilon>0$ and the one-point distribution function $\mathbf{F}_1$ is continuous, we find that

$$\limsup_{(\gamma',\tau',\beta')\to(\gamma,\tau,\beta)}|\mathbf{F}_m(\beta'_{(j)};\gamma'_{(j)},\tau'_{(j)})-\mathbf{F}_m(\beta'_{(j-1)};\gamma'_{(j-1)},\tau'_{(j-1)})|=0.$$

Summing over j, we conclude that

$$\limsup_{(\gamma',\tau',\beta')\to(\gamma,\tau,\beta)}|\mathbf{F}_m(\beta';\gamma',\tau')-\mathbf{F}_m(\beta;\gamma,\tau)|=0,$$

proving the desired continuity. □

B Formula of $D_{\mathbf{n}}(z)$

We state the formula of $D(\mathbf{z})$ given in [2, Lemma 2.10] and show that it can be written as the form (24) in Subsection 2.2. It is enough to check it when $p=1$ since the general p case follows from the property (9).

For complex vectors $W=(w_1,\cdots,w_n)$ and $W'=(w'_1,\cdots,w'_{n'})$, we denote

$$\Delta(W)=\prod_{1\le i<j\le n}(w_j-w_i)\quad\text{and}\quad\Delta(W;W')=\prod_{i=1}^{n}\prod_{j=1}^{n'}(w_i-w'_j).$$

We also use the notation that for a function g of a single variable and a vector $W=(w_1,\cdots,w_n)$,

$$g(W):=\prod_{i=1}^{n}g(w_i).$$

For $0<|z|<1$, the sets L_z and R_z are the discrete sets in the complex plane defined as

$$\mathrm{L}_z=\{w:e^{-w^2/2}=z,\,\mathrm{Re}(w)<0\}\quad\text{and}\quad\mathrm{R}_z=\{w:e^{-w^2/2}=z,\,\mathrm{Re}(w)>0\}.$$

The series formula of $D(\mathbf{z})$ given in [2, Lemma 2.10] is

$$D(\mathbf{z})=\sum_{\mathbf{n}\in\{0,1,\cdots\}^m}\frac{1}{(\mathbf{n}!)^2}D_{\mathbf{n}}(\mathbf{z})$$

where for $\mathbf{n}=(n_1,\cdots,n_m)$ and $0<|z_1|<\cdots<|z_m|<1$,

$$D_{\mathbf{n}}(\mathbf{z})=\left(1-\frac{z_{i-1}}{z_i}\right)^{n_i}\left(1-\frac{z_i}{z_{i-1}}\right)^{n_{i-1}}\sum_{\substack{U^{(i)}\in\mathrm{L}_{z_i}^{n_i}\\ V^{(i)}\in\mathrm{R}_{z_i}^{n_i}\\ i=1,\cdots,m}}\prod_{i=1}^{m}\frac{\Delta(U^{(i)})^2\Delta(V^{(i)})^2}{\Delta(U^{(i)};V^{(i)})^2}\hat{f}_i(U^{(i)})\hat{f}_i(V^{(i)})$$
$$\times\prod_{i=2}^{m}\frac{\Delta(U^{(i)};V^{(i-1)})\Delta(V^{(i)};U^{(i-1)})}{\Delta(U^{(i)};U^{(i-1)})\Delta(V^{(i)};V^{(i-1)})}\frac{e^{-h(V^{(i)},z_{i-1})-h(V^{(i-1)},z_i)}}{e^{h(U^{(i)},z_{i-1})+h(U^{(i-1)},z_i)}}\tag{174}$$

with

$$h(w,z)=\begin{cases}-\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{w}\mathrm{Li}_{1/2}(ze^{(w^2-y^2)/2})\mathrm{d}y, & \mathrm{Re}(w)<0,\\ -\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{-w}\mathrm{Li}_{1/2}(ze^{(w^2-y^2)/2})\mathrm{d}y, & \mathrm{Re}(w)>0,\end{cases}$$

$$f_i(w) = \begin{cases} e^{-\frac{1}{3}(\tau_i-\tau_{i-1})w^3+\frac{1}{2}(\gamma_i-\gamma_{i-1})w^2+(\beta_i-\beta_{i-1})w}, & \mathrm{Re}(w)<0, \\ e^{\frac{1}{3}(\tau_i-\tau_{i-1})w^3-\frac{1}{2}(\gamma_i-\gamma_{i-1})w^2-(\beta_i-\beta_{i-1})w}, & \mathrm{Re}(w)>0, \end{cases}$$

and

$$\hat{f}_i(w) = \frac{1}{w} f_i(w) e^{2h(w,z_i)}.$$

Note that $w \in \mathrm{R}_z$ if and only if $-w \in \mathrm{L}_z$. Thus, setting $\hat{U}^{(i)} = -V^{(i)}$, the sum in (174) can be written as

$$\sum_{\substack{U^{(i)},\hat{U}^{(i)} \in \mathrm{L}_{z_i}^{n_i} \\ i=1,\cdots,m}} \prod_{i=1}^{m} \frac{\Delta(U^{(i)})^2 \Delta(-\hat{U}^{(i)})^2}{\Delta(U^{(i)}; -\hat{U}^{(i)})^2} \hat{f}_i(U^{(i)}) \hat{f}_i(-\hat{U}^{(i)})$$
$$\times \prod_{i=2}^{m} \frac{\Delta(U^{(i)}; -\hat{U}^{(i-1)}) \Delta(-\hat{U}^{(i)}; U^{(i-1)})}{\Delta(U^{(i)}; U^{(i-1)}) \Delta(-\hat{U}^{(i)}; -\hat{U}^{(i-1)})} \frac{e^{-h(-\hat{U}^{(i)},z_{i-1})-h(-\hat{U}^{(i-1)},z_i)}}{e^{h(U^{(i)},z_{i-1})+h(U^{(i-1)},z_i)}}.$$

Since $h(-w,z) = h(w,z)$, after inserting the formula $\hat{f}_i(w) = \frac{1}{w} f_i(w) e^{2h(w,z_i)}$ and using the notation $E^{i,\pm}$ of (31) instead of f_i, we can express the above sum as

$$\sum_{\substack{U^{(i)},\hat{U}^{(i)} \in \mathrm{L}_{z_i}^{n_i} \\ i=1,\cdots,m}} \prod_{i=1}^{m} \frac{e^{2h(\hat{U}^{(i)},z_i)+2h(\hat{U}^{(i)},z_i)}}{e^{h(U^{(i-1)},z_i)+h(U^{(i+1)},z_i)+h(\hat{U}^{(i-1)},z_i)+h(\hat{U}^{(i+1)},z_i)}} \prod_{i=1}^{m} \prod_{j=1}^{n_i} E^{i,+}(u_j^{(i)}) E^{i,-}(\hat{u}_j^{(i)})$$
$$\times \prod_{i=1}^{m} \prod_{j=1}^{n_i} \frac{-1}{u_j^{(i)} \hat{u}_j^{(i)}} \prod_{i=1}^{m} \frac{\Delta(U^{(i)})^2 \Delta(-\hat{U}^{(i)})^2}{\Delta(U^{(i)}; -\hat{U}^{(i)})^2} \prod_{i=2}^{m} \frac{\Delta(U^{(i)}; -\hat{U}^{(i-1)}) \Delta(-\hat{U}^{(i)}; U^{(i-1)})}{\Delta(U^{(i)}; U^{(i-1)}) \Delta(-\hat{U}^{(i)}; -\hat{U}^{(i-1)})}$$

where we set $U^{(0)} = \hat{U}^{(0)} = U^{(m+1)} = \hat{U}^{(m+1)} = \emptyset$ so that $e^{h(U^{(0)};z_1)} = 1$, and so on. The product involving the function h is $H_{\mathbf{n}}(U, \hat{U})$ of (28) and the next product involving $E^{i,\pm}$ is $E_{\mathbf{n}}(U, \hat{U})$ of (31). Finally, again with the convention $U^{(0)} = \hat{U}^{(0)} = U^{(m+1)} = \hat{U}^{(m+1)} = \emptyset$, we have

$$\begin{aligned} &\prod_{i=1}^{m} \frac{\Delta(U^{(i)})^2 \Delta(-\hat{U}^{(i)})^2}{\Delta(U^{(i)}; -\hat{U}^{(i)})^2} \prod_{i=2}^{m} \frac{\Delta(U^{(i)}; -\hat{U}^{(i-1)}) \Delta(-\hat{U}^{(i)}; U^{(i-1)})}{\Delta(U^{(i)}; U^{(i-1)}) \Delta(-\hat{U}^{(i)}; -\hat{U}^{(i-1)})} \\ &= \prod_{i=1}^{m+1} \frac{\Delta(U^{(i-1)}) \Delta(-\hat{U}^{(i-1)}) \Delta(U^{(i)}; -\hat{U}^{(i-1)}) \Delta(-\hat{U}^{(i)}; U^{(i-1)}) \Delta(U^{(i)}) \Delta(-\hat{U}^{(i)})}{\Delta(U^{(i-1)}; -\hat{U}^{(i-1)}) \Delta(U^{(i)}; U^{(i-1)}) \Delta(-\hat{U}^{(i)}; -\hat{U}^{(i-1)}) \Delta(U^{(i)}; -\hat{U}^{(i)})} \\ &= (-1)^{n_1+\cdots+n_m} \prod_{i=1}^{m+1} \mathsf{K}(U^{(i-1)}, -\hat{U}^{(i)}; \hat{U}^{(i-1)}, -U^{(i)}) \end{aligned}$$

in terms of the Cauchy determinant (29). This is a factor of $R_{\mathbf{n}}(U, \hat{U})$ of (30), and we thus find that $D_{\mathbf{n}}(\mathbf{z})$ is equal to the form (24) in Subsection 2.2.

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