Fluctuations of the free energy of the spherical Sherrington–Kirkpatrick model with ferromagnetic interaction

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Abstract

We consider a spherical spin system with pure 2-spin spherical Sherrington–Kirkpatrick Hamiltonian with ferromagnetic Curie–Weiss interaction. The system shows a two-dimensional phase transition with respect to the temperature and the coupling constant. We compute the limiting distributions of the free energy for all parameters away from the critical values. The zero temperature case corresponds to the well-known phase transition of the largest eigenvalue of a rank 1 spiked random symmetric matrix. As an intermediate step, we establish a central limit theorem for the linear statistics of rank 1 spiked random symmetric matrices.

1 Introduction

1.1 Model

Let $A = (A_{ij})_{i,j=1}^N$ be a real symmetric matrix where A_{ij}, $1 \le i < j \le N$, are independent random variables with mean 0 and variance 1, and the diagonal entries $A_{ii} = 0$. The pure 2-spin spherical Sherrington–Kirkpatrick (SSK) model with no external field is a disordered system defined by the random Hamiltonian

$$H_N^{\mathrm{SSK}}(\boldsymbol{\sigma}) := \frac{1}{\sqrt{N}} \langle \boldsymbol{\sigma}, A\boldsymbol{\sigma} \rangle = \frac{1}{\sqrt{N}} \sum_{i,j=1}^N A_{ij} \sigma_i \sigma_j \tag{1.1}$$

for the spin variables on the sphere, $\boldsymbol{\sigma} \in S_{N-1}$, where $S_{N-1} := \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\|^2 = N\}$. For the history and the existing results on the model, including the proof of the Parisi formula, we refer to [28, 17, 40, 32] and references therein.

We are interested in the spherical spin system with (random) Hamiltonian

$$H_N(\boldsymbol{\sigma}) = H_N^{\mathrm{SSK}}(\boldsymbol{\sigma}) + H_N^{\mathrm{CW}}(\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} \in S_{N-1}, \tag{1.2}$$

where the Curie–Weiss (CW) Hamiltonian with coupling constant J is defined by

$$H_N^{\mathrm{CW}}(\boldsymbol{\sigma}) := \frac{J}{N} \sum_{i,j=1}^N \sigma_i \sigma_j = \frac{J}{N} \left(\sum_{i=1}^N \sigma_i \right)^2 . \tag{1.3}$$

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Note that $H_N^{\rm CW}$ is large in magnitude when all σ_i have the same sign. The Hamiltonian H_N is similar to the SSK model with external field,

$$H_N^{\rm ext}(\boldsymbol{\sigma}) = H_N^{\rm SSK}(\boldsymbol{\sigma}) + h\sum_{i=1}^{N}\sigma_i. \tag{1.4}$$

See [13] for a relation between these two Hamiltonians.

The main result of this paper is a limit theorem for the free energy at positive temperature $1/\beta > 0$ with positive coupling constant J. This paper is an extension of our previous paper [6] in which we obtained limit theorems for pure 2-spin SSK model (with $J = 0$).

Before we state our result, we first summarize the known limit theorems for the free energy of SSK and also the Sherrington–Kirkpatrick (SK) model. Results for (3)–(7) were established in the same year 2015. We indicate the limiting distribution and the order of fluctuations of the free energy. These results assume that A_{ij} are standard Gaussian. However, the result (1) was extended to non-Gaussian A_{ij} in [25, 12], and the results (3) and (4) were obtained for general normalized random variables.

No external field.

When there is no external field ($h = 0$), the following are known for pure p-spin models:

(1) Pure 2-spin SK model for $\beta \in (0, \beta_c)$: Gaussian, $O(N^{-1})$ [1, 23, 16]

(2) Pure p-spin SK model for $\beta \in (0, \beta_c^{(p)})$: Gaussian, $O(N^{-p/2})$ [10]

(3) Pure 2-spin SSK model for $\beta \in (0, \beta_c)$: Gaussian, $O(N^{-1})$ [6]

(4) Pure 2-spin SSK model for $\beta \in (\beta_c, \infty)$: $\rm TW_1$, $O(N^{-2/3})$ [6]

(5) Pure p-spin SSK model for $p \geq 3$ at $\beta = \infty$: Gumbel, $O(N^{-1})$ [39]

Here, $\rm TW_1$ denotes the GOE Tracy-Widom distribution. The numbers $\beta_c^{(p)}$, $p \geq 3$, and β_c are certain critical values. A results for pure p-spin SSK model with $p \geq 3$ for low temperature is given in [38].

We also remark that the free energy of the pure 2-spin SSK model at zero temperature, $\beta = \infty$, is, after modifying the definition slightly, equal to the rescaled largest eigenvalue of symmetric random matrix A. Hence, from the well-known result in the random matrix theory [36, 41, 21], this case also corresponds to $\rm TW_1$ with $O(N^{-2/3})$ fluctuation. Comparing with (5), we find that at zero temperature the free energy fluctuates differently for $p = 2$ and $p \geq 3$. There is an important difference of $p = 2$ case and $p \geq 3$ case: the number of critical points for the Hamiltonian (subject to the constraint $\|\boldsymbol{\sigma}\|^2 = N$) is $2N$ for $p = 2$, but is exponential in N for $p \geq 3$ as proved in [2] (for upper bound) and [37] (for lower bound). The critical points are the eigenvectors of A for $p = 2$, hence strongly correlated, whereas the extremal process of critical points converges in distribution to a Poisson point process for $p \geq 3$. See Theorem 1 of [39] for more detail.

Positive external field.

The behavior of the free energy changes drastically under the presence of an external field ($h > 0$). For this case, the more complicated model with mixed p-spin interactions were also studied.

(6) Mixed p-spin SK and SSK models (without odd p-interactions for $p \geq 3$) with $h > 0$ for all $\beta \in (0, \infty)$: Gaussian, $O(N^{-1/2})$ [14]

(7) Mixed p-spin SSK model with $h > 0$ at $\beta = \infty$: Gaussian, $O(N^{-1/2})$ [15]

Note that the fluctuations are significantly increased from the $h = 0$ case.

It is interesting to scale $h \to 0$ with N and consider a transition from (6) and (7) to (4) or (5). By matching the variance when $h = 0$ and $h > 0$, it is expected that the transitional scaling is $h = O(N^{-1/6})$ for $p = 2$. For the large deviation analysis for the pure 2-spin SSK model and discussions for such h, we refer to [24] for deterministic h and [18] for random h.

1.2 Definitions

We first define a Hamiltonian that generalizes H_N in (1.2).

Definition 1.1 (Interactions)**.** Let A_{ij}, $1 \le i \le j$, be independent real random variables satisfying the following conditions:

- All moments of A_{ij} are finite and $\mathbb{E}[A_{ij}] = 0$.
- For all $i < j$, $\mathbb{E}[A_{ij}^2] = 1$, $\mathbb{E}[A_{ij}^3] = W_3$, and $\mathbb{E}[A_{ij}^4] = W_4$ for some constants $W_3 \in \mathbb{R}$ and $W_4 > 0$.
- For all i, $\mathbb{E}[A_{ii}^2] = w_2$ for a constant $w_2 \ge 0$.

Set $A_{ji} = A_{ij}$ for $i < j$, and set $A = (A_{ij})_{i,j=1}^N$. Let

$$M_{ij} = \frac{A_{ij}}{\sqrt{N}} + \frac{J}{N} \quad (i \neq j), \qquad M_{ii} = \frac{A_{ii}}{\sqrt{N}} + \frac{J'}{N} \tag{1.5}$$

for some (N-independent) non-negative constants J and J'. Set $M = (M_{ij})_{i,j=1}^N$. We call M a Wigner matrix with non-zero mean.

The Hamiltonian in (1.2) is obtained by setting $A_{ii} = 0$ and $J' = 0$.

Definition 1.2 (Free energy)**.** Define the Hamiltonian $H_N(\boldsymbol{\sigma}) = \langle \boldsymbol{\sigma}, M\boldsymbol{\sigma} \rangle$ on sphere $\|\boldsymbol{\sigma}\| = \sqrt{N}$. For $\beta > 0$, define the partition function and the free energy as

$$Z_N = Z_N(\beta) = \int_{S_{N-1}} e^{\beta H_N(\boldsymbol{\sigma})} \mathrm{d}\omega_N(\boldsymbol{\sigma}), \qquad F_N = F_N(\beta) = \frac{1}{N} \log Z_N, \tag{1.6}$$

where $\mathrm{d}\omega_N$ is the normalized uniform measure on the sphere $S_{N-1} = \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\|^2 = N\}$.

Remark 1.3. We may also consider complex matrix M. In this case, the real and the complex entries are independent and we add an extra condition that $\mathbb{E}A_{ij}^2 = 0$. The results in this paper have corresponding results for complex M, but we do not state them here.

1.3 Results

The following is the main result. The case when $J = 0$ was proved previously in [6].

Theorem 1.4. *The following holds as $N \to \infty$ where all the convergences are in distribution. The notation $\mathcal{N}(a, b)$ denotes Gaussian distribution with mean a and variance b and TW_1 is the GOE Tracy–Widom distribution.*

(i) (Spin glass regime) If $\beta > \frac{1}{2}$ and $J < 1$, then

$$\frac{1}{\beta - \frac{1}{2}} N^{2/3} \left(F_N - F(\beta)\right) \Rightarrow \mathrm{TW}_1 . \tag{1.7}$$

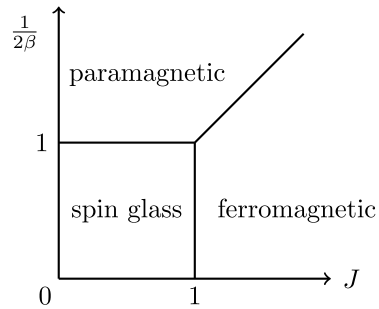

Figure 1: Phase diagram

(ii) (Paramagnetic regime) If $\beta < \frac{1}{2}$ *and* $\beta < \frac{1}{2J}$*, then*

$$N\left(F_N - F(\beta)\right) \Rightarrow \mathcal{N}\left(f_1, \alpha_1\right). \tag{1.8}$$

(iii) (Ferromagnetic regime) If $J > 1$ *and* $\beta > \frac{1}{2J}$*, then*

$$\sqrt{N}\left(F_N - F(\beta)\right) \Rightarrow \mathcal{N}\left(0, \alpha_2\right). \tag{1.9}$$

The leading order limit of the free energy is given by

$$F(\beta) = \begin{cases} 2\beta - \frac{1}{2}\log(2\beta) - \frac{3}{4} & \text{for (i)} \\ \beta^2 & \text{for (ii)} \\ \beta\left(J + \frac{1}{J}\right) - \frac{1}{2}\log(2\beta J) - \frac{1}{4J^2} - \frac{1}{2}, & \text{for (iii).} \end{cases} \tag{1.10}$$

The parameters for case (ii) in (1.8) *are*

$$f_1 = \frac{1}{4}\log(1-4\beta^2) + \beta^2(w_2 - 2) + 2\beta^4(W_4 - 3) - \beta J - \frac{1}{2}\log(1 - 2\beta J) + \beta J' \tag{1.11}$$

and

$$\alpha_1 = -\frac{1}{2}\log(1-4\beta^2) + \beta^2(w_2 - 2) + 2\beta^4(W_4 - 3), \tag{1.12}$$

and the parameter for case (iii) in (1.9) *is*

$$\alpha_2 = 2\left(1 - \frac{1}{J^2}\right)\left(\beta - \frac{1}{2J}\right)^2. \tag{1.13}$$

If we set $T = \frac{1}{2\beta}$, then the trichotomy corresponds to the cases $\max\{T, J, 1\} = 1$, $\max\{T, J, 1\} = T$, and $\max\{T, J, 1\} = J$, respectively. See Figure 1 for the phase diagram.

The above result implies that

$$F_N(\beta) \to F(\beta) \tag{1.14}$$

in probability for (J, β) not on the critical lines. The formula (1.10) of $F(\beta)$, and hence also the phase diagram, were obtained by Kosterlitz, Thouless, and Jones [28]. Their proof is not completely rigorous but can be made rigorous by using the estimates that were developed later in random matrix theory. In this paper, we make their analysis rigorous and improve it to obtain the results on the fluctuations. Note that even though the paramagnetic regime and the ferromagnetic regime both have a Gaussian as the limiting distribution, the order of the fluctuations are different. The reason for this can be seen from the following theorem of which Theorem 1.4 is a consequence.

Theorem 1.5. *Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$ be the eigenvalues of Wigner matrix with non-zero mean M in Definition 1.1. For every $\epsilon > 0$ and $D > 0$, the following holds as $N \to \infty$ with probability higher than $1 - N^{-D}$.*

(i) (Spin glass regime) If $\beta > \frac{1}{2}$ and $J < 1$, then

$$F_N = F(\beta) + \left(\beta - \frac{1}{2}\right)(\mu_1 - 2) + O(N^{-1+\epsilon}). \tag{1.15}$$

(ii) (Paramagnetic regime) If $\beta < \frac{1}{2}$ and $\beta < \frac{1}{2J}$, then

$$F_N = 2\beta^2 - \frac{1}{2}\log(2\beta) - \frac{1}{2N}\sum_i g(\mu_i) + \frac{1}{N}\left(\log(2\beta) - \frac{1}{2}\log\left(-\frac{1}{N}\sum_i g''(\mu_i)\right)\right) + O(N^{-2+\epsilon}) \tag{1.16}$$

where

$$g(x) = \log\left(2\beta + \frac{1}{2\beta} - x\right). \tag{1.17}$$

(iii) (Ferromagnetic regime) If $J > 1$ and $\beta > \frac{1}{2J}$, then

$$F_N = F(\beta) + \left(\beta - \frac{1}{2J}\right)\left(\mu_1 - J - \frac{1}{J}\right) + O(N^{-1}\log N). \tag{1.18}$$

Intuitively, the free energy is dominated by the ground state, μ_1, at low temperature, and by all eigenvalues at high temperature. The above result makes this intuition precise: in the spin glass regime (i) and the ferromagnetic regime (iii), the fluctuations of the free energy are governed by the ground state, the largest eigenvalue μ_1, while in the ferromagnetic regime (ii), they are governed by all of the eigenvalues in the form of the linear statistics $\sum_i g(\mu_i)$ of a specific function g.

The Wigner matrix with non-zero mean M is a rank 1 case of so-called a spiked random matrix. A spiked random matrix is a random matrix perturbed additively by a deterministic matrix of fixed N-independent rank. Spiked random matrices were studied extensively in random matrix theory [5, 22, 11, 34, 27]. Since the perturbation has a rank independent of N, the semi-circle law (see (1.25) below) still holds. However, the top eigenvalues may have different limit theorems. For the rank 1 case M, it was shown in Theorem 1.3 of [34] that

$$\begin{cases} N^{2/3}(\mu_1 - 2) \Rightarrow \mathrm{TW}_1, & J < 1 \\ N^{1/2}\left(\mu_1 - (J + \frac{1}{J})\right) \Rightarrow \mathcal{N}(0, 2(1 - \frac{1}{J^2})), & J > 1. \end{cases} \tag{1.19}$$

(See also Theorem 3.4 of [11].) For Hermitian matrix, (1.19) was first proved in [5]. When $J < 1$, then the perturbation has little effect on μ_1. But when $J > 1$, μ_1 becomes an "outlier" in the sense that it is separated from the support of the semi-circle and as a consequence, becomes "freer" to fluctuate; the fluctuation order $N^{-1/2}$ is bigger in this case. Theorem 1.4 (i) and (iii) follow directly from Theorem 1.5 and (1.19).

1.4 Linear statistics for Wigner matrix with non-zero mean

In order to prove Theorem 1.4 (ii) from Theorem 1.5 (ii), we need a limit theorem for the linear statistic $\sum_i g(\mu_i)$. It is a well-known result in random matrix theory that for mean-zero Wigner matrices (i.e. $J = 0$

case), the linear statistics converge to Gaussian distributions with scale $O(1)$ instead of the classical diffusive $O(N^{1/2})$ scale for the sum of independent random variables [26, 35, 3, 4, 30]. The main technical component of this paper is the central limit theorem for the linear statistics of Wigner matrix with non-zero mean (i.e. $J > 0$ case). The next theorem shows that the spike (i.e. $J > 0$) only changes the mean of the limiting Gaussian distribution; the variance of the Gaussian distribution is same for all $J \geq 0$. We remark that the change of the mean due to the spike is already known for spiked sample covariance matrices [42, 33].

We prove the following result for $J > 0$. Set

$$\tau_\ell(\varphi) = \frac{1}{\pi}\int_{-2}^{2} \varphi(x) \frac{T_\ell(x/2)}{\sqrt{4-x^2}}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(2\cos\theta)\cos(\ell\theta)\,\mathrm{d}\theta \tag{1.20}$$

for $\ell = 0, 1, 2, \dots,$ where $T_\ell(t)$ are the Chebyshev polynomials of the first kind; $T_0(t) = 1$, $T_1(t) = t$, $T_2(t) = 2t^2 - 1$, $T_3(t) = 4t^3 - 3t$, $T_4(t) = 8t^4 - 8t^2 + 1$, etc.

Theorem 1.6 (Linear statistics of Wigner matrix with non-zero mean)**.** *Let M be an $N \times N$ Wigner matrix with non-zero mean as in Definition 1.1. Denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$ the eigenvalues of M. Set*

$$\widehat{J} = \begin{cases} J + J^{-1} & \text{if } J > 1\,, \\ 2 & \text{if } J \leq 1\,. \end{cases} \tag{1.21}$$

Then, for any function $\varphi : \mathbb{R} \to \mathbb{R}$ that is analytic in an open neighborhood of $[-2, \widehat{J}\,]$ and has compact support, the random variable

$$T_N(\varphi) := \sum_{i=1}^{N} \varphi(\mu_i) - N\int_{-2}^{2} \varphi(x)\frac{\sqrt{4-x^2}}{2\pi}\mathrm{d}x \tag{1.22}$$

converges in distribution to the Gaussian distribution with mean $M(\varphi)$ and variance $V(\varphi)$, where

$$\begin{aligned} M(\varphi) = \frac{1}{4}\left(\varphi(2) + \varphi(-2)\right) - \frac{1}{2}\tau_0(\varphi) + J'\tau_1(\varphi) + (w_2 - 2)\tau_2(\varphi) + (W_4 - 3)\,\tau_4(\varphi) \\ + \frac{1}{2\pi \mathrm{i}}\oint \varphi\left(-s - \frac{1}{s}\right)\frac{J^2 s}{1 + Js}\mathrm{d}s \end{aligned} \tag{1.23}$$

and

$$V(\varphi) = (w_2 - 2)\tau_1(\varphi)^2 + (W_4 - 3)\tau_2(\varphi)^2 + 2\sum_{\ell=1}^{\infty} \ell\tau_\ell(\varphi)^2. \tag{1.24}$$

The contour for the integral in (1.23) *is any simple closed contour containing* 0 *inside in the slit disk $\{|s| < 1\} \setminus [-1, -1/J]$ in which $\varphi\left(-s - \frac{1}{s}\right)$ is analytic. (The analyticity condition of φ implies that there is such a contour.)*

Note that the variance does not depend on J and J' but the mean does.

Among various methods of studying the linear statistics in random matrix theory, we follow the method of Bai and Silverstein, and Bai and Yao [3, 4] to prove the above result. Specifically, we extend the analysis of [4] to the $J > 0$ case. Let $\rho_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{\mu_j}$ be the empirical spectral distribution of M. As $N \to \infty$, ρ_N converges to the semicircle measure ρ, defined by

$$\rho(\mathrm{d}x) = \frac{1}{2\pi}\sqrt{4-x^2}_+\mathrm{d}x. \tag{1.25}$$

Let $s_N(z)$ and $s(z)$ be the Stieltjes transforms of ρ_N and ρ, respectively, for $z \in \mathbb{C}^+$. Then, $T_N(\varphi)$ admits an integral representation, which can be easily converted to a contour integral that contains $\xi_N(z) := s_N(z) - s(z)$ in its integrand. The problem then reduces to showing that $\xi_N(z)$ converges to a Gaussian process $\xi(z)$. Due to the non-zero mean of the entries M_{ij}, the proof of convergence of $\xi_N(z)$ and the evaluation of the mean and the covariance of $\xi(z)$ become complicated. The main technical input we use in the estimate is the local semicircle law obtained in [19].

Theorem 1.4 (ii) follows from Theorem 1.5 and Theorem 1.6 once we evaluate the mean and the variance of the limiting Gaussian distribution: see Section 2.

Remark 1.7. It is direct to check that the integral in (1.23) can also be expressed as:

$$\frac{1}{2\pi \mathrm{i}} \oint \varphi\left(-s - \frac{1}{s}\right) \frac{J^2 s}{1 + Js} \mathrm{d}s = \begin{cases} \sum_{\ell=2}^{\infty} J^{\ell} \tau_{\ell}(\varphi) & \text{if } J < 1, \\ \frac{1}{2}\varphi(2) - \frac{1}{2}\tau_0(\varphi) - \tau_1(\varphi) & \text{if } J = 1, \\ \varphi(\widehat{J}) - \tau_0(\varphi) - \widehat{J}\tau_1(\varphi) - \sum_{\ell=2}^{\infty} J^{-\ell}\tau_{\ell}(\varphi) & \text{if } J > 1. \end{cases} \tag{1.26}$$

1.5 Transitions

It is interesting to consider the phase transition and the near-critical behaviors in Theorem 1.4 and 1.5. We have the following result for the transition between the spin glass regime (i) and the ferromagnetic regime (iii). For fixed $\beta > 1/2$, consider J depending on N as

$$J = 1 + wN^{-1/3}. \tag{1.27}$$

Then for each $w \in \mathbb{R}$, the asymptotic result (1.15) still holds. Now in the theory of spiked random matrices, the distribution of μ_1 is known to have the transition under the scaling (1.27):

$$N^{2/3}(\mu_1 - 2) \Rightarrow \mathrm{TW}_{1,w} \tag{1.28}$$

where $\mathrm{TW}_{1,w}$ is a one-parameter family of random variables with the distribution function obtained in Theorems 1.5 and 1.7 of [9]. See also [31] for the Gaussian case and [22] for a more general class of Wigner matrices. See Section 3.2.4.

For other transitions, by matching the fluctuation scales, we expect that the critical window for the transition between the paramagnetic regime (ii) and the ferromagnetic regime (iii) is $J = \frac{1}{2\beta} + O(N^{-1/2})$ for each $\beta < 1$ and that of the transition between the spin glass regime (i) and the paramagnetic regime (ii) is $\beta = \frac{1}{2} + O(\frac{\sqrt{\log N}}{N^{1/3}})$ for each $J < 1$. However, the analysis of these transition regimes is yet to be done.

1.6 Organization

The rest of paper is organized as follows. In Section 2 and Section 3, we prove the main results, Theorem 1.4 and Theorem 1.5, respectively. Theorem 1.6 (linear statistics) is proved in Section 4 assuming Proposition 4.1 and Lemma 4.2. Proposition 4.1 is proved in Section 5–8. Lemma 4.2 is proved in Section 9. Certain technical large deviation estimates are proved in Section 10.

Notational Remark 1.8. Throughout the paper we use C or c in order to denote a constant that is independent of N. Even if the constant is different from one place to another, we may use the same notation C or c as long as it does not depend on N for the convenience of the presentation.

Notational Remark 1.9. The notation $\Rightarrow$ denotes the convergence in distribution as $N \to \infty$.

Notational Remark 1.10. For random variables X and Y depending on N, we use the notation $X \prec Y$ to mean that

$$\mathbb{P}(|X| > N^{\epsilon}|Y|) < N^{-D}$$

for any (small) $\epsilon > 0$ and (large) $D > 0$. The relation $\prec$ is transitive and satisfies the arithmetic rules, e.g., if $X_1 \prec Y_1$ and $X_2 \prec Y_2$ then $X_1+X_2 \prec Y_1+Y_2$ and $X_1X_2 \prec Y_1Y_2$. We will also use the notation $X = \mathcal{O}(N^p)$ if $X \prec N^p$ for a constant p.

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2 Proof of Theorem 1.4

We already discussed in Section 1.3 how Theorem 1.4 (i), (iii) follow from Theorem 1.5 and (1.19). We now check that Theorem 1.4 (iii) follows from Theorem 1.5 and Theorem 1.6.

In Theorem 1.6, we use the function $\varphi(x) = g(x) = \log\left(2\beta + \frac{1}{2\beta} - x\right)$. We first evaluate $M(\varphi)$ and $V(\varphi)$ in Theorem 1.6 for this function.

The variance $V(\varphi)$ does not depend on J and J', and hence it is the same as the $J = J' = 0$ case. The value $\sigma^2 = \frac{1}{4}V(\varphi)$ was evaluated (3.13) of [6] (see the second last sentence in Section 5 of [6]); this is equal to α_1 in (1.12).

Now consider $M(\varphi)$. For the function $\varphi = g$, it was shown in (A.17) of [6] that

$$\tau_0(\varphi) = -\log(2\beta), \quad \tau_1(\varphi) = -2\beta, \quad \tau_2(\varphi) = -2\beta^2, \quad \tau_4(\varphi) = -4\beta^4. \tag{2.1}$$

We now evaluate

$$\frac{1}{2\pi \mathrm{i}} \oint \varphi\left(-s - \frac{1}{s}\right) \frac{J^2 s}{1+Js} \mathrm{d}s = \frac{1}{2\pi \mathrm{i}} \oint \log\left(2\beta + \frac{1}{2\beta} + s + \frac{1}{s}\right) \frac{J^2 s}{1+Js} \mathrm{d}s. \tag{2.2}$$

Set $B = 2\beta$. Then $B < \min\{1, 1/J\}$ since we are in the paramagnetic regime. The above integral is

$$F(B) = \frac{1}{2\pi \mathrm{i}} \oint_{|s|=r} \log\left(B + \frac{1}{B} + s + \frac{1}{s}\right) \frac{J^2 s}{1+Js} \mathrm{d}s \tag{2.3}$$

where we can take r to be any number satisfying $B < r < \min\{1, 1/J\}$. Its derivative is

$$F'(B) = \frac{1}{2\pi \mathrm{i}} \oint_{|s|=r} \frac{(B^2-1)J^2 s^2}{B(B+s)(1+Bs)(1+Js)} \mathrm{d}s = -\frac{J^2 B}{1-JB} = J - \frac{J}{1-JB} \tag{2.4}$$

by the calculus of residue: the one pole inside the contour is $s = -B$. Hence $F(B) = JB + \log(1-JB) + C$ for a constant C for every B satisfying $0 < B < \min\{1, 1/J\}$. To find the constant C, note that

$$F(B) = \frac{1}{2\pi \mathrm{i}} \oint_{|s|=r} \log\left(\frac{(B+s)(1+Bs)}{Bs}\right) \frac{J^2 s}{1+Js} \mathrm{d}s = \frac{1}{2\pi \mathrm{i}} \oint_{|s|=r} \log\left(\frac{(B+s)(1+Bs)}{s}\right) \frac{J^2 s}{1+Js} \mathrm{d}s \tag{2.5}$$

since the integral of $\frac{J^2 s}{1+Js}$ over the circle $|s| = r$ is zero. Hence $F(B) \to 0$ as $B \to 0$. This implies that $C = 0$

and therefore $F(B) = JB + \log(1 - JB)$. This implies that

$$\frac{1}{2\pi i} \oint \varphi\left(-s - \frac{1}{s}\right) \frac{J^2 s}{1 + Js} ds = 2\beta J + \log(1 - 2\beta J). \tag{2.6}$$

Therefore,

$$M(\varphi) = \frac{1}{2} \log\left(1 - 4\beta^2\right) - 2\beta J' - 2\beta^2(w_2 - 2) - 4\beta^4\left(W_4 - 3\right) + 2\beta J + \log(1 - 2\beta J). \tag{2.7}$$

We also have (see (A.5) of [6])

$$\int_{-2}^{2} \log\left(2\beta + \frac{1}{2\beta} - x\right) \rho(dx) = 2\beta^2 - \log\left(2\beta\right). \tag{2.8}$$

Furthermore, applying Theorem 1.6 to function $-g''(x)$, we have

$$-\frac{1}{N} \sum_i g''(\mu_i) \to \int_{-2}^{2} \frac{1}{(2\beta + \frac{1}{2\beta} - x)^2} \rho(dx) = \frac{4\beta^2}{1 - 4\beta^2}. \tag{2.9}$$

in probability (see (A.8) of [6] for the equality). Therefore, Theorem 1.5 (ii) implies that

$$N\left(F_N - \beta^2\right) \Rightarrow \mathcal{N}\left(f_1, \alpha_1\right) \tag{2.10}$$

where

$$f_1 = -\frac{1}{2} M(\varphi) + \log(2\beta) - \frac{1}{2} \log\left(\frac{4\beta^2}{1 - 4\beta^2}\right), \qquad \alpha_1 = \frac{1}{4} V(\varphi). \tag{2.11}$$

These are same as (1.11) and (1.12). The proof is complete.

3 Proof of Theorem 1.5

As we mentioned before, the leading order limit of the free energy (1.10) was obtained in [28]. This is based on the following integral representation for the quenched case, i.e. for fixed matrix M.

Lemma 3.1 ([28]; also Lemma 1.3 of [6]). *Let M be an $N \times N$ symmetric matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$. Then*

$$\int_{S_{N-1}} e^{\beta\langle \boldsymbol{\sigma}, M\boldsymbol{\sigma}\rangle} d\omega_N(\boldsymbol{\sigma}) = C_N \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{N}{2} G(z)} dz, \quad G(z) = 2\beta z - \frac{1}{N} \sum_i \log(z - \mu_i), \tag{3.1}$$

where γ is any constant satisfying $\gamma > \mu_1$, the integration contour is the vertical line from $\gamma - i\infty$ to $\gamma + i\infty$, the log *function is defined in the principal branch, and*

$$C_N = \frac{\Gamma(N/2)}{2\pi i (N\beta)^{N/2-1}}. \tag{3.2}$$

Here $\Gamma(z)$ denotes the Gamma function.

Now for the spin system, the eigenvalues μ_i are random, but using random matrix theory, there are precise estimates on these random variables, and we can still apply the method of steepest-descent. A formal application of the method of steepest-descent was done in [28] and obtained the leading order term. In [6], we supply necessary estimates and made the result of [28] rigorous when $J = 0$. We furthermore, extended the analysis to the next order term and obtained limit theorems, Theorem 1.4 when $J = 0$. It is not explicitly stated in [6], but the analysis in it proved Theorem 1.5 for $J = 0$ as well. We now follow the similar approach and prove Theorem 1.5 for $J > 0$.

3.1 Rigidity estimates of the eigenvalues

Let M be a Wigner matrix M with non-zero mean in Definition 1.1. By definition,

(a) For $i \neq j$, $\mathbb{E}M_{ij} = JN^{-1}$, $\mathbb{E}|M_{ij}|^2 = N^{-1} + J^2N^{-2}$, $\mathbb{E}|A_{ij}|^4 = W_4N^{-2} + O(N^{-\frac{3}{2}})$. In addition, for Hermitian case, $\mathbb{E}M_{ij}^2 = J^2N^{-2}$.

(b) For $i = j$, $\mathbb{E}M_{ii} = J'N^{-1}$, $\mathbb{E}|M_{ii}|^2 = W_2N^{-1} + (J'N^{-1})^2$.

For M, we have the following precise rigidity estimate for all eigenvalues other than the largest one.

Lemma 3.2 (Theorem 2.13 of [19], rigidity)**.** *For a positive integer $k \in [1, N]$, let $\hat{k} := \min\{k, N+1-k\}$. Let γ_k be the classical location defined by*

$$\int_{\gamma_k}^{\infty} \mathrm{d}\rho_{sc} = \frac{1}{N}\left(k - \frac{1}{2}\right). \tag{3.3}$$

Then,

$$|\mu_k - \gamma_k| \prec \hat{k}^{-1/3}N^{-2/3} \tag{3.4}$$

for all $k = 2, 3, \dots, N$.

The largest eigenvalue μ_1 depends on J and we have the following Dichotomy:

Lemma 3.3 (Theorem 6.3 of [27])**.**

(a) If $J \leq 1$,

$$|\mu_1 - 2| \prec N^{-2/3} \tag{3.5}$$

(b) If $J > 1$,

$$\left|\mu_1 - (J + \frac{1}{J})\right| \prec \sqrt{\frac{J - 1 + N^{-1/3}}{N}}. \tag{3.6}$$

3.2 Proof

We apply the method of steepest-descent to the integral in Lemma 3.1. It is easy to check that $G'(z)$ is an increasing function of z on (μ_1, ∞), hence there exists a unique $\gamma \in (\mu_1, \infty)$ satisfying the equation $G'(\gamma) = 0$: see Lemma 4.1 of [6]. We see in the analysis below that in the spin glass regime and the ferromagnetic regime, γ is close to μ_1 with distance of order $O(N^{\epsilon-1})$. On the other hand, for the paramagnetic regime, γ is away from μ_1 with distance of order $O(1)$.

3.2.1 Spin glass regime: $\beta > \frac{1}{2}$ and $J < 1$

Proof of Theorem 1.5 (i). In Theorem 2.11 of [6], we obtained a Tracy-Widom limit theorem, Theorem 1.5 (i), for general symmetric random matrix M without assuming that the mean is zero. This theorem assumes three conditions, Condition 2.3 (Regularity of measure), Condition 2.4 (Rigidity of eigenvalues), and Condition 2.6 (Tracy-Widom limit of the largest eigenvalue). The proof actually establishes Theorem

1.5 (i) under Condition 2.3 and Condition 2.4 first, which then implies Theorem 1.4 (i) if we add Condition 2.6: See (6.3) in [6] and then the sentence below it. Now for Wigner matrix with non-zero mean M, Condition 2.3 and Condition 2.4 are satisfied clearly from Lemma 3.2 and Lemma 3.3, including the largest eigenvalue. Hence Theorem 1.5 (i) is proved. $\square$

3.2.2 Paramagnetic regime: $\beta < \frac{1}{2}$ and $\beta < \frac{1}{2J}$

Proof of Theorem 1.5 (ii). In Theorem 2.10 of [6], we proved a Gaussian limit theorem, Theorem 1.4 (ii), for general symmetric random matrix M without assuming that the mean is zero. This theorem assumes three conditions, Condition 2.3 (Regularity of measure), Condition 2.4 (Rigidity of eigenvalues), and Condition 2.5 (Linear statistics of the eigenvalues). Similar to the spin glass regime, the proof actually establishes Theorem 1.5 (ii) under Condition 2.3 and Condition 2.4 first, which then implies Theorem 1.4 (ii) if we add Condition 2.5: See (5.27) and (5.29) in [6]. Now for Wigner matrix with non-zero mean M, Condition 2.4 is not satisfied when $J > 1$ due to Lemma 3.3. However, we can easily modify the proof of Theorem 2.10 of [6] for the paramagnetic conditions as we see now. The case $J \le 1$ follows from Theorem 2.10 of [6] directly, but we consider this case as well here.

We choose γ in Lemma 3.1 as the unique critical value of $G(z)$ on the part of the real line $z \in (\mu_1, \infty)$. In order to evaluate the integral in (3.1), we introduce a deterministic function

$$\widehat{G}(z) = 2\beta z - \int_{-2}^{2} \log(z-x) \mathrm{d}\rho(x) \tag{3.7}$$

where ρ is the semicircle measure. Let $\widehat{\gamma}$ be the critical point of $\widehat{G}$ in the interval $(2, \infty)$. As in (A.4) of [6], it can be easily checked that

$$\widehat{\gamma} = 2\beta + \frac{1}{2\beta}. \tag{3.8}$$

Recall the definition of $\widehat{J}$ in (1.21). Since $\beta < 1/2$ and $\beta < \frac{1}{2J}$ in the paramagnetic regime, we find that

$$\widehat{\gamma} > \widehat{J}, \tag{3.9}$$

hence $\widehat{\gamma} > \mu_1$ with high probability.

Recall that γ_1 is the classical location of the largest eigenvalue as defined in (3.3). Since $|\mu_1 - \gamma_1| = O(1)$ with high probability, Lemma 5.1 and Corollary 5.2 of [6] hold for this case as well. Then, Corollary 5.3 and Lemma 5.4 of [6] also hold, which implies the calculations up to (5.27) and (5.29) of [6]. This proves Theorem 1.5 (ii). $\square$

3.2.3 Ferromagnetic regime: $J > 1$ and $\beta > \frac{1}{2J}$

In this case, $\widehat{\gamma}$ in (3.8) satisfies $\widehat{\gamma} < \widehat{J}$ since $\beta > \frac{1}{2J}$, and hence the proof for the paramagnetic regime does not apply. Instead, this case is similar to the spin glass regime and we modify the proof of Theorem 2.11 of [6]. The following lemma shows that γ is close to μ_1 up to order $1/N$. This is similar to Lemma 6.1 of [6].

Lemma 3.4. *Let $c > 0$ be a constant such that $2\beta - \frac{1}{J} > c$ and $J - 1 > c$. Then,*

$$\frac{1}{3\beta N} \le \gamma - \mu_1 \le \frac{2}{cN}. \tag{3.10}$$

with high probability.

Proof. Note that

$$G'(z) = 2\beta - \frac{1}{N}\sum_i \frac{1}{z-\mu_i}. \tag{3.11}$$

Since $G'(z) < 2\beta - \frac{1}{N(z-\mu_1)}$, we find that $G'(\mu_1 + \frac{1}{3\beta N}) < 0$.

Since $G'(z)$ is an increasing function of z on (μ_1, ∞), it suffices to show that $G'(\mu_1 + \frac{2}{cN}) > 0$. In order to show this, we first notice that

$$G'(z) = 2\beta - \frac{1}{N}\frac{1}{z-\mu_1} - \frac{1}{N}\sum_{i=2}^{N} \frac{1}{z-\mu_i} \geq 2\beta - \frac{1}{N}\frac{1}{z-\mu_1} - \frac{1}{N}\sum_{i=2}^{N} \frac{1}{\mu_1-\mu_i} \tag{3.12}$$

for $z \geq \mu_1$. From Lemma 3.2, we may assume that μ_k $(k \geq 2)$ satisfies the rigidity estimate (3.4). Thus, for any $\epsilon > 0$, if $z > \mu_1 > 2$,

$$\begin{aligned} G'(z) &\geq 2\beta - \frac{1}{N}\frac{1}{z-\mu_1} - \frac{1}{N}\sum_{i=2}^{N}\left(\frac{1}{\mu_1-\gamma_i} + \hat{i}^{-1/3}N^{-2/3+\epsilon}\right) \\ &\geq 2\beta - \frac{1}{N}\frac{1}{z-\mu_1} - \int_{-2}^{2}\frac{\mathrm{d}\rho(x)}{\mu_1 - x} - CN^{-1+\epsilon} = 2\beta - \frac{1}{N}\frac{1}{z-\mu_1} - \frac{\mu_1 - \sqrt{\mu_1^2-4}}{2} - CN^{-1+\epsilon}. \end{aligned} \tag{3.13}$$

From Lemma 3.3, we thus find that, for any $0 < \delta < \frac{c}{4}$,

$$\begin{aligned} G'\left(\mu_1 + \frac{2}{cN}\right) &\geq 2\beta - \frac{c}{2} - \frac{1}{2}\left(J + \frac{1}{J} - \sqrt{\left(J+\frac{1}{J}\right)^2 - 4}\right) - \delta - CN^{-1+\epsilon} \\ &\geq 2\beta - \frac{1}{J} - c > 0 \end{aligned} \tag{3.14}$$

with high probability. This proves the lemma. □

The following lemma is a modification of Lemma 6.2 of [6]. The proof is simpler here due to the fact that μ_1 is away from μ_2 by $O(1)$.

Lemma 3.5. *Assume that there exists a constant $c > 0$ such that $2\beta - \frac{1}{J} > c$ and $J - 1 > c$. Let γ be the solution of the equation $G'(\gamma) = 0$ in Lemma 3.4. Then, for any $0 < \epsilon < 1$,*

$$G(\gamma) = \widehat{G}(\mu_1) + O(N^{-1+\epsilon}) \tag{3.15}$$

with probability. (See (3.7) for the definition of $\widehat{G}$). Moreover, there exist constants $C_0, C_1 > 0$ such that

$$C_0 N^{\ell-1} \leq \frac{(-1)^\ell}{(\ell-1)!} G^{(\ell)}(\gamma) \leq C_1^\ell N^{\ell-1} \tag{3.16}$$

for all $\ell = 2, 3, \ldots$ with probability. Here, C_0 and C_1 do not depend on ℓ.

Proof. We assume that the eigenvalues μ_k $(k \geq 2)$ satisfies the rigidity estimate (3.4). Then, from Lemma 3.4,

$$G(\gamma) = 2\beta\mu_1 - \int_{-2}^{2} \log(\mu_1 - x)\mathrm{d}\rho(x) + O(N^{-1+\epsilon}) = \widehat{G}(\mu_1) + O(N^{-1+\epsilon}) \tag{3.17}$$

with probability. Thus, the first part of the lemma holds.

For the second part of the lemma, recall that there exists a constant $\delta > 0$ such that $\gamma - \mu_i > \delta$ for all $i = 2, 3, \ldots, N$. Since

$$G^{(\ell)}(\gamma) = \frac{(-1)^\ell(\ell-1)!}{N(\gamma-\mu_1)^\ell} + \frac{(-1)^\ell(\ell-1)!}{N} \sum_{i=2}^{N} \frac{1}{(\gamma-\mu_i)^\ell}, \tag{3.18}$$

we can conclude that the second part of the lemma holds. □

Proof of Theorem 1.5 (iii). Using the above two lemmas, the proof of Lemma 6.3 of [6] applies without any change, and we find that there exists $K \equiv K(N)$ satisfying $N^{-C} < K < C$ for some constant $C > 0$ such that

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)} \mathrm{d}z = \mathrm{i} e^{\frac{N}{2}G(\gamma)} K \tag{3.19}$$

with high probability. This implies that, as (6.61) of [6],

$$Z_N = \frac{\sqrt{N}\beta}{\mathrm{i}\sqrt{\pi}(2\beta e)^{N/2}} e^{\frac{N}{2}G(\gamma)} K(1 + O(N^{-1})) \tag{3.20}$$

with high probability. Recall that $\widehat{J} = J + \frac{1}{J}$. Then, using Lemma 3.5 and evaluating $\widehat{G}$ as in (A.5) of [6], we find that

$$\begin{aligned} F_N &= \frac{1}{2}[G(\gamma) - 1 - \log(2\beta)] + O(N^{-1}\log N) = \frac{1}{2}[\widehat{G}(\mu_1) - 1 - \log(2\beta)] + O(N^{-1}\log N) \\ &= \frac{1}{2}[\widehat{G}(\widehat{J}) - 1 - \log(2\beta)] + \frac{1}{2}\widehat{G}'(\widehat{J}) \cdot (\mu_1 - \widehat{J}) + O(N^{-1}\log N) \\ &= \beta\left(J + \frac{1}{J}\right) - \frac{1}{4J^2} - \frac{1}{2}\log(2\beta J) - \frac{1}{2} + \left(\beta - \frac{1}{2J}\right)(\mu_1 - \widehat{J}) + O(N^{-1}\log N), \end{aligned} \tag{3.21}$$

hence

$$F_N - F(\beta) = \left(\beta - \frac{1}{2J}\right)(\mu_1 - \widehat{J}) + O(N^{-1}\log N), \tag{3.22}$$

with high probability. This completes the proof. □

3.2.4 Transition between spin glass regime and ferromagnetic regime

Consider a fixed $\beta > 1/2$ and $J = 1 + wN^{-1/3}$. As in Section 3.2.1, we can prove Theorem 1.5 (i) assuming Condition 2.3 (Regularity of measure) and Condition 2.4 (Rigidity of eigenvalues) of [6], and Condition 2.3 and Condition 2.4 are satisfied from Lemma 3.2 and Lemma 3.3.

As discussed in Section 1.5, for the Gaussian case

$$N^{2/3}(\mu_1 - 2) \Rightarrow \mathrm{TW}_{1,w} \tag{3.23}$$

where $\mathrm{TW}_{1,w}$ is a one-parameter family of random variables with the distribution function obtained in Theorems 1.5 and 1.7 of [9]. For a non-Gaussian Wigner matrix with non-zero mean, the limit theorem can be proved by applying the Green function comparison method based on the Lindeberg replacement strategy in Theorems 2.4 and 6.3 in [21]. The proof of Theorem 6.3 in [21] can be reproduced by assuming the rigidity of eigenvalues and the local semicircle law, which hold also for a Wigner matrix with non-zero mean from Lemmas 3.2, 3.3, and 5.1 (See also Theorem 3.3 and Lemma 3.5 of [29] for more detail on the case that the variance of the diagonal entries does not match that of GOE.)

4 Linear statistics

4.1 Proof of Theorem 1.6

For a function φ that satisfies the assumptions of the theorem, we consider $T(\varphi)$, the weak limit of the random variable

$$T_N(\varphi) = \sum_{i=1}^{N} \varphi(\mu_i) - N \int_{-2}^{2} \varphi(x) \frac{\sqrt{4-x^2}}{2\pi} \mathrm{d}x = N \int_{-\infty}^{\infty} \varphi(x)[\rho_N - \rho](\mathrm{d}x). \tag{4.1}$$

Fix (N-independent) constants $a_- < -2$ and $a_+ > \widehat{J}$. Let Γ be the rectangular contour whose vertices are $(a_- \pm \mathrm{i}v_0)$ and $(a_+ \pm \mathrm{i}v_0)$ for some $v_0 \in (0,1]$. Then,

$$T_N(\varphi) = \frac{N}{2\pi \mathrm{i}} \int_{\mathbb{R}} \oint_{\Gamma} \frac{\varphi(z)}{z-x} [\rho_N - \rho](\mathrm{d}x)\mathrm{d}z = -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} \varphi(z)\xi_N(z)\mathrm{d}z \tag{4.2}$$

where

$$\xi_N(z) := N \int_{\mathbb{R}} \frac{1}{x-z} (\rho_N - \rho)(\mathrm{d}x). \tag{4.3}$$

Decompose Γ into $\Gamma = \Gamma_u \cup \Gamma_d \cup \Gamma_l \cup \Gamma_r \cup \Gamma_0$, where

$$\Gamma_u = \{z = x + \mathrm{i}v_0 : a_- \le x \le a_+\}, \tag{4.4}$$
$$\Gamma_d = \{z = x - \mathrm{i}v_0 : a_- \le x \le a_+\}, \tag{4.5}$$
$$\Gamma_l = \{z = a_- + \mathrm{i}y : N^{-\delta} \le |y| \le v_0\}, \tag{4.6}$$
$$\Gamma_r = \{z = a_+ + \mathrm{i}y : N^{-\delta} \le |y| \le v_0\}, \tag{4.7}$$
$$\Gamma_0 = \{z = a_- + \mathrm{i}y : |y| < N^{-\delta}\} \cup \{z = a_+ + \mathrm{i}y : |y| < N^{-\delta}\}, \tag{4.8}$$

for some sufficiently small $\delta > 0$.

In Sections 5–8, we prove the following result for $\xi_N(z)$.

Proposition 4.1. *Let*

$$s(z) = \int \frac{1}{x-z} \rho(\mathrm{d}x) = \frac{-z + \sqrt{z^2-4}}{2} \tag{4.9}$$

be the Stieltjes transform of the semicircle measure ρ. Fix a (small) constant $c > 0$ and a path $\mathcal{K} \subset \mathbb{C}^+$ such that $\operatorname{Im} z > c$ for any $z \in \mathcal{K}$. Then, the process $\{\xi_N(z) : z \in \mathcal{K}\}$ converges weakly to a Gaussian process $\{\xi(z) : z \in \mathcal{K}\}$ with the mean

$$b(z) = \frac{s(z)^2}{1-s(z)^2} \left(-J' + \frac{J^2 s(z)}{1+Js(z)} + (w_2 - 1)s(z) + s'(z)s(z) + (W_4 - 3)\, s(z)^3 \right) \tag{4.10}$$

and the covariance matrix

$$\Gamma(z_i, z_j) = s'(z_i)s'(z_j) \left((w_2 - 2) + 2(W_4 - 3)s(z_i)s(z_j) + \frac{2}{(1 - s(z_i)s(z_j))^2} \right). \tag{4.11}$$

On the other hand, the following lemma is proved in Section 9.

Lemma 4.2. *For sufficiently small $\delta > 0$,*

$$\lim_{v_0 \searrow 0} \limsup_{N \to \infty} \int_{\Gamma_\sharp} \mathbb{E}|\xi_N(z)|^2 \mathrm{d}z = 0, \tag{4.12}$$

where $\Gamma_\sharp$ *can be* Γ_r, Γ_l, *or* Γ_0.

From the explicit formulas (4.10) and (4.11), it is direct to check that

$$\lim_{v_0 \searrow 0} \int_{\Gamma_\sharp} \mathbb{E}|\xi(z)|^2 \mathrm{d}z = 0. \tag{4.13}$$

Combining Proposition 4.11, Lemma 4.2 and (4.13), we obtain that $T_N(\varphi)$ converges in distribution to a Gaussian random variable $T(\varphi)$ with mean and variance

$$\mathbb{E}[T(\varphi)] = -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma \varphi(z) b(z) \mathrm{d}z, \qquad \mathrm{var}[T(\varphi)] = \frac{1}{(2\pi \mathrm{i})^2} \oint_\Gamma \oint_\Gamma \varphi(z_1)\varphi(z_2)\Gamma(z_1, z_2) \mathrm{d}z_1 \mathrm{d}z_2. \tag{4.14}$$

These integrals are equal to $M(\varphi)$ and $V(\varphi)$ in (1.23) and (1.24): see Lemma 4.4 below. This completes the proof of Theorem 1.6.

Remark 4.3. The covariance matrix $\Gamma(z_i, z_j)$ in (4.11) coincides with the one obtained in Proposition 4.1 of [4]. On the other hand, the mean $b_N(z)$ is different from the one in Proposition 3.1 of [4].

4.2 Computation of the mean and variance of $T(\varphi)$

Lemma 4.4. *We have* $\mathbb{E}[T(\varphi)] = M(\varphi)$ *and* $\mathrm{var}[T(\varphi)] = V(\varphi)$.

Proof. Consider $\mathrm{var}[T(\varphi)]$. Since $\Gamma(z_1, z_2)$ is same as the $J = J' = 0$ case, $\mathrm{var}[T(\varphi)]$ is same as the one in [4], and we obtain the result. We note that it was further shown in [4] that

$$\mathrm{cov}[T(\varphi_1), T(\varphi_2)] = (w_2 - 2)\tau_1(\varphi_1)\tau_1(\varphi_2) + (W_4 - 3)\tau_2(\varphi_1)\tau_2(\varphi_2) + 2\sum_{\ell=1}^{\infty} \ell \tau_\ell(\varphi_1)\tau_\ell(\varphi_2). \tag{4.15}$$

Now let us consider $\mathbb{E}[T(\varphi)]$. Recall that (see (1.20)), for $\ell = 0, 1, 2, \ldots$,

$$\tau_\ell(\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(2\cos\theta)\cos(\ell\theta)\,\mathrm{d}\theta = \frac{(-1)^\ell}{2\pi \mathrm{i}} \oint_{|s|=1} \varphi\left(-s - \frac{1}{s}\right) s^{\ell-1}\,\mathrm{d}s \tag{4.16}$$

where we set $s = -e^{\mathrm{i}\theta}$ for the second equality.

We change the variable z to $s = s(z)$ in the first integral in (4.14). Note that (4.9) implies that $s + 1/s = -z$ and the map $z \mapsto s$ maps $\mathbb{C} \setminus [-2, 2]$ to the disk $|s| < 1$. Then Γ is mapped to a contour with negative orientation that contains 0 and lies in the slit disk $\Omega := \{|s| < 1\} \setminus [-1, -1/J]$. Changing the orientation of the contour, we obtain

$$\mathbb{E}[T(\varphi)] = \frac{1}{2\pi \mathrm{i}} \oint \varphi\left(-s - \frac{1}{s}\right) \left[-J' + \frac{J^2 s}{1 + Js} + (w_2 - 1)s + \frac{s^3}{1 - s^2} + (W_4 - 3)\, s^3\right] \mathrm{d}s \tag{4.17}$$

along a contour with positive orientation that contains 0 and lies in the slit disk $\Omega := \{|s| < 1\} \setminus [-1, -1/J]$. Note that $\varphi\left(-s - \frac{1}{s}\right)$ is analytic in a neighborhood of the boundary of Ω.

The first, third, and fifth terms in the integrand of (4.17) are, using analyticity, equal to

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \oint_{|s|=1} \varphi\left(-s - \frac{1}{s}\right) \left[-J' + (w_2 - 1)s + (W_4 - 3)\, s^3\right] \mathrm{d}s \\ &= J'\tau_1(\varphi) + (w_2 - 1)\tau_2(\varphi) + (W_4 - 3)\,\tau_4(\varphi). \end{aligned} \tag{4.18}$$

For the fourth term in the integrand of (4.17), when we deform the contour to the unit circle, then the two poles $s = -1$ and $s = 1$ on the circle yields the half of the residue terms and the integral becomes the principal value. The principal value integral is, after setting $-s = e^{\mathrm{i}\theta}$,

$$P.V. \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(2\cos\theta) \frac{e^{4\mathrm{i}\theta}}{1 - e^{2\mathrm{i}\theta}} \mathrm{d}\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(2\cos\theta) \left(-\frac{1}{2} - \cos 2\theta \right) \mathrm{d}\theta = -\frac{1}{2}\tau_0(\varphi) - \tau_2(\varphi). \tag{4.19}$$

Hence we obtain

$$\frac{1}{2\pi\mathrm{i}} \oint_{|s|=r} \varphi\left(-s - \frac{1}{s}\right) \frac{s^3}{1 - s^2} \mathrm{d}s = \frac{1}{4} \left(\varphi(-2) + \varphi(2)\right) - \frac{1}{2}\tau_0(\varphi) - \tau_2(\varphi). \tag{4.20}$$

From (4.17), (4.18), and (4.20), we proved that $\mathbb{E}[T(\varphi)] = M(\varphi)$.

□

5 Outline of the proof of Proposition 4.1

Sections 5–8 are dedicated to proving Proposition 4.1. From Theorem 8.1 of [7], we need to show (a) the finite-dimensional convergence of $\xi_N(z)$ to a Gaussian vector and (b) the tightness of $\xi_N(z)$. To establish the part (a), we compute the limits of the mean $\mathbb{E}[\xi_N(z)]$ and the covariance $\mathrm{cov}[\xi_N(z_1), \xi_N(z_2)]$ in Section 6 and 7, respectively. The part (a) is concluded in Section 8.1. The part (b) is proved in Section 8.2.

We use the following known results for the resolvent and large deviation estimates.

5.1 Local semicircle law and large deviation estimates

The Green function (resolvent) of M is $R(z) = (M - zI)^{-1}$. The normalized trace of the Green function is defined as

$$s_N(z) = \frac{1}{N} \operatorname{Tr} R(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_i - z}, \tag{5.1}$$

which is also the Stieltjes transform of ρ_N. Recall that

$$\xi_N(z) = N \int_{\mathbb{R}} \frac{1}{x - z} (\rho_N - \rho)(\mathrm{d}x) = N \left(s_N(z) - s(z)\right). \tag{5.2}$$

We also set

$$\zeta_N(z) := \xi_N(z) - \mathbb{E}\xi_N(z) = N \left(s_N(z) - \mathbb{E}s_N(z)\right). \tag{5.3}$$

Lemma 5.1 (Theorem 2.9 of [19], local semicircle law)**.** *Let $\Sigma \geq 3$ be a fixed but arbitrary constant and define the domain $D = \{z = E + \mathrm{i}\eta \in \mathbb{C} : |E| \leq \Sigma, \eta \in (0, 3)\}$. Set $\kappa = \min\{|E - 2|, |E + 2|\}$. Then, for any $z \in D$ with $\operatorname{Im} z = \eta$,*

$$|s_N(z) - s(z)| \prec \min\left\{ \frac{1}{N\sqrt{\kappa + \eta}}, \frac{1}{\sqrt{N}} \right\} + \frac{1}{N\eta} \tag{5.4}$$

and

$$\max_{i,j} |R_{ij}(z) - \delta_{ij} s(z)| \prec \frac{1}{\sqrt{N}} + \sqrt{\frac{\operatorname{Im} s(z)}{N\eta}} + \frac{1}{N\eta}. \tag{5.5}$$

For $\eta \sim 1$, we have the following corollary.

Corollary 5.2. *Let $\Sigma \geq 3$ be a fixed but arbitrary constant. For a fixed (small) constant $c > 0$, define $D_c = \{z = E + \mathrm{i}\eta \in \mathbb{C} : |E| \leq \Sigma, \eta \in (c, 3)\}$. Then, for any $z \in D_c$,*

$$|s_N(z) - s(z)| \prec N^{-1} \tag{5.6}$$

and

$$|R_{ii}(z) - s(z)| \prec N^{-\frac{1}{2}}, \qquad |R_{ij}(z)| \prec N^{-\frac{1}{2}} \quad (i \neq j). \tag{5.7}$$

Moreover, (5.6) *holds for any $z \in \Gamma_r \cup \Gamma_l \cup \Gamma_0$ and* (5.7) *holds for any $z \in \Gamma_r \cup \Gamma_l$.*

Proof. The bounds for $z \in D_c$ are straightforward since $\eta \sim 1$. For $z \in \Gamma_r \cup \Gamma_l \cup \Gamma_0$, from Lemma 3.2 and Lemma 3.3,

$$\begin{aligned} |s_N(z) - s(z)| &= \left| \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\mu_j - z} - \int_{\mathbb{R}} \frac{\rho(\mathrm{d}x)}{x - z} \right| = \left| \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\gamma_j - z} - \int_{\mathbb{R}} \frac{\rho(\mathrm{d}x)}{x - z} \right| + \mathcal{O}(N^{-1}) \\ &= \mathcal{O}(N^{-1}). \end{aligned} \tag{5.8}$$

To prove (5.7) for $z \in \Gamma_r$, we notice that

$$\sqrt{\frac{\operatorname{Im} s(z)}{N\eta}} \sim \sqrt{\frac{\eta}{\sqrt{\kappa + \eta}} \frac{1}{N\eta}} \sim \sqrt{\frac{1}{N}} \tag{5.9}$$

since $\kappa \sim 1$ and $N^{-\delta} \leq \eta \leq v_0$. Since $\frac{1}{N\eta} \leq N^{-1+\delta}$, from (5.5), we find that (5.7) holds for $z \in \Gamma_r$. The proof of (5.7) for $z \in \Gamma_r$ is the same. ☐

Let $M^{(a)}$ be the minor of M obtained by removing the a-th row and the a-th column. We denote by $R^{(a)}$ and $s_N^{(a)}$ the Green function and the averaged Green function of $M^{(a)}$, respectively. It is well known that

$$R_{ii} = \frac{1}{M_{ii} - z - \sum_{p,q}^{(i)} M_{ip} R_{pq}^{(i)} M_{qi}}, \qquad R_{ij} = -R_{ii} \sum_{p}^{(i)} M_{ip} R_{pj}^{(i)} \quad (i \neq j), \tag{5.10}$$

and

$$R_{ij} - R_{ij}^{(a)} = \frac{R_{ia} R_{aj}}{R_{aa}}. \tag{5.11}$$

Here, (i) in the summation notation means that the index $p = 1, 2, \ldots, N$ with $p \neq i$. From the second identity in (5.10), we also have an estimate

$$\left| \sum_{p}^{(i)} M_{ip} R_{pj}^{(i)} \right| = \left| \frac{R_{ij}}{R_{ii}} \right| \prec N^{-\frac{1}{2}} \tag{5.12}$$

for $i \neq j$.

We will also frequently use the following large deviation estimates, which will be proved in Section 10.

Lemma 5.3. *Let S be an $(N-1) \times (N-1)$ matrix independent of M_{ia} $(1 \leq a \leq N, a \neq i)$ with matrix norm $\|S\|$. Then, for $n = 1, 2$, there exists a constant C_n depending only on J and W_4 in Definition 1.1 and*

Condition 1.1 such that

$$\mathbb{E}\left|\sum_{p,q}^{(i)} M_{ip}S_{pq}M_{qi} - \frac{1}{N}\sum_{p}^{(i)} S_{pp}\right|^{2n} \le \frac{C_n\|S\|^{2n}}{N^n}. \tag{5.13}$$

Moreover,

$$\left|\sum_{p,q}^{(i)} M_{ip}S_{pq}M_{qi} - \frac{1}{N}\sum_{p}^{(i)} S_{pp}\right| \prec \frac{\|S\|}{\sqrt{N}}. \tag{5.14}$$

6 The mean function

In this section, we assume that $z \in \mathcal{K} \cup \Gamma_r \cup \Gamma_l$. The estimate for $z \in \Gamma_r \cup \Gamma_l$ will be used later in the proof of Lemma 4.2.

Let

$$b_N(z) = \mathbb{E}[\xi_N(z)] = N[\mathbb{E}s_N(z) - s(z)]. \tag{6.1}$$

From (5.10), if we set

$$Q_i := -M_{ii} + \sum_{p,q}^{(i)} M_{ip}R_{pq}^{(i)}M_{qi}, \tag{6.2}$$

we have

$$\begin{aligned} R_{ii} = \frac{1}{-z-Q_i} &= \frac{1}{-z-s} + \frac{Q_i - s}{(-z-s)^2} + \frac{(Q_i-s)^2}{(-z-s)^3} + O\left(\frac{|Q_i-s|^3}{|z+s|^4}\right) \\ &= s + s^2(Q_i - s) + s^3\,(Q_i-s)^2 + O\left(|s|^4|Q_i-s|^3\right) \end{aligned} \tag{6.3}$$

since $s = -1/(s+z)$. Using $R_{ii} = \frac{1}{-z-Q_i}$, we have

$$Q_i - s = -\frac{1}{R_{ii}} - z - s = -\frac{1}{R_{ii}} + \frac{1}{s} = \mathcal{O}(N^{-\frac{1}{2}}). \tag{6.4}$$

We thus find that

$$b_N = s^2\sum_i \mathbb{E}(Q_i - s) + s^3\sum_i \mathbb{E}(Q_i-s)^2 + O(N^{-\frac{1}{2}+\epsilon}). \tag{6.5}$$

6.1 $\sum_i \mathbb{E}(Q_i - s)$

We first consider

$$\begin{aligned} \sum_i \mathbb{E}(Q_i - s) &= \sum_i \mathbb{E}\left[-M_{ii} + \sum_{p,q}^{(i)} M_{ip}R_{pq}^{(i)}M_{qi} - s\right] \\ &= -J' + \frac{J^2}{N^2}\mathbb{E}\sum_i\sum_{p,q}^{(i)} R_{pq}^{(i)} + \frac{1}{N}\mathbb{E}\sum_i\sum_{p}^{(i)} R_{pp}^{(i)} - Ns. \end{aligned} \tag{6.6}$$

Naive power counting shows that the second term is $O(N^{\frac{1}{2}+\epsilon})$ and the third term is $O(N^{\epsilon})$. We show that the second term is actually $O(1)$ and the third term is Ns plus an $O(1)$ term.

6.1.1 $\frac{1}{N}\mathbb{E}\sum_i \sum_p^{(i)} R_{pp}^{(i)}$

From (5.11) and (5.7),

$$\sum_p^{(i)}(R_{pp}^{(i)} - R_{pp}) = -\sum_p^{(i)} \frac{R_{pi}R_{ip}}{R_{ii}} = -\frac{1}{s}\sum_p^{(i)} R_{pi}R_{ip} + \mathcal{O}(N^{-\frac{1}{2}}). \tag{6.7}$$

This implies

$$\sum_p^{(i)} R_{pp}^{(i)} = \left(\sum_p R_{pp}\right) - R_{ii} - \frac{1}{s}(R^2)_{pp} + \frac{1}{s}(R_{ii})^2 + \mathcal{O}(N^{-\frac{1}{2}}), \tag{6.8}$$

and hence

$$\frac{1}{N}\sum_i \sum_p^{(i)} R_{pp}^{(i)} = \frac{N-1}{N}\operatorname{Tr}(R) - \frac{1}{sN}\operatorname{Tr}(R^2) + \frac{1}{sN}\sum_i (R_{ii})^2 + \mathcal{O}(N^{-\frac{1}{2}}). \tag{6.9}$$

Note that by spectral decomposition,

$$\frac{1}{N}\operatorname{Tr} R^2 = \frac{1}{N}\sum_i \frac{1}{(\mu_i - z)^2} = \frac{\mathrm{d}}{\mathrm{d}z} s_N(z). \tag{6.10}$$

Since $|s_N(z) - s(z)| \prec N^{-1}$, we find from Cauchy integral formula that $|s_N'(z) - s'(z)| \prec N^{-1+\delta}$. Hence, using (5.7),

$$\begin{aligned} \frac{1}{N}\sum_i \sum_p^{(i)} R_{pp}^{(i)} &= (N-1)s_N(z) - \frac{s_N'}{s} + \frac{1}{sN}\sum_i (R_{ii})^2 + \mathcal{O}(N^{-\frac{1}{2}}) \\ &= N s_N(z) - \frac{s'}{s} + \mathcal{O}(N^{-\frac{1}{2}}) \end{aligned} \tag{6.11}$$

We therefore find that

$$\frac{1}{N}\mathbb{E}\sum_i \sum_p^{(i)} R_{pp}^{(i)} = Ns + b_N - \frac{s'}{s} + O(N^{-\frac{1}{2}+\epsilon}). \tag{6.12}$$

6.1.2 $\frac{J^2}{N^2}\mathbb{E}\sum_i \sum_{p,q}^{(i)} R_{pq}^{(i)}$

The case when $p = q$ follows from (6.12):

$$\frac{J^2}{N^2}\mathbb{E}\sum_i \sum_p^{(i)} R_{pp}^{(i)} = J^2 s + O(N^{-1+\epsilon}) \tag{6.13}$$

since a naive estimate shows $b_N = O(N^\epsilon)$ from the definition.

We now consider the case when $p \neq q$. We start with a lemma. The strategy of the proof of this lemma is used several places in the paper.

Lemma 6.1. *For $q \neq i$,*

$$\frac{1}{N}\mathbb{E}\sum_p^{(i,q)} R_{pi}R_{iq}^{(p)} = O(N^{-\frac{3}{2}+\epsilon}). \tag{6.14}$$

Proof. For distinct p, q, i, we have from (5.10) and (5.12) that

$$R_{pi}R_{iq}^{(p)} = -R_{pp}\left(\sum_r^{(p)} M_{pr}R_{ri}^{(p)}\right)R_{iq}^{(p)} = -s\left(\sum_r^{(p)} M_{pr}R_{ri}^{(p)}\right)R_{iq}^{(p)} + \mathcal{O}(N^{-\frac{3}{2}}). \tag{6.15}$$

Hence,

$$\mathbb{E}\left[R_{pi}R_{iq}^{(p)}\right] = -\frac{Js}{N}\mathbb{E}\sum_r^{(p)} R_{ri}^{(p)}R_{iq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.16}$$

Using $R_{ab}^{(c)} - R_{ab} = \mathcal{O}(N^{-1})$, which follows from (5.11) and (5.7), repeatedly, we find that for distinct p, q, i,

$$\sum_r^{(p)} R_{ri}^{(p)}R_{iq}^{(p)} = \sum_r R_{ri}R_{iq} + \mathcal{O}(N^{-\frac{1}{2}}) = \sum_r^{(i,q)} R_{ri}R_{iq}^{(r)} + \mathcal{O}(N^{-\frac{1}{2}}). \tag{6.17}$$

Summing (6.16) over p, this implies that

$$\frac{1}{N}\sum_p^{(i,q)}\mathbb{E}\left[R_{pi}R_{iq}^{(p)}\right] = -Js\frac{N-1}{N^2}\mathbb{E}\sum_r^{(i,q)} R_{ri}R_{iq}^{(r)} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.18}$$

Since the two sums on either side are the same, we obtain that

$$\frac{1+Js}{N}\mathbb{E}\sum_p^{(i,q)} R_{pi}R_{iq}^{(p)} = O(N^{-\frac{3}{2}+\epsilon}). \tag{6.19}$$

We now claim that $|1+Js| > c'$ uniformly on $\mathcal{K}\cup\Gamma_r\cup\Gamma_l$ for some (N-independent) constant $c' > 0$. Assuming the claim, it is obvious from (6.19) that the desired lemma holds.

To prove the claim, we first note that, for $J < 1$, the claim is trivial since $|1+Js| > 1 - J|s| > 1 - J$. Thus, we assume that $J > 1$.

Let $z = E + \mathrm{i}\eta$. It is straightforward to check that for $\operatorname{Im} z > 0$,

$$\operatorname{Im} s(z) \geq C\sqrt{||E|-2|+\eta} \qquad \text{if} \quad |E| < 2 \quad \text{or} \quad ||E|-2| < \eta \tag{6.20}$$

and

$$\operatorname{Im} s(z) \geq \frac{C\eta}{\sqrt{|E|-2|+\eta}} \qquad \text{if} \quad |E| \geq 2 \quad \text{and} \quad ||E|-2| \geq \eta \tag{6.21}$$

for some $C > 0$ independent of z. (See, e.g., Lemma 3.4 of [21].) Thus, $|1+Js| \geq |J|\operatorname{Im} s \sim 1$, for $z \in \mathcal{K}$.

Recall that $a_+ > \widehat{J} \geq 2$. From the definition of $s(z)$, it is direct to see that $s(a_+) > s(\widehat{J}) = -1/J$. Moreover,

$$\operatorname{Re} s(a_+ + \mathrm{i}\eta) = \operatorname{Re}\int_{-2}^{2}\frac{1}{x-a_+-\mathrm{i}\eta}\rho(\mathrm{d}x) = \int_{-2}^{2}\frac{x-a_+}{(x-a_+)^2+\eta^2}\rho(\mathrm{d}x), \tag{6.22}$$

hence $\operatorname{Re} s(a_+ + \mathrm{i}\eta)$ is an increasing function of η. Thus, for $z \in \Gamma_u$,

$$|1+Js| > 1 + J\operatorname{Re} s > 1 + Js(a_+) \sim 1. \tag{6.23}$$

For $z \in \Gamma_l$, it is easy to see that $\operatorname{Re} s > 0$, hence $|1+Js| \geq 1 + J\operatorname{Re} s > 1$. This completes the proof of the lemma. $\square$

From (5.11) and (5.7),

$$R_{pq}^{(i)} = R_{pq} - \frac{R_{pi}R_{iq}}{R_{ii}} = R_{pq} - \frac{R_{pi}R_{iq}}{s} + \mathcal{O}(N^{-\frac{3}{2}}) = R_{pq} - \frac{R_{pi}R_{iq}^{(p)}}{s} + \mathcal{O}(N^{-\frac{3}{2}}). \tag{6.24}$$

Hence we conclude from (6.24) and (6.14) that

$$\begin{aligned} \frac{J^2}{N^2}\mathbb{E}\sum_i \sum_{p\neq q}^{(i)} R_{pq}^{(i)} &= \frac{J^2}{N^2}\mathbb{E}\sum_i \sum_{p\neq q}^{(i)} R_{pq} + O(N^{-\frac{1}{2}+\epsilon}) \\ &= \frac{J^2}{N^2}\mathbb{E}\sum_i \sum_{p\neq q} R_{pq} + O(N^{-\frac{1}{2}+\epsilon}) = \frac{J^2}{N}\mathbb{E}\sum_{p\neq q} R_{pq} + O(N^{-\frac{1}{2}+\epsilon}). \end{aligned} \tag{6.25}$$

We showed that the upper index (i) after adding a negligible term.

We now compute the right hand side of (6.25). From (5.10) and (6.3),

$$\begin{aligned} R_{pq} &= -R_{pp}\sum_r^{(p)} M_{pr}R_{rq}^{(p)} \\ &= -s\sum_r^{(p)} M_{pr}R_{rq}^{(p)} - s^2\left(Q_p - s\right)\sum_r^{(p)} M_{pr}R_{rq}^{(p)} + \mathcal{O}(N^{-\frac{3}{2}+\epsilon}). \end{aligned} \tag{6.26}$$

Taking expectation, the first term becomes,

$$-s\mathbb{E}\sum_r^{(p)} M_{pr}R_{rq}^{(p)} = -\frac{Js}{N}\mathbb{E}\sum_r^{(p)} R_{rq}^{(p)}. \tag{6.27}$$

Since

$$\mathbb{E}M_{pp}\sum_r^{(p)} M_{pr}R_{rq}^{(p)} = \frac{J'J}{N^2}\mathbb{E}\sum_r^{(p)} R_{rq}^{(p)} = O(N^{-\frac{3}{2}+\epsilon}), \tag{6.28}$$

the second term in (6.26) satisfies, also using (6.27),

$$\mathbb{E}\left[(Q_p - s)\sum_r^{(p)} M_{pr}R_{rq}^{(p)}\right] = \mathbb{E}\sum_{a,b}^{(p)} M_{pa}R_{ab}^{(p)}M_{bp}\sum_r^{(p)} M_{pr}R_{rq}^{(p)} - \frac{Js}{N}\mathbb{E}\sum_r^{(p)} R_{rq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.29}$$

We now evaluate the term

$$\mathbb{E}\sum_{a,b}^{(p)} M_{pa}R_{ab}^{(p)}M_{bp}\sum_r^{(p)} M_{pr}R_{rq}^{(p)},$$

by considering different choices of the indices a, b, r separately as follows.

1) When a, b, r are all distinct,

$$\mathbb{E}\sum^{(p)} M_{pa}R_{ab}^{(p)}M_{bp}M_{pr}R_{rq}^{(p)} = \frac{J^3}{N^3}\sum^{(p)}\mathbb{E}\left[R_{ab}^{(p)}R_{rq}^{(p)}\right], \tag{6.30}$$

where the summation is over all distinct a, b, r. The part of the sum in which the index r is equal to q is

$$\frac{J^3}{N^3}\sum_{a\neq b}^{(p)}\mathbb{E}\left[R_{ab}^{(p)}R_{qq}^{(p)}\right] = O(N^{-3/2+\epsilon}) \tag{6.31}$$

by naive estimate. Hence we assume that the index r satisfies $r \neq q$. Now similar to Lemma 6.1, for distinct a, b, r, q, p,

$$\begin{aligned}\mathbb{E}\left[R_{ab}^{(p)}R_{rq}^{(p)}\right] &= \mathbb{E}\left[R_{ab}R_{rq}^{(a)}\right] + O(N^{-\frac{3}{2}+\epsilon}) = \mathbb{E}\left[-R_{aa}\sum_{t}^{(a)}M_{at}R_{tb}^{(a)}R_{rq}^{(a)}\right] + O(N^{-\frac{3}{2}+\epsilon}) \\ &= -\frac{Js}{N}\mathbb{E}\sum_{t}^{(a)}R_{tb}^{(a)}R_{rq}^{(a)} + O(N^{-\frac{3}{2}+\epsilon}) = -\frac{Js}{N}\mathbb{E}\sum_{t}^{(p)}R_{tb}^{(p)}R_{rq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}) \\ &= -\frac{Js}{N}\mathbb{E}\sum_{t:t\neq b,r,q}^{(p)}R_{tb}^{(p)}R_{rq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}).\end{aligned} \tag{6.32}$$

Summing over a,

$$\frac{1}{N}\mathbb{E}\sum_{a:a\neq b,r,q}^{(p)}R_{ab}^{(p)}R_{rq}^{(p)} = -Js\frac{N-4}{N^2}\mathbb{E}\sum_{t:t\neq b,r,q}^{(p)}R_{tb}^{(p)}R_{rq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}) \tag{6.33}$$

for distinct b, r, q, p. Hence, after adding three $\mathcal{O}(N^{-1/2})$ terms to the sum,

$$\frac{1}{N}\mathbb{E}\sum_{a}^{(p)}R_{ab}^{(p)}R_{rq}^{(p)} = O(N^{-\frac{3}{2}+\epsilon}) \tag{6.34}$$

for distinct b, r, q, p. Using this, we find that (6.30) with the summation over all distinct a, b, r with $r \neq q$ is $O(N^{-\frac{3}{2}+\epsilon})$. Since the case when $r = q$ has the same estimate in (6.31), we find that

$$\mathbb{E}\left[\sum^{(p)}M_{pa}R_{ab}^{(p)}M_{bp}M_{pr}R_{rq}^{(p)}\right] = O(N^{-\frac{3}{2}+\epsilon}), \tag{6.35}$$

where the summation is over all distinct a, b, r.

2) When $a = b \neq r$,

$$\begin{aligned}\mathbb{E}\sum_{a\neq r}^{(p)}M_{pa}R_{aa}^{(p)}M_{ap}M_{pr}R_{rq}^{(p)} &= \frac{J}{N}\left(\frac{1}{N}+\frac{J^2}{N^2}\right)\sum_{a\neq r}^{(p)}\mathbb{E}\left[R_{aa}^{(p)}R_{rq}^{(p)}\right] \\ &= \frac{J}{N^2}\sum_{a,r}^{(p)}\mathbb{E}\left[R_{aa}^{(p)}R_{rq}^{(p)}\right] + O(N^{-\frac{3}{2}+\epsilon}) = \frac{J}{N}\mathbb{E}\left[s_N^{(p)}\sum_{r}^{(p)}R_{rq}^{(p)}\right] + O(N^{-\frac{3}{2}+\epsilon})\end{aligned} \tag{6.36}$$

where we define

$$s_N^{(p)} = \frac{1}{N}\sum_{a}^{(p)}R_{aa}^{(p)}. \tag{6.37}$$

3) When $a = r \neq b$ (or $b = r \neq a$),

$$\mathbb{E}\sum_{a\neq b}^{(p)} M_{pa}R_{ab}^{(p)}M_{bp}M_{pa}R_{aq}^{(p)} = \frac{J}{N}\left(\frac{1}{N}+\frac{J^2}{N^2}\right)\sum_{a\neq b}^{(p)}\mathbb{E}\left[R_{ab}^{(p)}R_{aq}^{(p)}\right]. \tag{6.38}$$

The part of the sum in which either $a = q$ or $b = q$ is $O(N^{-\frac{3}{2}+\epsilon})$ from naive estimate. Now for $a \neq q$,

$$\frac{1}{N}\sum_{b}^{(p,a,q)} R_{ab}^{(p)}R_{aq}^{(p)} = \frac{1}{N}\sum_{b}^{(p,a,q)} R_{ab}R_{aq}^{(b)} + \mathcal{O}(N^{-3/2}) = \frac{1}{N}\sum_{b}^{(a,q)} R_{ab}R_{aq}^{(b)} + \mathcal{O}(N^{-3/2}) \tag{6.39}$$

Following the proof of (6.14), we can check that $\mathbb{E}\left[\frac{1}{N}\sum_b^{(a,q)} R_{ba}^{(p)}R_{aq}^{(b)}\right] = O(N^{-\frac{3}{2}+\epsilon})$. (This is easy to see for a real symmetric matrix since $R_{ab} = R_{ba}$.) Thus,

$$\mathbb{E}\sum_{a\neq b}^{(p)} M_{pa}R_{ab}^{(p)}M_{bp}M_{pa}R_{aq}^{(p)} = \frac{J}{N^2}\sum_{a}^{(p,q)}\mathbb{E}\left[\sum_{b}^{(a,q)} R_{ba}^{(p)}R_{aq}^{(b)}\right] + O(N^{-\frac{3}{2}+\epsilon}) = O(N^{-\frac{3}{2}+\epsilon}). \tag{6.40}$$

4) When $a = b = r$,

$$\mathbb{E}\sum_{r}^{(p)} M_{pr}R_{rr}^{(p)}M_{rp}M_{pr}R_{rq}^{(p)} = \left(\frac{W_3}{N^{\frac{3}{2}}}+\frac{J^3}{N^3}\right)\mathbb{E}\sum_{r}^{(p)} R_{rr}^{(p)}R_{rq}^{(p)} = \frac{W_3 s}{N^{\frac{3}{2}}}\mathbb{E}\sum_{r}^{(p)} R_{rq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.41}$$

Putting the above four cases into (6.29), we find that

$$\mathbb{E}\left[(Q_p - s)\sum_{r}^{(p)} M_{pr}R_{rq}^{(p)}\right] = \frac{J}{N}\mathbb{E}\left[\left(s_N^{(p)} - s\right)\sum_{r}^{(p)} R_{rq}^{(p)}\right] + \frac{W_3 s}{N^{\frac{3}{2}}}\mathbb{E}\sum_{r}^{(p)} R_{rq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.42}$$

Note that

$$s_N^{(p)} - s = \frac{1}{N}\sum_{a}^{(p)}\left(R_{aa}^{(p)} - R_{aa}\right) - \frac{1}{N}R_{pp} + (s_N - s) = \mathcal{O}(N^{-1}). \tag{6.43}$$

Hence,

$$\mathbb{E}\left[(Q_p - s)\sum_{r}^{(p)} M_{pr}R_{rq}^{(p)}\right] = \frac{W_3 s}{N^{\frac{3}{2}}}\mathbb{E}\sum_{r}^{(p)} R_{rq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.44}$$

From (6.26), (6.27), and (6.44), for $p \neq q$,

$$\mathbb{E}\left[R_{pq}\right] = -\left(\frac{Js}{N}+\frac{W_3 s^3}{N^{\frac{3}{2}}}\right)\mathbb{E}\sum_{r}^{(p)} R_{rq}^{(p)} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.45}$$

Using (6.24) and (6.14), this implies that

$$\mathbb{E}\left[R_{pq}\right] = -\left(\frac{Js}{N}+\frac{W_3 s^3}{N^{\frac{3}{2}}}\right)\mathbb{E}\sum_{r} R_{rq} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.46}$$

From this we find that

$$\frac{1}{N}\mathbb{E}\sum_p R_{pq} = \frac{1}{N}\mathbb{E}\left[\sum_p^{(q)} R_{pq} + R_{qq}\right] = -\left(\frac{Js}{N} + \frac{W_3 s^3}{N^{\frac{3}{2}}}\right)\mathbb{E}\sum_r R_{rq} + \frac{s}{N} + O(N^{-\frac{3}{2}+\epsilon}), \tag{6.47}$$

which implies that

$$\frac{1}{N}\mathbb{E}\sum_p R_{pq} = \frac{s}{(1+Js)N} + O(N^{-\frac{3}{2}+\epsilon}). \tag{6.48}$$

Therefore, we obtain

$$\frac{J^2}{N}\mathbb{E}\sum_{p\neq q} R_{pq} = \frac{J^2}{N}\mathbb{E}\sum_{p,q} R_{pq} - \frac{J^2}{N}\mathbb{E}\sum_p R_{pp} = \frac{J^2 s}{1+Js} - J^2 s + O(N^{-\frac{1}{2}+\epsilon}). \tag{6.49}$$

We obtain from (6.13), (6.25), and (6.49) that

$$\frac{J^2}{N^2}\mathbb{E}\sum_i \sum_{p,q}^{(i)} R_{pq}^{(i)} = \frac{J^2 s}{1+Js} + O(N^{-\frac{1}{2}+\epsilon}). \tag{6.50}$$

6.1.3 Conclusion for $\sum_i \mathbb{E}(Q_i - s)$

From (6.6), (6.12), and (6.50),

$$\sum_i \mathbb{E}(Q_i - s) = -J' + b_N - \frac{s'}{s} + \frac{J^2 s}{1+Js} + O(N^{-\frac{1}{2}+\epsilon}). \tag{6.51}$$

6.2 $\sum_i \mathbb{E}(Q_i - s)^2$

We next turn to the second term in (6.5). We begin with

$$\begin{aligned}\mathbb{E}(Q_i - s)^2 =& \frac{w_2}{N} + \frac{(J')^2}{N^2} + \frac{2J's}{N} + s^2 - 2\left(s + \frac{J'}{N}\right)\mathbb{E}\sum_{p,q}^{(i)} M_{ip}R_{pq}^{(i)}M_{qi} \\ &+ \mathbb{E}\sum_{p,q,r,t}^{(i)} M_{ip}R_{pq}^{(i)}M_{qi}M_{ir}R_{rt}^{(i)}M_{ti}.\end{aligned} \tag{6.52}$$

The first sum on the right hand side satisfies

$$\mathbb{E}\sum_{p,q}^{(i)} M_{ip}R_{pq}^{(i)}M_{qi} = \frac{1}{N}\mathbb{E}\sum_p^{(i)} R_{pp}^{(i)} + \frac{J^2}{N^2}\mathbb{E}\sum_{p,q}^{(i)} R_{pq}^{(i)} = \mathbb{E}s_N^{(i)} + \frac{J^2 s}{(1+Js)N} + O(N^{-\frac{3}{2}+\epsilon}), \tag{6.53}$$

using (6.48) (applied to the Green function of an $(N-1)\times(N-1)$ matrix).

6.2.1 Computation of $\mathbb{E}\big[\sum_{p,q,r,t}^{(i)} M_{ip}R_{pq}^{(i)}M_{qi}M_{ir}R_{rt}^{(i)}M_{ti}\big]$

In order to evaluate the last term in (6.52), we consider several cases separately.

1) When p, q, r, t are all distinct,

$$\sum^{(i)} \mathbb{E}\left[M_{ip}R^{(i)}_{pq}M_{qi}M_{ir}R^{(i)}_{rt}M_{ti}\right] = \frac{J^4}{N^4}\sum^{(i)}\mathbb{E}\left[R^{(i)}_{pq}R^{(i)}_{rt}\right] = O(N^{-\frac{3}{2}+\epsilon})$$

due to (6.34). Here the sum is taken over all distinct p, q, r, t.

2) When $|\{p, q, r, t\}| = 3$:

(a) If $p = q$,

$$\mathbb{E}\left[M_{ip}R^{(i)}_{pp}M_{pi}M_{ir}R^{(i)}_{rt}M_{ti}\right] = \frac{J^2}{N^2}\left(\frac{1}{N}+\frac{J^2}{N^2}\right)\mathbb{E}\left[R^{(i)}_{pp}R^{(i)}_{rt}\right] = \frac{J^2}{N^3}\mathbb{E}\left[R_{pp}R^{(i)}_{rt}\right] + O(N^{-\frac{9}{2}+\epsilon}).$$

Thus, using (5.7) and (6.49), we find that

$$\begin{aligned}\sum^{(i)}\mathbb{E}\left[M_{ip}R^{(i)}_{pp}M_{pi}M_{ir}R^{(i)}_{rt}M_{ti}\right] &= \frac{J^2}{N^2}\sum^{(i)}_{r\neq t}\mathbb{E}\left[s_N R^{(i)}_{rt}\right] + O(N^{-\frac{3}{2}+\epsilon}) \\ &= \frac{J^2 s}{N^2}\sum^{(i)}_{r\neq t}\mathbb{E}\left[R^{(i)}_{rt}\right] + O(N^{-\frac{3}{2}+\epsilon}) = -\frac{J^3 s^3}{(1+Js)N} + O(N^{-\frac{3}{2}+\epsilon}).\end{aligned} \tag{6.54}$$

where the first sum is over all distinct p, r, t.

(b) If $r = t$, the calculation is the same as the above.

(c) Other cases have negligible contributions, i.e., bounded by $N^{-\frac{3}{2}+\epsilon}$, due to unmatching off-diagonal terms using (6.14) and the derivation is similar to that of (6.40).

3) When $|\{p, q, r, t\}| = 2$:

(a) If there is a triplet, e.g., $p = q = r$, the contribution is $O(N^{-\frac{3}{2}+\epsilon})$. For example,

$$\begin{aligned}\mathbb{E}\sum^{(i)}_{p\neq t} M_{ip}R^{(i)}_{pp}M_{pi}M_{ip}R^{(i)}_{pt}M_{ti} &= \frac{J}{N}\left(\frac{W_3}{N^{\frac{3}{2}}}+\frac{J^3}{N^3}\right)\mathbb{E}\sum^{(i)}_{p\neq t}R^{(i)}_{pp}R^{(i)}_{pt} \\ &= \frac{W_3 Js}{N^{\frac{5}{2}}}\mathbb{E}\sum^{(i)}_{p\neq t}R^{(i)}_{pt} + O(N^{-\frac{3}{2}+\epsilon}) = O(N^{-\frac{3}{2}+\epsilon}),\end{aligned} \tag{6.55}$$

where we used (6.49).

(b) If $p = q$ and $r = t$,

$$\begin{aligned}&\mathbb{E}\sum^{(i)}_{p\neq r} M_{ip}R^{(i)}_{pp}M_{pi}M_{ir}R^{(i)}_{rr}M_{ri} = \left(\frac{1}{N}+\frac{J^2}{N^2}\right)^2\mathbb{E}\sum^{(i)}_{p\neq r}R^{(i)}_{pp}R^{(i)}_{rr} \\ &= \left(\frac{1}{N}+\frac{J^2}{N^2}\right)^2\mathbb{E}\sum^{(i)}_{p}R^{(i)}_{pp}\left(Ns^{(i)}_N - R^{(i)}_{pp}\right) = \mathbb{E}\left(s^{(i)}_N\right)^2 - \frac{s^2}{N} + \frac{2J^2s^2}{N} + O(N^{-\frac{3}{2}+\epsilon}).\end{aligned} \tag{6.56}$$

(c) If $p=t$ and $q=r$,

$$\begin{aligned}\mathbb{E}\sum_{p\neq q}^{(i)} M_{ip}R^{(i)}_{pq}M_{qi}M_{iq}R^{(i)}_{qp}M_{pi} &= \left(\frac{1}{N}+\frac{J^2}{N^2}\right)^2 \mathbb{E}\sum_{p\neq q}^{(i)} R^{(i)}_{pq}R^{(i)}_{qp} \\ &= \left(\frac{1}{N}+\frac{J^2}{N^2}\right)^2 \left[\mathbb{E}\,\mathrm{Tr}(R^{(i)})^2 - \mathbb{E}\sum_{p}^{(i)}(R^{(i)}_{pp})^2\right] = \frac{s'}{N}-\frac{s^2}{N}+O(N^{-\frac{3}{2}+\epsilon}),\end{aligned} \tag{6.57}$$

where we used (6.10) (applied to an $(N-1)\times(N-1)$ matrix).

(d) If $p=r$ and $q=t$, the expectation $\mathbb{E}[M_{ip}R^{(i)}_{pq}M_{qi}M_{ip}R^{(i)}_{pq}M_{qi}]$ is negligible when M is complex Hermitian. When M is real symmetric, the calculation is the same as the above, since R is also symmetric and the contribution is

$$\frac{s'}{N}-\frac{s^2}{N}+O(N^{-\frac{3}{2}+\epsilon}). \tag{6.58}$$

4) When $p=q=r=t$,

$$\mathbb{E}\sum_{p}^{(i)} M_{ip}R^{(i)}_{pp}M_{pi}M_{ip}R^{(i)}_{pp}M_{pi} = \frac{W_4 s^2}{N}+O(N^{-\frac{3}{2}+\epsilon}). \tag{6.59}$$

Combining all cases together, we obtain

$$\begin{aligned}&\mathbb{E}\sum_{p,q,r,t}^{(i)} M_{ip}R^{(i)}_{pq}M_{qi}M_{ir}R^{(i)}_{rt}M_{ti} \\ &= -\frac{2J^3s^3}{(1+Js)N}+\mathbb{E}\left(s^{(i)}_N\right)^2-\frac{s^2}{N}+\frac{2J^2s^2}{N}+\frac{2s'}{N}-\frac{2s^2}{N}+\frac{W_4s^2}{N}+O(N^{-\frac{3}{2}+\epsilon})\end{aligned} \tag{6.60}$$

(when M is real symmetric. For complex Hermitian M, we have $\frac{s'}{N}-\frac{s^2}{N}$ instead of $\frac{2s'}{N}-\frac{2s^2}{N}$.)

6.2.2 Conclusion for $\sum_i \mathbb{E}\,(Q_i-s)^2$

From (6.52), (6.53), and (6.60),

$$\begin{aligned}\mathbb{E}\,(Q_i-s)^2 =& s^2+\mathbb{E}\left(s^{(i)}_N\right)^2-2\left(s+\frac{J'}{N}\right)\mathbb{E}s^{(i)}_N+\frac{2J's}{N} \\ &+\frac{1}{N}\left(w_2-3s^2+2s'+W_4s^2\right)+O(N^{-\frac{3}{2}+\epsilon}).\end{aligned} \tag{6.61}$$

Using $|s^{(i)}_N-s|\prec N^{-1}$ and summing over i, we obtain

$$\sum_i \mathbb{E}\,(Q_i-s)^2 = w_2-3s^2+2s'+W_4s^2+O(N^{-\frac{1}{2}+\epsilon}). \tag{6.62}$$

6.3 Formula of b_N

Inserting (6.51) and (6.62) into (6.5), we obtain

$$b_N = -s^2J'-s's+b_Ns^2+\frac{J^2s^3}{1+Js}+w_2s^3+2s's^3+(W_4-3)\,s^5+O(N^{-\frac{1}{2}+\epsilon}). \tag{6.63}$$

Therefore,

$$b_N = \frac{s^2}{1-s^2}\left(-J' - \frac{s'}{s} + \frac{J^2 s}{1+Js} + w_2 s + 2s's + (W_4-3)\,s^3\right) + O(N^{-\frac{1}{2}+\epsilon}). \tag{6.64}$$

Using the algebraic identity $s' = \frac{s^2}{1-s^2}$, we can express

$$b_N = \frac{s^2}{1-s^2}\left(-J' + \frac{J^2 s}{1+Js} + (w_2-1)s + s's + (W_4-3)\,s^3\right) + O(N^{-\frac{1}{2}+\epsilon}). \tag{6.65}$$

This converges to $b(z)$ in Proposition 4.1. We remark that, when $J' = J = 0$, this reduces to

$$b = (1+s')s^3\left((w_2-1) + s' + (W_4-3)\,s^2\right), \tag{6.66}$$

which is the same as Proposition 3.1 of [4].

7 The covariance function

7.1 Martingale decomposition

Following [4], we consider the filtration

$$\mathcal{F}_k = \sigma(M_{ij},\, k < i, j \le N) \tag{7.1}$$

for $k = 0, 1, \dots, N$ and the conditional expectation

$$\mathbb{E}_k(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_k). \tag{7.2}$$

Recall that

$$\zeta_N = \xi_N - \mathbb{E}\xi_N = \operatorname{Tr} R - \mathbb{E}\operatorname{Tr} R. \tag{7.3}$$

We use the following martingale decomposition:

$$\zeta_N = \sum_{k=1}^{N}(\mathbb{E}_{k-1}\operatorname{Tr} R - \mathbb{E}_k \operatorname{Tr} R) = \sum_{k=1}^{N}(\mathbb{E}_{k-1} - \mathbb{E}_k)\operatorname{Tr} R = \sum_{k=1}^{N}(\mathbb{E}_{k-1} - \mathbb{E}_k)(\operatorname{Tr} R - \operatorname{Tr} R^{(k)}). \tag{7.4}$$

From (5.11) and (5.10),

$$\operatorname{Tr} R - \operatorname{Tr} R^{(k)} = R_{kk} + \sum_{i}^{(k)} \frac{R_{ik}R_{ki}}{R_{kk}} = R_{kk} + \sum_{i}^{(k)} R_{kk} \sum_{p,q}^{(k)} M_{kp} R^{(k)}_{pi} R^{(k)}_{iq} M_{qk}. \tag{7.5}$$

Hence

$$\zeta_N = \sum_{k=1}^{N}(\mathbb{E}_{k-1} - \mathbb{E}_k)\left[R_{kk}\left(1 + \sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq} M_{qk}\right)\right]. \tag{7.6}$$

As in the previous section, we expand R_{kk} using Schur formula. Since

$$\left|\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq} M_{qk}\right| \prec 1 \tag{7.7}$$

from (5.14), it is tempting to speculate that one needs to expand R_{kk} up to third order term, i.e., up to the term of order N^{-1}. However, for any random variables X_k and X_ℓ with $k > \ell$ adapted to the filtration,

$$\begin{aligned} &\mathbb{E}\left[(\mathbb{E}_{k-1}-\mathbb{E}_k)X_k \cdot (\mathbb{E}_{\ell-1}-\mathbb{E}_\ell)\overline{X}_\ell\right] = \mathbb{E}\left[\mathbb{E}_{k-1}[(\mathbb{E}_{k-1}-\mathbb{E}_k)X_k \cdot (\mathbb{E}_{\ell-1}-\mathbb{E}_\ell)\overline{X}_\ell]\right] \\ &= \mathbb{E}\left[(\mathbb{E}_{k-1}-\mathbb{E}_k)X_k \cdot \mathbb{E}_{k-1}[(\mathbb{E}_{\ell-1}-\mathbb{E}_\ell)\overline{X}_\ell]\right] = 0. \end{aligned} \tag{7.8}$$

Thus,

$$\mathbb{E}\left|\sum_{k=1}^N (\mathbb{E}_{k-1}-\mathbb{E}_k)X_k\right|^2 = \mathbb{E}\sum_{k=1}^N \left|(\mathbb{E}_{k-1}-\mathbb{E}_k)X_k\right|^2. \tag{7.9}$$

This implies, in particular, that if a random variable $Y_k = \mathcal{O}(N^{-1})$, then

$$\sum_{k=1}^N (\mathbb{E}_{k-1}-\mathbb{E}_k)(X_k+Y_k) = \sum_{k=1}^N (\mathbb{E}_{k-1}-\mathbb{E}_k)X_k + \mathcal{O}_p(N^{-\frac{1}{2}}), \tag{7.10}$$

where $\mathcal{O}_p(N^{-\frac{1}{2}})$ means that the other terms are bounded by $N^{-\frac{1}{2}+\epsilon}$ in probability. Applying the argument to the expansion (6.3) of R_{kk} in (7.6), we find that

$$\begin{aligned} \zeta_N = s\sum_{k=1}^N (\mathbb{E}_{k-1}-\mathbb{E}_k)\left[\left(1+\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right)\right] \\ + s^2\sum_{k=1}^N (\mathbb{E}_{k-1}-\mathbb{E}_k)\left[(Q_k-s)\left(1+\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right)\right] + \mathcal{O}_p(N^{-\frac{1}{2}}). \end{aligned} \tag{7.11}$$

where $Q_k = -M_{kk} + \sum_{r,t}^{(k)} M_{kr}R^{(k)}_{rt}M_{tk}$ as in (6.2).

7.1.1 First term

The first term on the right hand side of (7.11) is given by

$$\begin{aligned} &(\mathbb{E}_{k-1}-\mathbb{E}_k)\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk} \\ &= \mathbb{E}_{k-1}\left[\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right] - \mathbb{E}_{k-1}\left[\frac{J^2}{N^2}\sum_{p,q}^{(k)}(R^{(k)})^2_{pq} + \frac{1}{N}\sum_{p}^{(k)}(R^{(k)})^2_{pp}\right] \\ &= \mathbb{E}_{k-1}\left[\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk} - \frac{J^2}{N^2}\sum_{p,q}^{(k)}(R^{(k)})^2_{pq} - s'\right] + \mathcal{O}(N^{-1}). \end{aligned} \tag{7.12}$$

This corresponds to b_k of [4].

7.1.2 Second term

In order to compute the second term in the right hand side of (7.11), note that

$$|Q_k - s|\left|\left(1+\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right) - \left(1+\frac{1}{N}\sum_{p}^{(k)}(R^{(k)})^2_{pp}\right)\right| \prec \frac{1}{N}. \tag{7.13}$$

from (6.4) and (5.14), since $\|R^{(k)}\| \leq \frac{1}{\operatorname{Im} z}$ and $z \in \mathcal{K}$. Thus, the summand in the second term is given by

$$(\mathbb{E}_{k-1} - \mathbb{E}_k) \left[(Q_k - s) \left(1 + \frac{1}{N} \sum_p^{(k)} (R^{(k)})^2_{pp} \right) \right] + \mathcal{O}(N^{-1}). \tag{7.14}$$

Now

$$\begin{aligned}
&\mathbb{E}_k \left[(Q_k - s) \left(1 + \frac{1}{N} \sum_p^{(k)} (R^{(k)})^2_{pp} \right) \right] \\
&= \mathbb{E}_{k-1} \left[\left(-\frac{J'}{N} + \frac{J^2}{N^2} \sum_{r,t}^{(k)} R^{(k)}_{rt} + \frac{1}{N} \sum_r^{(k)} R^{(k)}_{rr} - s \right) \left(1 + \frac{1}{N} \sum_p^{(k)} (R^{(k)})^2_{pp} \right) \right] \\
&= \mathbb{E}_{k-1} \left[\left(\frac{J^2}{N^2} \sum_{r,t}^{(k)} R^{(k)}_{rt} \right) \left(1 + \frac{1}{N} \sum_p^{(k)} (R^{(k)})^2_{pp} \right) \right] + \mathcal{O}(N^{-1}).
\end{aligned} \tag{7.15}$$

Hence, (7.14) becomes

$$\mathbb{E}_{k-1} \left(-M_{kk} + \sum_{r,t}^{(k)} M_{kr} R^{(k)}_{rt} M_{tk} - \frac{J^2}{N^2} \sum_{r,t}^{(k)} R^{(k)}_{rt} - s \right) (1 + s') + \mathcal{O}(N^{-1}). \tag{7.16}$$

7.1.3 Simplified formula of the martingale decomposition

From (7.11), (7.12), and (7.16), we find that

$$\zeta_N = \sum_{k=1}^N \mathbb{E}_{k-1} \phi_k + \mathcal{O}_p(N^{-\frac{1}{2}}), \tag{7.17}$$

where

$$\begin{aligned}
\phi_k := s &\left(\sum_{p,q}^{(k)} M_{kp} (R^{(k)})^2_{pq} M_{qk} - \frac{J^2}{N^2} \sum_{p,q}^{(k)} (R^{(k)})^2_{pq} - s' \right) \\
&+ s^2 (1 + s') \left(-M_{kk} + \sum_{p,q}^{(k)} M_{kp} R^{(k)}_{pq} M_{qk} - \frac{J^2}{N^2} \sum_{p,q}^{(k)} R^{(k)}_{pq} - s \right).
\end{aligned} \tag{7.18}$$

Since $\frac{\mathrm{d}}{\mathrm{d}z} R^{(k)} = (R^{(k)})^2$ and $s' = s^2(1 + s')$, this can also be written as

$$\phi_k = \frac{\partial}{\partial z} \left[s \left(-M_{kk} + \sum_{p,q}^{(k)} M_{kp} R^{(k)}_{pq} M_{qk} - \frac{J^2}{N^2} \sum_{p,q}^{(k)} R^{(k)}_{pq} - s \right) \right]. \tag{7.19}$$

Note that $\phi_k \prec N^{-\frac{1}{2}}$.

7.2 Covariance

Let $z_1, z_2, \ldots, z_p$ are p distinct points in $\mathcal{K}$. In order to prove the finite dimensional convergence of ξ_N, it suffices to show that the random vector $(\zeta_N(z_1), \zeta_N(z_2), \ldots, \zeta_N(z_p))$ converges weakly to a p-dimensional mean-zero Gaussian distribution with the covariance matrix $\Gamma(z_i, z_j)$ defined in (4.11). To prove it, we use the martingale CLT for $\sum_k \mathbb{E}_{k-1}\phi_k$.

Let z_1 and z_2 be two distinct points in $\mathcal{K}$. Following [4], we consider

$$\Gamma_N(z_1, z_2) = \sum_{k=1}^{N} \mathbb{E}_k \left[\mathbb{E}_{k-1}[\phi_k(z_1)] \cdot \mathbb{E}_{k-1}[\phi_k(z_2)]\right]. \tag{7.20}$$

For simplicity, we introduce the notations

$$s_1 = s(z_1), \qquad s_2 = s(z_2). \tag{7.21}$$

Let

$$\begin{aligned}\widetilde{\Gamma}_N(z_1, z_2) = \sum_{k=1}^{N} \mathbb{E}_k \Bigg[\mathbb{E}_{k-1}\Bigg[-M_{kk} + \sum_{p,q}^{(k)} M_{kp} R_{pq}^{(k)}(z_1) M_{qk} - \frac{J^2}{N^2} \sum_{p,q}^{(k)} R_{pq}^{(k)}(z_1) - s_1\Bigg] \\ \times \mathbb{E}_{k-1}\Bigg[-M_{kk} + \sum_{p,q}^{(k)} M_{kp} R_{pq}^{(k)}(z_2) M_{qk} - \frac{J^2}{N^2} \sum_{p,q}^{(k)} R_{pq}^{(k)}(z_2) - s_2\Bigg]\Bigg]\end{aligned} \tag{7.22}$$

so that

$$\Gamma_N(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \left[s_1 s_2 \widetilde{\Gamma}_N(z_1, z_2)\right]. \tag{7.23}$$

7.2.1 The easy parts of $\widetilde{\Gamma}_N(z_1, z_2)$

We now find the limit of $\widetilde{\Gamma}_N(z_1, z_2)$. In order to simplify notations, let us write

$$S_k(z) := \sum_{p,q}^{(k)} M_{kp} R_{pq}^{(k)}(z) M_{qk}, \qquad T_k(z) := \frac{J^2}{N^2} \sum_{p,q}^{(k)} R_{pq}^{(k)}(z). \tag{7.24}$$

Then a summand in the formula of $\widetilde{\Gamma}_N(z_1, z_2)$ is

$$\mathbb{E}_k \left[\mathbb{E}_{k-1}\left[-M_{kk} + S_k(z_1) - T_k(z_1) - s_1\right] \cdot \mathbb{E}_{k-1}\left[-M_{kk} + S_k(z_2) - T_k(z_2) - s_2\right]\right]. \tag{7.25}$$

We first estimate $S_k(z) - T_k(z) - s(z)$. By definition,

$$S_k(z) - T_k(z) = \sum_{p,q}^{(k)} A_{kp} R_{pq}^{(k)}(z) A_{qk} + \frac{J}{N} \sum_{p,q}^{(k)} A_{kp} R_{pq}^{(k)}(z) + \frac{J}{N} \sum_{p,q}^{(k)} R_{pq}^{(k)}(z) A_{qk}. \tag{7.26}$$

Using Lemma 5.3 with $J = 0$ (or the second part and the third part of Lemma 10.1), we find that

$$\left|\sum_{p,q}^{(k)} A_{kp} R_{pq}^{(k)}(z) A_{qk} - \frac{1}{N} \sum_{p}^{(k)} R_{pp}^{(k)}(z)\right| \prec \frac{\|R^{(k)}\|}{\sqrt{N}}. \tag{7.27}$$

Moreover, from the first part of Lemma 10.1,

$$\frac{1}{N}\left|\sum_{p,q}^{(k)} A_{kp}R^{(k)}_{pq}(z)\right| \prec \frac{1}{N}\sum_{q}^{(k)}\left(\frac{1}{N}\sum_{p}^{(k)}|R^{(k)}_{pq}(z)|^2\right)^{\frac{1}{2}} \le \left(\sum_{q}^{(k)}\frac{1}{N}\sum_{p}^{(k)}|R^{(k)}_{pq}(z)|^2\right)^{\frac{1}{2}} = \frac{\|R^{(k)}\|}{\sqrt{N}}. \tag{7.28}$$

Since $\|R^{(k)}\| \le \frac{1}{\operatorname{Im} z}$ and $z \in \mathcal{K}$, and $|s^{(k)}(z) - s(z)| \prec N^{-1}$, we obtain that

$$|S_k(z) - T_k(z) - s(z)| \prec N^{-\frac{1}{2}}. \tag{7.29}$$

Now we consider (7.25). We note that

$$\mathbb{E}_k\left(\mathbb{E}_{k-1}[M_{kk}]\right)^2 = \frac{w_2}{N} + O(N^{-2}) \tag{7.30}$$

and

$$\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[M_{kk}\right]\cdot\mathbb{E}_{k-1}\left[S_k(z_2) - T_k(z_2) - s_2\right]\right] = \mathcal{O}(N^{-\frac{3}{2}}). \tag{7.31}$$

We also have

$$\begin{aligned}
&\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[S_k(z_1)\right]\cdot\mathbb{E}_{k-1}\left[T_k(z_2) + s_2\right]\right] \\
&= \sum_{p,q}^{(k)}\mathbb{E}[M_{kp}M_{qk}]\cdot\mathbb{E}_k\left[\mathbb{E}_{k-1}[R^{(k)}_{pq}(z_1)]\cdot\mathbb{E}_{k-1}\left[\frac{J^2}{N^2}\sum_{r,t}^{(k)}R^{(k)}_{rt}(z_2) + s_2\right]\right] \\
&= \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[\frac{J^2}{N^2}\sum_{p,q}^{(k)}R^{(k)}_{pq}(z_1) + s^{(k)}_N(z_1)\right]\cdot\mathbb{E}_{k-1}\left[\frac{J^2}{N^2}\sum_{r,t}^{(k)}R^{(k)}_{rt}(z_2) + s_2\right]\right] \\
&= \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[T_k(z_1) + s^{(k)}_N(z_1)\right]\cdot\mathbb{E}_{k-1}\left[T_k(z_2) + s_2\right]\right].
\end{aligned} \tag{7.32}$$

Similar estimates hold if z_2 in (7.31) and (7.32) is replaced by z_1. Noting the similarity of the formula of (7.32) with $\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[T_k(z_1) + s_1\right]\cdot\mathbb{E}_{k-1}\left[T_k(z_2) + s_2\right]\right]$, (7.25) becomes

$$\begin{aligned}
&\frac{w_2}{N} + \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[S_k(z_1)\right]\cdot\mathbb{E}_{k-1}\left[S_k(z_2)\right]\right] - \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[T_k(z_1) + s^{(k)}_N(z_1)\right]\cdot\mathbb{E}_{k-1}\left[T_k(z_2) + s^{(k)}_N(z_2)\right]\right] \\
&+ \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[s_1 - s^{(k)}_N(z_1)\right]\cdot\mathbb{E}_{k-1}\left[s_2 - s^{(k)}_N(z_2)\right]\right] + \mathcal{O}(N^{-\frac{3}{2}}).
\end{aligned} \tag{7.33}$$

7.2.2 $\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[S_k(z_1)\right]\cdot\mathbb{E}_{k-1}\left[S_k(z_2)\right]\right]$

We compute

$$\begin{aligned}
&\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[S_k(z_1)\right]\cdot\mathbb{E}_{k-1}\left[S_k(z_2)\right]\right] \\
&= \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[\sum_{p,q}^{(k)}\left(A_{kp} + \frac{J}{N}\right)R^{(k)}_{pq}(z_1)\left(A_{qk} + \frac{J}{N}\right)\right]\cdot\mathbb{E}_{k-1}\left[\sum_{r,t}^{(k)}\left(A_{kr} + \frac{J}{N}\right)R^{(k)}_{rt}(z_2)\left(A_{tk} + \frac{J}{N}\right)\right]\right].
\end{aligned} \tag{7.34}$$

We rearrange it in descending order of J and calculate the conditional expectations.

1) For J^4-terms, we get

$$\frac{J^4}{N^4}\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[\sum_{p,q}^{(k)} R_{pq}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[\sum_{r,t}^{(k)} R_{rt}^{(k)}(z_2)\right]\right] = \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[T_k(z_1)\right]\cdot\mathbb{E}_{k-1}\left[T_k(z_2)\right]\right].$$

2) For J^3-terms, the conditional expectation vanishes because it always contains a factor $\mathbb{E}[A_{k\cdot}]$ or $\mathbb{E}[A_{\cdot k}]$.

3) For J^2-terms, we get

$$\begin{aligned}&\frac{J^2}{N^2}\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[s_N^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[\sum_{r,t}^{(k)} R_{rt}^{(k)}(z_2)\right]+\mathbb{E}_{k-1}\left[\sum_{p,q}^{(k)} R_{pq}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[s_N^{(k)}(z_2)\right]\right]\\&=\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[s_N^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[T_k(z_2)\right]+\mathbb{E}_{k-1}\left[T_k(z_1)\right]\cdot\mathbb{E}_{k-1}\left[s_N^{(k)}(z_2)\right]\right].\end{aligned}$$

We also have other terms, but they are all negligible, i.e., of order $\mathcal{O}(N^{-\frac{3}{2}})$. (After summing over k, the contribution from such terms will be $N^{-\frac{1}{2}}$.) For example, consider

$$\begin{aligned}X_k &= \frac{J^2}{N^2}\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[\sum_{p,q}^{(k)} R_{pq}^{(k)}(z_1)A_{qk}\right]\cdot\mathbb{E}_{k-1}\left[\sum_{r,t}^{(k)} A_{kr}R_{rt}^{(k)}(z_2)\right]\right]\\&=\frac{J^2}{N^3}\mathbb{E}_k\left[\sum_{q:q>k}\mathbb{E}_{k-1}\left[\sum_{p}^{(k)} R_{pq}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[\sum_{t}^{(k)} R_{qt}^{(k)}(z_2)\right]\right].\end{aligned} \tag{7.35}$$

By naive power counting, we see that $X_k = \mathcal{O}(N^{-1})$. Since the contribution from the case $p = q$ is $\mathcal{O}(N^{-\frac{3}{2}})$, we may assume that $p \neq q$. Expanding $R_{pq}^{(k)}$ by

$$R_{pq}^{(k)}(z_1) = -R_{pp}^{(k)}(z_1)\sum_{a}^{(k,p)} M_{pa}R_{aq}^{(k,p)}(z_1) = -s_1\sum_{a}^{(k,p)} M_{pa}R_{aq}^{(k,p)}(z_1)+\mathcal{O}(N^{-1}), \tag{7.36}$$

we obtain that

$$\begin{aligned}X_k &= \frac{-J^2 s_1}{N^3}\mathbb{E}_k\left[\sum_{q:q>k}\mathbb{E}_{k-1}\left[\sum_{p}^{(k)}\sum_{a}^{(k,p)} M_{pa}R_{aq}^{(k,p)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[\sum_{t}^{(k)} R_{qt}^{(k)}(z_2)\right]\right]+\mathcal{O}(N^{-\frac{3}{2}})\\&=\frac{-J^3 s_1}{N^4}\mathbb{E}_k\left[\sum_{q:q>k}\mathbb{E}_{k-1}\left[\sum_{p}^{(k)}\sum_{a}^{(k,p)} R_{aq}^{(k,p)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[\sum_{t}^{(k)} R_{qt}^{(k)}(z_2)\right]\right]+\mathcal{O}(N^{-\frac{3}{2}})\\&=\frac{-J^3 s_1}{N^4}\mathbb{E}_k\left[\sum_{q:q>k}\mathbb{E}_{k-1}\left[\sum_{p}^{(k)}\sum_{a}^{(k)} R_{aq}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[\sum_{t}^{(k)} R_{qt}^{(k)}(z_2)\right]\right]+\mathcal{O}(N^{-\frac{3}{2}})\\&=-Js_1X_k+\mathcal{O}(N^{-\frac{3}{2}}).\end{aligned} \tag{7.37}$$

Hence, $X_k = \mathcal{O}(N^{-\frac{3}{2}})$, which is negligible.

4) The J-terms can be computed as in the previous case and find that the contribution is negligible, i.e., $\mathcal{O}(N^{-\frac{3}{2}})$. Since the computation is similar to the previous case, we skip the proof.

5) For the terms with no J, the conditional expectation vanishes unless $|\{p,q,r,t\}| = 2$ or $p = q = r = t$.

(a) If $p = q \neq r = t$, we get

$$\begin{aligned} &\frac{1}{N^2}\mathbb{E}_k\left[\sum_{p\neq r}^{(k)}\mathbb{E}_{k-1}\left[R_{pp}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[R_{rr}^{(k)}(z_2)\right]\right] \\ &= \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[s_N^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[s_N^{(k)}(z_2)\right]\right] - \frac{s_1 s_2}{N} + \mathcal{O}(N^{-2}). \end{aligned} \tag{7.38}$$

(b) If $p = q = r = t$, we get

$$\begin{aligned} &\mathbb{E}_k\left[\sum_{p}^{(k)}\mathbb{E}_{k-1}\left[A_{kp}R_{pp}^{(k)}(z_1)A_{pk}\right]\cdot\mathbb{E}_{k-1}\left[A_{kp}R_{pp}^{(k)}(z_2)A_{pk}\right]\right] \\ &= \sum_{p:p<k}\frac{s_1 s_2}{N^2} + \sum_{p:p>k}\frac{W_4 s_1 s_2}{N^2} + \mathcal{O}(N^{-2}) = \frac{k}{N}\frac{s_1 s_2}{N} + \frac{N-k}{N}\frac{W_4 s_1 s_2}{N} + \mathcal{O}(N^{-2}). \end{aligned} \tag{7.39}$$

(c) If $p = t \neq q = r$, we get

$$\begin{aligned} &\mathbb{E}_k\left[\sum_{p\neq q}^{(k)}\mathbb{E}_{k-1}\left[A_{kp}R_{pq}^{(k)}(z_1)A_{qk}\right]\cdot\mathbb{E}_{k-1}\left[A_{kq}R_{qp}^{(k)}(z_2)A_{pk}\right]\right] \\ &= \mathbb{E}_k\left[\sum_{p,q:p,q>k,p\neq q}(A_{kp}A_{qk})^2\cdot\mathbb{E}_{k-1}\left[R_{pq}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[R_{qp}^{(k)}(z_2)\right]\right] \\ &= \frac{1}{N^2}\mathbb{E}_k\left[\sum_{p,q:p,q>k,p\neq q}\mathbb{E}_{k-1}\left[R_{pq}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[R_{qp}^{(k)}(z_2)\right]\right] =: Y_k. \end{aligned} \tag{7.40}$$

We note that $Y_k = \mathcal{O}(N^{-1})$. The idea in the estimate for Y_k is similar to that for X_k, except that we expand both $R_{pq}^{(k)}(z_1)$ and $R_{qp}^{(k)}(z_2)$. Then,

$$\begin{aligned} Y_k &= \frac{s_1 s_2}{N^2}\mathbb{E}_k\left[\sum_{p,q:p,q>k,p\neq q}\sum_{a,b}^{(k,p)}\mathbb{E}_{k-1}\left[M_{pa}R_{aq}^{(k,p)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[R_{qb}^{(k,p)}(z_2)M_{bp}\right]\right] + \mathcal{O}(N^{-\frac{3}{2}}) \\ &= \frac{s_1 s_2}{N^3}\mathbb{E}_k\left[\sum_{p,q:p,q>k,p\neq q}\sum_{a:a>k}^{(p)}\mathbb{E}_{k-1}\left[R_{aq}^{(k,p)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[R_{qa}^{(k,p)}(z_2)\right]\right] + \mathcal{O}(N^{-\frac{3}{2}}) \\ &= \frac{s_1 s_2}{N^3}\mathbb{E}_k\left[\sum_{p,q:p,q>k,p\neq q}\sum_{a:a>k}\mathbb{E}_{k-1}\left[R_{aq}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[R_{qa}^{(k)}(z_2)\right]\right] + \mathcal{O}(N^{-\frac{3}{2}}) \\ &= \frac{N-k-1}{N}\frac{s_1 s_2}{N^2}\mathbb{E}_k\left[\sum_{a,q:a,q>k}\mathbb{E}_{k-1}\left[R_{aq}^{(k)}(z_1)\right]\cdot\mathbb{E}_{k-1}\left[R_{qa}^{(k)}(z_2)\right]\right] + \mathcal{O}(N^{-\frac{3}{2}}). \end{aligned} \tag{7.41}$$

Thus, writing the last sum for $a \neq q$ and $a = q$ separately, we find that

$$Y_k = \frac{N-k-1}{N}s_1 s_2 Y_k + \frac{(N-k-1)(N-k)}{N^3}(s_1 s_2)^2 + \mathcal{O}(N^{-\frac{3}{2}}), \tag{7.42}$$

and we obtain that

$$Y_k = \left(1 - \frac{N-k}{N} s_1 s_2\right)^{-1} \frac{(N-k)^2}{N^3} (s_1 s_2)^2 + \mathcal{O}(N^{-\frac{3}{2}}). \tag{7.43}$$

(d) If $p = r \neq q = t$, the conditional expectation is the same as Y_k in the previous case. (When A is complex Hermitian, it vanishes.)

Altogether, we obtain that

$$\begin{aligned}&\mathbb{E}_k\left[\mathbb{E}_{k-1}\left[S_k(z_1)\right] \cdot \mathbb{E}_{k-1}\left[S_k(z_2)\right]\right] = \mathbb{E}_k\left[\mathbb{E}_{k-1}\left[T_k(z_1) + s_N^{(k)}(z_1)\right] \cdot \mathbb{E}_{k-1}\left[T_k(z_2) + s_N^{(k)}(z_2)\right]\right] \\ &- \frac{s_1 s_2}{N} + \frac{k}{N}\frac{s_1 s_2}{N} + \frac{N-k}{N}\frac{W_4 s_1 s_2}{N} + 2\left(1 - \frac{N-k}{N} s_1 s_2\right)^{-1} \frac{(N-k)^2}{N^3} (s_1 s_2)^2 + \mathcal{O}(N^{-\frac{3}{2}}).\end{aligned} \tag{7.44}$$

7.2.3 Conclusion for $\widetilde{\Gamma}_N(z_1, z_2)$ and $\Gamma_N(z_1, z_2)$

Combining (7.33) and (7.44), we find that (7.25) is equal to

$$\frac{w_2}{N} - \frac{s_1 s_2}{N} + \frac{k}{N}\frac{s_1 s_2}{N} + \frac{N-k}{N}\frac{W_4 s_1 s_2}{N} + 2\left(1 - \frac{N-k}{N} s_1 s_2\right)^{-1} \frac{(N-k)^2}{N^3} (s_1 s_2)^2 + \mathcal{O}(N^{-\frac{3}{2}}). \tag{7.45}$$

Summing over k, we obtain from (7.22) that

$$\widetilde{\Gamma}_N(z_1, z_2) = w_2 - 1 + \frac{1}{2}\left(W_4 - 3\right) s_1 s_2 - \frac{2\log(1 - s_1 s_2)}{s_1 s_2} + \mathcal{O}(N^{-\frac{1}{2}}), \tag{7.46}$$

where we used

$$\int_0^1 \frac{a^2 x^2}{1 - ax} \mathrm{d}x = -\frac{a}{2} - 1 - \frac{\log(1-a)}{a} \tag{7.47}$$

for $a \in \mathbb{C} \setminus [1, \infty)$. To check that $s_1 s_2 \in \mathbb{C} \setminus [1, \infty)$, we notice that $\operatorname{Im} s_1(z), \operatorname{Im} s_2(z) > 0$ for $z \in \mathcal{K}$. If $(\operatorname{Re} s_1)(\operatorname{Re} s_2) > 0$, $\operatorname{Im}(s_1 s_2) \neq 0$. If $(\operatorname{Re} s_1)(\operatorname{Re} s_2) \leq 0$, $\operatorname{Re}(s_1 s_2) < 0$. Thus, in any case, $s_1 s_2 \in \mathbb{C} \setminus [1, \infty)$. Therefore,

$$\begin{aligned}\Gamma_N(z_1, z_2) &= \frac{\partial^2}{\partial z_1 \partial z_2}\left[s_1 s_2 \widetilde{\Gamma}_N(z_1, z_2)\right] \\ &= s_1' s_2' \left((w_2 - 1) + 2\left(W_4 - 3\right) s_1 s_2 + \frac{2}{(1 - s_1 s_2)^2}\right) + \mathcal{O}(N^{-\frac{1}{2}}),\end{aligned} \tag{7.48}$$

which converges to $\Gamma(z_1, z_2)$ in probability.

8 Proof of Proposition 4.1

We conclude the proof of Proposition 4.1 by establishing the (a) the finite-dimensional convergence to Gaussian vectors and (b) the tightness of $\xi_N(z)$, as discussed in Section 5.

8.1 Finite-dimensional convergence

To prove the finite-dimensional convergence, we use Theorem 35.12 of [8] for Martingale central limit theorem. Recall the definition of ϕ_k in (7.17) and (7.18). Since we already proved the convergence of the variance in the previous section, it suffices to check that

$$\sum_{k=1}^{N} \mathbb{E}\left[|\mathbb{E}_{k-1}[\phi_k]|^2 \chi_{|\mathbb{E}_{k-1}[\phi_k]|\geq\epsilon}\right] \to 0 \tag{8.1}$$

for any (N-independent) $\epsilon > 0$, as $N \to \infty$. Since

$$\mathbb{E}\left[|\mathbb{E}_{k-1}[\phi_k]|^2 \chi_{|\mathbb{E}_{k-1}[\phi_k]|\geq\epsilon}\right] \leq \frac{1}{\epsilon^2}\mathbb{E}\left[|\mathbb{E}_{k-1}[\phi_k]|^4\right], \tag{8.2}$$

it is sufficient to prove that

$$\sum_{k=1}^{N} \mathbb{E}\left[|\mathbb{E}_{k-1}[\phi_k]|^4\right] \to 0 \tag{8.3}$$

as $N \to 0$, which is the Lyapunov condition in [4]. The Lypanov condition (8.3) is obvious from the estimate $\phi_k \prec N^{-\frac{1}{2}}$, which was established in the previous section.

8.2 Tightness of (ζ_N)

Since $\xi_N(z) = \zeta_N(z) + \mathbb{E}[\xi_N(z)]$ and the mean $\mathbb{E}[\xi_N(z)]$ converges, it is enough to check the tightness of the sequence $\zeta_N(z)$. From Theorem 12.3 of [7], it suffices to show that $(\zeta_N(z))$ is tight for a fixed z and the following Hölder condition as in [4]: for some (N-independent) constant $K > 0$,

$$\mathbb{E}|\zeta_N(z_1) - \zeta_N(z_2)|^2 \leq K|z_1 - z_2|^2, \qquad z_1, z_2 \in \mathcal{K}. \tag{8.4}$$

The fact that $(\zeta_N(z))$ is tight for a fixed z is obvious from that the variance is bounded uniformly on N as shown in (7.48).

We now check the Hölder condition. Note that since $R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2)$, we have

$$\begin{aligned}\mathbb{E}|\zeta_N(z_1) - \zeta_N(z_2)|^2 &= |z_1 - z_2|^2 \mathbb{E}|\operatorname{Tr} R(z_1)R(z_2) - \mathbb{E}\operatorname{Tr} R(z_1)R(z_2)|^2 \\ &= |z_1 - z_2|^2 \mathbb{E}\left|\sum_{k=1}^{N} (\mathbb{E}_{k-1} - \mathbb{E}_k)\left(\operatorname{Tr} R(z_1)R(z_2) - \operatorname{Tr} R^{(k)}(z_1)R^{(k)}(z_2)\right)\right|^2.\end{aligned} \tag{8.5}$$

We follow the arguments in Section 7 to estimate the right hand side of (8.5). When compared with (7.4), the main difference is that we do not need to precisely find the leading order term as in the covariance computation in Section 7.

For the ease of notation, we set

$$R \equiv R(z_1), \qquad S \equiv R(z_2). \tag{8.6}$$

We will frequently use the estimate

$$\|R\|, \|S\|, \|R^{(k)}\|, \|S^{(k)}\| \leq C. \tag{8.7}$$

for any $k = 1, 2, \dots, N$, uniformly for $z_1, z_2 \in \mathcal{K}$. For $i, j \neq k$,

$$\begin{aligned}R_{ij}S_{ji} - R_{ij}^{(k)}S_{ji}^{(k)} &= \left(R_{ij} - R_{ij}^{(k)}\right)S_{ji}^{(k)} + R_{ij}^{(k)}\left(S_{ji} - S_{ji}^{(k)}\right) + \left(R_{ij} - R_{ij}^{(k)}\right)\left(S_{ji} - S_{ji}^{(k)}\right) \\ &= \frac{R_{ik}R_{kj}}{R_{kk}}S_{ji}^{(k)} + R_{ij}^{(k)}\frac{S_{jk}S_{ki}}{S_{kk}} + \frac{R_{ik}R_{kj}}{R_{kk}}\frac{S_{jk}S_{ki}}{S_{kk}}.\end{aligned} \tag{8.8}$$

Thus, using (5.11),

$$\operatorname{Tr} RS - \operatorname{Tr}\left(R^{(k)}S^{(k)}\right) = \sum_{i,j}^{(k)} \left(\frac{R_{ik}R_{kj}}{R_{kk}} S_{ji}^{(k)} + R_{ij}^{(k)} \frac{S_{jk}S_{ki}}{S_{kk}} + \frac{R_{ik}R_{kj}}{R_{kk}} \frac{S_{jk}S_{ki}}{S_{kk}} \right) + 2(RS)_{kk} \tag{8.9}$$

and

$$\begin{aligned} &\mathbb{E}|\zeta_N(z_1) - \zeta_N(z_2)|^2 \\ &= |z_1 - z_2|^2 \mathbb{E} \left| \sum_{k=1}^{N} (\mathbb{E}_{k-1} - \mathbb{E}_k) \sum_{i,j}^{(k)} \left(\frac{R_{ik}R_{kj}}{R_{kk}} S_{ji}^{(k)} + R_{ij}^{(k)} \frac{S_{jk}S_{ki}}{S_{kk}} + \frac{R_{ik}R_{kj}}{R_{kk}} \frac{S_{jk}S_{ki}}{S_{kk}} \right) + 2(RS)_{kk} \right|^2 \\ &= |z_1 - z_2|^2 \mathbb{E} \sum_{k=1}^{N} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) \sum_{i,j}^{(k)} \left(\frac{R_{ik}R_{kj}}{R_{kk}} S_{ji}^{(k)} + R_{ij}^{(k)} \frac{S_{jk}S_{ki}}{S_{kk}} + \frac{R_{ik}R_{kj}}{R_{kk}} \frac{S_{jk}S_{ki}}{S_{kk}} \right) + 2(RS)_{kk} \right|^2, \end{aligned} \tag{8.10}$$

where we used (7.9) to get the last line.

To estimate the right hand side of (8.10), we rewrite the first term in the summand as

$$\sum_{i,j}^{(k)} \frac{R_{ik}R_{kj}}{R_{kk}} S_{ji}^{(k)} = \sum_{i,j}^{(k)} R_{kk} \sum_{p,q}^{(k)} M_{pk} R_{ip}^{(k)} R_{qj}^{(k)} M_{kq} S_{ji}^{(k)} = R_{kk} \sum_{p,q}^{(k)} M_{kq} \left(R^{(k)}S^{(k)}R^{(k)}\right)_{qp} M_{pk}. \tag{8.11}$$

Since

$$(\mathbb{E}_{k-1} - \mathbb{E}_k) \left[\frac{s_N^{(k)}(z_1)}{N} \sum_p \left(R^{(k)}S^{(k)}R^{(k)}\right)_{pp} \right] = 0, \tag{8.12}$$

we obtain

$$\begin{aligned} &\mathbb{E} \sum_{k=1}^{N} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) \sum_{i,j}^{(k)} \frac{R_{ik}R_{kj}}{R_{kk}} S_{ji}^{(k)} \right|^2 \\ &= \mathbb{E} \sum_{k=1}^{N} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) \left[R_{kk} \sum_{p,q}^{(k)} M_{kq} \left(R^{(k)}S^{(k)}R^{(k)}\right)_{qp} M_{pk} - \frac{s_N^{(k)}(z_1)}{N} \sum_p \left(R^{(k)}S^{(k)}R^{(k)}\right)_{pp} \right] \right|^2 \\ &\leq 4 \sum_{k=1}^{N} \mathbb{E} \left| R_{kk} \sum_{p,q}^{(k)} M_{kq} \left(R^{(k)}S^{(k)}R^{(k)}\right)_{qp} M_{pk} - \frac{s_N^{(k)}(z_1)}{N} \sum_p \left(R^{(k)}S^{(k)}R^{(k)}\right)_{pp} \right|^2. \end{aligned} \tag{8.13}$$

Using that $|R_{kk}| \leq \|R\| \leq C$, we get

$$\begin{aligned} &\mathbb{E} \left| R_{kk} \sum_{p,q}^{(k)} M_{kq} \left(R^{(k)}S^{(k)}R^{(k)}\right)_{qp} M_{pk} - \frac{R_{kk}}{N} \sum_p \left(R^{(k)}S^{(k)}R^{(k)}\right)_{pp} \right|^2 \\ &\leq C\, \mathbb{E} \left| \sum_{p,q}^{(k)} M_{kq} \left(R^{(k)}S^{(k)}R^{(k)}\right)_{qp} M_{pk} - \frac{1}{N} \sum_p \left(R^{(k)}S^{(k)}R^{(k)}\right)_{pp} \right|^2 \leq \frac{C\|R^{(k)}S^{(k)}R^{(k)}\|^2}{N} \leq \frac{C}{N}, \end{aligned} \tag{8.14}$$

where we used Lemma 5.3 to get the second inequality. Moreover, since

$$\left| \frac{1}{N} \sum_{p}^{(k)} \left(R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right| \leq \left\| R^{(k)} S^{(k)} R^{(k)} \right\| \leq C, \tag{8.15}$$

we also have that

$$\mathbb{E} \left| \frac{R_{kk}}{N} \sum_{p} \left(R^{(k)} S^{(k)} R^{(k)} \right)_{pp} - \frac{s_N^{(k)}(z_1)}{N} \sum_{p} \left(R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right|^2 \leq C\, \mathbb{E} \left| R_{kk} - s_N^{(k)}(z_1) \right|^2 . \tag{8.16}$$

Recall that we defined $Q_k = -M_{kk} + \sum_{p,q}^{(k)} M_{kp} R_{pq}^{(k)} M_{qk}$. Applying (6.3) to expand R_{kk} and using Corollary 5.2, we find that

$$R_{kk} - s_N^{(k)}(z_1) = s(z_1) - s_N^{(k)}(z_1) + s(z_1)^2 (Q_k - s(z_1)) + \mathcal{O}(N^{-1}) = s(z_1)^2 (Q_k - s_N^{(k)}(z_1)) + \mathcal{O}(N^{-1}). \tag{8.17}$$

Thus, from Lemma 5.3,

$$\mathbb{E} \left| R_{kk} - s_N^{(k)}(z_1) \right|^2 \leq C\, \mathbb{E} \left| -M_{kk} + \sum_{p,q}^{(k)} M_{kp} R_{pq}^{(k)} M_{qk} - \frac{1}{N} \sum_{p}^{(k)} R_{pp}^{(k)} \right|^2 \leq \frac{C}{N} \tag{8.18}$$

hence, together with (8.16), we get

$$\mathbb{E} \left| \frac{R_{kk}}{N} \sum_{p} \left(R^{(k)} S^{(k)} R^{(k)} \right)_{pp} - \frac{s_N^{(k)}(z_1)}{N} \sum_{p} \left(R^{(k)} S^{(k)} R^{(k)} \right)_{pp} \right|^2 \leq \frac{C}{N}. \tag{8.19}$$

Combining (8.14) and (8.19) with (8.13), we find that

$$\mathbb{E} \sum_{k=1}^{N} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) \sum_{i,j}^{(k)} \frac{R_{ik} R_{kj}}{R_{kk}} S_{ji}^{(k)} \right|^2 \leq C. \tag{8.20}$$

Similarly, we can also obtain a bound

$$\mathbb{E} \sum_{k=1}^{N} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) \sum_{i,j}^{(k)} R_{ij}^{(k)} \frac{S_{jk} S_{ki}}{S_{kk}} \right|^2 \leq C. \tag{8.21}$$

We expand the third term of the summand in (8.10) as

$$\begin{aligned} \sum_{i,j}^{(k)} \frac{R_{ik} R_{kj}}{R_{kk}} \frac{S_{jk} S_{ki}}{S_{kk}} &= R_{kk} S_{kk} \sum_{i,j}^{(k)} \sum_{p,q}^{(k)} M_{pk} R_{ip}^{(k)} R_{qj}^{(k)} M_{kq} \sum_{r,t}^{(k)} M_{rk} S_{jr}^{(k)} S_{ti}^{(k)} M_{kt} \\ &= R_{kk} S_{kk} \sum_{t,p}^{(k)} M_{kt} \left(S^{(k)} R^{(k)} \right)_{tp} M_{pk} \sum_{q,r}^{(k)} M_{kq} \left(R^{(k)} S^{(k)} \right)_{qr} M_{rk} \\ &= R_{kk} S_{kk} \left(\sum_{p,q}^{(k)} M_{kp} \left(R^{(k)} S^{(k)} \right)_{pq} M_{qk} \right)^2 \end{aligned} \tag{8.22}$$

since R and S commute. Following the decomposition idea we used in the proof of (8.20), we first observe

$$(\mathbb{E}_{k-1} - \mathbb{E}_k) \left[s_N^{(k)}(z_1) s_N^{(k)}(z_2) \left(\frac{1}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \right)^2 \right] = 0. \tag{8.23}$$

Thus,

$$\begin{aligned} &\mathbb{E} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) \sum_{i,j}^{(k)} \frac{R_{ik} R_{kj}}{R_{kk}} \frac{S_{jk} S_{ki}}{S_{kk}} R_{kk} \right|^2 \\ &= \mathbb{E} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) \left[R_{kk} S_{kk} \left(\sum_{p,q}^{(k)} M_{kp} \left(R^{(k)} S^{(k)} \right)_{pq} M_{qk} \right)^2 - s_N^{(k)}(z_1) s_N^{(k)}(z_2) \left(\frac{1}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \right)^2 \right] \right|^2 . \end{aligned} \tag{8.24}$$

Since $|R_{kk} S_{kk}| \le \|R\| \|S\| \le C$ and $\frac{1}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \le \|R\| \|S\| \le C$,

$$\begin{aligned} &\mathbb{E} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) \sum_{i,j}^{(k)} \frac{R_{ik} R_{kj}}{R_{kk}} \frac{S_{jk} S_{ki}}{S_{kk}} R_{kk} \right|^2 \\ &\le C \, \mathbb{E} \left| \left(\sum_{p,q}^{(k)} M_{kp} \left(R^{(k)} S^{(k)} \right)_{pq} M_{qk} \right)^2 - \left(\frac{1}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \right)^2 \right|^2 + C \, \mathbb{E} \left| R_{kk} S_{kk} - s_N^{(k)}(z_1) s_N^{(k)}(z_2) \right|^2 . \end{aligned} \tag{8.25}$$

Using a simple identity $A^2 - B^2 = (A-B)^2 + 2B(A-B)$, we estimate the first term in the right hand side of (8.25) by

$$\begin{aligned} &\mathbb{E} \left| \left(\sum_{p,q}^{(k)} M_{kp} \left(R^{(k)} S^{(k)} \right)_{pq} M_{qk} \right)^2 - \left(\frac{1}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \right)^2 \right|^2 \\ &= \mathbb{E} \left| \left(\sum_{p,q}^{(k)} M_{kp} \left(R^{(k)} S^{(k)} \right)_{pq} M_{qk} - \frac{1}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \right)^2 \right. \\ &\qquad \left. + \frac{2}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \left(\sum_{p,q}^{(k)} M_{kp} \left(R^{(k)} S^{(k)} \right)_{pq} M_{qk} - \frac{1}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \right) \right|^2 \\ &\le C \, \mathbb{E} \left| \sum_{p,q}^{(k)} M_{kp} \left(R^{(k)} S^{(k)} \right)_{pq} M_{qk} - \frac{1}{N} \sum_p \left(R^{(k)} S^{(k)} \right)_{pp} \right|^2 \le \frac{C}{N}, \end{aligned} \tag{8.26}$$

where we used Lemma 5.3 in the last inequality. From (8.19), we also find that

$$\begin{aligned}\mathbb{E}\left|R_{kk}S_{kk}-s_N^{(k)}(z_1)s_N^{(k)}(z_2)\right|^2&=\mathbb{E}\left|\left(R_{kk}-s_N^{(k)}(z_1)\right)S_{kk}+s_N^{(k)}(z_1)\left(S_{kk}-s_N^{(k)}(z_2)\right)\right|^2\\&\le C\,\mathbb{E}\left|R_{kk}-s_N^{(k)}(z_1)\right|^2+C\,\mathbb{E}\left|S_{kk}-s_N^{(k)}(z_2)\right|^2\le\frac{C}{N}.\end{aligned}\tag{8.27}$$

From (8.25), (8.26), and (8.27), we obtain a bound

$$\mathbb{E}\sum_{k=1}^{N}\left|(\mathbb{E}_{k-1}-\mathbb{E}_k)\sum_{i,j}^{(k)}\frac{R_{ik}R_{kj}}{R_{kk}}\frac{S_{jk}S_{ki}}{S_{kk}}\right|^2\le C.\tag{8.28}$$

Finally, the last term in (8.10) becomes

$$(RS)_{kk}=R_{kk}S_{kk}\sum_{p,q}^{(k)}M_{kq}\left(R^{(k)}S^{(k)}\right)_{qp}M_{pk},\tag{8.29}$$

and one can prove by following the same argument as in the derivation of (8.28) that

$$\mathbb{E}\sum_{k=1}^{N}\left|(\mathbb{E}_{k-1}-\mathbb{E}_k)\left[R_{kk}S_{kk}\sum_{p,q}^{(k)}M_{kq}\left(R^{(k)}S^{(k)}\right)_{qp}M_{pk}\right]\right|^2\le C.\tag{8.30}$$

From (8.10), (8.20), (8.21), (8.28), and (8.30), we find that the Hölder condition (8.4) holds, which concludes the proof for tightness of (ζ_N).

9 Proof of Lemma 4.2

For $z\in\Gamma_0$, we have $|s_N(z)-s(z)|\prec N^{-1}$ from Corollary 5.2. Thus, for any $\epsilon>0$,

$$\int_{\Gamma_0}\mathbb{E}|\xi_N(z)|^2\mathrm{d}z\le N^{-1+\epsilon}|\Gamma_0|=4N^{-1+\epsilon-\delta}.\tag{9.1}$$

Setting $\epsilon=\frac{\delta}{2}$, we find that (4.12) holds for Γ_0.

To prove (4.12) for Γ_r, it suffices to show that $\mathbb{E}|\xi_N(z)|^2<K$ for some (N-independent) constant $K>0$. In Section 6, we proved that

$$\mathbb{E}\xi_N=\frac{s^2}{1-s^2}\left(-J'+\frac{J^2s}{1+Js}+(w_2-1)s+s's+(W_4-3)\,s^3\right)+O(N^{-\frac{1}{2}+\epsilon}),\tag{9.2}$$

thus $|\mathbb{E}\xi_N|^2<C$ for $z\in\Gamma_r$.

We now estimate $\mathbb{E}|\zeta_N|^2=\mathbb{E}|\xi_N-\mathbb{E}\xi_N|^2$. Recall that we showed in (7.6) that

$$\zeta_N=\sum_{k=1}^{N}(\mathbb{E}_{k-1}-\mathbb{E}_k)\left[R_{kk}\left(1+\sum_{p,q}^{(k)}M_{kp}(R^{(k)})^2_{pq}M_{qk}\right)\right].\tag{9.3}$$

Following the idea in (8.14), we use

$$(\mathbb{E}_{k-1} - \mathbb{E}_k)\left[s_N^{(k)}\left(1 + \frac{1}{N}\sum_p^{(k)}(R^{(k)})^2_{pp}\right)\right] = 0, \tag{9.4}$$

hence

$$\begin{aligned}\mathbb{E}|\zeta_N|^2 &= \mathbb{E}\sum_{k=1}^N\left|(\mathbb{E}_{k-1} - \mathbb{E}_k)\left[R_{kk}\left(1 + \sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right) - s_N^{(k)}\left(1 + \frac{1}{N}\sum_p^{(k)}(R^{(k)})^2_{pp}\right)\right]\right|^2 \\ &\le 4\sum_{k=1}^N\mathbb{E}\left|R_{kk}\left(1 + \sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right) - s_N^{(k)}\left(1 + \frac{1}{N}\sum_p^{(k)}(R^{(k)})^2_{pp}\right)\right|^2.\end{aligned} \tag{9.5}$$

Define the event

$$\Omega_N := \{\mu_1 \le \widehat{J} + N^{-1/3}\}. \tag{9.6}$$

From Lemma 3.3, we find $\mathbb{P}(\Omega_N) < N^{-D}$ for any (large) fixed $D > 0$. On Ω_N,

$$|R_{kk}| \le \|R\| \le \frac{1}{a_+ - \widehat{J} - N^{-1/3}} \le C \tag{9.7}$$

for any $k = 1, 2, \dots, N$, uniformly for $z \in \Gamma_r$. Similarly, $\|R^{(k)}\| \le C$ for any $k = 1, 2, \dots, N$, uniformly for $z \in \Gamma_r$. Thus,

$$\begin{aligned}&\mathbb{E}\left|\mathbb{1}(\Omega_N)\left[R_{kk}\left(1 + \sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right) - R_{kk}\left(1 + \frac{1}{N}\sum_p^{(k)}(R^{(k)})^2_{pp}\right)\right]\right|^2 \\ &\le C\,\mathbb{E}\left|\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk} - \frac{1}{N}\sum_p^{(k)}(R^{(k)})^2_{pp}\right|^2 \le \frac{C\|R^{(k)}\|^4}{N} \le \frac{C}{N},\end{aligned} \tag{9.8}$$

Moreover, since

$$\left|\frac{1}{N}\sum_p^{(k)}(R^{(k)})^2_{pp}\right| \le \|R^{(k)}\|^2 \le C \tag{9.9}$$

on Ω_N, from (8.18), we get

$$\begin{aligned}&\mathbb{E}\left|\mathbb{1}(\Omega_N)\left[R_{kk}\left(1 + \frac{1}{N}\sum_p^{(k)}(R^{(k)})^2_{pp}\right) - s_N^{(k)}\left(1 + \frac{1}{N}\sum_p^{(k)}(R^{(k)})^2_{pp}\right)\right]\right|^2 \\ &\le C\,\mathbb{E}\left|\mathbb{1}(\Omega_N)\left[R_{kk} - s_N^{(k)}\right]\right|^2 \le \frac{C}{N}.\end{aligned} \tag{9.10}$$

On Ω_N^c, we use the trivial bound $\|R\|, \|R^{(k)}\| \leq \frac{1}{\operatorname{Im} z} \leq N^{\delta}$. Then,

$$\begin{aligned}
&\mathbb{E}\left|\mathbb{1}(\Omega_N^c)\left[R_{kk}\left(1+\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right) - R_{kk}\left(1+\frac{1}{N}\sum_{p}^{(k)}(R^{(k)})^2_{pp}\right)\right]\right|^2 \\
&\leq \left(\mathbb{E}\left[\mathbb{1}(\Omega_N^c)|R_{kk}|^2\right]\right)^{1/2}\left(\mathbb{E}\left|\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk} - \frac{1}{N}\sum_{p}^{(k)}(R^{(k)})^2_{pp}\right|^4\right)^{1/2} \leq C\mathbb{P}(\Omega_N^c)^{1/2}\frac{\|R\|\|R^{(k)}\|^4}{N} \leq \frac{C}{N}
\end{aligned} \tag{9.11}$$

and similarly,

$$\mathbb{E}\left|\mathbb{1}(\Omega_N^c)\left[R_{kk}\left(1+\frac{1}{N}\sum_{p}^{(k)}(R^{(k)})^2_{pp}\right) - s_N^{(k)}\left(1+\frac{1}{N}\sum_{p}^{(k)}(R^{(k)})^2_{pp}\right)\right]\right|^2 \leq \frac{C}{N}. \tag{9.12}$$

Combining (9.8), (9.10), (9.11), and (9.12), we obtain

$$\mathbb{E}\left|R_{kk}\left(1+\sum_{p,q}^{(k)} M_{kp}(R^{(k)})^2_{pq}M_{qk}\right) - s_N^{(k)}\left(1+\frac{1}{N}\sum_{p}^{(k)}(R^{(k)})^2_{pp}\right)\right|^2 \leq \frac{C}{N}, \tag{9.13}$$

thus, from (9.5),

$$\mathbb{E}|\xi_N|^2 \leq 2\,\mathbb{E}|\zeta_N|^2 + 2\,|\mathbb{E}\xi_N|^2 \leq C, \tag{9.14}$$

which proves the lemma for Γ_r. The proof of the lemma for Γ_l is the same.

10 Large deviation estimates

We prove Lemma 5.3 for the spiked random matrix M. The case of non-spiked random matrix is well-known and we adapt its proof. We use the following lemma, sometimes referred to as 'large deviation estimates'.

Lemma 10.1 (Lemma 8.1 and Lemma 8.2 of [20])**.** *Let $a_1, \dots, a_N$ be independent (complex) random variables with mean zero and variance* 1*. Suppose that $a_1, \dots, a_N$ satisfies the uniform subexponential decay condition. Then, for any deterministic complex numbers A_i and B_{ij} $(i, j = 1, 2, \dots, N)$,*

$$\begin{aligned}
\left|\sum_{i=1}^N A_i a_i\right| &\prec \left(\sum_{i=1}^N |A_i|^2\right)^{\frac{1}{2}} \\
\left|\sum_{i=1}^N A_i |a_i|^2 - \sum_{i=1}^N A_i\right| &\prec \left(\sum_{i=1}^N |A_i|^2\right)^{\frac{1}{2}} \\
\left|\sum_{i\neq j} a_i B_{ij} a_j\right| &\prec \left(\sum_{i\neq j} |B_{ij}|^2\right)^{\frac{1}{2}}
\end{aligned} \tag{10.1}$$

For the proof of Lemma 10.1, see Appendix B of [20].

Proof of Lemma 5.3. We consider the case $n = 1$ for the first part of the lemma. We first decompose

$$M_{ip}S_{pq}M_{qi} = A_{ip}S_{pq}A_{qi} + \frac{J}{N}S_{pq}A_{qi} + \frac{J}{N}A_{ip}S_{pq} + \frac{J^2}{N^2}S_{pq}. \tag{10.2}$$

Then,

$$\begin{aligned}&\left|\sum_{p,q}^{(i)} M_{ip}S_{pq}M_{qi} - \frac{1}{N}\sum_{p}^{(i)} S_{pp}\right|^2 \\ &\le 4\left|\sum_{p,q}^{(i)} A_{ip}S_{pq}A_{qi} - \frac{1}{N}\sum_{p}^{(i)} S_{pp}\right|^2 + \frac{4J^2}{N^2}\left|\sum_{p,q}^{(i)} S_{pq}A_{qi}\right|^2 + \frac{4J^2}{N^2}\left|\sum_{p,q}^{(i)} A_{ip}S_{pq}\right|^2 + \frac{4J^4}{N^4}\left|\sum_{p,q}^{(i)} S_{pq}\right|^2.\end{aligned} \tag{10.3}$$

Taking the expectation,

$$\begin{aligned}&\mathbb{E}\left|\sum_{p,q}^{(i)} A_{ip}S_{pq}A_{qi} - \frac{1}{N}\sum_{p}^{(i)} S_{pp}\right|^2 \\ &= \mathbb{E}\sum_{p,q,r,s}^{(i)} A_{ip}S_{pq}A_{qi}A_{ir}\overline{S_{rs}}A_{si} - \mathbb{E}\sum_{p,q,r} A_{ip}S_{pq}A_{qi}\overline{S_{rr}} - \mathbb{E}\sum_{p,q,r} A_{ip}\overline{S_{pq}}A_{qi}S_{rr} + \frac{1}{N^2}\sum_{p,q}^{(i)} \overline{S_{pp}}S_{qq} \\ &= \frac{1}{N^2}\sum_{p,q}^{(i)} |S_{pq}|^2 + \frac{1}{N^2}\sum_{p,q}^{(i)} S_{pq}\overline{S_{qp}} + \frac{W_4}{N^2}\sum_{p}^{(i)} |S_{pp}|^2.\end{aligned} \tag{10.4}$$

Since

$$\sum_{p,q}^{(i)} |S_{pq}|^2 = \|S\|_{HS}^2 \le N\|S\|^2 \tag{10.5}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Thus, we find that

$$\mathbb{E}\left|\sum_{p,q}^{(i)} A_{ip}S_{pq}A_{qi} - \frac{1}{N}\sum_{p}^{(i)} S_{pp}\right|^2 \le \frac{W_4 + 2}{N}\|S\|^2. \tag{10.6}$$

Similarly, for other terms in (10.3),

$$\left|\sum_{p,q}^{(i)} S_{pq}A_{qi}\right|^2 = \left|\sum_{p,q}^{(i)} A_{ip}S_{pq}\right|^2 \le N\|S\|^2 \tag{10.7}$$

and

$$\left|\sum_{p,q}^{(i)} S_{pq}\right|^2 \le N^2\sum_{p,q}^{(i)} |S_{pq}|^2 \le N^3\|S\|^2. \tag{10.8}$$

Altogether, we obtain that

$$\left|\sum_{p,q}^{(i)} M_{ip}S_{pq}M_{qi} - \frac{1}{N}\sum_{p}^{(i)} S_{pp}\right|^2 \le 4(W_4 + 2 + 2J^2 + J^4)\frac{\|S\|^2}{N}, \tag{10.9}$$

which proves the first part of the lemma for $n = 1$. The case $n = 2$ can be proved analogously.

Next, we prove the second part of the lemma. From the second inequality in Lemma 10.1,

$$\left| \sum_p^{(i)} A_{ip} S_{pp} A_{pi} - \frac{1}{N} \sum_p^{(i)} S_{pp} \right| \prec \frac{1}{N} \left(\sum_p^{(i)} |S_{pp}|^2 \right)^{\frac{1}{2}} . \tag{10.10}$$

From the third inequality in Lemma 10.1,

$$\left| \sum_{p \neq q}^{(i)} A_{ip} S_{pp} A_{pi} \right| \prec \frac{1}{N} \left(\sum_{p \neq q}^{(i)} |S_{pq}|^2 \right)^{\frac{1}{2}} . \tag{10.11}$$

Summing the inequalities above, we find that

$$\left| \sum_{p,q}^{(i)} A_{ip} S_{pq} A_{qi} - \frac{1}{N} \sum_p^{(i)} S_{pp} \right| \prec \frac{1}{N} \left(\sum_{p,q}^{(i)} |S_{pq}|^2 \right)^{\frac{1}{2}} = \frac{\|S\|_{HS}}{N} \leq \frac{\|S\|}{\sqrt{N}}. \tag{10.12}$$

For the second term in (10.2), we apply the first inequality in Lemma 10.1 and get

$$\left| \frac{J}{N} \sum_{p,q}^{(i)} S_{pq} A_{qi} \right| \prec \frac{1}{N\sqrt{N}} \sum_p^{(i)} \left(\sum_q^{(i)} |S_{pq}|^2 \right)^{\frac{1}{2}} \leq \frac{1}{N} \left(\sum_{p,q}^{(i)} |S_{pq}|^2 \right)^{\frac{1}{2}} \leq \frac{\|S\|}{\sqrt{N}}. \tag{10.13}$$

The same estimate holds for the third term in (10.2). Finally, for the last term in (10.2),

$$\left| \sum_{p,q}^{(i)} \frac{J^2}{N^2} S_{pq} \right| \leq \frac{J^2}{N} \left(\sum_{p,q}^{(i)} |S_{pq}|^2 \right)^{\frac{1}{2}} \leq \frac{\|S\|}{\sqrt{N}}. \tag{10.14}$$

Summing the estimates, we obtain (5.14). □

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