Fluctuations of the free energy of the spherical Sherrington–Kirkpatrick model

Jinho Baik* and Ji Oon Lee†

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Abstract

The spherical Sherrington–Kirkpatrick model is a spherical mean field model for spin glass. We consider the fluctuations of the free energy at arbitrary non-critical temperature for the 2-spin model with no magnetic field. We show that in the high temperature regime the law of the fluctuations converges to the Gaussian distribution just like in the Sherrington-Kirtkpatrick model. We show, on the other hand, that the law of the fluctuations is given by the GOE Tracy-Widom distribution in the low temperature regime. The orders of the fluctuations are markedly different in these two regimes. A universality of the limit law is also proved.

1 Introduction

The spherical Sherrington–Kirkpatrick (SSK) model (with 2-spin interaction and no magnetic field) is defined by the Hamiltonian

$$H_N(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}}\sum_{i\neq j} J_{ij}\sigma_i\sigma_j = -\frac{1}{\sqrt{N}}\langle \boldsymbol{\sigma}, J\boldsymbol{\sigma}\rangle \tag{1.1}$$

where the spin variables $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N) \in \mathbb{R}^N$ lie on the sphere $\|\boldsymbol{\sigma}\|^2 = \sum_{i=1}^N \sigma_i^2 = N$, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Here $J_{ij} = J_{ji}$, $1 \le i < j \le N$, are independent identically distributed random variables, representing the disorder of the system, and $J = (J_{ij})_{i,j=1}^N$ with $J_{ii} = 0$. We assume that J_{12} has mean 0 and variance 1. The free energy at inverse temperature β is defined by

$$F_N = F_N(\beta) = \frac{1}{N}\log Z_N, \qquad Z_N = \int_{S_{N-1}} e^{\beta H_N(\boldsymbol{\sigma})} \mathrm{d}\omega_N(\boldsymbol{\sigma}), \tag{1.2}$$

where $\mathrm{d}\omega_N$ is the normalized uniform measure on the sphere $S_{N-1} = \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\|^2 = N\}$. Note that since J_{ij} are random, F_N is a random variable. The subject of this paper is the fluctuations of F_N as $N \to \infty$.

*Department of Mathematics, University of Michigan, Ann Arbor, MI, 48109, USA
email: `baik@umich.edu`

†Department of Mathematical Sciences, KAIST, Daejeon, 305701, Korea
email: `jioon.lee@kaist.edu`

The usual Sherrington–Kirkpatrick (SK) model is a mean field version of the Edwards-Anderson model of spin glass, and is given by the same Hamiltonian as (1.1) but with the condition that the spin variables are on a lattice instead of the sphere: $\boldsymbol{\sigma} \in \{-1, 1\}^N$. (For this case many literatures use a different convention that the Hamiltonian is divided by 2.) The partition function is defined as $Z_N = \sum_{\boldsymbol{\sigma}\in\{-1,1\}^N} e^{\beta H_N(\boldsymbol{\sigma})}$. Among the numerous existing results on the SK model (see, for example, [49, 50, 37]), we here review a few about the free energy that are relevant to this paper. The non-random limit of the free energy, $\lim_{N\to\infty} F_N$, for the SK model was first predicted by Parisi [39] in the more general setting of p-spin interactions in the presence of external magnetic field. The Parisi formula was rigorously proved by Talagrand in his famous 2006 paper [48] in which he proved the convergence of the expectation to the Parisi formula when the disorder random variables J_{ij} are Gaussian. The universality of the limit of F_N independent of J is proved under the finite third moment condition, with mean 0 and variance 1 by Carmona and Hu [19], which improved the previous result by Guerra and Toninelli [28] for symmetric random variables with finite fourth moment. The Parisi formula is implicit and is given in terms of a variational problem. For a recent study on this variational problem, see [5]. An important feature here is the existence of the critical temperature $\beta_c = \frac{1}{2}$. The cardinality of the support of the measure that underlies in the Parisi formula changes at $\beta = \beta_c$. The phase transition is also understood by the fact that the difference between the quenched disorder free energy and the annealed free energy, $\frac{1}{N}\mathbb{E}[\log Z_N] - \frac{1}{N}\log\mathbb{E}[Z_N]$, tends to zero as $N \to \infty$ in the high temperature regime, $\beta < \beta_c$, but does not tend to zero in the low temperature regime, $\beta > \beta_c$ (see [1]).

The fluctuations of the free energy for the SK model in the high temperature regime was studied by Aizenman, Lebowtiz, and Ruelle [1]. They showed that if the disorder random variables J_{ij}, $1 \le i < j \le N$, are independent Gaussian with mean zero and variance 1, $\mathcal{N}(0, 1)$, then

$$N\left(F_N - \left(\log 2 + \beta^2\right)\right) \Rightarrow \mathcal{N}\big(-\frac{1}{2}\alpha, \alpha\big), \tag{1.3}$$

where

$$\alpha = -\frac{1}{2}\log(1 - 4\beta^2) - 2\beta^2, \tag{1.4}$$

and the convergence is in distribution as $N \to \infty$ (see also Section 11.4 in [50].) It was also shown in [1] that for non-Gaussian disorder, the same limit theorem holds with some changes on the formula of α. However, a limit theorem for the fluctuations in the low temperature regime still remains as an open question. The limit theorem is not known even for the zero temperature case.

The SSK model was introduced by Kosterlitz, Thouless, and Jones [31] as a model that is easier to analyze than the SK model. Indeed in their paper, the authors evaluated the limit of the free energy explicitly though a rigorous proof was not supplied. The analogue of Parisi formula for the SSK model was obtained by Crisanti and Sommers [24]. The Parisi formula for the SSK model was later proved rigorously by Talagrand [47] immediately after he proved the formula for the SK model. The Parisi formula can be evaluated explicitly for the Hamiltonian (1.1) (for the case with 2-spin interactions without magnetic field) [38] and the resulting formula is same as one obtained in [31]:

$$F_N \to F(\beta) = \begin{cases} \beta^2 & \text{if } 0 < \beta < 1/2, \\ 2\beta - \frac{\log(2\beta)+3/2}{2} & \text{if } \beta > 1/2 \end{cases} \tag{1.5}$$

in expectation and also in distribution. For a general class of random variables J_{ij}, a corresponding result for the limiting free energy $F(\beta)$ was obtained in [29]. The formula in [29] was given in terms of R-transform and one can check that it is same as the one in Definition 2.13 below. Note that $F(\beta)$ in (1.5) is C^2 but not C^3 at $\beta_c = \frac{1}{2}$; the critical temperature is same as that of the SK model. The third-order transition also holds for SSK model with general random variables J_{ij} as one can see in Definition 2.13.

We remark that paper [10] studied the so-called "soft" spherical Sherrington–Kirkpatrick (SSSK) model and evaluated the almost sure limit of the free energy explicitly. The limit of the free energy for the SSSK model also shows a third order phase transition.

In this paper, we obtain the limit theorem for the fluctuations of F_N for the SSK model. We first state the result when the disorder random variables are Gaussian. Non-Gaussian case will be stated in Theorem 1.2.

Theorem 1.1. *Let J_{ij}, $1 \le i < j \le N$, be independent Gaussian random variables with mean* 0 *and variance* 1*, and set $J_{ji} = J_{ij}$. Let*

$$F_N = F_N(\beta) = \frac{1}{N} \log \left[\int_{S_{N-1}} e^{-\frac{\beta}{\sqrt{N}} \sum_{i \neq j} J_{ij}\sigma_i\sigma_j} \mathrm{d}\omega_N(\boldsymbol{\sigma}) \right] \tag{1.6}$$

be the free energy of the SSK model at inverse temperature β, where $\mathrm{d}\omega_N$ is the normalized uniform measure on the sphere $S_{N-1} = \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\|^2 = N\}$. Then the following holds as $N \to \infty$. Here $F(\beta)$ is defined in (1.5)*, and all the convergences are in distribution.*

(i) In the high temperature regime $0 < \beta < \frac{1}{2}$,

$$N\left(F_N - F(\beta)\right) \Rightarrow \mathcal{N}\left(f, \alpha\right), \tag{1.7}$$

where

$$f = \frac{1}{4}\log(1-4\beta^2) - 2\beta^2, \qquad \alpha = -\frac{1}{2}\log(1-4\beta^2) - 2\beta^2. \tag{1.8}$$

(ii) In the low temperature regime $\beta > \frac{1}{2}$,

$$\frac{1}{\beta - \frac{1}{2}} N^{2/3}\left(F_N - F(\beta)\right) \Rightarrow TW_1, \tag{1.9}$$

where TW_1 is the GOE Tracy-Widom distribution.

Hence the order of the fluctuations changes from N^{-1} in the high temperature regime to $N^{-2/3}$ in the low temperature regime. In the high temperature regime, the fluctuations are asymptotically Gaussian with the same variance as in SK model (see (1.4)). On the other hand, in the low temperature regime, the fluctuations are asymptotically same as those of the largest eigenvalue of a large random matrix from GOE (Gaussian orthogonal ensemble). The connection to the GOE Tracy-Widom distribution is apparent at zero temperature: From (1.6), we may define the free energy at zero temperature (which is the formal limit of $\frac{1}{\beta}F_N(\beta)$ as $\beta \to \infty$) as

$$\widetilde{F}_N(\infty) = \sup_{\|\boldsymbol{\sigma}\|^2 = N} \frac{1}{N}\langle \boldsymbol{\sigma}, \frac{-J}{\sqrt{N}}\boldsymbol{\sigma}\rangle = \lambda_{\max}(N) \tag{1.10}$$

where $\lambda_{\max}(N)$ is the largest eigenvalue of the random symmetric matrix $M = -J/\sqrt{N}$, $J = (J_{ij})_{i,j=1}^N$. The random matrix M is almost exactly an $N \times N$ GOE matrix except that the diagonal terms are zero. The fluctuations of the smallest and the largest eigenvalues are still given by the GOE Tracy-Widom distribution in the limit $N \to \infty$ [45]: $N^{2/3}(\tilde{F}_N(\infty) - 2) \Rightarrow TW_1$. The theorem above shows that the same limit law for the fluctuations hold for all $\beta > \frac{1}{2}$ after the change by the multiplicative factor $\beta - \frac{1}{2}$. The proof of the theorem will show that for $\beta > \frac{1}{2}$, the main contribution to F_N comes from the largest eigenvalue of $-J$ and this holds not only for the leading asymptotic term of F_N but also for the second term corresponding to the fluctuations. On the other hand, we will see in the proof that for $\beta < \frac{1}{2}$, all of the eigenvalues of the random matrix $-J/\sqrt{N}$ contribute to F_N in the form of the linear statistic $\sum_{i=1}^{N} g(\lambda_i)$ where λ_i are the eigenvalues of $-J/\sqrt{N}$ for a specific function g. It is a well-known result in random matrix theory that if the function g is smooth in an open interval that contains the support of the limiting density function of the eigenvalues, then the linear statistic converges to the Gaussian distribution [30, 9, 7, 33].

For the SK model, there is no analytic result for the fluctuations in the low temperature regime. See [40] for some physical analysis and conjectures. Some numerical studies [4, 12] suggest that at zero temperature, the order of the fluctuations are smaller that $N^{-2/3}$ ([12] suggests $N^{-3/4}$) and the limiting distribution is not the GOE Tracy-Widom distribution. See [20] (also [21]) for a mathematical result on the upper bound of the order of the fluctuations.

It is interesting to consider the near critical case when β depends on N and satisfies $\beta \to \frac{1}{2}$ as $N \to \infty$. Even though we do not show in this paper, it is possible to improve the proof in this paper to show that (1.9) still holds when $\beta = \beta_N = \frac{1}{2} + N^{-\delta}$ with $\delta < \frac{1}{3}$. It is expected that (1.7) also holds when $\beta = \beta_N = \frac{1}{2} - N^{-\delta}$ with $\delta < \frac{1}{3}$. These will be discussed in a future paper. It is tempting to predict the critical window of β in which the transition from the Gaussian distribution to the GOE Tracy-Widom distribution occurs by using Theorem 1.1. The theorem indicates that the leading-order term of the variance of F_N is of order $-\frac{1}{N^2}\log(1-2\beta)$ for $\beta < 1/2$ and of order $\frac{1}{N^{4/3}}(\beta - \frac{1}{2})^2$ for $\beta > 1/2$, as $N \to \infty$ and $\beta \to \frac{1}{2}$. By matching these orders, we are lead to speculate that the critical window of the temperature is $\beta = \frac{1}{2} + O(\frac{\sqrt{\log N}}{N^{1/3}})$.

Another model in which the Tracy-Widom distribution appears is the directed polymer in random environment (DPRE) in $1+1$ dimension. Recent impressive developments in the field show that for some specific choices of disorders, the fluctuations of the free energy are given by the GUE Tracy-Widom distribution for all $\beta > 0$ [3, 13, 14, 23, 36]. (Here GUE stands for Gaussian unitary ensemble.) It was indeed shown previously in [18, 22] that the critical temperature for $1+1$ dimensional DPRE is $\beta_c = 0$. Recently it was shown by Alberts, Khanin, and Quastel [2] that the critical window is $\beta = O(N^{-1/4})$. More specifically if $\beta = BN^{-1/4}$, then the fluctuations of the free energy converge to a different distribution parametrized by B, called the crossover distribution that appears in the KPZ equation (see also [35]). This regime is called the intermediate disorder regime in [2].

For non-Gaussian disorder random variables, we have the following universality result. Note that the disordered random variables are not necessarily identically distributed.

Theorem 1.2. *Let J_{ij}, $1 \le i \le j \le N$, be independent random variables satisfying the following conditions:*

- *All moments of J_{ij} are finite and $\mathbb{E}[J_{ij}] = 0$ for all $1 \le i \le j \le N$.*
- *For all $i < j$, $\mathbb{E}[J_{ij}^2] = 1$, $\mathbb{E}[|J_{ij}|^3] = W_3$, and $\mathbb{E}[J_{ij}^4] = W_4$ for some constants $W_3, W_4 \ge 0$.*
- *For all $i = 1, \cdots, N$, $\mathbb{E}[J_{ii}^2] = w_2$ for a constant $w_2 \ge 0$.*

Set $J_{ji} = J_{ij}$ *for* $i < j$*. Define the free energy as* (1.6) *with the sum replaced by* $\sum_{i,j=1}^N J_{ij}\sigma_i\sigma_j$*. Then* (1.7) *still holds after the changes*

$$f = \frac{1}{4}\log(1-4\beta^2) + \beta^2(w_2-2) + 2\beta^4(W_4-3), \quad \alpha = -\frac{1}{2}\log(1-4\beta^2) + \beta^2(w_2-2) + 2\beta^4(W_4-3). \tag{1.11}$$

On the other hand, (1.9) *holds without any changes.*

The starting point of the proofs of the above theorems is a simple integral formula of the partition function.

Lemma 1.3. *Let* M *be an* $N \times N$ *symmetric matrix with eigenvalues* $\lambda_1 \geq \cdots \geq \lambda_N$*. Then*

$$\int_{S_{N-1}} e^{\beta\langle\boldsymbol{\sigma}, M\boldsymbol{\sigma}\rangle} \mathrm{d}\omega_N(\boldsymbol{\sigma}) = C_N \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)}\mathrm{d}z, \quad G(z) = 2\beta z - \frac{1}{N}\sum_i \log(z-\lambda_i), \tag{1.12}$$

where γ *is any constant satisfying* $\gamma > \lambda_1$*, the integration contour is the vertical line from* $\gamma - \mathrm{i}\infty$ *to* $\gamma + \mathrm{i}\infty$*, the* log *function is defined in the principal branch, and*

$$C_N = \frac{\Gamma(N/2)}{2\pi\mathrm{i}(N\beta)^{N/2-1}}. \tag{1.13}$$

Here $\Gamma(z)$ *denotes the Gamma function.*

We may apply the method of steepest-descent to analyze the asymptotic behavior of the above integral as $N \to \infty$. The difficulty is that $G(z)$ is random since $M = -J/\sqrt{N}$ is a random matrix. Now random matrix theory tells us that the eigenvalues of random symmetric matrix M have strong repulsions between them and as a consequence they are rigid in the sense that the eigenvalues are close to the deterministic locations determined by the quantiles of their limiting empirical distribution (i.e. semi-circle law). This rigidity of the eigenvalues allows us still to be able to apply the method of steepest-descent. The crucial technical ingredient here is precise estimate on the rigidity of the eigenvalues that was obtained recently by Erdos, Yin, and Yau [27]. This precise rigidity estimate is one of the central achievements of the recent surge of advancements of our understanding of random matrices. After we obtained the above integral representation (1.12), analyzed them asymptotically, and obtained the results in this paper, we learned that the same integral representation was already obtained in the paper of Kosterlitz, Thouless, and Jones [31] in which they obtained the leading order term of the asymptotics by using the method of steepest-descent but without supplying rigorous estimates. Our analysis makes their work rigorous, and goes a step further and obtains the second asymptotic term giving the law of the fluctuations. The paper [31] also considered the case when the mean of the disorder is not necessarily zero. We plan to study this case in a separate paper by the same method as in this paper. A similar formula to the above Lemma also appeared in [34] for the analysis of rank 1 real Wishart spiked model.

Since our analysis only relies on the above integral formula and the rigidity of the eigenvalues, the random matrix, M, corresponding to the disorder random variables does not necessarily have to have independent entries (hence corresponding to the real Wigner matrices). We indeed obtain similar results for M from orthogonal invariant ensembles or real sample covariance matrices, and for M from complex Hermitian matrices, since the rigidity of the eigenvalues was proved for a wide variety of random matrices. In the next section, we state general results assuming some spectral properties of random matrices, and in the subsequent section, we list a few of random matrices, including the one corresponding to the SSK model, for which the results may apply.

The rest of paper is organized as follows. In Section 2, we introduce general conditions and state general results. In Section 3, we illustrate the examples of random matrix ensembles that satisfy the general conditions. Theorems 1.1 and 1.2 follow from one of these examples. In Section 4, we prove Lemma 1.3. Sections 5 and 6 are the main technical part of this paper in which we analyze the integral representation in Lemma 1.3 asymptotically by using the method of steepest-descent. The high temperature regime is analyzed in Section 5 and the low temperature regime is analyzed in Section 6. In Section 7, we prove Theorem 2.9, on the third order phase transition of the free energy. Some technical details in Section 3 are collected in the Appendix.

Remark 1.4 (Notational Remark 1)*.* Throughout the paper we use C or c in order to denote a constant that is independent of N. Even if the constant is different from one place to another, we may use the same notation C or c as long as it does not depend on N for the convenience of the presentation.

Remark 1.5 (Notational Remark 2)*.* The notation $\Rightarrow$ denotes the convergence in distribution as $N \to \infty$.

We close this section by introducing the following terminology.

Definition 1.6 (High probability event)**.** We say that an N-dependent event Ω_N holds with high probability if, for any given $D > 0$, there exists $N_0 > 0$ such that

$$\mathbb{P}(\Omega_N^c) \le N^{-D}$$

for any $N > N_0$.

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2 General results

2.1 Definitions and conditions

Definition 2.1. For an $N \times N$ real random symmetric matrix $M = (M_{ij})_{i,j=1}^N$, we define the partition function at inverse temperature $\beta > 0$ by

$$Z_N = Z_N(\beta) = \int_{S_{N-1}} e^{\beta \langle \boldsymbol{\sigma}, M\boldsymbol{\sigma} \rangle} \mathrm{d}\omega_N(\boldsymbol{\sigma}), \qquad \langle \boldsymbol{\sigma}, M\boldsymbol{\sigma} \rangle = \sum_{i,j=1}^N M_{ij}\sigma_i\sigma_j, \tag{2.1}$$

where $\mathrm{d}\omega_N$ is the normalized uniform measure on the sphere $S_{N-1} = \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\|^2 = N\}$. The free energy F_N is defined by

$$F_N = F_N(\beta) = \frac{1}{N} \log Z_N. \tag{2.2}$$

The free energy (1.2) for the SSK model corresponds to the case when $M = -J/\sqrt{N}$ where J is a symmetric random matrix whose diagonal entries are zero and the entries below the diagonal are independent and identical random variables of mean 0 and variance 1. As mentioned in the previous section, we prove the limit theorem for the fluctuations for more general random symmetric matrices. Precise conditions on M in terms of its eigenvalues will be stated shortly below and these conditions are shown to be satisfied for Wigner matrices, invariant ensembles, and sample covariance matrices in Section 3.

We also consider Hermitian matrices.

Definition 2.2. For an $N \times N$ complex random Hermitian matrix M, we define

$$Z_N = Z_N(\beta) = \int_{\mathbb{C}S_{N-1}} e^{\beta\langle \boldsymbol{\sigma}, M\boldsymbol{\sigma}\rangle} \mathrm{d}\omega_N(\boldsymbol{\sigma}), \qquad \langle \boldsymbol{\sigma}, M\boldsymbol{\sigma}\rangle = \sum_{i,j=1}^{N} M_{ij}\overline{\sigma_i}\sigma_j, \tag{2.3}$$

where $\mathbb{C}S_{N-1} = \{\boldsymbol{\sigma} \in \mathbb{C}^N : \|\boldsymbol{\sigma}\|^2 = N\} \simeq S_{2N-1}$ and $\omega_N(\boldsymbol{\sigma})$ is the uniform measure on $\mathbb{C}S_{N-1}$. The free energy F_N is defined by the same formula (2.2).

For a symmetric or Hermitian matrix M, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ denote the eigenvalues of M. We now list four conditions for the eigenvalues of random matrix M under which the general theorems are proved.

Let $\nu_N := \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j}$ denote the empirical spectral measure of M. We assume that ν_N converges weakly to a probability measure ν. Our first condition is the regularity of the limiting spectral measure ν in the following sense.

Condition 2.3 (Regularity of measure)**.** *Suppose that the empirical spectral measure ν_N of M converges weakly to a probability measure ν that satisfies the following properties:*

- *ν is supported on an interval $[C_-, C_+]$ and is positive on (C_-, C_+).*
- *ν is absolutely continuous and $\frac{\mathrm{d}\nu}{\mathrm{d}x}$ exhibits square root decay at the upper edge, i.e.,*

$$\frac{\mathrm{d}\nu}{\mathrm{d}x}(x) = s_\nu\sqrt{C_+ - x}\,(1 + O(C_+ - x)) \qquad \text{as } x \nearrow C_+ \tag{2.4}$$

for some $s_\nu > 0$.

The second condition concerns the rigidity of the eigenvalues. This is the key assumption.

Condition 2.4 (Rigidity of eigenvalues)**.** *For a positive integer $k \in [1, N]$, let $\hat{k} := \min\{k, N+1-k\}$. Let γ_k be the classical location defined by*

$$\int_{\gamma_k}^{\infty} \mathrm{d}\nu = \frac{1}{N}\left(k - \frac{1}{2}\right). \tag{2.5}$$

Assume that for any $\epsilon > 0$

$$|\lambda_k - \gamma_k| \leq \hat{k}^{-1/3} N^{-2/3+\epsilon} \tag{2.6}$$

holds for all k with high probability.

We remark that under the assumption (2.4), the classical location γ_k satisfies the estimate

$$C^{-1}k^{2/3}N^{-2/3} \le |C_+ - \gamma_k| \le Ck^{2/3}N^{-2/3} \tag{2.7}$$

for some constant $C > 1$ that is independent of N.

The third condition is about the linear statistics of the eigenvalues and it is used in the analysis of the high temperature case $\beta < \beta_c$.

Condition 2.5 (Linear statistics of the eigenvalues)**.** *Assume that for every function $\varphi : \mathbb{R} \to \mathbb{R}$ that is analytic in an open neighborhood of $[C_-, C_+]$ and has compact support, the random variable*

$$\mathcal{N}_\varphi := \sum_i \varphi(\lambda_i) - N \int_{C_-}^{C_+} \varphi(\lambda) \mathrm{d}\nu(\lambda) \tag{2.8}$$

converges in distribution to a Gaussian random variable. The mean and the variance of this Gaussian random variable are denoted by $M(\varphi)$ and $V(\varphi)$, respectively, and they depend only on φ restricted on the support of ν.

The fourth condition is the convergence to the Tracy-Widom distribution of the largest eigenvalue. This will be used in the analysis of the supercritical case $\beta > \beta_c$.

Condition 2.6 (Tracy-Widom limit of the largest eigenvalue)**.** *Let s_ν be the constant appearing in* (2.4)*. Assume that the rescaled largest eigenvalue $(s_\nu \pi)^{-2/3} N^{2/3}(\lambda_1 - C_+)$ converges in distribution to the (GOE or GUE) Tracy-Widom distribution (depending on whether the matrices are symmetric or Hermitian).*

We denote by TW_1 and TW_2 GOE and GUE Tracy-Widom random variables and by F_1 and F_2 their cumulative distribution functions, respectively.

Remark 2.7. For invariant ensembles, the support of the measure ν in Condition 2.3 may consist of several disjoint intervals, In this case, Condition 2.5 does not hold in general; the variance of the linear statistics is a quasi-periodic function in N and does not converge. See [41]. On the other hand, Condition 2.6 is known to hold for multi-inverval cases as well. In this paper we choose to simplify the situation by assuming that the support consists of a single interval; one can still prove some parts of the theorems in the next subsection without assuming this condition.

2.2 Results for symmetric matrices

We first define the critical inverse temperature.

Definition 2.8 (Critical β)**.** Assume that Condition 2.3 holds. Define

$$\beta_c^{sym} = \beta_c = \frac{1}{2} \int_{C_-}^{C_+} \frac{\mathrm{d}\nu(x)}{C_+ - x}. \tag{2.9}$$

Note that β_c is finite due to the square root decay of $\frac{\mathrm{d}\nu}{\mathrm{d}x}$ at the upper edge. The exact values of β_c can be evaluated for some random matrix ensembles (see Section 3) as follows.

- $\beta_c = \frac{1}{2}$ for Wigner matrices (with the choice $C_+ = 2$): SSK model belongs to this example.

- $\beta_c = \frac{1}{4}Q'(C_+)$ for invariant ensembles with potential Q.
- $\beta_c = \frac{1}{2\sqrt{C_+}}$ for sample covariance matrices.

The first main result is the following.

Theorem 2.9 (Third order phase transition)**.** *Consider an ensemble of symmetric matrices satisfying Conditions 2.3, 2.4, 2.5, and 2.6. There is a function $F : (0, \infty) \to [0, \infty)$ such that the free energy F_N satisfies*

$$F_N(\beta) \Rightarrow F(\beta) \tag{2.10}$$

in distribution as $N \to \infty$, for $\beta \neq \beta_c$. The function $F(\beta)$ is C^2 but its third derivative $\partial_\beta^3 F(\beta)$ is discontinuous at $\beta = \beta_c$. This function is defined explicitly in Definition 2.13 below.

The next two results are about the fluctuations of F_N.

Theorem 2.10 (high temperature case)**.** *Consider an ensemble of symmetric random matrices satisfying Conditions 2.3, 2.4, and 2.5. Then for $\beta < \beta_c$,*

$$N\left(F_N(\beta) - F(\beta)\right) \Rightarrow \mathcal{N}(\ell, \sigma^2) \tag{2.11}$$

where the constants $\ell \equiv \ell(\beta)$ and $\sigma^2 \equiv \sigma^2(\beta)$ are defined in Definition 2.13 below.

Theorem 2.11 (low temperature case)**.** *Consider an ensemble of symmetric matrices satisfying Conditions 2.3, 2.4, and 2.6. Then for $\beta > \beta_c$,*

$$\frac{1}{(s_\nu \pi)^{2/3}(\beta - \beta_c)} N^{2/3}\left(F_N(\beta) - F(\beta)\right) \Rightarrow TW_1. \tag{2.12}$$

Here the constant s_ν is the one in Condition 2.3.

Remark 2.12. For a given random symmetric matrix M, $\frac{1}{Z_N} e^{\beta H(\boldsymbol{\sigma})}$ defines a probability measure on the sphere S_{N-1}. It is interesting to study how far a random point $\boldsymbol{\sigma}$ on S_{N-1} under this probability measure is from the eigenspace for the largest eigenvalue of the matrix. One such a measurement is the random variable

$$\mathbb{E}_M\left[|\langle \boldsymbol{\sigma}, \boldsymbol{v}_1\rangle|^2\right] = \frac{1}{N}\int_{S_{N-1}} |\langle \boldsymbol{\sigma}, \boldsymbol{v}_1\rangle|^2 \frac{e^{\beta H(\boldsymbol{\sigma})}}{Z_N} \mathrm{d}\omega_N(\boldsymbol{\sigma}) \tag{2.13}$$

where $\boldsymbol{v}_1$ is an ℓ_2-normalized eigenvector associated with λ_1. Note that due to the absolute value in $|\langle \boldsymbol{\sigma}, \boldsymbol{v}_1\rangle|$, (2.13) does not depend on the choice of the eigenvector $\boldsymbol{v}_1$. It is easy to check that $\mathbb{E}_M\left[|\langle \boldsymbol{\sigma}, \boldsymbol{v}_1\rangle|^2\right] = -\frac{1}{\beta N Z_N}\frac{\partial Z_N}{\partial \lambda_1}$ and, from this formula, it is straightforward to prove that $\mathbb{E}_M\left[|\langle \boldsymbol{\sigma}, \boldsymbol{v}_1\rangle|^2\right] = O(N^{-1})$ for $\beta < \beta_c$ by modifying the analysis in Section 5 for the proof of Theorem 2.10. This is consistent with the fact that for the high temperature case, all eigenvalues contribute to the free energy and the fluctuations of F_N come from the fluctuations of certain linear statistics of all of the eigenvalues. On the other hand, for the super-critical case when $\beta > \beta_c$, we expect that $\mathbb{E}_M\left[|\langle \boldsymbol{\sigma}, \boldsymbol{v}_1\rangle|^2\right]$ converges to a constant that depends on β.

The constants appearing in the above theorems are given as follows. Note that $h(s) := \frac{1}{2}\int_{C_-}^{C_+} \frac{d\nu(x)}{s-x}$ is a decreasing function for real $s > C_+$ and $h(s) \to 0$ as $s \to +\infty$. Moreover, by the definition 2.8 of β_c, $h(s) \to \beta_c$ as $s \searrow C_+$. Hence for $\beta < \beta_c$, there is a unique $\widehat{\gamma} \equiv \widehat{\gamma}(\beta) \in (C_+, \infty)$ that satisfies

$$\frac{1}{2}\int_{C_-}^{C_+} \frac{d\nu(x)}{\widehat{\gamma} - x} = \beta. \tag{2.14}$$

Note that $\widehat{\gamma}(\beta)$ is an decreasing function in $\beta \in (0, \beta_c)$ and $\widehat{\gamma}(\beta) \searrow C_+$ as $\beta \nearrow \beta_c$.

Definition 2.13. Define

$$F(\beta) = \beta\widehat{\gamma}(\beta) - \frac{1}{2}\left(\int_{C_-}^{C_+} \log(\widehat{\gamma}(\beta) - k)d\nu(k) + 1 + \log(2\beta)\right), \qquad \beta < \beta_c, \tag{2.15}$$

and

$$F(\beta) = \beta C_+ - \frac{1}{2}\left(\int_{C_-}^{C_+} \log(C_+ - k)d\nu(k) + 1 + \log(2\beta)\right), \qquad \beta \geq \beta_c. \tag{2.16}$$

Furthermore, for $\beta < \beta_c$, define

$$\ell(\beta) = \ell_1(\beta) - \frac{1}{2}M(\varphi), \qquad \sigma^2(\beta) = \frac{1}{4}V(\varphi), \tag{2.17}$$

where

$$\ell_1(\beta) = \log(2\beta) - \frac{1}{2}\log\left(\int_{C_-}^{C_+} \frac{d\nu(k)}{(\widehat{\gamma}(\beta) - k)^2}\right) \tag{2.18}$$

and $M(\varphi)$ and $V(\varphi)$ are defined in Condition 2.5 with

$$\varphi(x) = \log(\widehat{\gamma}(\beta) - x). \tag{2.19}$$

2.3 Results for Hermitian matrices

All of the previous results hold for Hermitian matrices after the following simple changes:

1. The critical value in Definition 2.8 is changed to $\beta_c^H = 2\beta_c^{sym}$.
2. Theorems 2.9 and 2.10 hold without any changes.
3. Theorem 2.11 holds with TW_1 replaced by TW_2 in (2.12).
4. In Definition 2.13, all terms remain the same after the change that β is replaced by $\beta/2$. For example, $F^H(\beta) = F(\beta/2)$.

3 Examples

We list some random matrix ensembles, which satisfy the conditions in Subsection 2.1, and hence to which the general results in Subsections 2.2 and 2.3 apply.

3.1 Wigner matrix

A real Wigner matrix is an $N \times N$ real symmetric matrix M whose upper triangle entries M_{ij} $(i \le j)$ are independent real random variables satisfying the following conditions:

- The entries are centered, i.e., $\mathbb{E}[M_{ij}] = 0$ for all i, j.
- Their variances satisfy that $\mathbb{E}[|M_{ij}|^2] = \frac{1}{N}$ for $i \neq j$ and $\mathbb{E}[|M_{ii}|^2] = \frac{w_2}{N}$ for a constant $w_2 \ge 0$.
- For any integer $p > 2$, $\mathbb{E}[|M_{ij}|^p] = O(N^{-p/2})$. Moreover, $\mathbb{E}[|M_{ij}|^3]$ and $\mathbb{E}[|M_{ij}|^4]$ do not depend on i, j.

A complex Wigner matrix is an $N \times N$ complex Hermtian matrix M whose real and imaginary parts of the entries are all independent, modulo the Hermitian condition, and satisfy the same moments conditions as above and an extra condition that $\mathbb{E}[(M_{ij})^2] = 0$ for $i \neq j$.

Remark 3.1. We note that some of the Conditions in Subsection 2.1 are still satisfied even if some of the conditions on the definition of Wigner matrices, such as the existence of all moments, are relaxed. However, we content with the above definition of Wigner matrices so that all of the four Conditions in the previous section are simultaneously satisfied. Similar remark also applies to the random matrix ensembles in the next two subsections.

For real and complex Wigner matrices, the following are known:

1. Condition 2.3 (Regularity)

   For both real and complex case, the limiting spectral measure is given by the semicircle law,

   $$\frac{\mathrm{d}\nu}{\mathrm{d}x}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \qquad -2 \le x \le 2. \tag{3.1}$$

   Hence Condition 2.3 is satisfied with $C_+ = 2$ and $s_\nu = \frac{1}{\pi}$.

2. Condition 2.4 (Rigidity)

   Condition 2.4 was proved in [27] for both real and complex cases.

3. Condition 2.5 (Linear statistics)

   Condition 2.5 was proved in [9]. See also [7] and [33] for non-analytic test functions. The mean $M(\varphi)$ and the variance $V(\varphi)$ for function φ are as follows. Let

   $$w_2 := N\mathbb{E}[|M_{11}|^2], \qquad W_4 := N^2\mathbb{E}[|M_{12}|^4]. \tag{3.2}$$

   We have $w_2 = 2, W_4 = 3$ for GOE, and $w_2 = 1, W_4 = 2$ for Gaussian unitary ensemble (GUE). Set

   $$\tau_\ell(\varphi) = \frac{1}{\pi}\int_{-2}^{2} \varphi(x)\frac{T_\ell(x/2)}{\sqrt{4 - x^2}}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(2\cos\theta)\cos(\ell\theta)\mathrm{d}\theta \tag{3.3}$$

   for $\ell = 0, 1, 2, \cdots$, where $T_\ell(t)$ are the Chebyshev polynomials of the first kind; $T_0(t) = 1$, $T_1(t) = t$, $T_2(t) = 2t^2 - 1$, $T_3(t) = 4t^3 - 3t$, $T_4(t) = 8t^4 - 8t^2 + 1$, etc.

   The mean and the variance for the real case are

   $$M(\varphi) = M_{GOE}(\varphi) + \tau_2(\varphi)(w_2 - 2) + \tau_4(\varphi)(W_4 - 3) \tag{3.4}$$

and

$$V(\varphi) = V_{GOE}(\varphi) + \tau_1(\varphi)^2(w_2 - 2) + 2\tau_2(\varphi)^2(W_4 - 3), \tag{3.5}$$

respectively, where

$$M_{GOE}(\varphi) = \frac{1}{4}\left[\varphi(2) + \varphi(-2)\right] - \frac{1}{2}\tau_0(\varphi) \tag{3.6}$$

and

$$\begin{aligned} V_{GOE}(\varphi) &= 2\sum_{\ell=1}^{\infty} \ell\tau_\ell(\varphi)^2 \\ &= \frac{1}{2\pi^2}\int_{-2}^{2}\int_{-2}^{2}\left(\frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2}\right)^2 \frac{4 - \lambda_1\lambda_2}{\sqrt{4-\lambda_1^2}\sqrt{4-\lambda_2^2}}\mathrm{d}\lambda_1\mathrm{d}\lambda_2. \end{aligned} \tag{3.7}$$

For the complex case,

$$M^H(\varphi) = M_{GUE}(\varphi) + \tau_2(\varphi)(w_2 - 1) + \tau_4(\varphi)(W_4 - 2) \tag{3.8}$$

and

$$V^H(\varphi) = V_{GUE}(\varphi) + \tau_1(\varphi)^2(w_2 - 1) + 2\tau_2(\varphi)^2(W_4 - 2), \tag{3.9}$$

respectively, where

$$M_{GUE}(\varphi) = 0, \qquad V_{GUE}(\varphi) = V_{GOE}(\varphi)/2. \tag{3.10}$$

4. Condition 2.6 (Tracy-Widom limit)

   The Tracy-Widom distribution limit of the largest eigenvalue was proved in [45, 51, 27] for both real and complex cases.

We can evaluate the various constants appearing in the theorems in Subsection 2.2 explicitly and obtain following for real Wigner matrices.

(i) $\beta_c = \frac{1}{2}$.

(ii) The limit of the free energy is

$$F(\beta) = \begin{cases} \beta^2 & \text{if } 0 < \beta < 1/2 \\ 2\beta - \frac{\log(2\beta)+3/2}{2} & \text{if } \beta > 1/2. \end{cases} \tag{3.11}$$

(iii) For $\beta < \frac{1}{2}$, $N\left(F_N(\beta) - F(\beta)\right) \Rightarrow \mathcal{N}(\ell, \sigma^2)$ where

$$\ell = \frac{1}{4}\left(\log(1 - 4\beta^2) + 4\beta^2(w_2 - 2) + 8\beta^4(W_4 - 3)\right) \tag{3.12}$$

and

$$\sigma^2 = \frac{1}{2}\left(-\log(1 - 4\beta^2) + 2\beta^2(w_2 - 2) + 4\beta^4(W_4 - 3)\right). \tag{3.13}$$

(iv) For $\beta > \frac{1}{2}$, $(\beta - \frac{1}{2})^{-1}N^{2/3}\left(F_N(\beta) - F(\beta)\right) \Rightarrow TW_1$.

See Appendix A.1 for the detail. This proves Theorem 1.1 and Theorem 1.2.

For complex Wigner matrices, we have the following changes: (i) $\beta_c^H = 1$ and (ii) $L^H(\beta) = L(\beta/2)$. For (iii),

$$\ell = \frac{1}{2}\left(\log(1-\beta^2) + \beta^2(w_2 - 1) + \frac{\beta^4}{2}(W_4 - 2)\right) \tag{3.14}$$

and

$$\sigma^2 = -\log(1-\beta^2) + \beta^2(w_2 - 1) + \frac{\beta^4}{2}(W_4 - 2). \tag{3.15}$$

for $\beta < 1$. For (iv), β is replaced by $\beta/2$ and TW_1 by TW_2.

3.2 Invariant ensemble

The orthogonal invariant ensemble associated with potential $Q : \mathbb{R} \to \mathbb{R}$ is defined by the density

$$\mathcal{P}(M) = \frac{1}{Z}\exp(-\frac{N}{2}\operatorname{Tr} Q(M)) \tag{3.16}$$

on the space of $N \times N$ real symmetric matrices and Z is the normalization constant. Similarly, the unitary invariant ensemble associated with potential $Q : \mathbb{R} \to \mathbb{R}$ is defined by the density

$$\mathcal{P}^H(M) = \frac{1}{Z^H}\exp(-N\operatorname{Tr} Q(M)) \tag{3.17}$$

on the space of complex Hermitian matrices, where Z^H is the normalization constant. The GOE and the GUE correspond to the choice $Q(x) = x^2/2$. We assume that Q is a polynomial of even degree with positive leading coefficient. Many of the results below hold true for more general Q but we restrict to this class of Q for the convenience of the presentation. Furthermore, we assume that the associated equilibrium measure ν is of form

$$\mathrm{d}\nu(x) = \mathbb{1}_{[C_-,C_+]}(x)h(x)\sqrt{(C_+ - x)(x - C_-)}\mathrm{d}x \tag{3.18}$$

for constants $C_+ > C_-$ and for a function $h(x)$ that is real analytic and positive in an open set containing the interval $[C_-, C_+]$. Recall that for general Q, the support of ν may consist of several intervals. Here we make the single interval assumption in order to use the central limit theorem for linear statistics. On the other hand, the square root behavior at the end points of the support holds for generic Q [32]. We remark that (3.18) is guaranteed if Q is convex.

The following are known:

1. Condition 2.3 (Regularity)

   This holds from the assumption (3.18) since the limiting spectral measure is given by the equilibrium measure. We remark that, from the variational condition on ν, the following relation holds:

$$\frac{1}{2}Q'(C_+) = \int_{C_-}^{C_+} \frac{\mathrm{d}\nu(x)}{C_+ - x}. \tag{3.19}$$

2. Condition 2.4 (Rigidity)

   Condition 2.4 is proved in [16] (see also [15, 17]).

3. Condition 2.5 (Linear statistics)

   The linear statistics of eigenvalues was proved in [30]. The mean $M(\varphi)$ and the variance $V(\varphi)$ for function φ are as follows. For orthogonal invariant ensemble, we have

$$M_{OE}(\varphi) = \frac{1}{4}\left(\varphi(C_-) + \varphi(C_+)\right) - \int_{C_-}^{C_+} \frac{\varphi(\lambda) U_Q(\lambda)}{\sqrt{(\lambda - C_-)(C_+ - \lambda)}} d\lambda \tag{3.20}$$

   for some rational function U_Q which depends on Q. The formula of U_Q is complicated and is given in (3.54) of [30]. It is, in particular, given by $U_Q(x) = \frac{1}{2\pi}$ when $Q(x) = \frac{1}{2}x^2$. On the other hand,

$$V_{OE}(\varphi) = V_{GOE}(\Phi), \qquad \Phi(x) = \varphi\left(\frac{C_- + C_+}{2} + \frac{C_+ - C_-}{4} x\right), \tag{3.21}$$

   where V_{GOE} is the variance (3.7) for the GOE case. The change from φ to Φ comes from the simple translation of the interval (C_-, C_+) to $(-2, 2)$. Observe that the variance $V_{OE}(\varphi)$ does not depend on the structure of the equilibrium measure except for the end points C_- and C_+.On the contrary, the mean $M_{OE}(\varphi)$ depends on the full structure of the equilibrium measure.

   For unitary invariant ensemble, we have

$$M_{UE}(\varphi) = 0, \qquad V_{UE}(\varphi) = V_{OE}(\varphi)/2. \tag{3.22}$$

   We remark that Condition 2.5 is not always valid if $\operatorname{supp}\nu$ consists of multiple disjoint intervals. See [41] for more detail.

4. Condition 2.6 (Tracy-Widom limit)

   Condition 2.6 was proved in [42, 11, 26, 25, 16].

We can easily check from (3.19) that the critical value is

$$\beta_c^{OE} = \frac{1}{2}\int_{C_-}^{C_+} \frac{\mathrm{d}\nu(x)}{C_+ - x} = \frac{1}{4}Q'(C_+), \qquad \beta_c^{UE} = \frac{1}{2}Q'(C_+) \tag{3.23}$$

for orthogonal ensembles and unitary ensembles, respectively. Other constants appearing in the main theorems of this paper in the previous section may be evaluated for a given potential once the equilibrium measure is obtained.

3.3 Sample covariance matrix

Let X be a $K \times N$ matrix whose entries are independent real random variables satisfying the following conditions:

- The entries are centered, i.e., $\mathbb{E}[X_{ij}] = 0$.
- Their variances satisfy that $\mathbb{E}[|X_{ij}|^2] = \frac{1}{N}$.

- For any integer $p > 2$, $\mathbb{E}[|X_{ij}|^p] = O(N^{-p/2})$. Moreover, $\mathbb{E}[|X_{ij}|^3]$ and $\mathbb{E}[|X_{ij}|^4]$ do not depend on i, j.

A sample covariance matrix M is a random matrix of the form $M = X^* X$. When X is a complex matrix, then we assume, in addition, that $\mathbb{E}[(X_{ij})^2] = 0$. We assume further that $K \equiv K(N)$ with

$$\frac{K}{N} \to d \in [1, \infty) \tag{3.24}$$

as $N \to \infty$.

The following are known.

1. Condition 2.3 (Regularity)

   The limiting spectral measure is the Marchenko-Pastur distribution given by

   $$\mathrm{d}\nu(x) = \frac{1}{2\pi} \frac{\sqrt{(C_+ - x)(x - C_-)}}{x} \mathrm{d}x, \tag{3.25}$$

   with support $[C_-, C_+] = [(\sqrt{d} - 1)^2, (\sqrt{d} + 1)^2]$. Hence $s_\nu = \frac{d^{1/4}}{\pi(\sqrt{d}+1)^2}$.

2. Condition 2.4 (Rigidity)

   Condition 2.4 was proved in [44].

3. Condition 2.5 (Linear statistics)

   Condition 2.5 was proved in [6]. For non-analytic test functions, see [8] and [33]. The mean $M(\varphi)$ and the variance $V(\varphi)$ for function φ are as follows: see (1.3)–(1.5) in [8], (5.13) of [6], and (4.28) of [33]. Let $W_4 = N^2 \mathbb{E}[|X_{11}|^4]$. Set

   $$\Phi(x) = \varphi\left(\frac{C_- + C_+}{2} + \frac{C_+ - C_-}{4} x\right) = \varphi(d + 1 + x\sqrt{d}). \tag{3.26}$$

   For real sample covariance matrix,

   $$M(\varphi) = M_{GOE}(\Phi) - (W_4 - 3)\tau_2(\Phi), \qquad V(\varphi) = V_{GOE}(\Phi) + (W_4 - 3)\tau_1(\Phi)^2. \tag{3.27}$$

   For complex sample covariance matrix,

   $$M^{comp}(\varphi) = -(W_4 - 2)\tau_2(\Phi), \qquad V^{comp}(\varphi) = V_{GUE}(\Phi) + (W_4 - 2)\tau_1(\Phi)^2. \tag{3.28}$$

4. Condition 2.6 (Tracy-Widom limit)

   Condition 2.6 was proved in [46, 43, 52, 44].

Various constants can be evaluated explicitly and we obtain the following. See Appendix A.2 for the detail.

(i) $\beta_c = \frac{1}{2\sqrt{C_+}} = \frac{1}{2(\sqrt{d}+1)}$.

(ii) The limit of the free energy per particle is

$$F(\beta) = \begin{cases} -\frac{d}{2\beta}\log(1-2\beta), & \beta < \frac{1}{2(\sqrt{d}+1)}, \\ (1+\sqrt{d})^2 - \frac{1}{2\beta}\left((1+\sqrt{d}) + d\log\frac{\sqrt{d}}{1+\sqrt{d}} - \log\frac{1}{1+\sqrt{d}} + \log(2\beta)\right), & \beta > \frac{1}{2(\sqrt{d}+1)}. \end{cases} \tag{3.29}$$

(iii) For $\beta < \beta_c$, $N(F_N(\beta) - F(\beta)) \Rightarrow \mathcal{N}(\ell, \sigma^2)$ where where L is given in (3.29) and

$$\ell = \frac{1}{4\beta}\left(\log(1-4B^2) - 4B^2(W_4-3)\right), \quad \sigma^2 = \frac{1}{2\beta^2}\left(-\log(1-4B^2) + B^2(W_4-3)\right) \tag{3.30}$$

where we set

$$B = \frac{\beta\sqrt{d}}{1-2\beta}. \tag{3.31}$$

Note that $B < 1/2$ for $\beta < \beta_c$ and $B > 1/2$ for $\beta > \beta_c$.

(iv) For $\beta > \beta_c$, $\frac{(1+\sqrt{d})^{4/3}}{d^{1/6}}(\beta-\beta_c)^{-1}N^{2/3}(F_N(\beta) - F(\beta)) \Rightarrow TW_1$.

For complex sample covariance matrices, $\beta_c^H = \frac{1}{\sqrt{d}+1}$, $L^H(\beta) = L(\beta/2)$, and

$$\ell = \frac{1}{2\beta}\left(\log(1-4B^2) - 4B^2(W_4-2)\right), \quad \sigma^2 = \frac{1}{\beta^2}\left(-\log(1-4B^2) + 4B^2(W_4-2)\right). \tag{3.32}$$

4 Integral representation of the partition Function

Our starting point in the analysis is the integral representation of the free energy given in Lemma 1.3. We prove it here. As mentioned in Introduction, this formula was also obtained in [31], and a similar formula appeared in [34].

Proof of Lemma 1.3. Let $S^{N-1} = \{\boldsymbol{x} \in \mathbb{R}^N : \|\boldsymbol{x}\| = 1\}$, the unit sphere in $\mathbb{R}^N$, and let $\mathrm{d}\Omega$ be the surface area measure on S^{N-1}. Hence $\frac{\mathrm{d}\Omega}{|S^{N-1}|}$ is the uniform measure on S^{N-1}. We denote the left-hand side of (1.12) by Z_N as in (2.1). By change of variables,

$$Z_N = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} e^{\beta N\langle \boldsymbol{x}, M\boldsymbol{x}\rangle}\mathrm{d}\Omega. \tag{4.1}$$

We diagonalize M and let $M = O^T D O$ for an orthogonal matrix O and a diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$. Since $\langle \boldsymbol{x}, M\boldsymbol{x}\rangle = \langle O\boldsymbol{x}, DO\boldsymbol{x}\rangle$ and O is orthogonal, we find after the changes of variables $\boldsymbol{x} \mapsto O^{-1}\boldsymbol{x}$ that

$$Z_N = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} e^{\beta N\langle \boldsymbol{x}, D\boldsymbol{x}\rangle}\mathrm{d}\Omega = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} e^{\beta N\sum \lambda_i x_i^2}\mathrm{d}\Omega. \tag{4.2}$$

In order to evaluate the integral, we consider

$$J(z) := \int_{\mathbb{R}^N} e^{\beta N\sum \lambda_i y_i^2} e^{-\beta N z\sum y_i^2}\mathrm{d}\boldsymbol{y}, \qquad z > \lambda_1. \tag{4.3}$$

We evaluate $J(z)$ the above integral in two different ways. First we evaluate it directly using Gaussian integral and second, we use polar coordinates. By evaluating the Gaussian integrals, we obtain

$$J(z) = \left(\frac{\pi}{\beta N}\right)^{N/2} \prod_i \frac{1}{\sqrt{z-\lambda_i}}, \qquad z > \lambda_1. \tag{4.4}$$

On the other hand, by using polar coordinates, we substitute $\boldsymbol{y} = r\boldsymbol{x}$, $r > 0$, with $\|\boldsymbol{x}\| = 1$ in (4.3), and then set $\beta N r^2 = t$ to find that

$$J(z) = \frac{1}{2(\beta N)^{N/2}} \int_0^\infty e^{-zt} t^{(N/2)-1} I(t) \mathrm{d}t, \qquad I(t) := \int_{S^{N-1}} e^{t \sum \lambda_i x_i^2} \mathrm{d}\Omega. \tag{4.5}$$

Note that $J(z)$ is, up to a constant factor, the Laplace transform of $t^{(N/2)-1} I(t)$. Taking inverse Laplace transform and using (4.4), we obtain

$$\frac{t^{N/2-1} I(t)}{2(\beta N)^{N/2}} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{zt} J(z) \mathrm{d}z = \left(\frac{\pi}{\beta N}\right)^{N/2} \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{zt} \prod_i \frac{1}{\sqrt{z-\lambda_i}} \mathrm{d}z \tag{4.6}$$

where γ is an arbitrary real number satisfying $\gamma > \lambda_1$ since $J(z)$ is defined for $z > \lambda_1$. Since $Z_N = \frac{1}{|S^{N-1}|} I(\beta N)$, we obtain the desired lemma by setting $t = \beta N$ and recalling that $\frac{1}{|S^{N-1}|} = \frac{\Gamma(N/2)}{2\pi^{N/2}}$. □

From Lemma 1.3, the partition function (2.1) satisfies

$$Z_N = C_N \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2} G(z)} \mathrm{d}z, \qquad C_N = \frac{\Gamma(N/2)}{2\pi \mathrm{i} (N\beta)^{N/2-1}}, \tag{4.7}$$

where $\gamma > \lambda_1$, and

$$G(z) = 2\beta z - \frac{1}{N} \sum_i \log(z - \lambda_i). \tag{4.8}$$

We use the method of steepest-descent to this integral. The following lemma shows that there is a critical value of $G(z)$ on the part of the real line $z \in (\lambda_1, \infty)$ and we choose γ as this critical point.

Lemma 4.1. *There exists a unique $\gamma \in (\lambda_1, \infty)$ satisfying the equation $G'(\gamma) = 0$.*

Proof. This is immediately obtained by noting that

$$G'(z) = 2\beta - \frac{1}{N} \sum_i \frac{1}{z - \lambda_i} \tag{4.9}$$

is an increasing function of $z \in \mathbb{R}$ on the interval (λ_1, ∞) with

$$\lim_{z \searrow \lambda_1} G'(z) = -\infty, \qquad \lim_{z \to \infty} G'(z) = 2\beta > 0,$$

□

We also remark that

$$G''(z) = \frac{1}{N}\sum_i \frac{1}{(z-\lambda_i)^2} > 0, \qquad \text{for } z > \lambda_1. \tag{4.10}$$

Hence $z=\gamma$ is a saddle point of the real part of the function $G(z)$, and $\mathrm{Re}(G(z))$ decays fastest along the vertical line $z=\gamma+\mathrm{i}y$ as $|y|$ increases for small y.

Remark 4.2. The analogue of Lemma 1.3 for Hermitian matrices is the following. For a Hermitian matrix M with eigenvalues $\lambda_1 \ge \cdots \ge \lambda_N$, we have

$$\int_{\mathbb{C}S_{N-1}} e^{\beta\langle\boldsymbol{\sigma}, M\boldsymbol{\sigma}\rangle}\mathrm{d}\omega_N(\boldsymbol{\sigma}) = C_N^H \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G_H(z)}\mathrm{d}z, \quad G_H(z) = \beta z - \frac{1}{N}\sum_i \log(z-\lambda_i), \tag{4.11}$$

where

$$C_N^H = \frac{\Gamma(N)}{2\pi\mathrm{i}(N\beta)^{N-1}}. \tag{4.12}$$

This can be obtained similarly. By diagonalizing M and changing variables, the partition function is

$$Z_N = \frac{1}{|S^{2N-1}|}\int_{S^{2N-1}} e^{\beta N\sum\lambda_i(x_{2i-1}^2+x_{2i}^2)}\mathrm{d}\Omega.$$

This is evaluated by considering

$$J_H(z) := \int_{\mathbb{R}^{2N}} e^{\beta N\sum\lambda_i(y_{2i-1}^2+y_{2i}^2)} e^{-\beta N z\sum y_i^2}\mathrm{d}\boldsymbol{y}.$$

5 High temperature case

In this section, we consider the case $\beta < \beta_c$ and prove Theorem 2.10 for symmetric ensembles; the proof for Hermitian ensembles can be done in a similar manner by using (4.11) and we skip its proof.

We use the method of steepest-descent in order to evaluate the integral (4.7) asymptotically. Since $G(z)$ is random (since λ_i are random), the critical point γ from Lemma 4.1 is a random variable. We approximate γ by a non-random number $\widehat{\gamma}$, which is the critical point in the interval (C_+,∞) of the function

$$\widehat{G}(z) = 2\beta z - \int_{C_-}^{C_+}\log(z-k)d\nu(k). \tag{5.1}$$

The function $\widehat{G}(z)$ is the version of $G(z)$ in which the random spectral measure is replaced by the limiting non-random spectral measure. From the discussions around (2.14), if $\beta\in(0,\beta_c)$, there is unique $\widehat{\gamma}$ satisfying

$$\widehat{G}'(\widehat{\gamma}) = 2\beta - \int_{C_-}^{C_+}\frac{\mathrm{d}\nu(k)}{\widehat{\gamma}-k} = 0, \qquad \widehat{\gamma}\in(C_+,\infty). \tag{5.2}$$

We start with the following lemma, which holds for all $\beta > 0$.

Lemma 5.1. *Assume Condition 2.3 and Condition 2.4. Fix $\delta > 0$. Then the following hold.*

(i) For every $\epsilon > 0$,

$$G'(z) - \widehat{G}'(z) = O(N^{-1+\epsilon}) \tag{5.3}$$

uniformly in $z \geq C_+ + \delta$ with high probability.

(ii) For each $\ell = 0, 1, 2, \cdots$, the derivative $G^{(\ell)}(z) = O(1)$ uniformly in $z \in \mathbb{C} \setminus B_\delta$ with high probability where $B_\delta = \{x + \mathrm{i}y : C_- - \delta < x < C_+ + \delta,\ -\delta < y < \delta\}$.

Proof. For (i), set

$$\widetilde{G}(z) = 2\beta z - \frac{1}{N} \sum_i \log(z - \gamma_i) \tag{5.4}$$

where γ_i is the classical location of the i-th eigenvalue as defined in (2.5). Then from the rigidity, Condition 2.4,

$$\left| G'(z) - \widetilde{G}'(z) \right| = \left| \frac{1}{N} \sum_i \frac{(\lambda_i - \gamma_i)}{(z - \lambda_i)(z - \gamma_i)} \right| \leq \frac{C}{N} \sum_i |\lambda_i - \gamma_i| \leq \frac{CN^\epsilon}{N} \tag{5.5}$$

uniformly in $z \geq C_+ + \delta$ with high probability. On the other hand, we claim that

$$\left| \widetilde{G}'(z) - \widehat{G}'(z) \right| = \left| \frac{1}{N} \sum_i \frac{1}{z - \gamma_i} - \int_{C_-}^{C_+} \frac{\mathrm{d}\nu(k)}{z - k} \right| \leq \frac{C}{N} \tag{5.6}$$

uniformly in $z \geq C_+ + \delta$. For this, we define $\widehat{\gamma}_j$ by

$$\int_{\widehat{\gamma}_j}^{\infty} \mathrm{d}\nu(k) = \frac{j}{N}, \qquad j = 1, 2, \cdots, N, \tag{5.7}$$

with $\widehat{\gamma}_0 = C_+$. Note that $\widehat{\gamma}_i \leq \gamma_i \leq \widehat{\gamma}_{i-1}$. We then have for $i = 2, 3, \cdots, N-1$ that

$$\int_{\widehat{\gamma}_{i+1}}^{\widehat{\gamma}_i} \frac{\mathrm{d}\nu(k)}{z - k} \leq \frac{1}{N} \frac{1}{z - \gamma_i} \leq \int_{\widehat{\gamma}_{i-1}}^{\widehat{\gamma}_{i-2}} \frac{\mathrm{d}\nu(k)}{z - k} \tag{5.8}$$

for $z \geq C_+ + \delta$. Summing over i and using the trivial estimates $\frac{1}{N}\frac{1}{z-\gamma_i} = O(N^{-1})$ and $\int_{\widehat{\gamma}_i}^{\widehat{\gamma}_{i-1}} \frac{\mathrm{d}\nu(k)}{z-k} = O(N^{-1})$, we find that the desired claim holds. The estimates (5.5) and (5.6) imply (5.3).

The part (ii) of the Lemma follows straightforwardly from the formula of $G^{(\ell)}(z)$ and the rigidity, Condition 2.4. □

Corollary 5.2. *Assume Condition 2.3 and Condition 2.4. Let $\beta < \beta_c$. Let γ be the number defined in Lemma 4.1 and let $\widehat{\gamma}$ be defined in (5.2). Then for every $\epsilon > 0$,*

$$|\gamma - \widehat{\gamma}| \leq \frac{N^{2\epsilon}}{N} \tag{5.9}$$

with high probability. In particular, there is a constant $c > 0$ such that

$$\gamma - \lambda_1 > c \tag{5.10}$$

with high probability.

Proof. Since $\widehat{\gamma} > C_+$, choosing $\delta \in (0, (\widehat{\gamma} - C_+)/2)$ in Lemma 5.1, we find that

$$G'(\widehat{\gamma} \pm N^{-1+2\epsilon}) = \widehat{G}'(\widehat{\gamma} \pm N^{-1+2\epsilon}) + O(N^{-1+\epsilon}) \tag{5.11}$$

with high probability. Now, since $\widehat{G}'(\widehat{\gamma}) = 0$ and $\widehat{G}'''(z) = O(1)$ for z near $\widehat{\gamma}$, the Taylor expansion of $\widehat{G}$ implies that

$$G'(\widehat{\gamma} \pm N^{-1+2\epsilon}) = \pm \widehat{G}''(\widehat{\gamma}) N^{-1+2\epsilon} + O(N^{-2+4\epsilon}) + O(N^{-1+\epsilon}). \tag{5.12}$$

Noting that $\widehat{G}''(\widehat{\gamma}) = \int_{C_-}^{C_+} \frac{\mathrm{d}\nu(k)}{(\widehat{\gamma}-k)^2} > 0$, this shows that

$$G'(\widehat{\gamma} - N^{-1+2\epsilon}) < 0, \qquad G'(\widehat{\gamma} + N^{-1+2\epsilon}) > 0 \tag{5.13}$$

with high probability. Since $G'(z)$ is an increasing function of z, this proves (5.9).

The estimate (5.10) is a consequence of (5.9), (2.6), (2.7), and the fact that $\widehat{\gamma}$ is a non-random number, independent of N, satisfying $\widehat{\gamma} > C_+$. □

Corollary 5.3. *Assume Condition 2.3 and Condition 2.4 and let $\beta < \beta_c$. Then for every $\epsilon > 0$*

$$G(\gamma) = G(\widehat{\gamma}) + O(N^{-2+4\epsilon}), \qquad G''(\gamma) = G''(\widehat{\gamma}) + O(N^{-1+2\epsilon}) \tag{5.14}$$

with high probability.

Proof. From the Taylor expansion, the definition of γ, and Lemma 5.1 (ii),

$$G(\widehat{\gamma}) = G(\gamma) + G'(\gamma)(\widehat{\gamma} - \gamma) + O(|\widehat{\gamma} - \gamma|^2) = G(\gamma) + O(|\widehat{\gamma} - \gamma|^2) \tag{5.15}$$

and

$$G''(\gamma) = G''(\widehat{\gamma}) + O(|\gamma - \widehat{\gamma}|). \tag{5.16}$$

The estimate (5.14) now follows from Corollary 5.2. □

We now evaluate the integral (4.7) using the method of steepest-descent.

Lemma 5.4. *Assume Condition 2.3 and Condition 2.4 and let $\beta < \beta_c$. Then for every $\epsilon > 0$,*

$$\int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2} G(z)} \mathrm{d}z = \mathrm{i} e^{\frac{N}{2} G(\widehat{\gamma})} \sqrt{\frac{4\pi}{N G''(\widehat{\gamma})}} \left(1 + O(N^{-1+6\epsilon})\right) \tag{5.17}$$

with high probability.

Proof. We had chosen γ as the critical point of $G(z)$ such that $\gamma > \lambda_1$. For this proof, it is enough to use the straight line $\gamma + \mathrm{i}\mathbb{R}$ for the contour instead of the path of steepest-descent. Changing the variables, we have

$$\begin{aligned} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2} G(z)} \mathrm{d}z &= \frac{\mathrm{i}}{\sqrt{N}} \int_{-\infty}^{\infty} \exp\left[\frac{N}{2} G\big(\gamma + \mathrm{i}\frac{t}{\sqrt{N}}\big)\right] \mathrm{d}t \\ &= \frac{\mathrm{i} e^{\frac{N}{2} G(\gamma)}}{\sqrt{N}} \int_{-\infty}^{\infty} \exp\left[\frac{N}{2} \left(G\big(\gamma + \mathrm{i}\frac{t}{\sqrt{N}}\big) - G(\gamma)\right)\right] \mathrm{d}t. \end{aligned} \tag{5.18}$$

We now estimate the integral in the right hand side of (5.18). First, we have

$$
\begin{aligned}
&\int_{-N^\epsilon}^{N^\epsilon} \exp\left[\frac{N}{2}\left(G\big(\gamma+\mathrm{i}\frac{t}{\sqrt{N}}\big)-G(\gamma)\right)\right]\mathrm{d}t \\
&=\int_{-N^\epsilon}^{N^\epsilon} \exp\left[\frac{N}{2}\left(\frac{G''(\gamma)}{2}\frac{(\mathrm{i}t)^2}{N}+\frac{G'''(\gamma)}{6}\frac{(\mathrm{i}t)^3}{N^{3/2}}+O(N^{-2+4\epsilon})\right)\right]\mathrm{d}t \\
&=\int_{-N^\epsilon}^{N^\epsilon} \exp\left[-\frac{G''(\gamma)}{4}t^2\right]\mathrm{d}t-\mathrm{i}\int_{-N^\epsilon}^{N^\epsilon}\frac{G'''(\gamma)}{12}\frac{t^3}{\sqrt{N}}\exp\left[-\frac{G''(\gamma)}{4}t^2\right]\mathrm{d}t+O(N^{-1+6\epsilon})
\end{aligned}
\tag{5.19}
$$

with high probability, where we used that $G''(\gamma)>0$ and $G^{(\ell)}(\gamma+\mathrm{i}t)=O(1)$ for any $t\in\mathbb{R}$ and $\ell=3,4$ (see Lemma 5.1 (ii)). The integral in the middle vanishes since the integrand is an odd function of t. From the estimate $\int_{N^\epsilon}^\infty e^{-t^2}\mathrm{d}t=O(N^{-\epsilon}e^{-N^{2\epsilon}})$, we obtain that

$$
\int_{-N^\epsilon}^{N^\epsilon} \exp\left[\frac{N}{2}\left(G\big(\gamma+\mathrm{i}\frac{t}{\sqrt{N}}\big)-G(\gamma)\right)\right]\mathrm{d}t=\sqrt{\frac{4\pi}{G''(\gamma)}}\left(1+O(N^{-1+6\epsilon})\right). \tag{5.20}
$$

Second, the tail part of the integral in the right hand side of (5.18) satisfies

$$
\begin{aligned}
&\left|\int_{N^\epsilon}^{\infty} \exp\left[\frac{N}{2}\left(G\big(\gamma+\mathrm{i}\frac{t}{\sqrt{N}}\big)-G(\gamma)\right)\right]\mathrm{d}t\right| \\
&\le \int_{N^\epsilon}^{\infty}\left|\exp\left[-\frac{1}{2}\sum_{j=1}^{N}\log\left(\frac{\gamma-\lambda_j+\mathrm{i}tN^{-1/2}}{\gamma-\lambda_j}\right)\right]\right|\mathrm{d}t \\
&\le \int_{N^\epsilon}^{\infty}\exp\left[-\frac{N}{4}\log\left(1+\frac{ct^2}{N}\right)\right]\mathrm{d}t \\
&\le \int_{N^\epsilon}^{N} e^{-\frac{c}{8}N^{2\epsilon}}\mathrm{d}t+\int_{N}^{\infty}(cN^{-1}t^2)^{-N/4}\mathrm{d}t=O(e^{-N^\epsilon})+O(N^{-N/8})
\end{aligned}
\tag{5.21}
$$

with high probability, where we used (5.10). Thus, the tail part is negligible, and hence we obtain from (5.18) that

$$
\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)}\mathrm{d}z=\frac{\mathrm{i}e^{\frac{N}{2}G(\gamma)}}{\sqrt{N}}\sqrt{\frac{4\pi}{G''(\gamma)}}\left(1+O(N^{-1+6\epsilon})\right). \tag{5.22}
$$

Finally, using Corollary 5.3, we conclude that

$$
\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}G(z)}\mathrm{d}z=\frac{\mathrm{i}e^{\frac{N}{2}G(\widehat{\gamma})}}{\sqrt{N}}\sqrt{\frac{4\pi}{G''(\widehat{\gamma})}}\left(1+O(N^{-1+6\epsilon})\right) \tag{5.23}
$$

with high probability. □

We now prove Theorem 2.10.

Proof of Theorem 2.10. From Lemmas 1.3 and 5.4, for every $\epsilon>0$,

$$
Z_N=\mathrm{i}C_N e^{\frac{N}{2}G(\widehat{\gamma})}\sqrt{\frac{4\pi}{NG''(\widehat{\gamma})}}\left(1+O(N^{-1+\epsilon})\right) \tag{5.24}
$$

with high probability. Since

$$C_N = \frac{\Gamma(N/2)}{2\pi \mathrm{i}(N\beta)^{N/2-1}} = \frac{\sqrt{N}\beta}{\mathrm{i}\sqrt{\pi}(2\beta e)^{N/2}}\left(1 + O(N^{-1})\right) \tag{5.25}$$

from the Stirling's formula, we find that

$$Z_N = e^{\frac{N}{2}G(\widehat{\gamma})}\frac{2\beta}{(2\beta e)^{N/2}\sqrt{G''(\widehat{\gamma})}}\left(1 + O(N^{-1+\epsilon})\right) \tag{5.26}$$

and hence

$$F_N = \frac{1}{N}\log Z_N = \frac{1}{2}\left(G(\widehat{\gamma}) - 1 - \log(2\beta)\right) + \frac{1}{N}\left(\log(2\beta) - \frac{1}{2}\log G''(\widehat{\gamma})\right) + O(N^{-2+\epsilon}) \tag{5.27}$$

with high probability.

Define the functions

$$\varphi(x) := \log(\widehat{\gamma} - x), \qquad \psi(x) := \frac{1}{(\widehat{\gamma} - x)^2} \tag{5.28}$$

for $x \in [C_+, C_-]$ and extend φ and ψ to bounded C^∞ functions with compact support on the real line. We then have

$$G(\widehat{\gamma}) = 2\beta\widehat{\gamma} - \frac{1}{N}\sum_i \varphi(\lambda_i), \qquad G''(\widehat{\gamma}) = \frac{1}{N}\sum_i \psi(\lambda_i) \tag{5.29}$$

with high probability. Regarding $\widehat{\gamma}$ as a function of β, set

$$f_0(\beta) := \int_{C_-}^{C_+} \log(\widehat{\gamma} - k)\mathrm{d}\nu(k), \qquad f_2(\beta) := \int_{C_-}^{C_+} \frac{\mathrm{d}\nu(k)}{(\widehat{\gamma} - k)^2}. \tag{5.30}$$

We find from Condition 2.5 that

$$\mathcal{N}_\varphi := \sum_i \varphi(\lambda_i) - N\int_{C_-}^{C_+} \varphi(k)\mathrm{d}\nu(k) = \sum_i \varphi(\lambda_i) - Nf_0(\beta) \tag{5.31}$$

converges in distribution to Gaussian random variable with mean $M(\varphi)$ and variance $V(\varphi)$. Similarly, we also have that

$$\mathcal{N}_\psi := \sum_i \psi(\lambda_i) - N\int_{C_-}^{C_+} \psi(k)\mathrm{d}\nu(k) = \sum_i \psi(\lambda_i) - Nf_2(\beta) \tag{5.32}$$

converges in distribution to Gaussian random variable with mean $M(\psi)$ and variance $V(\psi)$.

Thus, (5.27) becomes

$$\begin{aligned} F_N = &\left(\beta\widehat{\gamma} - \frac{f_0(\beta) + 1 + \log(2\beta)}{2}\right) - \frac{1}{2N}\mathcal{N}_\varphi \\ &+ \frac{2\log(2\beta) - \log(f_2(\beta))}{2N} - \frac{1}{2N}\log\left(1 + \frac{\mathcal{N}_\psi}{Nf_2(\beta)}\right) + O(N^{-1+\epsilon}) \end{aligned} \tag{5.33}$$

with high probability. Since $\mathcal{N}_\psi$ converges to a Gaussian random variable, we see, in particular, that the random variable $\log(1 + \frac{\mathcal{N}_\psi}{Nf_2(\beta)})$ in the right hand side of (5.33) converges in distribution to 0 as $N \to \infty$. Since the convergence in distribution to the constant 0 implies the convergence in probability to 0, using Slutsky theorem, we conclude that

$$NF_N - N\left(\beta\widehat{\gamma} - \frac{f_0(\beta) + 1 + \log(2\beta)}{2}\right) - \frac{2\log(2\beta) - \log(f_2(\beta))}{2}$$

converges in distribution to a Gaussian with mean $-\frac{1}{2}M(\varphi)$ and variance $\frac{1}{4}V(\varphi)$. This completes the proof of the theorem.

□

6 Low temperature case

In this section, we consider the case $\beta > \beta_c$ and prove Theorem 2.11. As in the last section, we only give a proof for symmetric random matrices; the proof for Hermitian random matrices is similar.

As we saw in the previous section, the location of γ, the critical point of the function G, is crucial in the asymptotic evaluation of the integral in (4.7). When $\beta < \beta_c$ we approximated γ by the deterministic number $\widehat{\gamma}$ that is the critical point of the deterministic approximation $\widehat{G}(z)$ of the function $G(z)$. However, when $\beta > \beta_c$, $\widehat{G}(z)$ does not have any critical point in $z > C_+$ and we cannot approximate γ by a deterministic number. The following lemma shows that γ is close to λ_1 up to order about $1/N$.

Lemma 6.1. *Assume Conditions 2.3 and 2.4. Let $\beta > \beta_c$. Then for every $0 < \epsilon < \frac{1}{4}$,*

$$\frac{1}{3\beta N} \le \gamma - \lambda_1 \le \frac{N^{4\epsilon}}{N} \tag{6.1}$$

with high probability.

Proof. Since

$$G'(z) = 2\beta - \frac{1}{N}\sum_i \frac{1}{z - \lambda_i} < 2\beta - \frac{1}{N}\frac{1}{z - \lambda_1}, \qquad z > \lambda_1, \tag{6.2}$$

we see that $G'(\lambda_1 + \frac{1}{3\beta N}) < 0$.

Since $G'(z)$ is an increasing function for $z > \lambda_1$, the Lemma is proved if we show that $G'(\lambda_1 + N^{-1+4\epsilon}) > 0$ with high probability. By Condition 2.4, we may assume that the eigenvalues λ_i's satisfy the rigidity (2.6): this event occurs with high probability. For such λ_i's we need to show that $G'(\lambda_1 + N^{-1+4\epsilon}) > 0$. In order to show this, we write

$$G'(z) = 2\beta - \frac{1}{N}\sum_{i=1}^{N^{3\epsilon}} \frac{1}{z - \lambda_i} - \frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \frac{1}{z - \lambda_i} - \frac{1}{N}\sum_{i=N-N^{3\epsilon}+1}^{N} \frac{1}{z - \lambda_i} \tag{6.3}$$

with $z = \lambda_1 + N^{-1+4\epsilon}$. For $1 \le i \le N^{3\epsilon}$, since $\lambda_1 \ge \lambda_i$,

$$\frac{1}{N}\sum_{i=1}^{N^{3\epsilon}} \frac{1}{\lambda_1 + N^{-1+4\epsilon} - \lambda_i} = O(N^{-\epsilon}). \tag{6.4}$$

For $N^{3\epsilon} < i \le N - N^{3\epsilon}$, we have from the rigidity condition (2.6) that $|\lambda_i - \gamma_i| \le N^{-2/3}$. We also note that $C_+ - \gamma_i \ge C^{-1}N^{-2/3+2\epsilon}$ from (2.7). On the other hand, since $\lambda_1 - \gamma_1 = O(N^{-2/3+\epsilon})$

from (2.6) and $C_+ - \gamma_1 = O(N^{-2/3})$ from (2.7), we have $\lambda_1 = C_+ + O(N^{-2/3+\epsilon})$. Therefore we find that

$$\frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \frac{1}{\lambda_1 + N^{-1+4\epsilon} - \lambda_i} = \frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \frac{1}{C_+ - \gamma_i}\left(1 + O(N^{-\epsilon})\right). \tag{6.5}$$

The right hand side can be estimated by applying the idea used in the proof of Lemma 5.1. Recall the definition of $\widehat{\gamma}_i$ in (5.7). Summing the inequalities (see (5.8))

$$\int_{\widehat{\gamma}_{i+1}}^{\widehat{\gamma}_i} \frac{\mathrm{d}\nu(k)}{C_+ - k} \leq \frac{1}{N}\frac{1}{C_+ - \gamma_i} \leq \int_{\widehat{\gamma}_{i-1}}^{\widehat{\gamma}_{i-2}} \frac{\mathrm{d}\nu(k)}{C_+ - k} \tag{6.6}$$

over i from $N^{3\epsilon}+1$ to $N - N^{3\epsilon}$, and recalling that $\beta_c = \frac{1}{2}\int_{C_-}^{C_+} \frac{\mathrm{d}\nu(k)}{C_+ - k}$ by Definition 2.8, we find that

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \frac{1}{C_+ - \gamma_i} - 2\beta_c\right| = \left|\frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \frac{1}{C_+ - \gamma_i} - \int_{C_-}^{C_+} \frac{\mathrm{d}\nu(k)}{C_+ - k}\right| \\
&\leq C\left(\int_{C_-}^{\widehat{\gamma}_{N-N^{3\epsilon}-2}} \frac{\mathrm{d}\nu(k)}{C_+ - k} + \int_{\widehat{\gamma}_{N^{3\epsilon}+2}}^{C_+} \frac{\mathrm{d}\nu(k)}{C_+ - k}\right) = O(N^{-1/3+\epsilon}).
\end{aligned} \tag{6.7}$$

Finally, for $N - N^{3\epsilon} < i \leq N$, since $\lambda_1 - \lambda_i \geq 1$, we find

$$\frac{1}{N}\sum_{i=N-N^{3\epsilon}+1}^{N} \frac{1}{\lambda_1 + N^{-1+4\epsilon} - \lambda_i} = O(N^{-1+3\epsilon}). \tag{6.8}$$

Combining the estimates, we find that

$$G'(\lambda_1 + N^{-1+4\epsilon}) = 2\beta - 2\beta_c + o(1) > 0. \tag{6.9}$$

This proves the lemma. □

We will show in Lemma 6.3 below that the method of steepest-descent still applies and the main contribution to the integral representation of Z_N comes from $G(\gamma)$. The value of $G(\gamma)$ is, heuristically,

$$\begin{aligned}
G(\gamma) &= 2\beta\gamma - \frac{1}{N}\sum_{i=1}^{N} \log(\gamma - \lambda_i) \\
&\approx 2\beta\gamma - \frac{1}{N}\sum_{i=1}^{N}\left[\log(C_+ - \lambda_i) + \frac{1}{C_+ - \lambda_i}(\gamma - C_+)\right] \\
&\approx 2\beta\gamma - \int_{C_-}^{C_+} \log(C_+ - z)d\nu(z) - \int_{C_-}^{C_+} \frac{d\nu(z)}{C_+ - z}(\gamma - C_+) \\
&\approx 2\beta\lambda_1 - \int_{C_-}^{C_+} \log(C_+ - z)d\nu(z) - 2\beta_c(\lambda_1 - C_+)
\end{aligned} \tag{6.10}$$

where we used (2.9) and Lemma 6.1 for the last line. We show this approximation rigorously and also estimate the derivatives of $G(\gamma)$ in the next Lemma.

Lemma 6.2. *Assume Conditions 2.3 and 2.4. Let $\beta > \beta_c$. Let γ be the solution of the equation $G'(\gamma) = 0$ in Lemma 6.1. Then for $0 < \epsilon < \frac{1}{4}$*

$$G(\gamma) = 2\beta\lambda_1 - \int_{C_-}^{C_+} \log(C_+ - z) d\nu(z) - 2\beta_c(\lambda_1 - C_+) + O(N^{-1+4\epsilon}) \tag{6.11}$$

with high probability. Moreover, for $0 < \epsilon < \frac{1}{4}$, there is a constant $C_0 > 0$ such that

$$N^{\ell-1-4\ell\epsilon} \leq \frac{(-1)^\ell}{(\ell-1)!} G^{(\ell)}(\gamma) \leq C_0^\ell N^{\ell-1+3\epsilon} \tag{6.12}$$

for all $\ell = 2, 3, \cdots$ with high probability. Here, C_0 does not depend on ℓ.

Proof. We may assume that the eigenvalues λ_i's satisfy the rigidity Condition 2.4 and Lemma 6.1 since these event occurs with high probability.

Since we have from Lemma 6.1 that $\gamma = \lambda_1 + O(N^{-1+4\epsilon})$, the first part of the lemma is proved if we show that

$$\left| \frac{1}{N} \sum_{i=1}^N \log(\gamma - \lambda_i) - \int_{C_-}^{C_+} \log(C_+ - k) \mathrm{d}\nu(k) - 2\beta_c(\gamma - C_+) \right| = O(N^{-1+4\epsilon}). \tag{6.13}$$

Mimicking the proof of Lemma 6.1, we first consider $1 \leq i \leq N^{3\epsilon}$. Since $\gamma - \lambda_1 \geq \frac{1}{3\beta N}$ with high probability by Lemma 6.1, we have the trivial estimate

$$\left| \frac{1}{N} \sum_{i=1}^{N^{3\epsilon}} \log(\gamma - \lambda_i) \right| \leq \frac{CN^{3\epsilon}}{N} \log N. \tag{6.14}$$

Similarly,

$$\left| \frac{1}{N} \sum_{i=N-N^{3\epsilon}+1}^{N} \log(\gamma - \lambda_i) \right| \leq \frac{CN^{3\epsilon}}{N}. \tag{6.15}$$

We now consider the case $N^{3\epsilon} < i < N - N^{3\epsilon}$. Note that

$$\begin{aligned} \log(\gamma - \lambda_i) - \log(C_+ - \gamma_i) &= \log\left(1 + \frac{\gamma - C_+}{C_+ - \gamma_i} + \frac{\gamma_i - \lambda_i}{C_+ - \gamma_i}\right) \\ &= \frac{\gamma - C_+}{C_+ - \gamma_i} + O\left(\left(\frac{\gamma - C_+}{C_+ - \gamma_i}\right)^2\right) + O\left(\frac{\lambda_i - \gamma_i}{C_+ - \gamma_i}\right). \end{aligned} \tag{6.16}$$

The first error term can be estimated by

$$\left(\frac{\gamma - C_+}{C_+ - \gamma_i}\right)^2 \leq C \frac{N^{-4/3+2\epsilon}}{i^{4/3} N^{-4/3}} = CN^{2\epsilon} i^{-4/3} \tag{6.17}$$

from (2.7) and $\gamma - C_+ = (\gamma - \lambda_1) + (\lambda_1 - C_+) = O(N^{-2/3+\epsilon})$ due to Lemma 6.1. For the second error term in (6.16), we consider two different cases. For $N^{3\epsilon} < i \leq N/2$, we use the estimate

$$\frac{|\lambda_i - \gamma_i|}{C_+ - \gamma_i} \leq C \frac{i^{-1/3} N^{-2/3+\epsilon}}{i^{2/3} N^{-2/3}} = CN^\epsilon i^{-1} \tag{6.18}$$

from (2.5) and (2.7). For $N/2 < i < N - N^{3\epsilon}$, we simply use

$$\frac{|\lambda_i - \gamma_i|}{C_+ - \gamma_i} \le CN^{-1+\epsilon}. \tag{6.19}$$

Since

$$\frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} N^{2\epsilon} i^{-4/3} \le CN^{-1+\epsilon}, \qquad \frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N/2} N^{\epsilon} i^{-1} \le CN^{-1+\epsilon}\log N, \tag{6.20}$$

we obtain, after summing (6.16) over i, that

$$\left| \frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \log(\gamma - \lambda_i) - \frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \log(C_+ - \gamma_i) - \frac{\gamma - C_+}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \frac{1}{C_+ - \gamma_i} \right| \le \frac{CN^{\epsilon}\log N}{N}. \tag{6.21}$$

From (6.14), (6.15), and (6.21), we conclude that

$$\left| \frac{1}{N}\sum_{i=1}^{N} \log(\gamma - \lambda_i) - \frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \log(C_+ - \gamma_i) - \frac{\gamma - C_+}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \frac{1}{C_+ - \gamma_i} \right| = O(N^{-1+4\epsilon}). \tag{6.22}$$

We proved in (6.7) that

$$\left| \frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \frac{1}{C_+ - \gamma_i} - 2\beta_c \right| = O(N^{-1/3+\epsilon}). \tag{6.23}$$

On the other hand, following the arguments in (6.6) and (6.7), we obtain

$$\begin{aligned}
&\left| \frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N-N^{3\epsilon}} \log(C_+ - \gamma_i) - \int_{C_-}^{C_+} \log(C_+ - k)\mathrm{d}\nu(k) \right| \\
&\le C\left(\int_{C_-}^{\widehat{\gamma}_{N-N^{3\epsilon}-2}} \log(C_+ - k)\mathrm{d}\nu(k) + \int_{\widehat{\gamma}_{N^{3\epsilon}+2}}^{C_+} \log(C_+ - k)\mathrm{d}\nu(k) \right) \\
&\le CN^{-1+3\epsilon} + C\int_0^{N^{-2/3+2\epsilon}} \sqrt{\kappa}\log\kappa\, \mathrm{d}\kappa \\
&\le CN^{-1+3\epsilon}\log N.
\end{aligned} \tag{6.24}$$

Inserting these estimates into (6.22), we obtain (6.13). Hence the first part of the lemma is proved.

We now prove the second part of the lemma. We have

$$G^{(\ell)}(\gamma) = \frac{(-1)^{\ell}(\ell-1)!}{N}\sum_{i=1}^{N} \frac{1}{(\gamma - \lambda_i)^{\ell}} \tag{6.25}$$

for $\ell = 2, 3, \cdots$. For the lower bound, we use Lemma 6.1 to obtain

$$\frac{1}{N}\sum_{i=1}^{N} \frac{1}{(\gamma - \lambda_i)^{\ell}} \ge \frac{1}{N}\frac{1}{(\gamma - \lambda_1)^{\ell}} \ge N^{\ell-1-4\ell\epsilon}. \tag{6.26}$$

For the upper bound, we use

$$\frac{1}{N}\sum_{i=1}^{N^{3\epsilon}}\frac{1}{(\gamma-\lambda_i)^\ell}\le N^{-1+3\epsilon}\frac{1}{(\gamma-\lambda_1)^\ell}\le(3\beta)^\ell N^{\ell-1+3\epsilon} \tag{6.27}$$

and

$$\frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N}\frac{1}{(\gamma-\lambda_i)^\ell}\le\frac{1}{N}\sum_{i=N^{3\epsilon}+1}^{N}\frac{2^\ell}{(C_+-\gamma_i)^\ell}\le\frac{2^\ell}{N}\sum_{i=N^{3\epsilon}+1}^{N}\frac{1}{(i^{2/3}N^{-2/3})^\ell}\le 3\cdot 2^\ell N^{(\frac{2}{3}\ell-1)(1-3\epsilon)}. \tag{6.28}$$

This completes the proof of the lemma.

□

In the supercritical case, we have the following lemma corresponding to Lemma 5.4 in the subcritical case.

Lemma 6.3. *Assume Conditions 2.3 and 2.4. Let $\beta>\beta_c$. Let γ be the unique solution to the equation $G'(\gamma)=0$ defined in Lemma 4.1. Then, there exists $K\equiv K(N)$ satisfying $N^{-C}<K<C$ for some constant $C>0$ such that*

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}e^{\frac{N}{2}G(z)}\mathrm{d}z=\mathrm{i}e^{\frac{N}{2}G(\gamma)}K \tag{6.29}$$

with high probability.

Proof. Define K by the relation (6.29):

$$K:=-\mathrm{i}e^{-\frac{N}{2}G(\gamma)}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}e^{\frac{N}{2}G(z)}\mathrm{d}z. \tag{6.30}$$

Then K is real and positive since $Z>0$ in (4.7) and $G(\gamma)$ is real. We now estimate K.

Fix $0<\epsilon<\frac{1}{26}$. Since $\epsilon<\frac{1}{4}$, we may assume that, as in the previous two lemmas, the eigenvalues λ_i's satisfy the rigidity Condition 2.4, Lemma 6.1 and Lemma 6.2 since these event occurs with high probability.

To prove the upper bound for K, we consider

$$\begin{aligned}K&=-\mathrm{i}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\exp\left(\frac{N}{2}\left[G(z)-G(\gamma)\right]\right)\mathrm{d}z=\int_{-\infty}^{\infty}\exp\left(\frac{N}{2}\left[G(\gamma+\mathrm{i}t)-G(\gamma)\right]\right)\mathrm{d}t\\&=\int_{-\infty}^{\infty}\exp\left(\mathrm{i}N\beta t-\frac{1}{2}\sum_{j=1}^{N}\log\left(1+\frac{\mathrm{i}t}{\gamma-\lambda_j}\right)\right)\mathrm{d}t\\&\le\int_{-\infty}^{\infty}\exp\left(-\frac{1}{4}\sum_{j=1}^{N}\log\left(1+\frac{t^2}{(\gamma-\lambda_j)^2}\right)\right)\mathrm{d}t\end{aligned} \tag{6.31}$$

where the last step was obtained by taking the absolute value of the integrand. Since $\gamma-\lambda_i\le\gamma-\lambda_N\le C$ for some C (with high probability), we find that

$$K\le\int_{-\infty}^{\infty}\exp\left(-\frac{N}{4}\log\left(1+Ct^2\right)\right)\mathrm{d}t=\int_{-\infty}^{\infty}(1+Ct^2)^{-N/4}\mathrm{d}t\le C \tag{6.32}$$

for some $C > 0$, if N is sufficiently large. This proves the upper bound for K.

In order to prove the lower bound for K, we consider the curve of steepest-descent that passes through the point γ in the complex plane. This curve, denoted by Γ, satisfies $\operatorname{Im} G(z) = 0$. (Note that the real axis also satisfies $\operatorname{Im} G(z) = 0$.) It is easy to check, using the formula (4.8) of $G(z)$, that

(i) $\Gamma \cap \mathbb{C}_+$ is a C^1 curve,

(ii) Γ intersects the real axis only at γ,

(iii) the tangent line of Γ at γ is parallel to the imaginary axis, and

(iv) the real axis is the asymptote in the $-\infty$ direction.

Since all solutions of the equation $G'(z) = 0$ are on the real axis, the function $G(z)$ is decreasing along the curve $\Gamma \cap \mathbb{C}^+$ as z moves from the point γ to the point $-\infty$.

We first have

$$\int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2} G(z)} \mathrm{d}z = \int_\Gamma e^{\frac{N}{2} G(z)} \mathrm{d}z. \tag{6.33}$$

This follows by noting that $\operatorname{Re} G(z) \le 2\beta\gamma - \log(R/2)$, for $|z| = R$ such that $\operatorname{Re} z \le \gamma$, and hence, if we let C_R be the circular arc $|z| = R$ such that $\operatorname{Re} z \le \gamma$, then

$$\left| \int_{C_R} e^{\frac{N}{2} G(z)} \mathrm{d}z \right| \le \frac{C_N}{R^{N/4 - 1}} \to 0 \tag{6.34}$$

as $R \to \infty$ where C_N is a constant depending on N. From (6.33), we have

$$K = -\mathrm{i} \int_\Gamma \exp\left(\frac{N}{2} \left[G(z) - G(\gamma) \right] \right) \mathrm{d}z. \tag{6.35}$$

Let

$$\Gamma^+ = \Gamma \cap \mathbb{C}^+, \qquad \Gamma^- = \Gamma \cap \mathbb{C}^-. \tag{6.36}$$

If substitute $\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}y$ in (6.35), we see that the integral with respect to $\mathrm{d}x$ vanishes since $K > 0$. (More precisely, the contribution from $\exp[(N/2)(G(z) - G(\gamma)](-\mathrm{d}x)$ on Γ^+ is cancelled by the contribution from $\exp[(N/2)(G(z) - G(\gamma)]\mathrm{d}x$ on Γ^-.) We thus find, using that $\overline{G(\bar z)} = G(z)$, that

$$K = 2 \int_{\Gamma^+} \exp\left(\frac{N}{2} \left[G(z) - G(\gamma) \right] \right) \mathrm{d}y. \tag{6.37}$$

Now let $B_{N^{-2}}$ be the ball of radius N^{-2} centered at γ. Since $G(z)$ is analytic for z such that $\operatorname{Re} z > \lambda_1$, $G(z)$ is analytic in $\overline{B}_{N^{-2}}$ from Lemma 6.1. Hence for $z \in \Gamma^+ \cap B_{N^{-2}}$,

$$G(z) - G(\gamma) = \sum_{j=2}^{\infty} \frac{G^{(j)}(\gamma)}{j!} (z - \gamma)^j. \tag{6.38}$$

Due to the second part of Lemma 6.2, this power series converges uniformly in $\overline{B}_{N^{-2}}$ for all $N > C_0$ where C_0 is the constant in Lemma 6.2. Recall that $G^{(j)}(\gamma)$ is a real number for all j; more precisely, $(-1)^j G^{(j)}(\gamma) > 0$. Comparing the imaginary parts of the both sides of (6.38), we find that for $z \in \Gamma_+ \cap \overline{B}_{N^{-2}}$,

$$0 = G''(\gamma) \operatorname{Re}(z - \gamma) \operatorname{Im} z + \frac{G'''(\gamma)}{2} \big(\operatorname{Re}(z - \gamma) \big)^2 \operatorname{Im} z - \frac{G'''(\gamma)}{6} (\operatorname{Im} z)^3 + \widetilde{\Omega} \tag{6.39}$$

with

$$\widetilde{\Omega} = \sum_{j=4}^{\infty} \frac{G^{(j)}(\gamma)}{j!} \operatorname{Im}\left((z-\gamma)^j\right). \tag{6.40}$$

Note that $\operatorname{Im}\left((z-\gamma)^j\right)$ divided by $\operatorname{Im}(z-\gamma)$ is a polynomial in $\operatorname{Re}(z-\gamma)$ and $\operatorname{Im} z$. Dividing (6.39) by $\operatorname{Im} z$ and $G''(\gamma)$, we find that $z = (X+\gamma) + \mathrm{i}Y \in \Gamma_+ \cap \overline{B}_{N^{-2}}$ solves the equation

$$F(X,Y) = 0, \qquad F(X,Y) := X - \frac{\alpha}{2}X^2 + \frac{\alpha}{6}Y^2 + \Omega(X,Y) \tag{6.41}$$

where

$$\alpha := -\frac{G'''(\gamma)}{G''(\gamma)} > 0, \tag{6.42}$$

and

$$\Omega(X,Y) = \sum_{j=4}^{\infty} \frac{G^{(j)}(\gamma)}{j!G''(\gamma)} \frac{\operatorname{Im}\left((X+\mathrm{i}Y)^j\right)}{Y}. \tag{6.43}$$

Using the general inequality

$$|\operatorname{Im}\left((a+\mathrm{i}b)^j\right)| \le j|a+\mathrm{i}b|^{j-1}|b| \tag{6.44}$$

for real numbers a, b, for $j \ge 1$, which can be checked easily by an induction and the trivial bound $|\operatorname{Re}\left((a+\mathrm{i}b)^j\right)| \le |a+\mathrm{i}b|^j$, and using the second part of Lemma 6.2, we find that

$$|\Omega| \le \sum_{j=4}^{\infty} \frac{C_0^j N^{j-1+3\epsilon}}{N^{1-8\epsilon}} |X+\mathrm{i}Y|^{j-1} \le 2C_0^4 N^{11\epsilon}\left(X^2+Y^2\right) \tag{6.45}$$

for $z = (X+\gamma)+\mathrm{i}Y \in \Gamma^+ \cap \overline{B}_{N^{-2}}$, for all $N > 2C_0$. From the second part of Lemma 6.2,

$$\frac{1}{C_0} N^{1-15\epsilon} \le \alpha \le C_0^3 N^{1+11\epsilon}, \tag{6.46}$$

and since $\epsilon < \frac{1}{26}$, we have

$$N^{11\epsilon}X^2 \ll \alpha X^2 \ll X, \qquad N^{11\epsilon}Y^2 \ll \alpha Y^2 \tag{6.47}$$

uniformly for $z = (X+\gamma)+\mathrm{i}Y \in \overline{B}_{N^{-2}}$ where the notation $a_N \ll b_N$ for sequences a_N and b_N means that $\frac{a_N}{b_N} \to 0$ as $N \to \infty$. Hence equation (6.41) becomes

$$X(1+o(1)) + \frac{\alpha}{6}Y^2(1+o(1)) = 0, \tag{6.48}$$

and hence

$$X = -\frac{\alpha}{6}Y^2(1+o(1)), \qquad \alpha = -\frac{G'''(\gamma)}{G''(\gamma)} > 0, \tag{6.49}$$

uniformly for $z = (X+\gamma)+\mathrm{i}Y \in \Gamma^+ \cap \overline{B}_{N^{-2}}$. This shows, in particular, that $\operatorname{Re} z < \gamma$ for $z \in \Gamma^+ \cap \overline{B}_{N^{-2}}$. Moreover, it is direct to check, by proceeding as above, that

$$\partial_X F = 1 + o(1), \qquad \partial_Y F = \frac{\alpha}{3}Y(1+o(1)), \tag{6.50}$$

uniformly for $z=(X+\gamma)+\mathrm{i}Y \in \overline{B}_{N^{-2}}$. (Here, for the estimate of $\partial_Y \Omega$, we use the inequality

$$|\operatorname{Re}\left((a+\mathrm{i}b)^j - a^j\right)| \leq \frac{j(j-1)}{2}|a+\mathrm{i}b|^{j-2}|b|^2 \tag{6.51}$$

for real numbers a,b, for $j \geq 2$, which can be checked easily by an induction and the bound (6.44).) Therefore, $\Gamma_+ \cap \overline{B}_{N^{-2}}$ is a graph, and $\mathrm{d}y=\mathrm{d}Y$ is positive on $\Gamma^+ \cap \overline{B}_{N^{-2}}$ and Γ^+ intersects $\partial B_{N^{-2}}$ at exactly one point.

Let $z_2 \in \mathbb{C}^+$ be the point where Γ^+ and $\partial B_{N^{-2}}$ intersect. Since Γ_+ is a path of steepest-descent, $\operatorname{Im} G(z_2) = \operatorname{Im} G(\gamma) = 0$ and $G(z_2) < G(\gamma)$. From (6.38) and Lemma 6.2,

$$G(\gamma)-G(z_2) = |G(z_2)-G(\gamma)| \geq \frac{1}{4}N^{1-8\epsilon}|z_2-\gamma|^2 - \sum_{j=3}^{\infty}\frac{1}{j}C_0^j N^{j-1+3\epsilon}|z_2-\gamma|^j \geq \frac{1}{8}N^{-3-8\epsilon} \tag{6.52}$$

for all large enough N. On the other hand, consider $B_{N^{-3}}$, the ball of radius N^{-3} centered at γ, and let $z_3 \in \mathbb{C}^+$ be the point where Γ^+ and $\partial B_{N^{-3}}$ intersect. Then, by a similar argument,

$$G(\gamma)-G(z_3) = |G(z_3)-G(\gamma)| \leq 2C_0^2 N^{-5+3\epsilon} \tag{6.53}$$

for all large enough N where C_0 is the constant in Lemma 6.2.

Next, we claim that, for any decreasing function $f:\Gamma^+ \to \mathbb{R}$,

$$\int_{\Gamma^+} e^{f(z)}\mathrm{d}y \geq 0. \tag{6.54}$$

To check the claim, we parametrize the curve $\Gamma^+ = \Gamma^+(t)$, $t \in [0,1]$, with $\Gamma^+(0)=\gamma$ and $\Gamma^+(1) = -\infty$. Note that $\operatorname{Im}\Gamma^+(0) = \operatorname{Im}\Gamma^+(1)$. Suppose that $\operatorname{Im}\Gamma^+(t)$ increases on $(0,t_0)$ and decreases on $(t_0,1)$. (In particular, it attains its maximum at $t=t_0$.) Then, we have

$$\begin{aligned}
\int_{\Gamma^+} e^{f(z)}\mathrm{d}y &= \int_0^1 e^{f(\Gamma^+(t))}\operatorname{Im}(\Gamma^+)'(t)\mathrm{d}t \\
&= \int_0^{t_0} e^{f(\Gamma^+(t))}\operatorname{Im}(\Gamma^+)'(t)\mathrm{d}t + \int_{t_0}^1 e^{f(\Gamma^+(t))}\operatorname{Im}(\Gamma^+)'(t)\mathrm{d}t \\
&\geq \int_0^{t_0} e^{f(\Gamma^+(t_0))}\operatorname{Im}(\Gamma^+)'(t)\mathrm{d}t + \int_{t_0}^1 e^{f(\Gamma^+(t_0))}\operatorname{Im}(\Gamma^+)'(t)\mathrm{d}t \\
&= e^{f(\Gamma^+(t_0))}\int_0^1 \operatorname{Im}(\Gamma^+)'(t)\mathrm{d}t = 0.
\end{aligned} \tag{6.55}$$

This shows that the claim (6.54) holds in this case. For the general case, suppose that there are real numbers $0<t_0<t_1<\cdots<t_{2k}<1$ such that $\operatorname{Im}\Gamma^+(t)$ increases on $(0,t_0),(t_1,t_2),\cdots,(t_{2k-1},t_{2k})$ and decreases on $(t_0,t_1),(t_2,t_3),\cdots,(t_{2k},1)$. (From the explicit formula of $\operatorname{Im} G(z)=0$, it is direct to check that Γ^+ approaches to $-\infty$ monotonically downward, and hence there are only finitely many such real numbers t_i's.) Set

$$\Delta_{2p} = \operatorname{Im}\Gamma^+(t_{2p}) - \operatorname{Im}\Gamma^+(t_{2p-1}), \qquad \Delta_{2p+1} = \operatorname{Im}\Gamma^+(t_{2p}) - \operatorname{Im}\Gamma^+(2p+1), \qquad (p=0,1,\cdots,k)$$

with $t_{-1}:=0$ and $t_{2k+1}:=1$. Note that $\Delta_{2p}\geq 0$, $\Delta_{2p+1}\geq 0$, and

$$\Delta_0+\Delta_2+\cdots+\Delta_{2p} \geq \Delta_1+\Delta_3+\cdots+\Delta_{2p+1}$$

for any $p = 0, 1, \cdots, k$ since Γ^+ is above the real axis. Applying the same strategy as in the case $k = 0$, we obtain

$$\begin{aligned}
&\int_0^1 e^{f(\Gamma^+(t))}\operatorname{Im}(\Gamma^+)'(t)\mathrm{d}t \\
&\geq e^{f(\Gamma^+(t_0))}(\Delta_0 - \Delta_1) + e^{f(\Gamma^+(t_2))}(\Delta_2 - \Delta_3) + \cdots + e^{f(\Gamma^+(t_{2k}))}(\Delta_{2k} - \Delta_{2k+1}) \\
&\geq e^{f(\Gamma^+(t_2))}(\Delta_0 - \Delta_1 + \Delta_2 - \Delta_3) + e^{f(\Gamma^+(t_4))}(\Delta_4 - \Delta_5) + \cdots + e^{f(\Gamma^+(t_{2k}))}(\Delta_{2k} - \Delta_{2k+1}) \\
&\geq e^{f(\Gamma^+(t_{2k}))}(\Delta_0 - \Delta_1 + \Delta_2 - \Delta_3 + \cdots + \Delta_{2k} - \Delta_{2k+1}) \geq 0.
\end{aligned} \tag{6.56}$$

This proves the claim (6.54).

We apply (6.54) to the function

$$f(z) = \begin{cases} \frac{N}{2}\left(G(z_2) - G(\gamma)\right) & \text{if } z \in \Gamma_+ \cap B_{N^{-2}} \\ \frac{N}{2}\left(G(z) - G(\gamma)\right) & \text{if } z \in \Gamma^+ \cap (B_{N^{-2}})^c, \end{cases}$$

and obtain

$$\int_{\Gamma^+ \cap B_{N^{-2}}} \exp\left(\frac{N}{2}\left[G(z_2) - G(\gamma)\right]\right)\mathrm{d}y + \int_{\Gamma^+ \cap (B_{N^{-2}})^c} \exp\left(\frac{N}{2}\left[G(z) - G(\gamma)\right]\right)\mathrm{d}y \geq 0. \tag{6.57}$$

Now, since $\mathrm{d}y$ is positive in $\Gamma^+ \cap B_{N^{-2}}$, we have the estimate (recall (6.37))

$$\begin{aligned}
\frac{1}{2}K &= \int_{\Gamma^+} \exp\left(\frac{N}{2}\left[G(z) - G(\gamma)\right]\right)\mathrm{d}y \\
&= \int_{\Gamma^+ \cap B_{N^{-3}}} \exp\left(\frac{N}{2}\left[G(z) - G(\gamma)\right]\right)\mathrm{d}y + \int_{\Gamma^+ \cap (B_{N^{-3}})^c} \exp\left(\frac{N}{2}\left[G(z) - G(\gamma)\right]\right)\mathrm{d}y \\
&\geq \int_{\Gamma^+ \cap B_{N^{-3}}} \exp\left(\frac{N}{2}\left[G(z_3) - G(\gamma)\right]\right)\mathrm{d}y + \int_{\Gamma^+ \cap (B_{N^{-3}})^c} \exp\left(\frac{N}{2}\left[G(z) - G(\gamma)\right]\right)\mathrm{d}y.
\end{aligned} \tag{6.58}$$

Subtracting (6.57) from (6.58), we find that

$$\begin{aligned}
\frac{1}{2}K &\geq \int_{\Gamma^+ \cap B_{N^{-3}}} \left[\exp\left(\frac{N}{2}\left[G(z_3) - G(\gamma)\right]\right) - \exp\left(\frac{N}{2}\left[G(z_2) - G(\gamma)\right]\right)\right]\mathrm{d}y \\
&\qquad + \int_{\Gamma^+ \cap (B_{N^{-3}})^c \cap B_{N^{-2}}} \left[\exp\left(\frac{N}{2}\left[G(z) - G(\gamma)\right]\right) - \exp\left(\frac{N}{2}\left[G(z_2) - G(\gamma)\right]\right)\right]\mathrm{d}y \\
&\geq \int_{\Gamma^+ \cap B_{N^{-3}}} \left[\exp\left(\frac{N}{2}\left[G(z_3) - G(\gamma)\right]\right) - \exp\left(\frac{N}{2}\left[G(z_2) - G(\gamma)\right]\right)\right]\mathrm{d}y.
\end{aligned} \tag{6.59}$$

Since $\int_{\Gamma^+ \cap B_{N^{-3}}} \mathrm{d}y = \operatorname{Im} z_3 \geq C|z_3 - \gamma| = CN^{-3}$ for some constant $C > 0$ (see (6.49) and (6.46)), we find, using the estimates (6.52) and (6.53), that

$$\frac{1}{2}K \geq CN^{-3}\left[\exp\left(-C_0^2 N^{-4+3\epsilon}\right) - \exp\left(-\frac{1}{16}N^{-2-8\epsilon}\right)\right] \geq CN^{-5-8\epsilon}. \tag{6.60}$$

This completes the proof of the lemma. □

We now prove Theorem 2.11.

Proof of Theorem 2.11. From (4.7), (5.25), and Lemma 6.3, we obtain

$$Z_N = \frac{\sqrt{N}\beta}{\mathrm{i}\sqrt{\pi}(2\beta e)^{N/2}} e^{\frac{N}{2}G(\gamma)} K(1 + O(N^{-1})) \tag{6.61}$$

with high probability, where K satisfies $N^{-C} \le K \le C$. Thus, using Lemma 6.2, we find that

$$\begin{aligned} F_N = \frac{1}{N}\log Z_N &= \frac{1}{2}\left[G(\gamma) - 1 - \log(2\beta)\right] + O(N^{-1}\log N) \\ &= \beta\lambda_1 - \beta_c(\lambda_1 - C_+) - \frac{1}{2}\left(\int_{C_-}^{C_+} \log(C_+ - z)d\nu(z) + 1 + \log(2\beta)\right) + O(N^{-1+4\epsilon}) \\ &= \frac{1}{2}\left[2\beta\lambda_1 - 2\beta_c(\lambda_1 - C_+) - \int_{C_-}^{C_+} \log(C_+ - z)d\nu(z) - 1 - \log(2\beta)\right] + O(N^{-1+4\epsilon}) \end{aligned} \tag{6.62}$$

with high probability. Hence, we obtain that

$$F_N - F(\beta) = (\beta - \beta_c)(\lambda_1 - C_+) + O(N^{-1+4\epsilon}). \tag{6.63}$$

where $F(\beta)$ is defined in (2.16). Now the proof of the theorem follows from the edge universality, Condition 2.6. □

7 Third order phase transition

In this section, we prove Theorem 2.9.

Proof of Theorem 2.9. The convergence of F_N to $F(\beta)$ in distribution follows from Theorem 2.10 and 2.11 since normal distribution and Tracy-Widom distribution have exponential tails.

We now prove that $F(\beta)$ is C^2 but not C^3 at $\beta = \beta_c$. It suffices to prove the statement for

$$\widetilde{F}(\beta) := \frac{1}{\beta}\left(F(\beta) + \frac{1 + \log(2\beta)}{2}\right). \tag{7.1}$$

Set

$$f_0(\beta) = \int_{C_-}^{C_+} \log(\widehat{\gamma} - k)\mathrm{d}\nu(k) \tag{7.2}$$

as in (5.30). Recall that $\widehat{\gamma} \equiv \widehat{\gamma}(\beta)$ is a function of β and and it has a limit as $\beta \nearrow \beta_c$: $\widehat{\gamma}(\beta_c) = C_+$ (see the discussion after (2.14)). Hence, if we set

$$f_0(\beta_c) = \int_{C_-}^{C_+} \log(C_+ - k)\mathrm{d}\nu(k), \tag{7.3}$$

then $f_0(\beta) \to f_0(\beta_c)$ as $\beta \nearrow \beta_c$. With this definition,

$$\widetilde{F}(\beta) = \begin{cases} \widehat{\gamma}(\beta) - \frac{1}{2\beta} f_0(\beta), & \beta < \beta_c, \\ C_+ - \frac{1}{2\beta} f_0(\beta_c), & \beta \ge \beta_c. \end{cases} \tag{7.4}$$

From this it easy to see that

$$\lim_{\beta\searrow\beta_c}\partial_\beta\widetilde{F}(\beta)=\frac{f_0(\beta_c)}{2\beta_c^2},\qquad \lim_{\beta\searrow\beta_c}\partial_\beta^2\widetilde{F}(\beta)=-\frac{f_0(\beta_c)}{\beta_c^3},\qquad \lim_{\beta\searrow\beta_c}\partial_\beta^3\widetilde{F}(\beta)=\frac{3f_0(\beta_c)}{\beta_c^4}. \tag{7.5}$$

We now calculate the limits as $\beta\nearrow\beta_c$. Since $\widehat{\gamma}(\beta)$ satisfies (see (2.14))

$$\int_{C_-}^{C_+}\frac{\mathrm{d}\nu(k)}{\widehat{\gamma}(\beta)-k}=2\beta, \tag{7.6}$$

we find that

$$\partial_\beta f_0(\beta)=(\partial_\beta\widehat{\gamma})\int_{C_-}^{C_+}\frac{\mathrm{d}\nu(k)}{\widehat{\gamma}-k}=(2\beta)\partial_\beta\widehat{\gamma}. \tag{7.7}$$

Thus,

$$\partial_\beta\widetilde{F}(\beta)=\partial_\beta\left(\widehat{\gamma}(\beta)-\frac{f_0(\beta)}{2\beta}\right)=\frac{f_0(\beta)}{2\beta^2}, \tag{7.8}$$

and hence

$$\lim_{\beta\nearrow\beta_c}\partial_\beta\widetilde{F}(\beta)=\frac{f_0(\beta_c)}{2\beta_c^2}=\lim_{\beta\searrow\beta_c}\partial_\beta\widetilde{F}(\beta). \tag{7.9}$$

Differentiating (7.8) again, we obtain

$$\partial_\beta^2\widetilde{F}(\beta)=\frac{\partial_\beta\widehat{\gamma}}{\beta}-\frac{f_0(\beta)}{\beta^3}. \tag{7.10}$$

We claim that $\partial_\beta\widehat{\gamma}\to 0$ as $\beta\nearrow\beta_c$. This can be checked by differentiating (7.6),

$$-(\partial_\beta\widehat{\gamma})\int_{C_-}^{C_+}\frac{\mathrm{d}\nu(k)}{(\widehat{\gamma}-k)^2}=2, \tag{7.11}$$

and noting that the integral tends to ∞ as $\beta\nearrow\beta_c$ due to the square root decay of $\frac{\mathrm{d}\nu}{dk}$. Therefore,

$$\lim_{\beta\nearrow\beta_c}\partial_\beta^2\widetilde{F}(\beta)=-\frac{f_0(\beta)}{\beta^3}=\lim_{\beta\searrow\beta_c}\partial_\beta^2\widetilde{F}(\beta). \tag{7.12}$$

Finally, after differentiating (7.10),

$$\partial_\beta^3\widetilde{F}(\beta)=\frac{\partial_\beta^2\widehat{\gamma}}{\beta}-\frac{\partial_\beta\widehat{\gamma}}{\beta^2}+\frac{3f_0(\beta)}{\beta^4}. \tag{7.13}$$

Since $\partial_\beta\widehat{\gamma}\to 0$ as $\beta\nearrow\beta_c$,

$$\lim_{\beta\nearrow\beta_c}\partial_\beta^3\widetilde{F}(\beta)=\frac{\lim_{\beta\nearrow\beta_c}\partial_\beta^2\widehat{\gamma}}{\beta_c}+\frac{3f_0(\beta_c)}{\beta_c^4}. \tag{7.14}$$

Comparing with (7.5), it suffices to show that $\lim\limits_{\beta\nearrow\beta_c}\partial_\beta^2\widehat{\gamma}\neq 0$ to prove the theorem. Differentiating (7.11) once more, we obtain

$$-(\partial_\beta^2\widehat{\gamma})\int_{C_-}^{C_+}\frac{\mathrm{d}\nu(k)}{(\widehat{\gamma}-k)^2}+2(\partial_\beta\widehat{\gamma})^2\int_{C_-}^{C_+}\frac{\mathrm{d}\nu(k)}{(\widehat{\gamma}-k)^3}=0. \tag{7.15}$$

Since $\frac{\mathrm{d}\nu}{dk}$ has the square root decay, we have

$$\frac{2}{\partial_\beta \widehat{\gamma}} = -\int_{C_-}^{C_+} \frac{\mathrm{d}\nu(k)}{(\widehat{\gamma}-k)^2} \sim (\widehat{\gamma}-C_+)^{-1/2}, \qquad \int_{C_-}^{C_+} \frac{\mathrm{d}\nu(k)}{(\widehat{\gamma}-k)^3} \sim (\widehat{\gamma}-C_+)^{-3/2}, \tag{7.16}$$

as $\beta \nearrow \beta_c$. We thus conclude from (7.15) that $\partial_\beta^2 \widehat{\gamma} \sim 1$. This completes the proof of the desired theorem. □

A Appendix

In this appendix, we evaluate various constants stated in Section 3. As a main tool of the evaluation, we use the Stieltjes transform of the measure ν, defined by

$$m(z) = \int_{C_-}^{C_+} \frac{\mathrm{d}\nu(x)}{x-z}, \qquad z \in \mathbb{C} \setminus [C_-, C_+). \tag{A.1}$$

Note that due to the square root decay of $\frac{\mathrm{d}\nu(x)}{dx}$ at $x = C_+$, $m(C_+)$ is well-defined.

A.1 Wigner matrix

We only consider real Wigner matrices. The complex case can be evaluated by the same way and we skip the details. The limiting spectral measure for real Wigner matrices is $\frac{\mathrm{d}\nu(x)}{dx} = \frac{1}{2\pi}\sqrt{4-x^2}$, $-2 \le x \le 2$. Hence

$$m(z) = \frac{-z + \sqrt{z^2-4}}{2}, \tag{A.2}$$

and we find that

$$\beta_c = \frac{1}{2}\int_{C_-}^{C_+} \frac{\mathrm{d}\nu(x)}{C_+ - x} = -\frac{1}{2} m(2) = \frac{1}{2}. \tag{A.3}$$

We now evaluate $\widehat{\gamma}$ defined in (2.14). This equation is equivalent to the equation $-m_\nu(\widehat{\gamma}) = 2\beta$. Solving this equation, we find that

$$\widehat{\gamma} = 2\beta + \frac{1}{2\beta}. \tag{A.4}$$

To evaluate (2.15) and (2.16), we note that

$$\int_{-2}^{2} \log(z-x)\mathrm{d}\nu(x) = \frac{1}{4} z\left(z - \sqrt{z^2-4}\right) + \log\left(z + \sqrt{z^2-4}\right) - \log 2 - \frac{1}{2}. \tag{A.5}$$

for $z \in \mathbb{C} \setminus (-\infty, 2]$. This follows by noting that

$$-m(z) = \frac{d}{dz}\left[\frac{1}{4} z\left(z - \sqrt{z^2-4}\right) + \log\left(z + \sqrt{z^2-4}\right)\right] \tag{A.6}$$

and $\int_{-2}^{2} \log(z-x)\mathrm{d}\nu(x) = \log z + O(z^{-1})$ as $z \to \infty$. Thus, using (A.4), we obtain

$$F(\beta) = \begin{cases} \beta^2 & \text{if } \beta < 1/2 \\ 2\beta - \frac{\log(2\beta)+3/2}{2} & \text{if } \beta > 1/2. \end{cases} \tag{A.7}$$

We now evaluate $\ell(\beta)$ and $\sigma^2(\beta)$ for the case when $\beta < \beta_c$. First, since

$$\int_{C_-}^{C_+} \frac{\mathrm{d}\nu(x)}{(z-x)^2} = m'(z) = -\frac{1}{2} + \frac{z}{2\sqrt{z^2-4}}, \qquad z \in \mathbb{C} \setminus [-2,2], \tag{A.8}$$

(2.18) becomes

$$\ell_1(\beta) = \log(2\beta) - \frac{1}{2}\log\left(m'(\widehat{\gamma}(\beta))\right) = \frac{1}{2}\log(1-4\beta^2). \tag{A.9}$$

Second, we evaluate (see (3.3)) $\tau_\ell(\varphi) = t_\ell(\widehat{\gamma}(\beta))$ for $\ell = 0, 1, 2,$ and 4, where

$$t_\ell(z) = \frac{1}{\pi}\int_{-2}^{2} \log(z-x)\frac{T_\ell(x/2)}{\sqrt{4-x^2}}dx, \qquad z \in \mathbb{C} \setminus (-\infty, 2), \tag{A.10}$$

We start with evaluating

$$t'_\ell(z) = \frac{1}{\pi}\int_{-2}^{2} \frac{T_\ell(x/2)}{(z-x)\sqrt{4-x^2}}dx. \tag{A.11}$$

Recall the recursions $T_{\ell+2}(y) = 2yT_{\ell+1}(y) - T_\ell(y)$, $\ell \ge 0$, and the orthogonality relation $\int_{-1}^{1} \frac{y^k T_\ell(y)}{\sqrt{1-y^2}}dy = 0$ for all integers $0 \le k < \ell$, for the Chebyshev polynomials. From this we obtain the recursions

$$t'_{\ell+2}(z) = zt'_{\ell+1}(z) - t'_\ell(z) \tag{A.12}$$

for $\ell \ge 0$. Since $T_0(x/2) = 1$ and $T_1(x/2) = x/2$, it is easy to check from residue calculus that

$$t'_0(z) = \frac{1}{\sqrt{z^2-4}}, \qquad t'_1(z) = \frac{z}{2\sqrt{z^2-4}} - \frac{1}{2}. \tag{A.13}$$

Hence

$$t'_2(z) = \frac{z^2-2}{2\sqrt{z^2-4}} - \frac{z}{2}, \qquad t'_4(z) = \frac{z^4-4z^2+2}{2\sqrt{z^2-4}} - \frac{z^3}{2} + z. \tag{A.14}$$

Taking the anti-derivaties and noting from (A.10) that $\tau_0(z) = \log z + O(1/z)$ and $t_\ell(z) = O(1/z)$, $\ell \ge 1$, as $z \to \infty$ using the orthogonality relations of the Chebyshev polynomials, we find that

$$t_0(z) = \log\left(z + \sqrt{z^2-4}\right) - \log 2, \qquad t_1(z) = \frac{1}{2}\sqrt{z^2-4} - \frac{z}{2}, \tag{A.15}$$

and

$$t_2(z) = \frac{z}{4}\sqrt{z^2-4} - \frac{z^2}{4} + \frac{1}{2}, \qquad t_4(z) = \frac{1}{8}(z^3-2z)\sqrt{z^2-4} - \frac{z^4}{8} + \frac{z^2}{2} - \frac{1}{4}. \tag{A.16}$$

Hence,

$$\tau_0(\varphi) = -\log(2\beta), \quad \tau_1(\varphi) = -2\beta, \quad \tau_2(\varphi) = -2\beta^2, \quad \tau_4(\varphi) = -4\beta^4. \tag{A.17}$$

Third, $V_{GOE}(\varphi)$ (see (3.7)) is evaluated from the lemma A.1 below. Hence we find that

$$V_{GOE}(\varphi) = \frac{1}{2\pi^2}L(\widehat{\gamma}(\beta), \widehat{\gamma}(\beta)) = -2\log(1-4\beta^2) \tag{A.18}$$

where $F(z,w)$ is defined in (A.20). We also have (see (3.6))

$$M_{GOE}(\varphi) = \frac{1}{2}\log(1-4\beta^2). \tag{A.19}$$

Therefore, combining with (A.9), (A.17), we obtain (3.12) and (3.13).

It remains to prove following Lemma.

Lemma A.1. *Set*

$$L(z,w)=\int_{-2}^{2}\int_{-2}^{2}\left(\log(z-x)-\log(z-y)\right)\left(\log(w-x)-\log(w-y)\right)Q(x,y)\mathrm{d}x\mathrm{d}y, \tag{A.20}$$

for $z,w\in\mathbb{C}\setminus(-\infty,2]$*, where*

$$Q(x,y)=\frac{4-xy}{(x-y)^2\sqrt{4-x^2}\sqrt{4-y^2}}. \tag{A.21}$$

Then

$$L(z,w)=2\pi^2\log\left[\frac{(z+R(z))(w+R(w))}{2(zw-4+R(z)R(w))}\right],\qquad R(z)=\sqrt{z^2-4}, \tag{A.22}$$

for $z,w\in\mathbb{C}\setminus(-\infty,2]$*, where* $R(z)$ *is defined with branch cut* $[-2,2]$.

Proof of Lemma A.1. Consider the second derivative

$$L_{zw}(z,w)=\int_{-2}^{2}\left[\int_{-2}^{2}\left(\frac{1}{z-x}-\frac{1}{z-y}\right)\left(\frac{1}{w-x}-\frac{1}{w-y}\right)\frac{4-xy}{(x-y)^2\sqrt{4-x^2}\sqrt{4-y^2}}\mathrm{d}y\right]\mathrm{d}x. \tag{A.23}$$

Recall that $R(x)=\sqrt{x^2-4}$ is defined with the branch cut $[-2,2]$. Using this, it is easy to check that

$$L_{zw}(z,w)=-\frac{1}{4}\int_{\Sigma}\left[\int_{\Sigma'}\left(\frac{1}{z-x}-\frac{1}{z-y}\right)\left(\frac{1}{w-x}-\frac{1}{w-y}\right)\frac{4-xy}{(x-y)^2R(x)R(y)}\mathrm{d}y\right]\mathrm{d}x \tag{A.24}$$

where Σ and Σ' are simple closed contours, oriented positively, that contain the interval $[-2,2]$ in its interior and the points z and w in its exterior, and Σ' lies in the interior of Σ. By residue Calculus, we obtain

$$L_{zw}(z,w)=\frac{-2\pi^2}{(z-w)^2}\left(1+\frac{4-zw}{R(z)R(w)}\right). \tag{A.25}$$

Integrating with respect to z,

$$L_w(z,w)=\frac{-2\pi\left(R(z)-R(w)\right)}{(z-w)R(w)}+C_1(w) \tag{A.26}$$

for some function $C_1(w)$ of w. The function $C_1(w)$ is determined by noting that, from (A.26),

$$L_w(z,w)=\frac{-2\pi}{R(w)}+C_1(w)+O(z^{-1}) \tag{A.27}$$

as $z\to\infty$ with fixed w, while from (A.20)

$$L_w(z,w)=L(z,w)=\int_{-2}^{2}\int_{-2}^{2}\left(\log(z-x)-\log(z-y)\right)\left(\frac{1}{w-x}-\frac{1}{w-y}\right)Q(x,y)\mathrm{d}x\mathrm{d}y, \tag{A.28}$$

is $O(z^{-1})$. Hence $C_1(w)=\frac{2\pi^2}{R(w)}$. Integrating with respect to w, we find that

$$L(z,w)=2\pi^2\log\left[\frac{(w+R(w))}{2(zw-4+R(z)R(w))}\right]+C_2(z) \tag{A.29}$$

for some function $C_2(z)$ of z. By considering the asymptotics as $w\to\infty$, we find that $C_2(z)=2\pi^2\left[\log(z+R(z))-\log w\right]$, and this completes the proof of Lemma. □

A.2 Sample covariance matrix

We again only consider real case here. Complex case is similar. The Stieltjes transform of the Marchenko-Pastur distribution $\mathrm{d}\nu$ (3.25) is given by

$$m_\nu(z) = \frac{-z + \sqrt{C_+ C_-} + \sqrt{(z - C_+)(z - C_-)}}{2z}, \qquad z \in \mathbb{C} \setminus [C_-, C_+]. \tag{A.30}$$

Here $C_+ = (\sqrt{d}+1)^2$ and $C_- = (\sqrt{d}-1)^2$. Hence, for real sample covariance matrices, we have

$$\beta_c = \frac{1}{2}\int_{C_-}^{C_+} \frac{\mathrm{d}\nu(x)}{C_+ - x} = -\frac{m_\nu(C_+)}{2} = \frac{1}{2(\sqrt{d}+1)}. \tag{A.31}$$

In the high temperature case, $\beta < \beta_c$, we get from the definition of $\widehat{\gamma}$ that

$$-m_\nu(\widehat{\gamma}) = 2\beta. \tag{A.32}$$

Solving the equation, we find that

$$\widehat{\gamma} = \frac{d}{1-2\beta} + \frac{1}{2\beta}. \tag{A.33}$$

We note that $\widehat{\gamma}$ is a decreasing function in β and satisfies $\widehat{\gamma} > C+$ for $0 < \beta < \beta_c$. In order to evaluate $L(\beta)$ in (2.15), note that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\beta}\int_{-2}^{2} \log(\widehat{\gamma} - \lambda)\mathrm{d}\nu(\lambda) = \left(\frac{\mathrm{d}\widehat{\gamma}}{\mathrm{d}\beta}\right)\frac{\mathrm{d}}{\mathrm{d}\widehat{\gamma}}\int_{-2}^{2}\log(\widehat{\gamma} - \lambda)\mathrm{d}\nu(\lambda) \\ &= -m_\nu(\widehat{\gamma})\left(\frac{\mathrm{d}\widehat{\gamma}}{\mathrm{d}\beta}\right) = 2\beta\left(\frac{2d}{(1-2\beta)^2} - \frac{1}{2\beta^2}\right) \end{aligned} \tag{A.34}$$

using (A.32) and (A.33). Hence, by taking the antiderivative,

$$\int_{-2}^{2} \log(\widehat{\gamma}(\beta) - \lambda)\mathrm{d}\nu(\lambda) = \frac{2\beta}{1-2\beta}d + d\log(1-2\beta) - \log(2\beta), \tag{A.35}$$

where the constant of integration is determined by the asymptotics that the left hand side behaves as

$$\log\widehat{\gamma} + O(\widehat{\gamma}^{-1}) = -\log(2\beta) + O(\beta)$$

as $\widehat{\gamma} \to \infty$, or $\beta \to 0$. Hence

$$L(\beta) = -\frac{d}{2\beta}\log(1-2\beta)$$

for $\beta < \beta_c$.

The limit of the free energy per particle $L(\beta)$ for $\beta > \beta_c$ is obtained easily using (A.35) and noting that $C_+ = \widehat{\gamma}(\beta_c)$.

We now evaluate ℓ and σ^2 for $\beta < \beta_c$. From (2.18), we find that

$$\ell_1 = \log(2\beta) - \frac{1}{2}\log(m_\nu'(\widehat{\gamma})) = \frac{1}{2}\log(1 - 4B^2) \tag{A.36}$$

where we set

$$B = \frac{\beta\sqrt{d}}{1-2\beta}. \tag{A.37}$$

On the other hand, in other to evaluate $M(\varphi)$ and $V(\varphi)$ for $\varphi(x) = \log(\widehat{\gamma} - x)$, we observe that (3.26) implies that

$$\Phi(x) = \log(\sqrt{d}) + \psi(x), \qquad \psi(x) = \log\left(2B + \frac{1}{2B} - x\right) \tag{A.38}$$

where B is given above. Note that $\psi(x)$ is the same function as $\varphi(x) = \log(\widehat{\gamma} - x)$ for Wiger matrices with the change that β is replaced by B. Moreover, we have $\tau_\ell(c + f) = \tau_\ell(f)$ for $\ell > 0$ and $\tau_0(c + f) = c + \tau_0(f)$ for a constant c and function f, from the definition of τ_ℓ and the orthogonality of Chebyshev polynomials. From this and the results obtained for Wigner matrices, we can easily find $M(\varphi)$ and $V(\varphi)$ for sample covariance matrices.

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