Random Matrix Central Limit Theorems for Non-Intersecting Random Walks

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April 9, 2006

Abstract

We consider non-intersecting random walks satisfying the condition that the increments have a finite moment generating function. We prove that in a certain limiting regime where the number of walks and the number of time steps grow to infinity, several limiting distributions of the walks at the mid-time behave as the eigenvalues of random Hermitian matrices as the dimension of the matrices grows to infinity.

1 Introduction

It is known that various limiting local statistics arising in random matrix theory are independent of the precise structure of the randomness of the ensemble [41, 11, 21, 45, 29, 17, 18]. For example, consider the set of Hermitian matrices equipped with a probability measure invariant under unitary conjugation. For a very general class of measures, as the size of the matrix becomes large, the largest eigenvalue converges in distribution to the Tracy-Widom distribution, while the gap probability in the 'bulk scaling limit' converges to a (different) universal distribution.

It has been discovered that the limiting distributions arising in random matrix theory also describe limit laws of a number of specific models in combinatorics, probability theory, and statistical physics; apparently, these models are not expressible in terms of random matrix ensembles. Examples include the longest increasing subsequence of random permutations [5, 40, 13, 28], random Aztec and Hexagon tiling models [30, 9], last passage percolation models with geometric and exponential random variables [27], polynuclear growth models [42, 31], and vicious walker models [25, 3]. For these models, the distribution function of interest was computed explicitly in terms of certain determinantal formulae and the asymptotic analysis of these determinants yielded the desired limit law. Nevertheless, it is believed that such limit laws should hold for a class of models much wider than the explicitly computable ("integrable") models. One such universality result for models "outside random matrices" was obtained in [12, 10, 47] for thin last passage percolation models with general random variables.

This paper studies non-intersecting random walks and proves random matrix central limit theorems in a certain limiting regime. The motivations for this study come from two sources. The first one is the fact that the eigenvalue density function of Gaussian unitary ensemble can be described in terms of a non-intersecting Brownian bridge process [22, 30]. Namely, consider n standard Brownian bridge processes $(B_t^{(1)}, \cdots, B_t^{(n)})$ conditioned not to intersect during the time interval $(0, 2)$ (i.e. $B_t^{(1)} > \cdots > B_t^{(n)}$ for $0 < t < 2$), all starting from and ending at the origin. A simple computation shows that the distribution of $\{B_1^{(1)}, \ldots, B_1^{(n)}\}$ at time 1 is the same as the distribution of the eigenvalues of the $n \times n$ Gaussian unitary ensemble. See subsection 1.1.1 below for the computation. Hence, it is a natural question to ask if the same limit laws hold for general non-intersecting random walks. The second motivation is that a number of the mentioned specific probability models for which the random matrix central limit theorem was obtained are indeed interpreted in terms of non-intersecting random walks. We mention a few of them in the following subsection.

1.1 Motivating Examples

We begin by introducing two distribution functions. Define the kernels

$$\mathbb{A}(a,b) = \frac{\mathrm{Ai}(a)\,\mathrm{Ai}'(b) - \mathrm{Ai}'(a)\,\mathrm{Ai}(b)}{a-b}, \qquad \mathbb{S}(a,b) = \frac{\sin(\pi(a-b))}{\pi(a-b)}. \tag{1}$$

Set

$$F_{TW}(\xi) = \det(1 - \mathbb{A}|_{(\xi,\infty)}), \qquad F_{Sine}(\eta) = \det(1 - \mathbb{S}|_{[-\eta,\eta]}). \tag{2}$$

The Tracy-Widom distribution, F_{TW}, is the limiting distribution of the largest eigenvalue and F_{Sine} is the limiting distribution for the gap probability of the eigenvalues 'in the bulk' in Hermitian random matrix theory.

1.1.1 Non-intersecting Brownian bridge process

Let $B_t = (B_t^{(1)}, \dots, B_t^{(n)})$ be an n-dimensional standard Brownian motion. We compute the density function of B_1 conditioned that $B_t^{(1)} > B_t^{(2)} > \cdots > B_t^{(n)}$ for $0 < t < 2$ and $B_0 = B_2 = (0, \dots, 0)$. Let $p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$. The argument of Karlin and McGregor [33] implies that the density function of n one-dimensional non-intersecting Brownian motions at time t which start from $(x_1, \dots, x_n)$ where $x_1 > \cdots > x_n$ is given by

$$f_t(b_1, \dots, b_n) = \det\left(p_t(x_i, b_j)\right)_{i,j=1}^n, \qquad b_1 > \cdots > b_n. \tag{3}$$

Hence, the density function of B_1 equals for $b_1 > \cdots > b_n$,

$$\begin{aligned} f(b_1, \dots, b_n) &= \lim_{x,y\to 0} \frac{\det\left(p_1(x_i, b_j)\right)_{i,j=1}^n \cdot \det\left(p_1(b_i, y_j)\right)_{i,j=1}^n}{\det\left(p_2(x_i, y_j)\right)_{i,j=1}^n} \\ &= \frac{2^{n(n-1)/2}}{\pi^{n/2} \prod_{j=1}^{n-1} j!} \prod_{1\le i<j\le n} |b_i - b_j|^2 \prod_{j-1}^{n} e^{-b_j^2}, \end{aligned} \tag{4}$$

which is the density function of the eigenvalues of $n \times n$ Hermitian matrices from the Gaussian unitary ensemble. Therefore, combined with the well-known results of random matrix theory,

$$\lim_{n\to\infty} \mathbb{P}\big((B_1^{(1)} - \sqrt{2n})\sqrt{2}n^{1/6} \le x\big) = F_{TW}(x). \tag{5}$$

1.1.2 Longest Increasing Subsequence and Plancherel Measure on Partitions

The longest increasing subsequence problem can be formulated in the following manner. Denote by S_n the symmetric group on n symbols endowed with uniform measure. Given $\pi \in S_n$, a subsequence $\pi(i_1), ..., \pi(i_r)$ is called an increasing subsequence if $i_1 < \cdots < i_r$ and $\pi(i_1) < \cdots \pi(i_r)$. Denote by $\ell_n(\pi)$ the length of the longest increasing subsequence (this subsequence need not be unique). For applications of ℓ_n and activities around the asymptotic behavior of ℓ_n see [2, 5, 16], for example . In particular, in [5] the following limit theorem is proven:

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\ell_n(\pi) - 2\sqrt{n}}{n^{1/6}} < x\right) = F_{TW}(x). \tag{6}$$

A closely related object is the uniform measure on the set of pairs of standard Young tableaux having the same shape (equivalently, the so-called Plancherel measure on the set of partitions). Given a partition of n, $\lambda = (\lambda_1, ..., \lambda_r)$, where $\lambda_1 \ge \cdots \ge \lambda_r > 0$ and $\lambda_1 + \cdots + \lambda_r = n$, a standard Young tableaux of shape λ consists of r rows of boxes with distinct entries from $\{1, .., n\}$ such that the rows are left-justified, the i^{th} row has λ_i boxes, and the entries are constrained to increase along rows and columns from left to right and top to bottom, respectively. These objects will be called row increasing Young tableaux if the rows increase but the columns do not necessarily increase. The Robinson-Schensted bijection implies that the number of boxes in the top row of the pair of standard Young tableaux corresponding to $\pi \in S_n$ is equal to $\ell_n(\pi)$ [46]. Therefore, the distribution of ℓ_n is same as the distribution of the number of boxes in the top row of the pair of standard Young tableaux having the same shape chosen uniformly. This correspondence provides a representation of ℓ_n which is computable in terms of explicit formulae if the number of standard Young tableaux of a given shape is computable.

One way (among many) to compute the number of standard Young tableaux of shape λ is by a non-intersecting path argument [32]. Let $N_t^1, ..., N_t^r$ be independent rate 1 Poisson processes with initial conditions $N_0^i = 1 - i$ for $i = 1, 2, ..., r$. Define A_λ to be the event that $N_1^i = \lambda_i + (1 - i)$ for all $i = 1, 2, ..., r$. For almost every element of A_λ (the elements of A_λ where no two jumps of these processes occurs at the same time) there is a natural map to a row increasing Young tableaux. The map is defined as follows: If N^i jumps first then place a 1 in the leftmost box in row i; if N^j jumps second then place a 2 in the first box of row j if $j \neq i$ and a 2 in the second box of row i if $j = i$; continue in this fashion to produce a row increasing Young tableaux of shape λ. It is not hard to show that this map induces the uniform probability measure (when properly normalized by $\mathbb{P}(A_\lambda)$) on the row increasing Young tableaux. The subset $B_\lambda \subset A_\lambda$ which is mapped to the standard Young tableaux of shape λ correspond to the realizations whose paths do not intersect each other for all $t \in [0, 1]$. Since the mapping described induces uniform measure on the row increasing Young tableaux of shape λ and the standard Young tableaux correspond to non-intersecting path realizations, B_λ, the number of standard Young tableaux of shape λ can be computed by evaluating:

$$|\text{row increasing Young tableaux of shape } \lambda| \frac{\mathbb{P}(B_\lambda)}{\mathbb{P}(A_\lambda)}. \tag{7}$$

The denominator of (7) is $e^{-r} \prod_{i=1}^r \frac{1}{\lambda_i!}$ by definition of Poisson processes and the independence of the N^i while $|\text{row increasing Young tableaux of shape } \lambda| = \frac{n!}{\lambda_1! \cdots \lambda_r!}$ by elementary combinatorics. On the other hand, via the Karlin-McGregor formula [33],

$$\mathbb{P}(B_\lambda) = \det \left(\frac{e^{-1}}{(\lambda_i - i + j)!} \right)_{i,j=1}^r . \tag{8}$$

Hence, the number of standard Young tableaux of shape λ is $n! \det \left(\frac{1}{(\lambda_i - i + j)!} \right)_{i,j=1}^r$. In tandem with the RSK correspondence this formula leads to an algebraic formula for the number of $\pi \in S_n$ for which $\ell_n(\pi) \leq m$. Moreover, a slight extension of this argument shows that the result (6) can be stated in terms of the top curve of the nonintersecting Poisson processes if these process where forced to return to their initial locations at time 2 by imposing that their dynamics between times 1 and 2 have negative rather than positive jumps. The asymptotic behavior of other curves can also be studied [6, 40, 13, 28, 7].

1.1.3 Symmetric Simple Random Walks and Random Rhombus Tilings of Hexagon

Consider n symmetric simple random walks $S(m) = (S^{(1)}(m), \ldots, S^{(n)}(m))$, conditioned not to intersect and $S(0) = (2(n-1), 2(n-2), \ldots, 0) = S(2k)$. Any realization of such walks is in one-to-one correspondence to a rhombus tiling of a hexagon with sides of lengths k, k, n, k, k, n. Again, using the argument of Karlin and McGregor, the distribution of $S(k)$ can be expressed in terms of a determinant. This determinant was significantly simplified and was shown to be related to the so-called Hahn orthogonal polynomials by Johansson [30]. A further asymptotic analysis of the Hahn polynomials [8, 9] shows that as $n, k \to \infty$ such that $k = O(n)$, the top walk $S^{(1)}(k)$ converges to F_{TW} and the gap distribution in the 'bulk' converges to a discrete version of F_{Sine}. A similar asymptotic result was obtained also for domino tilings of an Aztec diamond [30].

Certain polynuclear growth models, last passage percolation problems, and a bus system problem [43, 31, 39, 4] have also been analyzed in depth using non-intersecting path techniques. In each of the cases described above, the random walks are very specific and the analysis relies heavily on their particular properties.

1.2 Statement of Theorems

Let k be a positive integer. Let

$$x_i = \frac{2i-k}{k}, \qquad i \in \{0,...,k\}. \tag{9}$$

Note that $x_i \in [-1,1]$ for all i. Let $\{Y_l^j\}_{j=0,l=1}^{k,N_k}$ be a family of independent identically distributed random variables where N_k is a positive integer. Assume that $\mathbb{E}Y_l^j = 0$ and $Var(Y_l^j) = 1$. Further assume that there is $\lambda_0 > 0$ such that $\mathbb{E}\big(e^{\lambda Y_l^j}\big) < \infty$ for all $|\lambda| < \lambda_0$.

Define the random walk process $S(t) = (S_0(t), ..., S_k(t))$ by

$$S_j(t) = x_j + \sqrt{\frac{2}{N_k}}\Bigg(\sum_{i=1}^{\left\lfloor \frac{tN_k}{2}\right\rfloor} Y_i^j + \left(\frac{tN_k}{2} - \left\lfloor\frac{tN_k}{2}\right\rfloor\right) Y^j_{\left\lfloor \frac{tN_k}{2}\right\rfloor+1}\Bigg), \qquad \text{for } t \in [0,2], \tag{10}$$

which starts at $S_j(0) = x_j$. For N_k equally spaced times S_j is given by

$$S_j\left(\frac{2}{N_k}l\right) = x_j + \sqrt{\frac{2}{N_k}}\big(Y_1^j + \cdots + Y_l^j\big), \qquad l = 1, 2, \ldots, N_k. \tag{11}$$

For t between $\frac{2}{N_k}l$ and $\frac{2}{N_k}(l+1)$, $S_j(t)$ is simply defined by linear interpolation.

Let $(C([0,2];\mathbb{R}^{k+1}), \mathcal{C})$ be the family of measurable spaces constructed from the continuous functions on $[0,2]$ taking values in $\mathbb{R}^{k+1}$ equipped with their Borel sigma algebras (generated by the sup norm). Let $A_k, B_k \in \mathcal{C}$ be the events defined by

$$A_k = \{y_0(t) < \cdots < y_k(t) \text{ for } t \in [0,2]\}, \tag{12}$$

$$B_k = \{y_i(2) \in [x_i - h_k, x_i + h_k] \text{ for } i \in \{0,...,k\}\} \tag{13}$$

where $h_k > 0$. The results of this paper focus on the process $S(t)$ conditioned on the event $A_k \cap B_k$ where $h_k \ll \frac{2}{k}$. In other words, the particles never intersect and all particles essentially return to their original location at the final time 2.

The main results of this paper state that under certain technical conditions on h_k and N_k, as $k \to \infty$, the locations of the particles at the half time ($t = 1$) behave, after suitable scaling, statistically like the eigenvalues of a large random Hermitian matrix from the Gaussian unitary ensemble. The conditions for h_k and N_k are that $\{h_k\}_{k>0}$ is a sequence of positive numbers and $\{N_k\}_{k>0}$ is a sequence of positive integers satisfying

$$h_k \le (2k)^{-2k^2} \quad \text{and} \quad N_k \ge h_k^{-4(k+2)}. \tag{14}$$

Let $C_k, D_k \in \mathcal{C}$ be defined by

$$C_k = \left\{y_k(1) \le \sqrt{2k} + \frac{\xi}{\sqrt{2}k^{1/6}}\right\}, \tag{15}$$

$$D_k = \left\{y_i(1) \notin \left[-\frac{\pi\eta}{\sqrt{2k}}, \frac{\pi\eta}{\sqrt{2k}}\right] \text{ for all } i \in \{0,...,k\}\right\}, \tag{16}$$

where ξ and $\eta > 0$ are fixed real numbers. The event C_k is a constraint on the location of the right-most particle, and D_k is the event that no particle is in a small neighborhood of the origin at time 1.

Theorem 1 (Edge)**.** *Let $\mathbb{P}_k$ be the probability measure induced on $(C([0,2];\mathbb{R}^{k+1}), \mathcal{C})$ by the random walks $\{S(t) : t \in [0,2]\}$. Let $\{h_k\}_{k>0}$ and $\{N_k\}_{k>0}$ satisfy* (14)*. Then*

$$\lim_{k\to\infty} \mathbb{P}_k(C_k | A_k \cap B_k) = F_{TW}(\xi). \tag{17}$$

A similar theorem holds for the bulk.

Theorem 2 (Bulk). *Let $\mathbb{P}_k$ be the probability measure induced on $(C([0,2];\mathbb{R}^{k+1}),\mathcal{C})$ by the random walks $\{S(t):t\in[0,2]\}$, and let $\{h_k\}_{k>0}$ and $\{N_k\}_{k>0}$ satisfy* (14). *Then*

$$\lim_{k\to\infty}\mathbb{P}_k(D_k|A_k\cap B_k)=F_{Sine}(\eta). \tag{18}$$

The proofs have a two step strategy. The first step is to show that under the conditions of the theorems the process $S(t)$ is well approximated by non-intersecting Brownian bridge processes starting at the same positions. This proof relies on the Komlos, Major, Tusnady (KMT) theorem. The second step is to compute the limiting distributions of the non-intersecting Brownian bridge processes and prove that these distributions are indeed F_{TW} or F_{Sine}. This Brownian bridge process is quite similar to the one discussed in subsection 1.1.1 above, but with the minor difference in the starting and ending positions. This results, using the Karlin-McGregor formula, in a density function for the particles analogous to the Coulomb gas density with a potential different from GUE, namely the Stieltjes-Wigert potential. The asymptotics of such Coulomb system is analyzed using Riemann-Hilbert methods.

The above theorems are proven under the condition that N_k is large compared to the number, $k+1$, of particles. This assumption ensures that the Brownian approximation of the random walks has a smaller effect than the non-intersecting condition. Although it is believed that the condition on N_k is technical, it is not clear under which conditions on the random variables one has $N_k=O(k)$. For example, when $\{Y_l^j\}$ are Bernoulli, these results were proven even when $N_k=O(k)$ (see subsection 1.1.3 above). This is because there is an integrability in this problem: The Karlin-McGregor argument applies directly because intersecting paths must be incident at some time. It is a challenge to find the optimal scaling such that a result of this nature holds for more general random variables. In other words, in what scaling regime does the exact Karlin-McGregor calculation essentially not matter?

This paper is organized as follows. The approximation by the Brownian bridge process is proven in Section 2. The asymptotic analysis of the Brownian bridge process (appearing in section 2) is carried out in Section 3. Some other considerations such as finite dimensional distributions and the modifications necessary to study random variables without finite moment generating functions are discussed in section 4.

Acknowledgments. The authors would like to thank Percy Deift and Harold Widom for useful discussions. The work of Baik was supported in part by NSF grants DMS #0457335 and the Sloan Fellowship. The work of Suidan was supported in part by NSF grants DMS #0202530/0553403.

2 Approximation by a Brownian bridge process

Let $\{X_t\}_{t\geq 0}$ be the $\mathbb{R}^{k+1}$-valued stochastic process $X_t=(X_0(t),...X_k(t))$ where $X_j(t)=x_j+B_t^j$ for a family of $k+1$ independent standard Brownian motions B_t^j. The proof in this section relies on the Komlos, Major, Tusnady coupling of Brownian motions and random walks [35, 36] which can be stated in our setting as follows: With increments of the form $\{Y_l^j\}_{j=0,l=1}^{k,N_k}$ described in the introduction, there exists a coupling such that

$$\mathbb{P}\left(\sup_{0\leq l\leq N_k}\left|S_i\left(\frac{2l}{N_k}\right)-X_i\left(\frac{2l}{N_k}\right)\right|>\frac{1}{\sqrt{N_k}}(c\log N_k+x)\right)\leq e^{-ax}, \tag{19}$$

for some fixed $a,c>0$ which depend only on the properties of the moment generating functions of the $\{Y_l^j\}_{j=0,l=1}^{k,N_k}$. Alternatively, (19) can be written as

$$\mathbb{P}\left(\sup_{0\leq l\leq N_k}\left|S_i\left(\frac{2l}{N_k}\right)-X_i\left(\frac{2l}{N_k}\right)\right|>\frac{c\log N_k}{\sqrt{N_k}}+y\right)\leq e^{-ay\sqrt{N_k}}. \tag{20}$$

This fact immediately implies that

$$\mathbb{P}\left(\sup_{0\leq i\leq k}\sup_{0\leq l\leq N_k}\left|S_i\left(\frac{2l}{N_k}\right)-X_i\left(\frac{2l}{N_k}\right)\right|>\frac{c\log N_k}{\sqrt{N_k}}+y\right)\leq (k+1)e^{-ay\sqrt{N_k}}. \tag{21}$$

Let $\{S(t)\}_{t\in[0,2]}$ be the $k+1$-dimensional random walk process defined in Introduction and let $\{X_t\}$ be the KMT coupled $k+1$-dimensional Brownian process on the same probability spaces $(\Omega^{(k)}, \mathcal{F}^{(k)}, \mathbb{P}^{(k)})$. We can assume that the probability space which holds S and X is large enough to hold a third process $Z_t = (Z_0(t), ..., Z_k(t))$ where the $Z_i(t)$ are standard Brownian bridge processes with initial and terminal conditions specified by $Z_i(0) = Z_i(2) = x_i$. Let $F_k^S, F_k^X, F_k^Z : C([0,2], \mathbb{R}^{k+1}) \to \mathbb{R}$ be defined by

$$F_k^S(y) = \frac{1\!\!1_{A_k\cap B_k\cap C_k}(y)}{\mathbb{E}1\!\!1_{A_k\cap B_k}(S)}, \tag{22}$$

$$F_k^X(y) = \frac{1\!\!1_{A_k\cap B_k\cap C_k}(y)}{\mathbb{E}1\!\!1_{A_k\cap B_k}(X)}, \tag{23}$$

$$F_k^Z(y) = \frac{1\!\!1_{A_k\cap B_k\cap C_k}(y)}{\mathbb{E}1\!\!1_{A_k\cap B_k}(Z)}. \tag{24}$$

Let $G_k^S, G_k^X, G_k^Z : C([0,2], \mathbb{R}^{k+1}) \to \mathbb{R}$ be defined by

$$G_k^S(y) = \frac{1\!\!1_{A_k\cap B_k\cap D_k}(y)}{\mathbb{E}1\!\!1_{A_k\cap B_k}(S)}, \tag{25}$$

$$G_k^X(y) = \frac{1\!\!1_{A_k\cap B_k\cap D_k}(y)}{\mathbb{E}1\!\!1_{A_k\cap B_k}(X)}, \tag{26}$$

$$G_k^Z(y) = \frac{1\!\!1_{A_k\cap B_k\cap D_k}(y)}{\mathbb{E}1\!\!1_{A_k\cap B_k}(Z)}. \tag{27}$$

Theorems 1 and 2 will be proven in two steps. The first step is to show that under the conditions given in the Introduction,

Proposition 1. *As $k \to \infty$, the random variables F_k^S, F_k^Z, G_k^S, and G_k^Z satisfy*

$$\mathbb{E}(F_k^S(S) - F_k^Z(Z)) \to 0, \tag{28}$$

$$\mathbb{E}(G_k^S(S) - G_k^Z(Z)) \to 0. \tag{29}$$

As $\mathbb{E}(F_k^S(S)) = \mathbb{P}_k(C_k|A_k \cap B_k)$ and $\mathbb{E}(G_k^S(S)) = \mathbb{P}_k(D_k|A_k \cap B_k)$, it is enough to prove

Proposition 2.

$$\lim_{k\to\infty} \mathbb{E}F_k^Z(Z) = F_{TW}(\xi) \tag{30}$$

$$\lim_{k\to\infty} \mathbb{E}G_k^Z(Z) = F_{Sine}(\eta). \tag{31}$$

The proof of proposition 2 is given in section 3 below. The rest of this section focuses on the proof of Proposition 1. Proposition 1 is proven in two steps: First, $\mathbb{E}(F_k^S(S))$ is approximated by $\mathbb{E}(F_k^X(X))$; second, $\mathbb{E}(F_k^X(X))$ is approximated by $\mathbb{E}(F_k^Z(Z))$. The proof of (29) is handled in a similar way.

Three preliminary lemmas are needed in order to prove Proposition 1. Recall (9) that

$$x_i = \frac{2i-k}{k}, \qquad i = 0, ..., k. \tag{32}$$

Lemma 1. *Let $a, c > 0$ be the constants in KMT approximation* (21). *For any $\rho \geq \frac{3c\log N_k}{\sqrt{N_k}}$,*

$$\mathbb{E}|1\!\!1_{A_k\cap B_k}(S) - 1\!\!1_{A_k\cap B_k}(X)| \leq 2(k+1)e^{-\frac{1}{2}a\sqrt{N_k}\rho} + \frac{32(k+1)}{\rho}\sqrt{\frac{N_k}{\pi}}e^{-\frac{\rho^2 N_k}{64}} + 8(2k+1)\rho \tag{33}$$

$$\mathbb{E}|1\!\!1_{B_k}(S) - 1\!\!1_{B_k}(X)| \leq (k+1)e^{-\frac{1}{2}a\sqrt{N_k}\rho} + \frac{16(k+1)}{\rho}\sqrt{\frac{N_k}{\pi}}e^{-\frac{\rho^2 N_k}{64}} + 8(k+1)\rho \tag{34}$$

$$\mathbb{E}|1\!\!1_{C_k}(S) - 1\!\!1_{C_k}(X)| \leq (k+1)e^{-\frac{1}{2}a\sqrt{N_k}\rho} + \frac{16(k+1)}{\rho}\sqrt{\frac{N_k}{\pi}}e^{-\frac{\rho^2 N_k}{64}} + 8(k+1)\rho \tag{35}$$

$$\mathbb{E}|1\!\!1_{D_k}(S) - 1\!\!1_{D_k}(X)| \leq (k+1)e^{-\frac{1}{2}a\sqrt{N_k}\rho} + \frac{16(k+1)}{\rho}\sqrt{\frac{N_k}{\pi}}e^{-\frac{\rho^2 N_k}{64}} + 8(k+1)\rho. \tag{36}$$

Proof. Note that

$$\begin{aligned}\mathbb{E}|1\!\!1_{A_k\cap B_k}(S) - 1\!\!1_{A_k\cap B_k}(X)| &= \mathbb{E}|1\!\!1_{A_k}(S)1\!\!1_{B_k}(S) - 1\!\!1_{A_K}(X)1\!\!1_{B_k}(X)| \\ &= \mathbb{E}|(1\!\!1_{A_k}(S) - 1\!\!1_{A_k}(X))1\!\!1_{B_k}(S) + (1\!\!1_{B_k}(S) - 1\!\!1_{B_k}(X))1\!\!1_{A_k}(X)| \\ &\leq \mathbb{E}|1\!\!1_{A_k}(S) - 1\!\!1_{A_k}(X)| + \mathbb{E}|1\!\!1_{B_k}(S) - 1\!\!1_{B_k}(X)|. \end{aligned} \tag{37}$$

We first estimate $\mathbb{E}|1\!\!1_{A_k}(S) - 1\!\!1_{A_k}(X)| = \mathbb{P}(\mathcal{E})$ where $\mathcal{E} = \{S \in A_k, X \notin A_k\} \cup \{S \notin A_k, X \in A_k\}$. Recall that $A_k = \{y_0(t) < \cdots < y_k(t) \text{ for } t \in [0,2]\}$. Let $\rho \geq \frac{3c\log N_k}{\sqrt{Nk}}$ where c is the KMT coupling constant. The event $\mathcal{E}$ can be expressed as the disjoint union of the following three events $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$. The first event $\mathcal{E}_1$ is the subset of $\mathcal{E}$ consisting of "bad" paths satisfying $\sup_{0\leq i\leq k}\sup_{t\in[0,2]}|S_i(t) - X_i(t)| > \dfrac{c\log N_k}{\sqrt{N_k}} + \rho$. The second event $\mathcal{E}_2$ is the subset of $\mathcal{E}\setminus\mathcal{E}_1$ consisting of paths satisfying $\min_{t\in[0,2]}(S_i(t)-S_{i-1}(t)) < 0$ for some $1 \leq i \leq k$ while $X_0(t) < \cdots < X_k(t)$ for all $t \in [0,2]$. The third event $\mathcal{E}_3$ is the subset of $\mathcal{E}\setminus\mathcal{E}_1$ consisting of paths such that $\min_{t\in[0,2]}(X_i(t) - X_{i-1}(t)) < 0$ for some $1 \leq i \leq k$ while $S_0(t) < \cdots < S_k(t)$ for all $t \in [0,2]$. In order to estimate $\mathbb{P}(\mathcal{E}_1)$, note that the KMT theorem couples random walks to Brownian motion at discrete times. Hence, even when X and S are close at discrete times, "bad paths" may occur if X fluctuates too much in $(\frac{2}{N_k}l, \frac{2}{N_k}(l+1))$ for some l. (Note that S is simply linearly interpolated for times not integral multiple of $\frac{2}{N_k}$.) Thus, from (21) and a standard estimates for Brownian motions,

$$\begin{aligned}\mathbb{P}(\mathcal{E}_1) &\leq \mathbb{P}\left(\sup_{0\leq i\leq k}\sup_{0\leq l\leq N_k}\left|S_i\left(\frac{2l}{N_k}\right) - X_i\left(\frac{2l}{N_k}\right)\right| > \frac{c\log N_k}{\sqrt{N_k}} + \frac{\rho}{2}\right) \\ &\quad + \mathbb{P}\Bigg(\left\{\sup_{0\leq i\leq k}\sup_{0\leq l\leq N_k}\left|S_i\left(\frac{2l}{N_k}\right) - X_i\left(\frac{2l}{N_k}\right)\right| \leq \frac{c\log N_k}{\sqrt{N_k}} + \frac{\rho}{2}\right\} \\ &\qquad \cap\left\{\max_{s,t\in(\frac{2l}{N_k},\frac{2(l+1)}{N_k})}|X_i(t) - X_i(s)| > \frac{\rho}{2} \text{ for some } 0 \leq i \leq k \text{ and for some } 0 \leq l < N_k\right\}\Bigg) \\ &\leq (k+1)e^{-\frac{1}{2}a\sqrt{N_k}\rho} + (k+1)N_k\mathbb{P}\Bigg(\max_{t,s\in[0,\frac{2}{N_k}]}|X_1(t) - X_1(s)| > \frac{\rho}{2}\Bigg) \\ &\leq (k+1)e^{-\frac{1}{2}a\sqrt{N_k}\rho} + \frac{16(k+1)}{\rho}\sqrt{\frac{N_k}{\pi}}e^{-\frac{\rho^2 N_k}{64}}. \end{aligned} \tag{38}$$

Note that this estimate does not use the fact that $\mathcal{E}_1$ is a subset of $\mathcal{E}$. For a path in the event $\mathcal{E}_2$, there is $i \in \{1, 2, \ldots, k\}$ such that $\min_{t\in[0,2]}(S_i(t) - S_{i-1}(t)) < 0$, but $X_{i-1}(t) < X_i(t)$ for all $t \in [0,2]$ and $|S_j(t) - X_j(t)| \leq \frac{c\log N_k}{\sqrt{N_k}} + \rho$ all $t \in [0,2]$ and $j \in \{0, 1, \ldots, N_k\}$. Therefore, for a path in $\mathcal{E}_2$, $0 < \min_{t\in[0,2]}(X_i(t) -$

$X_{i-1}(t)) < \frac{2c\log N_k}{\sqrt{N_k}} + 2\rho \le 4\rho$. Thus, from a standard Brownian motion argument,

$$\begin{aligned}\mathbb{P}(\mathcal{E}_2) &\le \mathbb{P}\left(0 \le \min_{t\in[0,2]}(X_i(t) - X_{i-1}(t)) < 4\rho \text{ for some } 1 \le i \le k\right)\\ &\le k\mathbb{P}\Big(0 \le \min_{t\in[0,2]}(X_1(t) - X_0(t)) < 4\rho\Big)\\ &\le 4k\rho.\end{aligned} \tag{39}$$

A similar argument yields that

$$\begin{aligned}\mathbb{P}(\mathcal{E}_3) &\le \mathbb{P}\left(-4\rho < \min_{t\in[0,2]}(X_i(t) - X_{i-1}(t)) < 0 \text{ for some } 1 \le i \le k\right)\\ &\le k\mathbb{P}\Big(-4\rho < \min_{t\in[0,2]}(X_1(t) - X_0(t)) < 0\Big)\\ &\le 4k\rho.\end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned}\mathbb{E}|\mathbb{1}_{A_k}(S) - \mathbb{1}_{A_k}(X)| &= \mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_3)\\ &\le (k+1)e^{-\frac{1}{2}a\sqrt{N_k}\rho} + \frac{16(k+1)}{\rho}\sqrt{\frac{N_k}{\pi}}e^{-\frac{\rho^2 N_k}{64}} + 8k\rho.\end{aligned} \tag{41}$$

Now we estimate $\mathbb{E}|\mathbb{1}_{B_k}(S) - \mathbb{1}_{B_k}(X)| = \mathbb{P}(\mathcal{F})$ where $\mathcal{F} = \{S \in B_k, X \notin B_k\} \cup \{S \notin B_k, X \in B_k\}$. As before, we express $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, a disjoint union. The first event $\mathcal{F}_1$ is the subset of $\mathcal{F}$ consisting of the same bad paths as in $\mathcal{E}_1$. The event $\mathcal{F}_2$ is the intersection of $\mathcal{F} \setminus \mathcal{F}_1$ and $\{S \in B_k, X \notin B_k\}$ and the event $\mathcal{F}_3$ is the intersection of $\mathcal{F} \setminus \mathcal{F}_1$ and $\{S \notin B_k, X \in B_k\}$. The argument for $\mathcal{E}_1$ implies that same bound (38) to $\mathbb{P}(\mathcal{F}_1)$. For a path in $\mathcal{F}_2$, there is $i \in \{0, 1, \dots, k\}$ such that $X_i(2) \notin [x_i - h_k, x_i + h_k]$. But as $S_i(2) \in [x_i - h_k, x_i + h_k]$ and $|S_i(2) - X_i(2)| \le \frac{2\log N_k}{\sqrt{N_k}} + \rho \le 2\rho$, we find that $X_i(2) \in (x_i + h_k, x_i + h_k + 2\rho]$ or $X_i(2) \in [x_i - h_k - 2\rho, x_i - h_k)$. Therefore,

$$\mathbb{P}(\mathcal{F}_2) \le (k+1)\mathbb{P}(X_0(2) \in [-2\rho, 2\rho]) \le 4(k+1)\rho. \tag{42}$$

A similar argument yields the same bound for $\mathbb{P}(\mathcal{F}_3)$. Therefore, (34) is proven and so is (33) by using (37) and (41). An almost identical argument proves (35) and (36). ☐

Denote by $p_t(a,b) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{(a-b)^2}{2t}}$ the standard heat kernel in one dimension. The theorem of Karlin and McGregor [33] for the non-intersecting Brownian motions implies that the joint probability density function $f_t(y_0, \dots, y_k)$ of $(k+1)$-dimensional Brownian motion $X(t)$ at time t satisfying $X_0(s) < X_1(s) \cdots < X_k(s)$ for $s \in [0,t]$ is equal to

$$f_t(y_0, \dots, y_k) = \det\big(p_t(x_i, y_j)\big)_{i,j=0}^{k} \tag{43}$$

where $x_i = X_i(0)$. The following Lemma establishes a lower bound of this density when $y_i = x_i$ for all i.

Lemma 2. *For $t > 0$,*

$$\det\big(p_t(x_i, x_j)\big)_{i,j=0}^{k} \ge \frac{1}{(2\pi t)^{\frac{k+1}{2}}}e^{-\frac{2(k+1)(k+2)}{3tk}}\left(\frac{2}{\sqrt{t}k}\right)^{k(k+1)}. \tag{44}$$

In particular, for all sufficiently large k,

$$\det\big(p_2(x_i, x_j)\big)_{i,j=0}^{k} \ge k^{-k^2}. \tag{45}$$

Proof. As $x_i = \frac{2i-k}{k}$,

$$\begin{aligned}\det\left(p_t(x_i,x_j)\right)_{i,j=0}^k &= \det\left(\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(x_i-x_j)^2}\right)_{i,j=0}^k \\ &= \frac{e^{-2\sum_{j=0}^k j^2}}{(2\pi t)^{\frac{k+1}{2}}}\det\left(e^{\frac{2ij}{tk^2}}\right)_{i,j=0}^k\end{aligned} \tag{46}$$

It is an exercise to show that for $k \geq 1$,

$$\det\left(e^{\frac{2ij}{tk^2}}\right)_{i,j=0}^k = \left[\prod_{l=1}^k \delta^{\frac{l(l-1)}{2}}\right]\left[\prod_{j=1}^k (\delta^j - 1)^{k+1-j}\right], \tag{47}$$

where $\delta = e^{\frac{4}{tk^2}}$. Using (47) and the fact that $\delta - 1 > \frac{4}{tk^2} > 0$,

$$\begin{aligned}\det\left(p_t(x_i,x_j)\right)_{i,j=0}^k &= \frac{1}{(2\pi t)^{\frac{k+1}{2}}}e^{-\frac{2(k+1)(k+2)}{3tk}}\prod_{j=1}^k(\delta^j-1)^{k+1-j} \\ &\geq \frac{1}{(2\pi t)^{\frac{k+1}{2}}}e^{-\frac{2(k+1)(k+2)}{3tk}}(\delta-1)^{\frac{k(k+1)}{2}} \\ &\geq \frac{1}{(2\pi t)^{\frac{k+1}{2}}}e^{-\frac{2(k+1)(k+2)}{3tk}}\left(\frac{4}{tk^2}\right)^{\frac{k(k+1)}{2}}.\end{aligned} \tag{48}$$

This completes the proof of Lemma 2. □

The following lemma will be used to control the difference between a conditioned version of the process X and the process Z.

Lemma 3. *If $h_k \leq (2k)^{-2k^2}$, then for sufficiently large k,*

$$\left|\frac{\det(p_1(y_i,x_j)}{\det(p_2(x_i,x_j))} - \frac{\int_{-h_k}^{h_k}\cdots\int_{-h_k}^{h_k}\det(p_1(y_i,x_j+s_j))ds_0\cdots ds_k}{\int_{-h_k}^{h_k}\cdots\int_{-h_k}^{h_k}\det(p_2(x_i,x_j+s_j))ds_0\cdots ds_k}\right| \leq \frac{1}{k} \tag{49}$$

uniformly in $(y_0,...,y_k) \in \mathbb{R}^{k+1}$.

Proof. The conclusion of this lemma is a consequence of several elementary determinant estimates. First note that if $A = (a_{ij})_{i,j=0}^k$ is a $(k+1)\times(k+1)$ matrix with entries $|a_{ij}| \leq 1$, then for the matrix I^{ij} given by $(I^{ij})_{mn} = \delta_{im}\delta_{jn}$,

$$|\det(A) - \det(A + \epsilon I^{ij})| \leq \epsilon k!. \tag{50}$$

Using a Lipschitz estimate for the Gaussian density, equation (50) implies that for any $t \geq \frac{1}{\sqrt{2\pi e}}$, any $h > 0$, and any $(a_0,\ldots,a_k),(b_0,\ldots,b_k) \in \mathbb{R}^{k+1}$,

$$\left|\det(p_t(a_i,b_j)) - \frac{1}{(2h)^{k+1}}\int_{-h}^h\cdots\int_{-h}^h\det(p_t(a_i+s_i,b_j))ds_0\cdots ds_k\right| \leq 2h(k+1)^2k!. \tag{51}$$

Now a simple algebraic manipulation yields that the left-hand-side of (49) equals

$$\begin{aligned}&\left|\frac{\det(p_1(y_i,x_j))}{\det(p_2(x_i,x_j))}\cdot\frac{\frac{1}{(2h_k)^{k+1}}\int_{[-h_k,h_k]^{k+1}}\left(\det(p_2(x_i,x_j+s_j)) - \det(p_2(x_i,x_j))\right)ds_0\cdots ds_k}{\det(p_2(x_i,x_j)) + \frac{1}{(2h_k)^{k+1}}\int_{[-h_k,h_k]^{k+1}}\left[\det(p_2(x_i,x_j+s_j)) - \det(p_2(x_i,x_j))\right]ds_0\cdots ds_k}\right. \\ &\left.+\frac{\frac{1}{(2h_k)^{k+1}}\int_{[-h_k,h_k]^{k+1}}\left[\det(p_1(y_i,x_j)) - \det(p_1(y_i,x_j+s_j))\right]ds_0\cdots ds_k}{\det(p_2(x_i,x_j)) + \frac{1}{(2h_k)^{k+1}}\int_{[-h_k,h_k]^{k+1}}\left[\det(p_2(x_i,x_j+s_j)) - \det(p_2(x_i,x_j))\right]ds_0\cdots ds_k}\right|.\end{aligned} \tag{52}$$

Using the estimates (45) and (51), the denominator on the second fraction in (52) is bounded below by

$$k^{-k^2} - 2h_k(k+1)^2 k! \geq \frac{1}{2} k^{-k^2} \tag{53}$$

for sufficiently large k. Hence, using (51) again, the absolute value of second fraction of (52) is less than or equal to

$$2h_k(k+1)^2 k! k^{k^2}. \tag{54}$$

On the other hand, as $\det(p_1(x_i, y_j))$ is the density function (for $(y_0, \ldots, y_k) \in \mathbb{R}^{k+1}_{>}$ where $\mathbb{R}^{k+1}_{>} = \{(y_0, ..., y_k) \in \mathbb{R}^{k+1} : y_0 < \cdots < y_k\}$) corresponding to the probability of $k+1$ Brownian motions starting from $(x_0, \ldots, x_k)$ and ending at $(y_0, \ldots, y_k)$ at time 1 without having intersected, it is clearly less than the same type of probability density function when a non-intersection condition is not imposed. Therefore,

$$\det(p_1(x_i, y_j)) \leq \prod_{i=0}^{k} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i - y_i)^2} \leq 1. \tag{55}$$

The absolute value of the first term in the brackets of (52) is less than or equal to

$$2h_k(k+1)^2 k! k^{2k^2}. \tag{56}$$

Since h_k is assumed to be less than or equal to $(2k)^{-2k^2}$, (49) follows. □

The proof of Proposition 1 follows.

Proof of Proposition 1. Two estimates will be needed. Note that

$$|\mathbb{E}(F_k^S(S) - F_k^Z(Z))| \leq \mathbb{E}|F_k^S(S) - F_k^X(X)| + |\mathbb{E}(F_k^X(X) - F_k^Z(Z))|. \tag{57}$$

The first term on the right side of (57) is estimated as follows:

$$\begin{aligned}
&\mathbb{E}|F_k^S(S) - F_k^X(X)| \\
&= \mathbb{E}\left| \frac{\mathbb{1}_{A_k \cap B_k \cap C_k}(S)}{\mathbb{E}\mathbb{1} A_k \cap B_k(S)} - \frac{\mathbb{1}_{A_k \cap B_k \cap C_k}(X)}{\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X)} \right| \\
&= \mathbb{E}\left| \frac{\mathbb{1}_{A_k \cap B_k \cap C_k}(S)(\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X) - \mathbb{E}\mathbb{1}_{A_k \cap B_k}(S)) + (\mathbb{1}_{A_k \cap B_k \cap C_k}(S) - \mathbb{1}_{A_k \cap B_k \cap C_k}(X))\mathbb{E}\mathbb{1}_{A_k \cap B_k}(S)}{\mathbb{E}\mathbb{1}_{A_k \cap B_k}(S)\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X)} \right| \\
&\leq \frac{|\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X) - \mathbb{E}\mathbb{1}_{A_k \cap B_k}(S)|}{\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X)} + \mathbb{E}\left| \frac{(\mathbb{1}_{A_k \cap B_k \cap C_k}(S) - \mathbb{1}_{A_k \cap B_k \cap C_k}(X))}{\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X)} \right| \\
&\leq \frac{2\mathbb{E}|\mathbb{1}_{A_k \cap B_k}(S) - \mathbb{1}_{A_k \cap B_k}(X)| + \mathbb{E}|\mathbb{1}_{C_k}(S) - \mathbb{1}_{C_k}(X)|}{\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X)}.
\end{aligned} \tag{58}$$

By setting $\rho = N_k^{-1/4}$ in the Lemma 1, for large enough k, it is easy to check that

$$\mathbb{E}|\mathbb{1}_{A_k \cap B_k}(S) - \mathbb{1}_{A_k \cap B_k}(X)| \leq \frac{20k}{N_k^{1/4}}, \qquad \mathbb{E}|\mathbb{1}_{C_k}(S) - \mathbb{1}_{C_k}(X)| \leq \frac{20k}{N_k^{1/4}}. \tag{59}$$

On the other hand, by using (45) and the argument leading to (53), for large enough k,

$$\begin{aligned}
&\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X) \\
&= (2h_k)^{k+1} \det(p_2(x_i, x_j)) + \left(\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X) - (2h_k)^{k+1} \det(p_2(x_i, x_j))\right) \\
&= (2h_k)^{k+1} \det(p_2(x_i, x_j)) + \int_{[-h_k, h_k]^{k+1}} (\det(p_2(x_i, x_j + s_j)) - \det(p_2(x_i, x_j)))\, ds_0 \cdots ds_k \\
&\geq \frac{(2h_k)^{k+1}}{2k^{k^2}}.
\end{aligned} \tag{60}$$

Hence, from (59), for large enough k,

$$\mathbb{E}|F_k^S(S) - F_k^X(X)| \leq \frac{120 k^{k^2+1}}{(2h_k)^{k+1} N_k^{1/4}} \to 0 \tag{61}$$

as $k \to \infty$. For the second term of (57), note that the Karlin-McGregor formula for the non-intersecting Brownian motions implies that (cf. (43) above) the density function of the non-intersecting Brownian bridge process Z evaluated at time 1 is equal to

$$f(y_0, \dots, y_k) = \frac{\det(p_1(x_i, y_j))_{i,j=0}^k \det(p_1(y_i, x_j))_{i,j=0}^k}{\det(p_2(x_i, x_j))_{i,j=0}^k}. \tag{62}$$

Similarly, the density of the non-intersecting Brownian motion X evaluated at time t is equal to

$$f(y_0, \dots, y_k) = \frac{\int_{[-h_k,h_k]^{k+1}} \det(p_1(x_i, y_j))_{i,j=0}^k \det(p_1(y_i, x_j))_{i,j=0}^k ds_0 \cdots ds_k}{\int_{[-h_k,h_k]^{k+1}} \det(p_2(x_i, x_j + s_j)) ds_0 \cdots ds_k}. \tag{63}$$

Therefore,

$$\begin{aligned} |\mathbb{E}\left(F_k^X(X) - F_k^Z(Z)\right)| &= \left|\mathbb{E}\left(\frac{\mathbb{1}_{A_k \cap B_k \cap C_k}(X)}{\mathbb{E}\mathbb{1}_{A_k \cap B_k}(X)} - \frac{\mathbb{1}_{A_k \cap B_k \cap C_k}(Z)}{\mathbb{E}\mathbb{1}_{A_k \cap B_k}(Z)}\right)\right| \\ &\leq \int_{\mathbb{R}_>^{k+1}} \left| \frac{\int_{[-h_k,h_k]^{k+1}} \det(p_1(x_i, y_j)) \det(p_1(y_i, x_j + s_j)) ds_0 \cdots ds_k}{\int_{[-h_k,h_k]^{k+1}} \det(p_2(x_i, x_j + s_j)) ds_0 \cdots ds_k} \right. \\ &\qquad \left. - \frac{\det(p_1(x_i, y_j)) \det(p_1(y_i, x_j))}{\det(p_2(x_i, x_j))} \right| dy_0 \cdots dy_k. \end{aligned} \tag{64}$$

By using Lemma 3 and (55), this implies that

$$\begin{aligned} |\mathbb{E}\left(F_k^X(X) - F_k^Z(Z)\right)| &\leq \frac{1}{k} \int_{\mathbb{R}_>^{k+1}} |\det(p_1(x_i, y_j))| dy_0 \cdots dy_k \\ &\leq \frac{1}{k} \int_{\mathbb{R}_>^{k+1}} \left[\prod_{i=0}^k \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i - y_i)^2}{2}} \right] dy_0 \cdots dy_k \leq \frac{1}{k}. \end{aligned} \tag{65}$$

The proof of (29) is exactly the same. This completes the proof of Proposition 1. □

3 Asymptotics of a Brownian bridge process

We prove Proposition 2 in this section. Together with the results of Section 2, this completes the proof of Theorem 1 and Theorem 2.

From the density formula of Karlin and McGregor for non-intersecting Brownian bridge process [33] (cf. (43)),

$$\mathbb{E}\big(F_k^Z(Z)\big) = \frac{1}{\det\left(p_2(x_i, x_j)\right)_{i,j=0}^k} \int_{\mathbb{R}_>^{k+1}} \left[\det\left(p_1(x_i, y_j)\right)_{i,j=0}^k\right]^2 \prod_{j=0}^k (1 - \mathcal{H}_1(y_j)) dy_j \tag{66}$$

where $x_i = \frac{2i-k}{k}$ and

$$\mathcal{H}_1(y) = \mathbb{1}_{\left(\sqrt{2k} + \frac{\xi}{\sqrt{2}k^{1/6}}, \infty\right)}(y). \tag{67}$$

Also

$$\mathbb{E}\big(G_k^Z(Z)\big) = \frac{1}{\det\left(p_2(x_i, x_j)\right)_{i,j=0}^k} \int_{\mathbb{R}_>^{k+1}} \left[\det\left(p_1(x_i, y_j)\right)_{i,j=0}^k\right]^2 \prod_{j=0}^k (1 - \mathcal{H}_2(y_j)) dy_j \tag{68}$$

where

$$\mathcal{H}_2(y) = \mathbb{1}_{[-\frac{\eta}{\sqrt{k+1}}, \frac{\eta}{\sqrt{k+1}}]}(y). \tag{69}$$

We need the limit of (66) and (68) as $k \to \infty$.

In the discussion below, $\mathcal{H}(y)$ denotes either of $\mathcal{H}_1$ or $\mathcal{H}_2$. Indeed, the algebra below works for arbitrary bounded functions $\mathcal{H}(y)$. Using the formula of p_t and the definition of x_i, an elementary algebraic manipulation using Vandermode determinants yields that (66) and (68) are equal to

$$C'_k \cdot \int_{\mathbb{R}^{k+1}} \prod_{0\le i<j\le k} (e^{\frac{2y_j}{k}} - e^{\frac{2y_i}{k}})^2 \prod_{j=0}^{k} (1-\mathcal{H}(y_j)) e^{-y_j^2-2y_j} dy_j, \tag{70}$$

where C'_k is the normalization constant so that (70) becomes 1 when $\mathcal{H}(y) \equiv 0$:

$$C'_k = \frac{e^{-\frac{(k+1)(k+2)}{3k}}}{\det\big(p_2(x_i,x_j)\big)_{i,j=0}^{k} (k+1)!(2\pi)^{k+1}}. \tag{71}$$

Note that the integration domain is changed to $\mathbb{R}^{k+1}$ by using the symmetry of the integrand. Changing the variables by $y_j = \frac{k}{2}\log u_j - 1$, (70) equals

$$C_k \cdot \int_{\mathbb{R}_+^{k+1}} \prod_{0\le i<j\le k} (u_j - u_i)^2 \prod_{j=0}^{k} (1-\hat{\mathcal{H}}(u_j)) \frac{1}{u_j} e^{-\frac{k^2}{4}(\log u_j)^2} du_j \tag{72}$$

where

$$\hat{\mathcal{H}}(u) = \mathcal{H}\big(\frac{k}{2}\log u - 1\big) \tag{73}$$

and the normalization constant

$$C_k = \frac{k^{k+1} e^{-\frac{(k+1)(k+2)}{3k}}}{\det\big(p_2(x_i,x_j)\big)_{i,j=0}^{k} (k+1)!(2\pi)^{k+1}}. \tag{74}$$

This is the standard $\beta = 2$ ensemble in the random matrix theory on the half real line $\mathbb{R}_+$ with the weight

$$w(u) = \frac{1}{u} e^{-\frac{k^2}{4}(\log u)^2} = e^{-\frac{k^2}{4}(\log u)^2 - \log u}. \tag{75}$$

Note that $w(u) = o(u^{-m})$ for any $m \ge 0$ as $u \to +\infty$, and $w(u) = o(u^m)$ for any $m \ge 0$ as $u \downarrow 0$.

With the change of variables $u = e^{-\frac{2}{k^2}} x$, (75) equals

$$w(u)du = c \cdot e^{-\frac{k^2}{4}(\log x)^2} dx, \qquad c = e^{\frac{1}{k^2}}. \tag{76}$$

This is, up to a constant, the Stieltjes-Wigert weight which is defined as

$$\pi^{-1/2} k e^{-k^2(\log x)^2} \tag{77}$$

(see e.g. section 2.7 of [48] or section 3.27 of [34]). The moments for the Stieltjes-Wigert weight is an example of indeterminate moment problem; hence, there are several weights that have the same moments as the weight (77). Another interesting feature of the Stieltjes-Wigert weight (77) is that the corresponding orthogonal polynomials (called Stieltjes-Wigert polynomials) are an example of so-called q-polynomials with $q = e^{-\frac{1}{2k^2}}$ (see e.g. section 3.27 of [34]). It seems that the above non-intersecting bridge process Z is the first example where the q-polynomials appear in random matrix theory context.

Various $\beta = 2$ matrix ensembles of the form (72) (on both the real line and a subset of the real line) have been analyzed asymptotically and it has been proven that the local statistics of the 'eigenvalues', or the particles $u_0, \dots, u_k$, are generically independent of the potential w. For example, such 'universality' is proven when $w(x) = e^{-(k+1)V(x)}$ for an analytic weight V on $\mathbb{R}$ or $\mathbb{R}_+$ satisfying certain growth conditions as $x \to \pm\infty$ (and as $x \to 0$ for weights on $\mathbb{R}_+$) (e.g. [41, 11, 21, 37]) and when $w(x) = e^{-Q(x)}$ where $Q(x)$ is a polynomial (e.g. [20]). However, the asymptotic analysis of the ensemble with the weight given in (75) above does not seem to be in a literature. It is well-known (see (78) below) that the asymptotics of $\beta = 2$ ensemble amounts to the asymptotic analysis of the corresponding orthogonal polynomials. For our case, we need the asymptotics of the orthogonal polynomials of degree k and $k+1$ with respect to the weight (75) as $k \to \infty$; note that the weight also varies as k increases. The asymptotics of Stieltjes-Wigert polynomials are studied recently in [52] and [26] but in different asymptotic regimes: the degree goes to infinity while the weight is fixed. Therefore, the analysis of this section seems to yield new results for asymptotics of Stieltjes-Wigert polynomials. Nevertheless, the asymptotics analysis of the orthogonal polynomials and the ensemble (72) with varying weight (75) can be done in a very similar way to the analysis in [20, 21] using the Deift-Zhou steepest-descent method for related Riemann-Hilbert problems (RHP's), which is now one of standard tools for asymptotic analysis for orthogonal polynomials. We note that the paper [52] also used Deift-Zhou method (for a different asymptotic regime), and our analysis has some overlap with the analysis of [52]. In this section, we only present a sketch of the analysis.

It is a standard result in random matrix theory (see e.g. [38, 49]) that (72) equals

$$\det(1 - \mathbf{K}_k \hat{\mathcal{H}}) \tag{78}$$

where

$$\mathbf{K}_k(x,y) = \sqrt{w(x)w(y)} \frac{\gamma_k}{\gamma_{k+1}} \frac{p_{k+1}(x)p_k(y) - p_k(x)p_{k+1}(y)}{x-y} \tag{79}$$

is the Christoffel-Darboux kernel in which $p_n(x) = \gamma_n x^n + \cdots$ is the nth orthonormal polynomial with respect to w. Hence

$$\begin{aligned} \mathbb{E}\big(F_k^Z(Z)\big) &= \det(1 - \mathbf{K}_k \hat{H}_1) \\ \mathbb{E}\big(G_k^Z(Z)\big) &= \det(1 - \mathbf{K}_k \hat{H}_2). \end{aligned} \tag{80}$$

Let $\mathbf{Y}(z)$ be the solution to the following Riemann-Hilbert problem; $\mathbf{Y}(x)$ is the 2×2 matrix-valued function on $\mathbb{C} \setminus \overline{R_+}$ satisfying

- $\mathbf{Y}(z)$ is analytic for $z \in \mathbb{C} \setminus \overline{\mathbb{R}_+}$, $\mathbf{Y}_\pm(z) = \lim_{\epsilon \downarrow 0} \mathbf{Y}(z \pm i\epsilon)$ is continuous for $z \in \mathbb{R}_+$, and $\mathbf{Y}(z)$ is bounded as $z \to 0$.
- For $z \in \mathbb{R}_+$,
$$\mathbf{Y}_+(z) = \mathbf{Y}_-(z) \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}. \tag{81}$$
- $\mathbf{Y}(z) z^{-(k+1)\sigma_3} = (\mathbf{I} + O(z^{-1}))$ uniformly as $z \to \infty$ such that $z \in \mathbb{C} \setminus \overline{\mathbb{R}_+}$ where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

There is a unique $\mathbf{Y}$ to this RHP and in particular, the (11) and (21) entries of $\mathbf{Y}(z)$ are given by $\mathbf{Y}_{11}(z) = \gamma_{k+1}^{-1} p_{k+1}(z)$ and $\mathbf{Y}_{21}(z) = -2\pi i \gamma_k p_k(z)$ [24] . Note that the existence of $\mathbf{Y}$ under the condition that $\mathbf{Y}(z)$ is bounded as $z \to 0$ (rather than, for example, that $\mathbf{Y}_{12}(z) = O(z^{-1})$ as in, say, [51]) is due to the fact that $w(x) \to 0$ faster than any polynomials as $x \to 0$. Thus, the Christoffel-Darboux kernel can be written as, by using the fact that $\det \mathbf{Y}(z) = 1$,

$$\mathbf{K}_k(x,y) = \sqrt{w(x)w(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{Y}^{-1}(y) \mathbf{Y}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{82}$$

One of the main ingredient in analyzing the RHP for orthogonal polynomials asymptotically is the so-called equilibrium measure and the corresponding 'g-function'. Let $\psi(x)dx$ be a measure on $\mathbb{R}_+ = \operatorname{supp}(w)$ with total mass

$$\int \psi(x)dx = k+1. \tag{83}$$

Define the 'G-function'

$$G(z)=\int\log(z-x)\psi(x)dx,\qquad z\in\mathbb{C}\setminus\overline{\mathbb{R}_+},\tag{84}$$

where log represents the log function on the standard branch so that $\log u=\log|u|+i\arg(u)$ where $|\arg(u)|<\pi$. It is customary to define ψ to be the probability measure and define g-function as in (84) (hence $G(z)=(k+1)g(z)$), but in this paper we use the above convention since it simplifies some formulas below. Note that

$$G_+(x)+G_-(x)=2\int\log|x-y|\psi(y)dy,\qquad x\in\mathbb{R}_+.\tag{85}$$

We look for G satisfying the following two conditions; there is a constant ℓ such that

- $G_+(x)+G_-(x)+\log(w(x))-\ell=0$ for $x\in\operatorname{supp}(\psi)$,
- $G_+(x)+G_-(x)+\log(w(x))-\ell<0$ for $x\in\mathbb{R}_+\setminus\operatorname{supp}(\psi)$.

For such G, the measure ψ is called the equilibrium measure.

Using the standard procedure to solve this variational problem (see e.g. [44, 19]), one can compute the equilibrium measure for the weight (75).

Lemma 4. *For the weight* (75)*, the support of the equilibrium measure is* $[\mathbf{a},\mathbf{b}]$ *where*

$$\begin{aligned}\sqrt{\mathbf{a}}&=e^{\frac{2k+1}{k^2}}-\sqrt{e^{\frac{4k+2}{k^2}}-e^{\frac{2}{k}}},\\ \sqrt{\mathbf{b}}&=e^{\frac{2k+1}{k^2}}+\sqrt{e^{\frac{4k+2}{k^2}}-e^{\frac{2}{k}}}.\end{aligned}\tag{86}$$

The equilibrium measure is for $x\in[\mathbf{a},\mathbf{b}]$,

$$\psi(x)=\frac{1}{2\pi}\sqrt{(\mathbf{b}-x)(x-\mathbf{a})}h(x),\qquad h(z)=\frac{1}{2\pi i}\oint_C\frac{-(\log(w(s)))'}{(s-z)R(s)}ds\tag{87}$$

where $R(z)=((z-\mathbf{a})(z-\mathbf{b}))^{1/2}$ *denotes the principal branch of the square-root function and the simple closed contour* C *contains* z *and* $[\mathbf{a},\mathbf{b}]$ *inside, does not touch* $(-\infty,0]$ *and is oriented counter-clockwise. A residue calculation yields that*

$$\psi(x)=\frac{k^2}{2\pi x}\arctan\left(\frac{\sqrt{(\mathbf{b}-x)(x-\mathbf{a})}}{\sqrt{\mathbf{ab}}+x}\right),\qquad x\in[\mathbf{a},\mathbf{b}].\tag{88}$$

We remark that $\mathbf{a}$ and $\mathbf{b}$ are sometimes called the Mhaskar-Rakhmanov-Saff numbers. The above $\mathbf{a}$ and $\mathbf{b}$ are obtained in [52]: with α_n and β_n in (2.2) and (2.3) of [52],

$$\mathbf{a}=\left(e^{-\frac{1}{2k^2}}\alpha_n\right)|_{k\mapsto\frac{k}{2},n=k+1},\qquad \mathbf{b}=\left(e^{-\frac{1}{2k^2}}\beta_n\right)|_{k\mapsto\frac{k}{2},n=k+1}.\tag{89}$$

Given this ψ, $G(z)$ is defined as in (84) and ℓ defined as $\ell=2G(\mathbf{b})-\log(w(\mathbf{b}))=2G(\mathbf{a})-\log(w(\mathbf{a}))$. The function $h(z)$ in (87) is analytic in $z\in\mathbb{C}\setminus(-\infty,0]$. A residue calculation yields that

$$h(z)=\frac{k^2}{2zR(z)}\log\left(\frac{\sqrt{ab}+z-R(z)}{\sqrt{ab}+z+R(z)}\right),\qquad z\in\mathbb{C}\setminus(-\infty,0]\tag{90}$$

where log denotes the principal branch of logarithm. For a computation below, we note that as $k\to\infty$,

$$\begin{aligned}\sqrt{\mathbf{a}}&=1-\sqrt{\frac{2}{k}}+\frac{2}{k}+O(k^{-3/2}),\\ \sqrt{\mathbf{b}}&=1+\sqrt{\frac{2}{k}}+\frac{2}{k}+O(k^{-3/2}).\end{aligned}\tag{91}$$

We also remark that with $x = 1 + \frac{2w}{\sqrt{k}}$, for $w = O(1)$, as $k \to \infty$, at least formally,

$$\psi(x)dx \sim \frac{k}{\pi}\sqrt{2 - w^2}dw, \qquad w \in [-\sqrt{2}, \sqrt{2}], \tag{92}$$

which is precisely the Wigner's semicircle. This last calculation is not going to be used below, but it gives an intuitive reason why the ensemble (72) (and (70)) has the same asymptotics as the Gaussian unitary ensemble, not only locally, but also globally.

Set

$$\mathbf{M}(z) = e^{-\frac{1}{2}\ell\sigma_3}\mathbf{Y}(z)e^{-G(z)\sigma_3}e^{\frac{1}{2}\ell\sigma_3} \tag{93}$$

for $z \in \mathbb{C} \setminus \overline{\mathbb{R}_+}$. Using the analyticity of G for $z \in \mathbb{R}_+ \setminus [\mathbf{a}, \mathbf{b}]$ and the variational conditions, $\mathbf{M}(z)$ solves the following equivalent RHP:

- $\mathbf{M}(z)$ is analytic for $z \in \mathbb{C} \setminus \overline{\mathbb{R}_+}$, $\mathbf{M}_\pm(z)$ is continuous for $z \in \mathbb{R}_+$, and $\mathbf{M}(z)$ is bounded as $z \to 0$.
- For $z \in \mathbb{R}_+$, $\mathbf{M}_+(z) = \mathbf{M}_-(z)\mathbf{V}_M(z)$ where

$$\mathbf{V}_M(z) = \begin{pmatrix} e^{G_-(z)-G_+(z)} & 1 \\ 0 & e^{G_+(z)-G_-(z)} \end{pmatrix}, \qquad z \in (\mathbf{a}, \mathbf{b}) \tag{94}$$

$$\mathbf{V}_M(z) = \begin{pmatrix} 1 & e^{2G(z)+\log(w(z))-\ell} \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{R}_+ \setminus (\mathbf{a}, \mathbf{b}). \tag{95}$$

- $\mathbf{M}(z) = \mathbf{I} + O(z^{-1})$ as $z \to \infty$.

The non-unit terms in the jump matrix can be expressed in a unifying way. Set

$$H(z) = G(z) + \frac{1}{2}\log(w(z)) - \frac{1}{2}\ell, \qquad z \in \mathbb{C} \setminus \big((-\infty, 0] \cup [\mathbf{a}, \mathbf{b}]\big). \tag{96}$$

Noting the variational condition, we find that for $z \in (\mathbf{a}, \mathbf{b})$,

$$\begin{aligned} G_+(z) - G_-(z) &= 2G_+(z) + \log(w(z)) - \ell = 2H_+(z) \\ &= -(2G_-(z) + \log(w(z)) - \ell) = -2H_-(z). \end{aligned} \tag{97}$$

Hence $G_+(z) - G_-(z)$ has an analytic continuation for both above and the below the real axis. Therefore, the jump matrix $\mathbf{V}_M$ equals

$$\mathbf{V}_M(z) = \begin{pmatrix} e^{-2H_+(z)} & 1 \\ 0 & e^{-2H_-(z)} \end{pmatrix}, \qquad z \in (\mathbf{a}, \mathbf{b}) \tag{98}$$

$$\mathbf{V}_M(z) = \begin{pmatrix} 1 & e^{2H(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{R}_+ \setminus (\mathbf{a}, \mathbf{b}). \tag{99}$$

Using the definition of G and Lemma 4, one can check that

$$H'(z) = \frac{1}{2}R(z)h(z). \tag{100}$$

Now we scale the RHP for $\mathbf{M}$ so that the the interval $(\mathbf{a}, \mathbf{b})$ becomes $(-1, 1)$. In other words, instead of moving interval as the support of the equilibrium measure, we will fix the support. In that way, we can use the analysis of [20, 21] more directly. Define

$$\mathbf{N}(z) = \mathbf{M}\left(\frac{\mathbf{b}-\mathbf{a}}{2}z + \frac{\mathbf{b}+\mathbf{a}}{2}\right). \tag{101}$$

Set $\Sigma = (-\frac{\mathbf{b}+\mathbf{a}}{\mathbf{b}-\mathbf{a}}, \infty)$, and set

$$\hat{H}(z) = H\left(\frac{\mathbf{b}-\mathbf{a}}{2}z + \frac{\mathbf{b}+\mathbf{a}}{2}\right). \tag{102}$$

The matrix $\mathbf{N}$ solves the following RHP:

- $\mathbf{N}(z)$ is analytic for $z \in \mathbb{C} \setminus \overline{\Sigma}$, $\mathbf{N}_\pm(z)$ is continuous for $z \in \Sigma$, and $\mathbf{N}(z)$ is bounded as $z \to -\frac{\mathbf{b+a}}{\mathbf{b-a}}$.
- For $z \in \Sigma$, $\mathbf{N}_+(z) = \mathbf{N}_-(z)\mathbf{V}_N(z)$ where

$$\mathbf{V}_N(z) = \begin{pmatrix} e^{-2\hat{H}_+(z)} & 1 \\ 0 & e^{-2\hat{H}_-(z)} \end{pmatrix}, \qquad z \in (-1,1) \tag{103}$$

$$\mathbf{V}_N(z) = \begin{pmatrix} 1 & e^{2\hat{H}(z)} \\ 0 & 1 \end{pmatrix}, \qquad z \in \Sigma \setminus (-1,1). \tag{104}$$

- $\mathbf{N}(z) = \mathbf{I} + O(z^{-1})$ as $z \to \infty$.

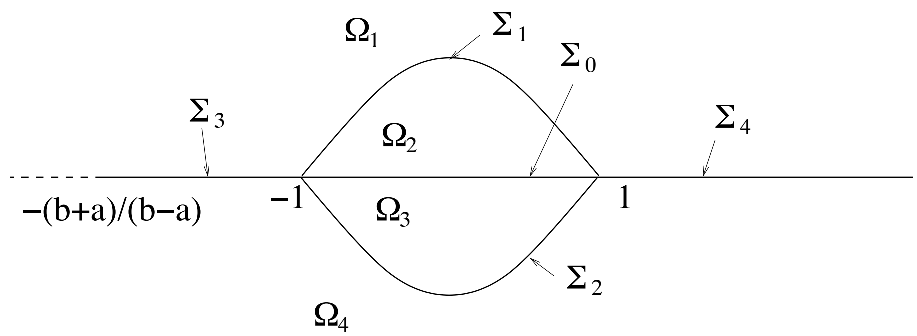

Figure 1: Contours for $\mathbf{N}$

Note the factorization for $z \in (-1,1)$

$$\begin{pmatrix} e^{-2\hat{H}_+(z)} & 1 \\ 0 & e^{-2\hat{H}_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-2\hat{H}_-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-2\hat{H}_+(z)} & 1 \end{pmatrix} \tag{105}$$

where we use the fact that $\hat{H}_+(z) + \hat{H}_-(z) = 0$ for $z \in (-1,1)$. Let Σ_j, $j = 0, 1, \ldots, 4$ and Ω_j, $j = 1, \ldots, 4$ be the contours and open regions given in 3. Contours are oriented from the left to the right. Define

$$\mathbf{Q}(z) = \begin{cases} \mathbf{N}(z) & z \in \Omega_1 \cup \Omega_4 \\ \mathbf{N}(z) \begin{pmatrix} 1 & 0 \\ -e^{-2\hat{H}(z)} & 1 \end{pmatrix} & z \in \Omega_2 \\ \mathbf{N}(z) \begin{pmatrix} 1 & 0 \\ e^{-2\hat{H}(z)} & 1 \end{pmatrix} & z \in \Omega_3. \end{cases} \tag{106}$$

Then $\mathbf{Q}_+(z) = \mathbf{Q}_-(z)\mathbf{V}_Q(z)$ for z in $\Sigma_0, \ldots, \Sigma_4$ where

$$\mathbf{V}_Q(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad z \in \Sigma_0 \tag{107}$$

$$\mathbf{V}_Q(z) = \begin{pmatrix} 1 & 0 \\ e^{-2\hat{H}(z)} & 1 \end{pmatrix} \qquad z \in \Sigma_1 \cup \Sigma_2 \tag{108}$$

$$\mathbf{V}_Q(z) = \begin{pmatrix} 1 & e^{2\hat{H}(z)} \\ 0 & 1 \end{pmatrix} \qquad z \in \Sigma_3 \cup \Sigma_4. \tag{109}$$

The off-diagonal terms of $\mathbf{V}_Q$ on $\Sigma_1 \cup \cdots \cup \Sigma_4$ converges to 0 as the following Lemma implies.

Lemma 5. *There are $\delta_0 > 0$ and $k_0 > 0$ such that for $k \geq k_0$,*

$$Re\big[\hat{H}(x+iy)\big] \geq 2k|y|\sqrt{1-x^2} \qquad \text{for } -1 \leq x \leq 1 \text{ and } -\delta_0 \leq y \leq \delta_0. \tag{110}$$

For any $\delta > 0$,

$$\hat{H}(x) \leq -k\delta^{3/2} \qquad \text{for } -\frac{\mathbf{b}+\mathbf{a}}{\mathbf{b}-\mathbf{a}} < x \leq -1-\delta \text{ and } x \geq 1+\delta \tag{111}$$

when $k \geq k_0$, and

$$\lim_{k\to\infty} \int_{(\Sigma_3\cup\Sigma_4)\cap\{|z-1|>\delta\}\cap\{|z+1|>\delta\}} e^{2\hat{H}(x)}dx = 0. \tag{112}$$

Hence $\mathbf{V}_Q \to \mathbf{V}_\infty$ for a constant matrix $\mathbf{V}_\infty$ defined as

$$\mathbf{V}_\infty(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad z \in \Sigma_0 \tag{113}$$

and $\mathbf{V}_\infty(z) = \mathbf{I}$ for $z \in \Sigma_1 \cup \cdots \cup \Sigma_4$, where the convergence $\mathbf{V}_Q \to \mathbf{V}_\infty$ is in $L^\infty(\Sigma_0 \cup \cdots \cup \Sigma_4)$ and also in $L^2((\Sigma_0 \cup \cdots \cup \Sigma_4) \cap \{|z-1| > \delta\} \cap \{|z+1| > \delta\})$ for an arbitrary but fixed $\delta > 0$. Let

$$\beta(z) = \left(\frac{z-1}{z+1}\right)^{1/4} \tag{114}$$

where the branch cut is $[-1,1]$ and $\beta(z) \sim 1$ as $z \to +\infty$ on the real line, and define

$$\mathbf{Q}^\infty(z) = \frac{1}{2}\begin{pmatrix} \beta+\beta^{-1} & -i(\beta-\beta^{-1}) \\ i(\beta-\beta^{-1}) & \beta+\beta^{-1} \end{pmatrix} \tag{115}$$

for $z \in \mathbb{C} \setminus \Sigma_0$. Then $\mathbf{Q}^\infty(z)$ is the solution to the RHP for the $\mathbf{Q}^\infty_+ = \mathbf{Q}^\infty_-\mathbf{V}_\infty$ and $\mathbf{Q}^\infty(z) \to \mathbf{I}$ as $z \to \infty$. As the convergence $\mathbf{V}_Q \to \mathbf{V}_\infty$ is not uniform near the points $z = \pm 1$, and hence it is not true that $\mathbf{Q}(z) \to \mathbf{Q}^\infty(z)$ for all z, and one needs local parametrix for z in a neighborhood of ± 1.

Let $\Psi(z)$ be the matrix-valued function constructed from the Airy function and its derivatives as defined in Proposition 7.3 of [20]. Let $\epsilon > 0$. For $z \in U_r := \{z : |z-1| < \epsilon\}$, set

$$\mathbf{S}_r(z) = \mathbf{E}(z)\Psi\Big(\big(-\frac{3}{2}\hat{H}(z)\big)^{2/3}\Big)e^{-\hat{H}(z)\sigma_3} \tag{116}$$

where

$$\mathbf{E}(z) = \sqrt{\pi}e^{\frac{\pi}{6}i}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}\begin{pmatrix} \big(-\frac{3}{2}\hat{H}(z)\big)^{1/6}\beta(z)^{-1} & 0 \\ 0 & \big(-\frac{3}{2}\hat{H}(z)\big)^{-1/6}\beta(z) \end{pmatrix}. \tag{117}$$

Note that $\mathbf{E}(z)$ is analytic in U_r if ϵ is chosen small enough. The matrix $\mathbf{S}_l(z)$ is defined in a similar way for $z \in U_l := \{z : |z+1| < \epsilon\}$. Define

$$\mathbf{Q}_{par}(z) = \begin{cases} \mathbf{Q}^\infty(z) & z \in \mathbb{C} \setminus U_r \cup U_l \cup \Sigma \\ \mathbf{S}_r(z) & z \in U_r \setminus \Sigma \\ \mathbf{S}_l(z) & z \in U_l \setminus \Sigma. \end{cases} \tag{118}$$

¿From the basic theorem of RHP, the estimate in Lemma 5, and from the same argument of [20], one can check that the jump matrix for $\mathbf{Q}_{par}^{-1}\mathbf{Q}$ converges to the identity in $L^2 \cap L^\infty$. Hence

$$\mathbf{Q}(z) = \big(\mathbf{I} + O(k^{-1})\big)\mathbf{Q}_{par}(z). \tag{119}$$

This holds uniformly for z outside an open neighborhood of the contours $\Sigma \cup \partial U_r \cup \partial U_l$. But a simple deformation argument implies that the result is extended to z on the contours (see [20]). Hence by reversing the transformations $\mathbf{Y} \to \mathbf{M} \to \mathbf{N} \to \mathbf{Q}$ (see (93), (101) and (106)), the asymptotics of $\mathbf{Y}(z)$ for all $z \in \mathbb{C}$ are obtained.

By plugging in the asymptotics of $\mathbf{Y}$ into (82), edge and bulk scaling limits of the $\mathbf{K}_k$ is obtained. See [15, 21, 18] for details. For x_0 such that $\sqrt{k}(x_0 - 1)$ lies in a compact subset of $(\sqrt{k}(\mathbf{a}-1), \sqrt{k}(\mathbf{b}-1))$, for all ξ, η in a compact subset of $\mathbb{R}$,

$$\frac{1}{\psi(x_0)}\mathbf{K}_k\left(x_0 + \frac{\xi}{\psi(x)}, x + \frac{\eta}{\psi(x_0)}\right) \to \mathbb{S}(\xi, \eta) \tag{120}$$

in trace norm for $\xi, \eta \in$ where

$$\mathbb{S}(\xi, \eta) = \frac{\sin(\pi(\xi - \eta))}{\pi(\xi - \eta)}. \tag{121}$$

Here we may replace $\psi(x_0)$ by $\mathbf{K}_k(x_0, x_0)$. The error is $O(k^{-1})$ uniformly for ξ, η in a compact set. The convergence is also in trace norm in the Hilbert space $L^2((-\eta, \eta))$ for a fixed $\eta > 0$. From (80), by taking $x_0 = e^{\frac{2}{k}}$, the limit (30) in Proposition 2 is obtained.

At the edge of the support of $\psi(x)$, set

$$B_k = \left[-\frac{1}{2}\sqrt{\mathbf{b}-\mathbf{a}}h(b)\right]^{2/3} \sim \frac{k^{7/6}}{\sqrt{2}}. \tag{122}$$

As $k \to \infty$,

$$\frac{1}{B_k}\mathbf{K}_k\left(b + \frac{\xi}{B_k}, b + \frac{\eta}{B_k}\right) \to \mathbb{A}(\xi, \eta) \tag{123}$$

in trace norm in the Hilbert space $L^2((\xi, \infty))$ for a fixed ξ, where

$$\mathbb{A}(\xi, \eta) = \frac{\mathrm{Ai}(\xi)\,\mathrm{Ai}'(\eta) - \mathrm{Ai}'(\xi)\,\mathrm{Ai}(\eta)}{\xi - \eta} \tag{124}$$

is the Airy kernel. Hence from (80), the limit (31) in Proposition 2 is obtained.

4 Generalizations and Discussions

We comment on three issues in this section: The case in which the moment generating function does not exist, finite dimensional distributions, and the connections of this work to q-orthogonal polynomials.

No moment generating function

In this paper we assumed the existence of the moment generating function for the random variable increments of the non-intersecting random walks. This is simply to improve the estimates. For the case $\mathbb{E}|X_i^j|^{2+\delta} < \infty$, $\delta > 0$, there is a version of the KMT theorem which gives analogous estimates to those used in Section 2. As one would expect for this case, N_k has to grow more quickly in k. Another method of achieving similar results to those of this paper is by using Skorohod embedding in order to imbed the non-intersecting random walks into Brownian motions. In order to achieve this one must assume that $\mathbb{E}|X_i^j|^4 < \infty$.

Finite dimensional distributions

The results of this paper focus on the limiting distributions of the non-intersecting random walks at the fixed time $t = 1$. It is also interesting to consider finite dimensional distributions of the process, i.e. in the correct scaling $t_1, ..., t_n \in [1 - Ak^{-\frac{1}{3}}, 1 + Ak^{-\frac{1}{3}}]$ the finite dimensional distributions of the fluctuations of the top random walk should converge to those of the Airy process. A similar but differently scaled result should also be true in the bulk. (See for example, [43, 50, 1] and references in them about Airy process and other processes from random matrix theory.) The methods of Section 2 are certainly applicable to this problem, however, the convergence of the finite dimensional distributions of the non-intersecting Brownian bridges to Airy/sine processes does not follow immediately from the analysis of Section 3. However, one can use a different approach based on the method of Eynard and Mehta [23, 14]. In this approach, an inversion of a matrix is crucial. After the completion of this paper, Widom had communicated to us how to invert the matrix. A work in this direction will appear in a future paper.

Stieltjes-Wigert weight and q-orthogonal polynomials

In section 3, the Riemann-Hilbert problems for the the orthogonal polynomials with respect to the Stieltjes-Wigert weight (77) was analyzed in the Plancherel-Rotach asymptotic regime. The analysis yields the asymptotics of the Stieltjes-Wigert polynomials in the entire complex plane. Since Stieltjest-Wigert polynomials are examples of q-polynomials, this result also yields an asymptotic result for certain q-polynomials.

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