

# Eigenvalues of Large Sample Covariance Matrices of Spiked Population Models

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## Abstract

We consider a spiked population model, proposed by Johnstone, whose population eigenvalues are all unit except for a few fixed eigenvalues. The question is to determine how the sample eigenvalues depend on the non-unit population ones when both sample size and population size become large. This paper completely determines the almost sure limits for a general class of samples.

## 1 Introduction

The sample covariance matrix is fundamental to multivariate statistics. When the population size is not large and for a sufficient number of samples, the sample covariance matrix is a good approximate of the population covariance matrix. However when the population size is large and comparable with the sample size, as is in many contemporary data, it is known that the sample covariance matrix is no longer a good approximation to the covariance matrix. The Marchenko-Pastur theorem [15] states that with  $n =$  the sample size,  $p =$  the population size, as  $n = n(p) \rightarrow \infty$  such that  $\frac{p}{n} \rightarrow c$ , the eigenvalues  $s_j^{(p)}$ ,  $j = 1, \dots, p$ , of the sample covariance matrix of normalized i.i.d. Gaussian samples satisfy for any real  $x$

$$\frac{1}{p} \#\{s_j^{(p)} : s_j^{(p)} < x\} \rightarrow F(x) \quad (1.1)$$

almost surely where

$$F'(x) = \frac{1}{2\pi xc} \sqrt{(b-x)(x-a)}, \quad a < x < b, \quad (1.2)$$

and  $a = (1 - \sqrt{c})^2$  and  $b = (1 + \sqrt{c})^2$  when  $0 < c \leq 1$ . When  $c > 1$ , there is an additional Dirac measure at  $x = 0$  of mass  $1 - \frac{1}{c}$ . Moreover, there are no stray eigenvalues in the sense that the top and bottom eigenvalues converge to the edges of the support of  $F$  [8]:

$$s_1^{(p)} \rightarrow (1 + \sqrt{c})^2 \quad (1.3)$$

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almost surely and [20]

$$s_{\min\{p,n\}}^{(p)} \rightarrow (1 - \sqrt{c})^2 \tag{1.4}$$

almost surely ( $s_{n+1}^{(p)} = \dots s_p^{(p)} = 0$  when  $n < p$ ). One can extract from this some information of the population covariance matrix even though the sample covariance matrix is not a good approximate. For example, if there are non-zero eigenvalues of the sample covariance matrix well separated from the rest of the eigenvalues, one finds, assuming the Gaussian entries, that the samples are not i.i.d..

There are indeed many cases in which a few eigenvalues of the sample covariance matrix are separated from the rest of the eigenvalues, the latter being packed together as in the support of the Marchenko-Pastur function (1.2). The examples include speech recognition [6, 12], mathematical finance [18], [13], [14], wireless communication [25], physics of mixture [19], and data analysis and statistical learning [10].

As a possible explanation for such features, Johnstone [12] proposed the ‘spiked population model’ where all but finitely many eigenvalues of the population covariance matrix are the same, say equal to 1. The question is how the eigenvalues of the sample covariance matrix would depend on the non-unit population eigenvalues as  $p, n \rightarrow \infty$ . It is known [15, 22] that the Marchenko-Pastur result (1.1) still holds for the spiked model. But (1.3) and (1.4) are not guaranteed and some of the eigenvalues are not necessarily in the support of (1.2).

For example, consider the case when the population covariance matrix has one non-unit eigenvalue, denoted by  $\sigma_1$ . If  $\sigma_1$  is close to 1, one would expect that as the dimension  $p$  becomes large, the population covariance matrix would be close to a large identity matrix, and hence  $\sigma_1$  would have little effect on the eigenvalues of the sample covariance matrix. On the other hand, if  $\sigma_1$  is much bigger than 1, then even if  $p$  becomes large,  $\sigma_1$  might still pull up the eigenvalues of the sample covariance matrix. How big should  $\sigma_1$  be in order to have any effect, how many eigenvalues of the sample covariance matrix would be pulled up and exactly where would the pulled-up eigenvalues be? We will see in the results below that the answers are  $\sigma_1 > 1 + \sqrt{c}$  (where  $\frac{p}{n} \rightarrow c$ ), one eigenvalue at most, and  $\sigma_1 + \frac{c\sigma_1}{\sigma_1 - 1}$ , respectively.

For *complex Gaussian* samples, the papers [17, 5] study the *largest* eigenvalue of the sample covariance matrix. The authors determine the transition behavior and the limiting distributions are also obtained. The purpose of this paper is a complete study of the spiked model for *both real and complex samples which are not necessarily Gaussian*. We obtain almost sure limit results. A general study of ‘non-null’ covariance matrices was done in [2, 3]. We will show in this paper how to extract the desired results from the work of [3]. While this paper was being prepared, the authors learned that Debashis Paul [16] was also studying the spiked model independently at the same time, which has some overlap with this work. Paul considers the real Gaussian samples for  $c < 1$ , and obtains the almost sure limits as in (1.10) and (1.11) below for large sample eigenvalues. Moreover, when all non-unit population eigenvalues are simple, the limiting distribution is found to be Gaussian (see Subsection 1.3 below for more detail). On the other hand, our paper (i) is concerned with more general samples, not necessarily Gaussian, (ii) includes all choices of  $c$  and (iii) studies both large and small sample eigenvalues. We remark that a complete study of limiting distributions is still an open question.

## 1.1 Model

Let  $T_p$  be a fixed  $p \times p$  non-negative definite Hermitian matrix. Let  $Z_{ij}$ ,  $i, j = 1, 2, \dots$ , be independent and identically distributed complex valued random variables satisfying

$$\mathbb{E}(Z_{11}) = 0, \quad \mathbb{E}(|Z_{11}|^2) = 1, \quad \text{and} \quad \mathbb{E}(|Z_{11}|^4) < \infty, \quad (1.5)$$

and set  $Z_p = (Z_{ij})$ ,  $1 \leq i \leq p$ ,  $1 \leq j \leq n$ . We take the sampled vectors to be the columns of  $T_p^{1/2} Z_p$ , where  $T_p^{1/2} Z_p$  is an Hermitian square root of  $T_p$ . Hence  $T_p$  is the population covariance matrix. Of course, not all random vectors are realized as such, but this model is still very general. When  $Z_{ij}$  are i.i.d (real or complex) Gaussian, the model becomes the Gaussian samples with population covariance matrix  $T_p$ . Outside the Gaussian case we see that these vectors cover a broad range of random vectors, completely real or complex, with arbitrary population covariance matrix.

Let

$$B_p := \frac{1}{n} T_p^{1/2} Z_p Z_p' T_p^{1/2} \quad (1.6)$$

be the sample covariance matrix, where  $Z_p'$  denotes conjugate transpose. Denote the eigenvalues of  $B_p$  by  $s_1^{(p)}, \dots, s_p^{(p)}$ : for some unitary matrix  $U_B$ ,

$$U_B B_p U_B^{-1} = \begin{pmatrix} s_1^{(p)} & & & \\ & s_2^{(p)} & & \\ & & \ddots & \\ & & & s_p^{(p)} \end{pmatrix} = \text{diag}(s_1^{(p)}, s_2^{(p)}, \dots, s_p^{(p)}). \quad (1.7)$$

For definiteness, we order the eigenvalues as  $s_1^{(p)} \geq s_2^{(p)} \geq \dots \geq s_p^{(p)} \geq 0$ .

Let  $\alpha_1 > \dots > \alpha_M > 0$  be fixed real numbers for some fixed  $M \geq 0$ , which is independent of  $p$  and  $n$ . Let  $k_1, \dots, k_M$  be fixed non-negative integers and set  $r = k_1 + \dots + k_M$ , which are also independent of  $p$  and  $n$ . We assume that all the eigenvalues of  $T_p$  are 1 except for, say, the first  $r$  eigenvalues. This is the ‘spiked population model’ proposed in [12]. Let the first  $r$  eigenvalues be equal to  $\alpha_1, \dots, \alpha_M$  with multiplicity  $k_1, \dots, k_M$ , respectively: for some unitary matrix  $U_T$ ,

$$U_T T_p U_T^{-1} = \text{diag}(\underbrace{\alpha_1, \dots, \alpha_1}_{k_1}, \underbrace{\alpha_2, \dots, \alpha_2}_{k_2}, \dots, \underbrace{\alpha_M, \dots, \alpha_M}_{k_M}, \underbrace{1, \dots, 1}_{p-r}). \quad (1.8)$$

We set  $k_0 = 0$ .

## 1.2 Results

**Theorem 1.1 (case  $c < 1$ ).** *Assume that  $n = n(p)$  and  $p \rightarrow \infty$  such that*

$$\frac{p}{n} \rightarrow c \quad (1.9)$$

*for a constant  $0 < c < 1$ . Let  $M_0$  be the number of  $j$ 's such that  $\alpha_j > 1 + \sqrt{c}$ , and let  $M - M_1$  be the number of  $j$ 's such that  $\alpha_j < 1 - \sqrt{c}$ . Then the following holds.*

- For each  $1 \leq j \leq M_0$ ,

$$s_{k_1+\dots+k_{j-1}+i}^{(p)} \rightarrow \alpha_j + \frac{c\alpha_j}{\alpha_j - 1}, \quad 1 \leq i \leq k_j. \quad (1.10)$$

almost surely.

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$$s_{k_1+\dots+k_{M_0}+1}^{(p)} \rightarrow (1 + \sqrt{c})^2 \quad (1.11)$$

almost surely.

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$$s_{p-r+k_1+\dots+k_{M_1}}^{(p)} \rightarrow (1 - \sqrt{c})^2 \quad (1.12)$$

almost surely (recall  $r = k_1 + \dots + k_M$ ).

- For each  $M_1 + 1 \leq j \leq M$ ,

$$s_{p-r+k_1+\dots+k_{j-1}+i}^{(p)} \rightarrow \alpha_j + \frac{c\alpha_j}{\alpha_j - 1}, \quad 1 \leq i \leq k_j \quad (1.13)$$

almost surely.

Therefore, when  $c < 1$ , in order for a population eigenvalue to contribute a non-trivial effect to the eigenvalues of the sample covariance matrix, it should sufficiently big (larger than  $1 + \sqrt{c}$ ) or sufficiently small (less than  $1 - \sqrt{c}$ ). As an example, when  $r = 1$ , by denoting the only non-unit eigenvalue by  $\sigma_1$ , the largest sample eigenvalue  $s_1^{(p)}$  satisfies

$$s_1^{(p)} \rightarrow \begin{cases} (1 + \sqrt{c})^2, & \sigma_1 \leq 1 + \sqrt{c} \\ \sigma_1 + \frac{c\sigma_1}{\sigma_1 - 1}, & \sigma_1 > 1 + \sqrt{c} \end{cases} \quad (1.14)$$

almost surely. When  $r = 2$ , by denoting the two non-unit eigenvalues by  $\sigma_1, \sigma_2$ , the largest sample eigenvalue  $s_1^{(p)}$  satisfies

$$s_1^{(p)} \rightarrow \begin{cases} (1 + \sqrt{c})^2, & \max\{\sigma_1, \sigma_2\} \leq 1 + \sqrt{c} \\ \max\{\sigma_1, \sigma_2\} + \frac{c \max\{\sigma_1, \sigma_2\}}{\max\{\sigma_1, \sigma_2\} - 1}, & \max\{\sigma_1, \sigma_2\} > 1 + \sqrt{c} \end{cases} \quad (1.15)$$

almost surely.

The results (1.10) and (1.11) are also independently obtained in [16] under the assumption that the samples are Gaussian.

**Theorem 1.2 (case  $c > 1$ ).** Assume that  $n = n(p)$  and  $p \rightarrow \infty$  such that

$$\frac{p}{n} \rightarrow c \quad (1.16)$$

for a constant  $c > 1$ . Let  $M_0$  be the number of  $j$ 's such that  $\alpha_j > 1 + \sqrt{c}$ . Then the following holds.

- For each  $1 \leq j \leq M_0$ ,

$$s_{k_1+\dots+k_{j-1}+i}^{(p)} \rightarrow \alpha_j + \frac{c\alpha_j}{\alpha_j - 1}, \quad 1 \leq i \leq k_j. \quad (1.17)$$

almost surely.

- 

$$s_{k_1+\dots+k_{M_0}+1}^{(p)} \rightarrow (1 + \sqrt{c})^2 \quad (1.18)$$

*almost surely.*

- 

$$s_n^{(p)} \rightarrow (1 - \sqrt{c})^2 \quad (1.19)$$

*almost surely.*

- For all  $p$ ,

$$s_{n+1}^{(p)} = \dots = s_p^{(p)} = 0. \quad (1.20)$$

Thus, unlike the case of  $c < 1$ , small eigenvalues of  $T_p$  do not affect the eigenvalues of  $B_p$  when  $c > 1$ .

**Theorem 1.3 (case  $c = 1$ ).** *Assume that  $n = n(p)$  and  $p \rightarrow \infty$  such that*

$$\frac{p}{n} \rightarrow 1. \quad (1.21)$$

*Let  $M_0$  be the number of  $j$ 's such that  $\alpha_j > 2$ . Then the following holds.*

- For each  $1 \leq j \leq M_0$ ,

$$s_{k_1+\dots+k_{j-1}+i}^{(p)} \rightarrow \alpha_j + \frac{\alpha_j}{\alpha_j - 1}, \quad 1 \leq i \leq k_j. \quad (1.22)$$

*almost surely.*

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$$s_{k_1+\dots+k_{M_0}+1}^{(p)} \rightarrow 4 \quad (1.23)$$

*almost surely.*

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$$s_{\min\{n,p\}}^{(p)} \rightarrow 0 \quad (1.24)$$

*almost surely.*

### 1.3 Discussion

As mentioned earlier, the limiting density of the eigenvalues of spiked population models is given by the Marchenko-Pastur theorem (1.2) as in the identity population matrix case, and for the top eigenvalue  $s_1^{(p)}$  in the *complex Gaussian* case, the results Theorem 1.1 and Theorem 1.3 were first obtained in [17, 5]. The paper [5] (see section 6) contains an interesting heuristic argument for the critical value  $1 + \sqrt{c}$  and the value

$$\alpha_j + \frac{c\alpha_j}{\alpha_j - 1} \quad (1.25)$$

for  $1 \leq j \leq M_0$  above: they come from a competition between a 1-dimensional last passage time and a 2-dimensional last passage time. It would be interesting to have such a heuristic reasoning for the general case.

When  $T_p$  is the identity matrix (the ‘null case’), under the Gaussian assumption, the limiting distribution for the largest eigenvalue is obtained for the complex case in [7, 11] and for the real case in [12]. [24] shows the Gaussian assumption is not necessary when  $c = 1$ . The limiting distributions are the Tracy-Widom distributions [26, 27] in the random matrix theory in mathematical physics. For the spiked model with complex Gaussian samples when  $c \leq 1$ , the limiting distributions of the largest eigenvalue are obtained in [17, 5]. The paper [5] determines the limiting distribution of  $s_1^{(p)}$  for complete choices of the largest population eigenvalue  $\alpha_1$  and its multiplicity  $k_1$ : the distribution is (i) the Tracy-Widom distribution when  $\alpha_1 < 1 + \sqrt{c}$ , (ii) certain generalizations of the Tracy-Widom distribution (see also [4]) when  $\alpha_1 = 1 + \sqrt{c}$ , and (iii) the Gaussian distribution ( $k_1 = 1$ ) and its generalization ( $k_1 \geq 2$ , the Gaussian unitary ensemble) when  $\alpha_1 > 1 + \sqrt{c}$ . For real Gaussian samples [16] showed that when  $c < 1$ ,  $M_0 \geq 1$  and  $k_1 = \dots = k_{M_0} = 1$ , the limiting distribution of  $s_j^{(p)}$ ,  $1 \leq j \leq M_0$ , is Gaussian. It is an interesting open question to determine the limiting distribution for the general case of real samples. See section 1.3 of [5] for a conjecture for the scaling.

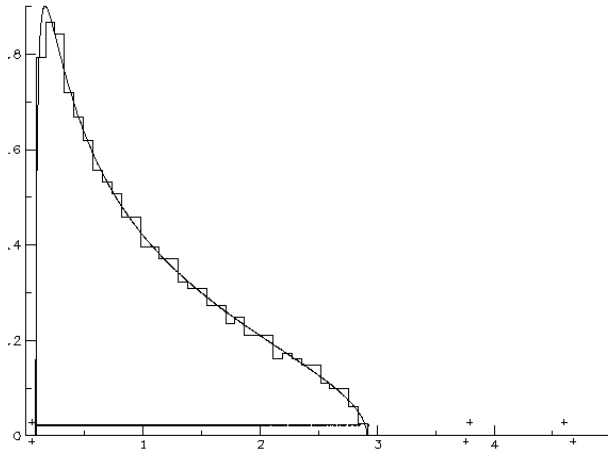


Figure 1: Gaussian samples when  $p = 1000, n = 2000$

We include several plots for the case when  $c = 0.5$  and there are three non-unit population eigenvalues given by 0.1, 3 and 4 (of multiplicity 1 each). In this case, the critical values of the eigenvalues are  $1 + \sqrt{c} \simeq 1.70711$  and  $1 - \sqrt{c} \simeq 0.29289$ . Hence theoretically we expect that three sample covariance eigenvalues of values  $\alpha_j + \frac{c\alpha_j}{\alpha_j - 1} \simeq 0.04444, 3.75$  and  $4.66667$  are away from the interval  $[(1 - \sqrt{c})^2, (1 + \sqrt{c})^2] \simeq [0.08578, 2.91422]$ . The histogram and the scatterplot of Figure 1 is from Gaussian samples when  $p = 1000, n = 2000$ . The smooth curve is the theoretical limiting density and the theoretical locations of the three separated eigenvalues are plotted with + signs below the horizontal axis. The smallest and largest two sample eigenvalues are plotted with + signs about the horizontal axis. Figure 2 is from Gaussian samples when  $p = 100, n = 200$  while Figures 3 and 4 from samples of Bernoulli variables taking values  $-1$  or  $1$  when  $p = 1000, n = 2000$ , and  $p = 100, n = 200$  respectively. The observed values of the four separated eigenvalues in each case are as follows:

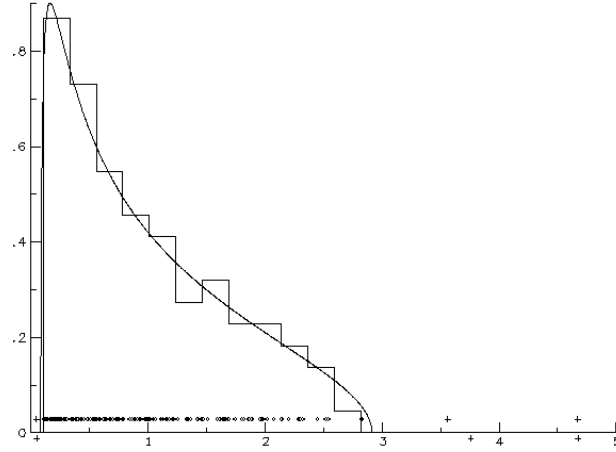


Figure 2: Gaussian samples when  $p = 100, n = 200$

	smallest eigenvalue	2nd largest eigenvalue	largest eigenvalue
theoretical	0.04444	3.75	4.66667
Gaussian $p = 1000$	0.04369	3.78400	4.59127
Gaussian $p = 100$	0.03979	3.55388	4.66192
Bernoulli $p = 1000$	0.04555	3.75706	4.66594
Bernoulli $p = 100$	0.05015	3.62337	4.70786

Figure 5 and Figure 6 are the cases when  $c = 2, p = 2000, n = 1000$  with Gaussian and Bernoulli samples, respectively. Again three non-unit population eigenvalues are chosen 0.1, 3 and 4. The critical value of the eigenvalues is  $1 + \sqrt{c} \simeq 2.41421$  and the theory predicts that the two largest sample eigenvalues given by  $\alpha_j + \frac{c\alpha_j}{\alpha_j - 1} \simeq 6$  and 6.66667 are separated from the interval  $[(1 - \sqrt{c})^2, (1 + \sqrt{c})^2] \simeq [0.17157, 3.41209]$ . Only non-zero eigenvalues are plotted in Figure 5 and Figure 6. The observed values of the separated eigenvalues in each case are as follows:

	2nd largest eigenvalue	largest eigenvalue
theoretical	6	6.66667
Gaussian $p = 2000$	5.8523	6.4013
Bernoulli $p = 2000$	6.01065	6.725

The paper is organized as follows. In section 2, we summarize the work of Z. D. Bai and J. W. Silverstein on which we heavily rely to prove our results. It turns out that the determination of the support of a Stieltjes transform plays the crucial role. This is obtained in section 3. The proofs of the main theorems are given in section 4.

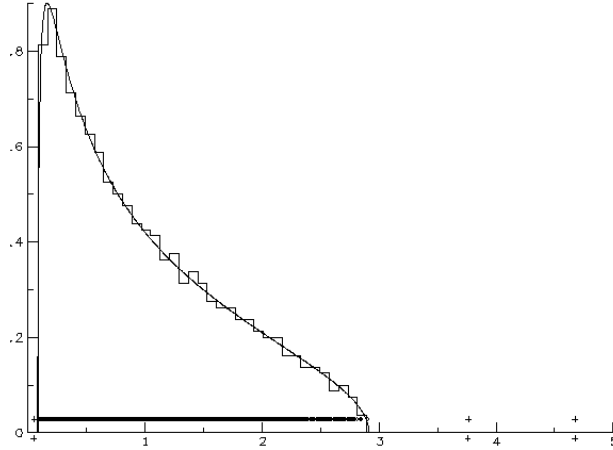


Figure 3: Bernoulli samples taking values  $-1$  or  $1$  when  $p = 1000, n = 2000$

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## 2 Results of Z. D. Bai and J. W. Silverstein

Our analysis relies heavily on the work [2, 3] of Bai and Silverstein. In this section, we summarize the necessary results from [2, 3].

*Notational Remark.* We denote by  $p$  the population size and by  $n$  the sample size. The notations  $n$  and  $N$  are used in [3] for  $p$  and  $n$ , respectively.

For a distribution function  $G(\lambda)$ , its Stieltjes transform  $m_G(z)$  is defined by

$$m_G(z) = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} dG(\lambda), \quad z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im}(z) > 0\}. \quad (2.1)$$

Also recall the inversion formula

$$G([a, b]) = \frac{1}{\pi} \lim_{\eta \downarrow 0} \int_a^b \text{Im}(m_G(\xi + i\eta)) d\xi \quad (2.2)$$

for continuity points  $a, b$  of  $G$ .

Assume the following:

- (a)  $Z_{ij}$  are i.i.d. random variables in  $\mathbb{C}$  with  $\mathbb{E}(Z_{11}) = 0$ ,  $\mathbb{E}|Z_{11}|^2 = 1$  and  $\mathbb{E}|Z_{11}|^4 < \infty$ .
- (b)  $n = n(p)$  with  $c_p := \frac{p}{n} \rightarrow c > 0$  as  $p \rightarrow \infty$ .



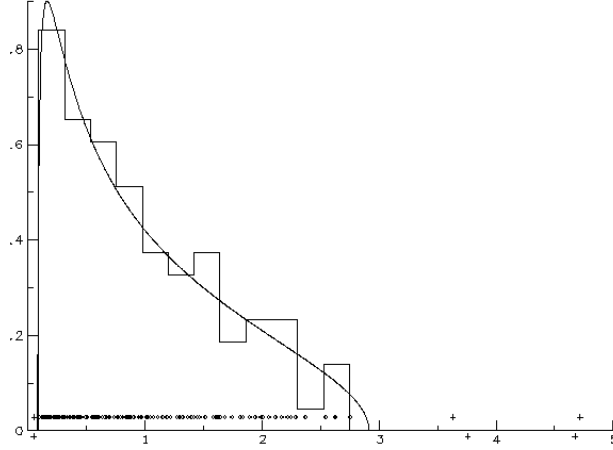


Figure 4: Bernoulli samples taking values  $-1$  or  $1$  when  $p = 100, n = 200$

- (c) For each  $p$ ,  $U_T T_p U_T^{-1} = \text{diag}(\sigma_1^{(p)}, \dots, \sigma_p^{(p)})$  for some unitary matrix  $U_T$  such that  $H_p \rightarrow H_\infty$  in distribution for some distribution function  $H_\infty$  where  $H_p$  is the empirical distribution function of the eigenvalues of  $T_p$  defined by

$$dH_p(\lambda) = \frac{1}{p} \sum_{j=1}^p \delta_{\sigma_j^{(p)}}(\lambda). \quad (2.3)$$

- (d)  $\max\{\sigma_1^{(p)}, \dots, \sigma_p^{(p)}\}$  is bounded in  $p$ .
- (e) Set  $Z_p = (Z_{ij})$ ,  $1 \leq i \leq p$ ,  $1 \leq j \leq n$  and  $B_p = \frac{1}{n} T_p^{1/2} Z_p Z_p^* T_p^{1/2}$ .
- (f) Set

$$z_p(m) = -\frac{1}{m} + c_p \int \frac{t}{1+tm} dH_p(t). \quad (2.4)$$

From [22] and [21], it is known that there is a unique inverse function  $m_p(z)$  such that  $m_p(z) \in \mathbb{C}^+$  for  $z \in \mathbb{C}^+$ . It is also known [22, 21] that  $m_p(z)$  is the Stieltjes transform of a distribution, which will be denoted by  $F_p$ :

$$m_p(z) = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} dF_p(\lambda), \quad z \in \mathbb{C}^+. \quad (2.5)$$

Suppose that the interval  $[a, b]$  with  $a > 0$  lies in an open interval outside the support of  $F_p$  for all large  $p$ .

Now we will state the main result of [3] which we need for our analysis. It is easy to check that  $F_p$  converges to some distribution function  $F_\infty$ . Then [21]  $F_\infty(\lambda)$  is the almost sure limit of the empirical spectral distribution of  $\underline{B}_p := \frac{1}{n} Z_p^* T_p Z_p$  and

$$F(\lambda) := \frac{1}{c} (F_\infty(\lambda) - (1-c)1_{[0, \infty)}) \quad (2.6)$$

is the almost sure limit of the empirical spectral distribution of  $B_p = \frac{1}{n} T_p^{1/2} Z_p Z_p^* T_p^{1/2}$ .

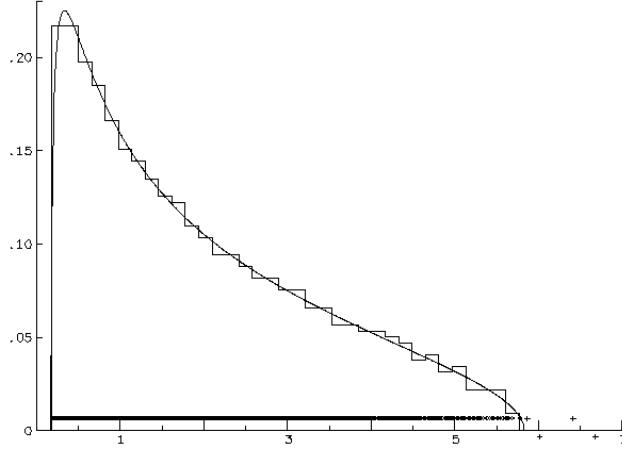


Figure 5: Gaussian samples when  $p = 2000, n = 1000$

*Remark.* The function  $F_p$  is not the empirical distribution of  $B_p$ . The distribution function  $F_p$  is defined only through (2.5).

Moreover [2], the Stieltjes transform of  $F_\infty$ ,

$$m_\infty(z) = \int_{-\infty}^{\infty} \frac{1}{\lambda - z} dF_\infty(\lambda), \quad z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \text{Im}(z) > 0\}, \quad (2.7)$$

is invertible, with the inverse given by

$$z_\infty(m) = -\frac{1}{m} + c \int \frac{t}{1 + tm} dH_\infty(t). \quad (2.8)$$

On the other hand, given  $H_\infty$ ,  $F_\infty$  is determined from (2.8) and (2.7). Note that  $m_\infty(z)$  is well-defined not only on  $\mathbb{C}_+$  but also up to the real line outside  $\text{supp}(F_\infty)$  and its inverse exists on  $m_\infty(\mathbb{C}_+ \cup \text{supp}(F_\infty)^c)$ .

*Remark.* If  $[a, b]$  satisfies condition (f) above, it is easy to check that  $[a, b] \subset \text{supp}(F_\infty)^c$ .

Given an interval  $[a, b]$  satisfying condition (f) above and  $m_\infty(b) < 0$ , it is shown in [3] that there is an integer  $i_p \geq 0$  satisfying the conditions

$$\sigma_{i_p}^{(p)} > -\frac{1}{m_\infty(b)}, \quad \sigma_{i_p+1}^{(p)} < -\frac{1}{m_\infty(a)} \quad (2.9)$$

for large  $p$ . (Here  $\sigma_0^{(p)} := \infty$ .)

**Proposition 2.1 (Theorem 1.2 [3]).** *Assume (a)-(f) above.*

(i) *If  $c(1 - H_\infty(0)) > 1$ , then  $x_0$ , the smallest value in the support of  $F_\infty$ , is positive, and  $s_n^{(p)} \rightarrow x_0$  with probability 1. The value  $x_0$  is the maximum of the function  $z_\infty(m)$  for  $m \in \mathbb{R}_+$ .*

(ii) *If  $c(1 - H_\infty(0)) \leq 1$  or  $c(1 - H_\infty(0)) > 1$  but  $[a, b]$  is not contained in  $[0, x_0]$ , then  $m_\infty(b) < 0$  and*

$$\mathbb{P}(s_{i_p}^{(p)} > b \text{ and } s_{i_p+1}^{(p)} < a \text{ for all large } p) = 1 \quad (2.10)$$

*with  $i_p$  defined in (2.9). (Here  $s_0^{(p)} := \infty$ .)*

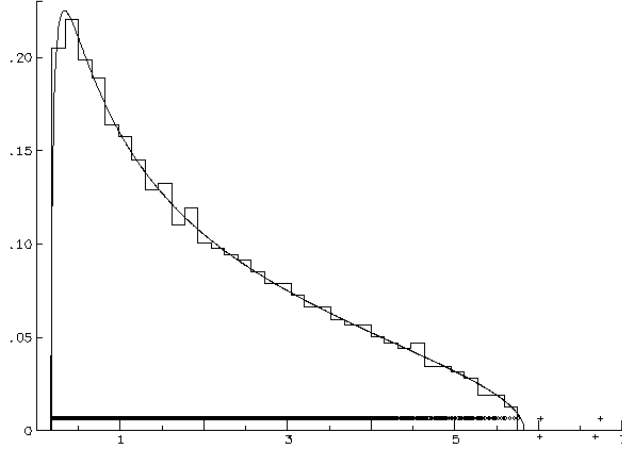


Figure 6: Bernoulli samples taking values  $-1$  or  $1$  when  $p = 2000, n = 1000$

### 3 Determination of $\text{supp}(F_p)$

The key part in applying Proposition 2.1 turns out to be determining the support of  $F_p$ . This can be extracted from the following result due to Silverstein and Choi.

**Lemma 3.1** ([23]; see also Lemma 1.3 [3]). *If  $x \notin \text{supp}(F_p)$ , then  $m := m_p(x)$  satisfies*

(i)  $m \in \mathbb{R} \setminus \{0\}$

(ii)  $-\frac{1}{m} \notin \text{supp}(H_p)$

(iii)  $z'_p(m) > 0$ .

*Conversely, if  $m$  satisfies (i)-(iii), then  $x = z_p(m) \notin \text{supp}(F_p)$ .*

*Remark.* Lemma 1.3 of [3] is stated for  $H_\infty$ . But the proof of Lemma 1.3 in [23] applies also to the finite  $p$  case of  $H_p$  without any change. Indeed, the proposition applies to any distribution defined by its Stieltjes transform as in (2.5).

*Remark.* It is also shown in [23] that  $F_p$  has continuous density on  $\mathbb{R}_+$ .

When  $T_p$  is as in (1.8),

$$dH_p(x) = \frac{1}{p} \sum_{j=1}^M k_j \delta_{\alpha_j}(x) + \left(1 - \frac{r}{p}\right) \delta_1(x) \quad (3.1)$$

and

$$z_p(m) = -\frac{1}{m} + \frac{c_p}{1+m} + \frac{1}{n} \left[ \sum_{j=1}^M \frac{k_j \alpha_j}{1 + \alpha_j m} - \frac{r}{1+m} \right], \quad (3.2)$$

where we recall that  $r = k_1 + \dots + k_M$ . We first determine the set of real  $m$  such that  $z'_p(m) > 0$ .

Now

$$\begin{aligned} z'_p(m) &= \frac{1}{m^2} - \frac{c_p}{(1+m)^2} + \frac{1}{n} \left[ \sum_{j=1}^M \frac{-k_j \alpha_j^2}{(1+\alpha_j m)^2} + \frac{r}{(1+m)^2} \right] \\ &= \frac{f(m) + \frac{1}{n} g(m)}{m^2 (1+m)^2 \prod_{\ell=1}^M (1+\alpha_\ell m)^2}, \end{aligned} \quad (3.3)$$

where

$$f(m) := ((1+m)^2 - c_p m^2) \prod_{\ell=1}^M (1+\alpha_\ell m)^2 \quad (3.4)$$

and

$$g(m) := \left[ \sum_{j=1}^M \frac{-k_j \alpha_j^2}{(1+\alpha_j m)^2} + \frac{r}{(1+m)^2} \right] m^2 (1+m)^2 \prod_{\ell=1}^M (1+\alpha_\ell m)^2. \quad (3.5)$$

We need the following basic lemmas of complex variables to determine the solution of  $z'_p(m) = 0$ .

**Lemma 3.2.** *Let  $h(z)$  be an analytic function in a closed disk  $\overline{D(z_0, r)}$  of radius  $r > 0$  centered at  $z_0$ . Then there is  $\epsilon_0 > 0$  such that for  $0 \leq \epsilon < \epsilon_0$ , the equation*

$$z - z_0 = \epsilon h(z) \quad (3.6)$$

has a unique solution in  $D(z_0, r)$ , which satisfies

$$z = z_0 + \epsilon h(z_0) + O(\epsilon^2). \quad (3.7)$$

Furthermore, if  $z_0$  is real and  $h(z)$  is real for real  $z$ , the solution (3.7) is real.

*Proof.* As  $h$  is continuous, there is a constant  $C > 0$  such that  $|h(z)| \leq C$  for  $|z - z_0| \leq r$ . When  $|\epsilon| < \frac{r}{C}$ , for  $|z - z_0| = r$ ,

$$|z - z_0| = r > |\epsilon| C \geq |\epsilon h(z)|. \quad (3.8)$$

Hence from Rouché's theorem, the number of zeros of  $z - z_0 - \epsilon h(z)$  inside  $D(z_0, r)$  is equal to the number of zeros of  $z - z_0$  inside  $D(z_0, r)$ , which is one. The zero  $z_\epsilon$  satisfies  $z_\epsilon - z_0 = \epsilon h(z_\epsilon) = O(\epsilon)$ . Thus

$$z_\epsilon - z_0 - \epsilon h(z_0) = \epsilon (h(z_\epsilon) - h(z_0)) = O(\epsilon^2). \quad (3.9)$$

If  $h(z)$  is real for real  $z$ , then by taking complex conjugate of (3.6), we find that  $\overline{z_\epsilon}$  is also a solution. Since there is only one solution, we find that  $z_\epsilon$  is real.  $\square$

**Lemma 3.3.** *Let  $h(z)$  be an analytic function in a closed disk  $\overline{D(z_0, r)}$  of radius  $r > 0$  centered at  $z_0$  such that  $h(z_0) \neq 0$ . Then there are  $0 < r_0 \leq r$  and  $\epsilon_0 > 0$  such that for  $0 \leq \epsilon < \epsilon_0$ , the equation*

$$(z - z_0)^2 = \epsilon h(z) \quad (3.10)$$

have precisely two distinct solutions in  $D(z_0, r_0)$ , which satisfy

$$z = z_0 \pm \sqrt{\epsilon} \sqrt{h(z_0)} + O(\epsilon) \quad (3.11)$$

where  $\sqrt{h(z_0)}$  is an arbitrary branch. Furthermore, suppose that  $z_0$  is real and  $h(z)$  is real for real  $z$ . Then if  $h(z_0) > 0$ , both solutions (3.11) are real. On the other hand, if  $h(z_0) < 0$ , both solutions (3.11) are non-real.

*Proof.* The proof of (3.11) follows from Lemma 3.2 by taking the square root of (3.10). When  $z_0$  is real and  $h(z)$  is real for real  $z$ , the complex conjugate of a solution of (3.10) is also a solution. Thus the two solutions(3.11) of (3.10) are either complex conjugates of each other or both real since there are precisely two distinct solutions. Hence the Lemma follows.  $\square$

For the remainder of this section, we assume that  $c \neq 1$  and none of  $\alpha_j$ 's are equal to  $1 \pm \sqrt{c}$ . We further assume that  $p$  and  $n$  are sufficiently large so that  $c_p \neq 1$  and none of  $\alpha_j$ 's are equal to  $1 \pm \sqrt{c_p}$ . Then the numerator of (3.3) is a polynomial of degree exactly  $2M + 2$ , and we now determine all the solutions of  $z'_p(m) = 0$ .

For  $f$  defined in (3.4), the equation  $f(m) = 0$  has distinct solutions

$$m = \frac{-1}{1 + \sqrt{c_p}} =: m_+, \quad m = \frac{-1}{1 - \sqrt{c_p}} =: m_- \quad (3.12)$$

of multiplicity 1 and

$$m = \frac{-1}{\alpha_j}, \quad j = 1, 2, \dots, M, \quad (3.13)$$

of multiplicity 2. The roots of  $z'_p(m)$  are expected to be perturbations of the roots of  $f(m)$ , which we will find. First consider  $m_+$ . Dividing the equation  $f(m) + \frac{1}{n}g(m) = 0$  by  $\frac{f(m)}{m - m_+}$ , we obtain the equation

$$m - m_+ + \frac{1}{n} \frac{m^2(1+m)^2}{(1-c_p)(m-m_-)} \left[ \sum_{j=1}^M \frac{-k_j \alpha_j^2}{(1+\alpha_j m)^2} + \frac{r}{(1+m)^2} \right] = 0. \quad (3.14)$$

Lemma 3.2 implies that there is a solution of  $z'_p(m) = 0$  of the form

$$m = m_+ + O\left(\frac{1}{n}\right), \quad (3.15)$$

which is real. Similarly, there is a real solution of  $z'_p(m) = 0$  of the form

$$m = m_- + O\left(\frac{1}{n}\right). \quad (3.16)$$

Now consider the root  $m = \frac{-1}{\alpha_j}$  of  $f(m) = 0$ . Dividing  $f + \frac{1}{n}g = 0$  by  $\frac{f(m)}{(m + \frac{1}{\alpha_j})^2}$ , we obtain the equation

$$\left(m + \frac{1}{\alpha_j}\right)^2 = \frac{1}{n} G_j(m) \quad (3.17)$$

where

$$G_j(m) = \frac{-(1 + \alpha_j m)^2 m^2 (1+m)^2}{\alpha_j^2 (1-c_p)(m-m_+)(m-m_-)} \left[ \sum_{\ell=1}^M \frac{-k_\ell \alpha_\ell^2}{(1 + \alpha_\ell m)^2} + \frac{r}{(1+m)^2} \right]. \quad (3.18)$$

Note that

$$G_j\left(-\frac{1}{\alpha_j}\right) = \frac{k_j (\alpha_j - 1)^2}{\alpha_j^4 (1-c_p) \left(\frac{-1}{\alpha_j} - m_+\right) \left(\frac{-1}{\alpha_j} - m_-\right)} \quad (3.19)$$

is not zero and also  $G_j(m)$  is real for real  $m$ . Thus Lemma 3.3 implies that there are precisely two solutions of  $z'_p(m) = 0$  of the form

$$m = -\frac{1}{\alpha_j} \pm \frac{1}{\sqrt{n}} \sqrt{G_j\left(-\frac{1}{\alpha_j}\right)} + O\left(\frac{1}{n}\right), \quad j = 1, \dots, M, \quad (3.20)$$

where the pair for each  $j$  are either both real or both non-real depending on the sign of  $G_j(-\frac{1}{\alpha_j})$ .

Now when  $c_p < 1$ , the condition  $G_j(-\frac{1}{\alpha_j}) > 0$  is equivalent to

$$\frac{-1}{\alpha_j} > m_+ \quad \text{or} \quad \frac{-1}{\alpha_j} < m_-, \quad (3.21)$$

which is the same as

$$\alpha_j > 1 + \sqrt{c_p} \quad \text{or} \quad \alpha_j < 1 - \sqrt{c_p}. \quad (3.22)$$

On the other hand, when  $c_p > 1$ , we note that  $m_+ < 0 < m_-$ . The condition  $G_j(-\frac{1}{\alpha_j}) > 0$  is now equivalent to

$$m_+ < \frac{-1}{\alpha_j} < m_-, \quad (3.23)$$

which is the same as (since  $\alpha_j > 0$ )

$$\alpha_j > 1 + \sqrt{c_p}. \quad (3.24)$$

We summarize the above calculations.

**Lemma 3.4.** *The solutions of  $z'_p(m) = 0$  are*

$$m = -\frac{1}{1 + \sqrt{c_p}} + O\left(\frac{1}{n}\right) =: m_+^{(n)}, \quad m = -\frac{1}{1 - \sqrt{c_p}} + O\left(\frac{1}{n}\right) =: m_-^{(n)}. \quad (3.25)$$

and

$$m = -\frac{1}{\alpha_j} \pm \frac{1}{\sqrt{n}} \sqrt{G_j(-\frac{1}{\alpha_j})} + O\left(\frac{1}{n}\right) =: m_{j,\pm}^{(n)}, \quad j = 1, \dots, M, \quad (3.26)$$

all of multiplicity 1. Furthermore, the following holds.

- When  $c_p < 1$ ,  $m_-^{(n)} < m_+^{(n)} < 0$ , and  $m_{j,\pm}^{(n)}$  are real if and only if  $\alpha_j > 1 + \sqrt{c_p}$  or  $\alpha_j < 1 - \sqrt{c_p}$ . If  $1 - \sqrt{c_p} < \alpha_j < 1 + \sqrt{c_p}$ ,  $m_{j,+}^{(n)}$  and  $m_{j,-}^{(n)}$  are complex conjugates of each other.
- When  $c_p > 1$ ,  $m_+^{(n)} < 0 < m_-^{(n)}$ , and  $m_{j,\pm}^{(n)}$  are real if and only if  $\alpha_j > 1 + \sqrt{c_p}$ . If  $\alpha_j < 1 + \sqrt{c_p}$ ,  $m_{j,+}^{(n)}$  and  $m_{j,-}^{(n)}$  are complex conjugates of each other.

We now consider the cases when  $c < 1$  and when  $c > 1$  separately.

### 3.1 When $c < 1$

Let the indices  $0 \leq M_0, M_1 \leq M$  be defined as in Theorem 1.1 (recall that we assume that none of the  $\alpha_j$ 's are equal to  $1 \pm \sqrt{c}$ ), so that

$$\alpha_1 > \dots > \alpha_{M_0} > 1 + \sqrt{c} > \alpha_{M_0+1} > \dots > \alpha_{M-M_1} > 1 - \sqrt{c} > \alpha_{M-M_1+1} > \dots > \alpha_M. \quad (3.27)$$

We now find the intervals in which  $z'_p(m) > 0$ .

The denominator of (3.3) is non-negative. From Lemma 3.4, the numerator of (3.3) is factored as

$$\text{const} \cdot (m - m_-^{(n)})(m - m_+^{(n)}) \prod_{j=1}^M (m - m_{j,-}^{(n)})(m - m_{j,+}^{(n)}) \quad (3.28)$$

The constant prefactor is, from (3.4) and (3.5),

$$(1 - c_p) \prod_{j=1}^M \alpha_j^2 + O\left(\frac{1}{n}\right), \quad (3.29)$$

which is positive when  $n$  is large enough. On the other hand, among the terms in the product of (3.28),  $m_{j,\pm}^{(n)}$  corresponding to the indices  $M_0 + 1 \leq j \leq M_1$  are complex conjugates of each other. Thus

$$\prod_{j=M_0+1}^{M_1+1} (m - m_{j,-}^{(n)})(m - m_{j,+}^{(n)}) \geq 0. \quad (3.30)$$

Hence using the fact that

$$0 > m_{1,+}^{(n)} > m_{1,-}^{(n)} > \cdots > m_{M_0,+}^{(n)} > m_{M_0,-}^{(n)} > m_+^{(n)} \quad (3.31)$$

and

$$m_-^{(n)} > m_{M_1+1,+}^{(n)} > m_{M_1+1,-}^{(n)} > \cdots > m_{M,+}^{(n)} > m_{M,-}^{(n)}, \quad (3.32)$$

we find that the numerator of (3.3) is positive in the intervals

$$(-\infty, m_{M,-}^{(n)}) \cup (m_{M,+}^{(n)}, m_{M-1,-}^{(n)}) \cup \cdots \cup (m_{M_1+2,+}^{(n)}, m_{M_1+1,-}^{(n)}) \cup (m_{M_1+1,+}^{(n)}, m_-^{(n)}) \quad (3.33)$$

union

$$(m_+^{(n)}, m_{M_0,-}^{(n)}) \cup (m_{M_0,+}^{(n)}, m_{M_0-1,-}^{(n)}) \cup \cdots \cup (m_{2,+}^{(n)}, m_{1,-}^{(n)}) \cup (m_{1,+}^{(n)}, \infty). \quad (3.34)$$

The singular points of (3.3) are not contained in any of the above intervals except for the singular point  $m = 0$ . Hence the set of  $m$  such that  $z'_p(m) > 0$  is equal to (3.33) union

$$(m_+^{(n)}, m_{M_0,-}^{(n)}) \cup (m_{M_0,+}^{(n)}, m_{M_0-1,-}^{(n)}) \cup \cdots \cup (m_{2,+}^{(n)}, m_{1,-}^{(n)}) \cup (m_{1,+}^{(n)}, 0) \cup (0, \infty). \quad (3.35)$$

Now Lemma 3.1 determines  $\text{supp}(F_p)$ .

**Proposition 3.5.** *Suppose that  $c < 1$  and none of  $\alpha_j$  is equal to  $1 \pm \sqrt{c}$ . With the indices  $M_0$  and  $M_1$  defined in Theorem 1.1, for  $n$  sufficiently large,*

$$\begin{aligned} \text{supp}(F_p)^c = & (-\infty, 0) \cup (0, z_{M,-}^{(n)}) \cup (z_{M,+}^{(n)}, z_{M-1,-}^{(n)}) \cup \cdots \cup (z_{M_1+1,+}^{(n)}, z_-^{(n)}) \\ & \cup (z_+^{(n)}, z_{M_0,-}^{(n)}) \cup (z_{M_0,+}^{(n)}, z_{M_0-1,-}^{(n)}) \cup \cdots \cup (z_{2,+}^{(n)}, z_{1,-}^{(n)}) \cup (z_{1,+}^{(n)}, \infty) \end{aligned} \quad (3.36)$$

where

$$z_{\pm}^{(n)} = (1 \pm \sqrt{c_p})^2 + O\left(\frac{1}{n}\right) \quad (3.37)$$

and

$$z_{j,\pm}^{(n)} = \alpha_j + \frac{c_p \alpha_j}{\alpha_j - 1} \pm \frac{A_j}{\sqrt{n}} + O\left(\frac{1}{n}\right), \quad j = 1, \dots, M_0, \quad j = M_1 + 1, \dots, M, \quad (3.38)$$

for some constant  $A_j > 0$ . The intervals in (3.36) are disjoint.

*Proof.* We will first see that the intervals (3.33) union (3.35) satisfy the conditions (i)-(iii) of Lemma 3.1. The condition (iii) is clearly satisfied. Also 0 is not contained in (3.33) and (3.35), and so condition (i) is fulfilled. Finally, as  $\text{supp}(H_p) = \{\alpha_1, \dots, \alpha_M, 1\}$  and

$$m_-^{(n)} < -1 < m_+^{(n)}, \quad m_{j,-}^{(n)} < -\frac{1}{\alpha_j} < m_{j,+}^{(n)}, \quad (3.39)$$

the condition (ii) is satisfied for  $m$  in (3.33) union (3.35).

We now need to find the image of the above intervals under  $z_p$ . Clearly,  $z_p(-\infty) = 0$ ,  $z_p(0-) = +\infty$ ,  $z_p(0+) = -\infty$  and  $z_p(+\infty) = 0$ . A direct computation yields

$$z_p(m_{\pm}^{(n)}) = (1 \pm \sqrt{c_p})^2 + O\left(\frac{1}{n}\right). \quad (3.40)$$

and

$$z_p(m_{j,\pm}^{(n)}) = \alpha_j + \frac{c_p \alpha_j}{\alpha_j - 1} \pm \frac{A_j}{\sqrt{n}} + O\left(\frac{1}{n}\right) \quad (3.41)$$

where

$$A_j = \frac{1}{C_j} \left\{ C_j^2 \alpha_j^2 \left( 1 - \frac{c_p}{(\alpha_j - 1)^2} \right) + k_j \right\}, \quad C_j := \sqrt{G(-1/\alpha_j)}. \quad (3.42)$$

Note that  $A_j > 0$  for  $1 \leq j \leq M_0$  and  $M_1 + 1 \leq j \leq M$  since  $\alpha_j > 1 + \sqrt{c_p}$  or  $\alpha_j < 1 - \sqrt{c_p}$ . Also it is straightforward to check from the graph of the function

$$x + \frac{c_p x}{x - 1} \quad (3.43)$$

that

$$\begin{aligned} 0 < \alpha_M + \frac{c_p \alpha_M}{\alpha_M - 1} < \dots < \alpha_{M_1+1} + \frac{c_p \alpha_{M_1+1}}{\alpha_{M_1+1} - 1} < (1 - \sqrt{c_p})^2 \\ < (1 + \sqrt{c_p})^2 < \alpha_{M_0} + \frac{c_p \alpha_{M_0}}{\alpha_{M_0} - 1} < \dots < \alpha_1 + \frac{c_p \alpha_1}{\alpha_1 - 1}. \end{aligned} \quad (3.44)$$

This implies the Proposition. □

### 3.2 When $c > 1$

This case is similar to the previous case when  $c < 1$ . We indicate only the difference.

We again assume that  $p$  and  $n$  are large enough so that the set of  $j$ 's satisfying  $\alpha_j > 1 + \sqrt{c_p}$  is the same as the set of  $j$ 's satisfying  $\alpha_j > 1 + \sqrt{c}$ . Let the index  $0 \leq M_0 \leq M$  be defined, as in Theorem 1.1. We further assume that none of  $\alpha_j$  is equal to  $1 + \sqrt{c}$  so that

$$\alpha_{M_0} > 1 + \sqrt{c} > \alpha_{M_0+1}. \quad (3.45)$$

The denominator of (3.3) is non-negative and as before, the numerator of (3.3) is equal to (3.28). But this time, the constant prefactor (3.29) is negative when  $n$  is large enough. Also as in (3.30),

$$\prod_{j=M_0+1}^M (m - m_{j,-}^{(n)})(m - m_{j,+}^{(n)}) \geq 0. \quad (3.46)$$



Now using the fact that

$$m_-^{(n)} > 0 > m_{1,+}^{(n)} > m_{1,-}^{(n)} > \cdots > m_{M_0,+}^{(n)} > m_{M_0,-}^{(n)} > m_+^{(n)}, \quad (3.47)$$

we find that the numerator of (3.3) is positive in the intervals

$$(m_+^{(n)}, m_{M_0,-}^{(n)}) \cup (m_{M_0,+}^{(n)}, m_{M_0-1,-}^{(n)}) \cup \cdots \cup (m_{2,+}^{(n)}, m_{1,-}^{(n)}) \cup (m_{1,+}^{(n)}, m_-^{(n)}). \quad (3.48)$$

Hence taking into accounts of the singular point  $m = 0$  of  $z_p'(m)$ , the intervals where  $z_p'(m) > 0$  is

$$(m_+^{(n)}, m_{M_0,-}^{(n)}) \cup (m_{M_0,+}^{(n)}, m_{M_0-1,-}^{(n)}) \cup \cdots \cup (m_{2,+}^{(n)}, m_{1,-}^{(n)}) \cup (m_{1,+}^{(n)}, 0) \cup (0, m_-^{(n)}). \quad (3.49)$$

The proof of the following proposition is parallel to Proposition 3.5.

**Proposition 3.6.** *Suppose that  $c > 1$  and none of  $\alpha_j$  is equal to  $1 + \sqrt{c}$ . With the index  $M_0$  defined in Theorem 1.2, for  $n$  sufficiently large,*

$$\text{supp}(F_p)^c = (-\infty, z_-^{(n)}) \cup (z_+^{(n)}, z_{M_0,-}^{(n)}) \cup (z_{M_0,+}^{(n)}, z_{M_0-1,-}^{(n)}) \cup \cdots \cup (z_{2,+}^{(n)}, z_{1,-}^{(n)}) \cup (z_{1,+}^{(n)}, \infty) \quad (3.50)$$

where

$$z_{\pm}^{(n)} = (1 \pm \sqrt{c_p})^2 + O\left(\frac{1}{n}\right) \quad (3.51)$$

and

$$z_{j,\pm}^{(n)} = \alpha_j + \frac{c_p \alpha_j}{\alpha_j - 1} \pm \frac{A_j}{\sqrt{n}} + O\left(\frac{1}{n}\right), \quad j = 1, \dots, M_0, \quad (3.52)$$

for some constant  $A_j > 0$ . The intervals in (3.50) are disjoint.

## 4 Proof of Theorems 1.1, 1.2 and 1.3

When  $T_p$  is (1.8), as  $H_p$  is equal to (3.1),

$$dH_\infty(x) = \delta_1(x). \quad (4.1)$$

Hence

$$z_\infty(m) = -\frac{1}{m} + c \int \frac{t}{1+tm} dH_\infty(t) = -\frac{1}{m} + \frac{c}{1+m}. \quad (4.2)$$

It is well-known that ([22], see also Theorem 3.4 of [1]) in this case,

$$dF_\infty(\lambda) = \begin{cases} \frac{1}{2\pi\lambda} \sqrt{((1+\sqrt{c})^2 - \lambda)(\lambda - (1-\sqrt{c})^2)} 1_{[(1-\sqrt{c})^2, (1+\sqrt{c})^2]}(\lambda), & c > 1 \\ \frac{1}{2\pi\lambda} \sqrt{((1+\sqrt{c})^2 - \lambda)(\lambda - (1-\sqrt{c})^2)} 1_{[(1-\sqrt{c})^2, (1+\sqrt{c})^2]}(\lambda) + (1-c)\delta_0, & 0 < c \leq 1. \end{cases} \quad (4.3)$$

### 4.1 When $c < 1$

We first assume that none of  $\alpha_j$  is equal to  $1 \pm \sqrt{c}$  so that Proposition 3.5 is applicable. The case when some of  $\alpha_j$  are equal to  $1 \pm \sqrt{c}$  will be discussed at the end of this subsection.

When  $T_p$  is (1.8), all the conditions (a)-(e) of Proposition 2.1 are satisfied or are defined accordingly.

Now suppose  $[a, b]$  is an interval satisfying condition (f). Since

$$z_+^{(n)} \rightarrow (1 + \sqrt{c})^2, \quad z_-^{(n)} \rightarrow (1 - \sqrt{c})^2, \quad (4.4)$$

and for any  $i$ ,

$$z_{i,+}^{(n)}, \quad z_{i,-}^{(n)} \rightarrow \alpha_i + \frac{c\alpha_i}{\alpha_i - 1}, \quad (4.5)$$

we see that

$$\begin{aligned} [a, b] \subset & (-\infty, 0) \cup \left(0, \alpha_M + \frac{c\alpha_M}{\alpha_M - 1}\right) \cup \left(\alpha_M + \frac{c\alpha_M}{\alpha_M - 1}, \alpha_{M-1} + \frac{c\alpha_{M-1}}{\alpha_{M-1} - 1}\right) \\ & \cup \cdots \cup \left(\alpha_{M_1+1} + \frac{c\alpha_{M_1+1}}{\alpha_{M_1+1} - 1}, (1 - \sqrt{c})^2\right) \\ & \cup \left((1 + \sqrt{c})^2, \alpha_{M_0} + \frac{c\alpha_{M_0}}{\alpha_{M_0} - 1}\right) \cup \cdots \cup \left(\alpha_2 + \frac{c\alpha_2}{\alpha_2 - 1}, \alpha_1 + \frac{c\alpha_1}{\alpha_1 - 1}\right) \cup \left(\alpha_1 + \frac{c\alpha_1}{\alpha_1 - 1}, \infty\right). \end{aligned} \quad (4.6)$$

On the other hand,

$$\text{supp}(F_\infty)^c = (-\infty, 0) \cup (0, (1 - \sqrt{c})^2) \cup ((1 + \sqrt{c})^2, \infty). \quad (4.7)$$

Hence  $[a, b] \subset \text{supp}(F_\infty)^c$ . Also from definition (2.7), it is easy to see that  $m'_\infty(z) > 0$  for  $z \in \text{supp}(F_\infty)^c$ . The first consequence of (ii) of Proposition 2.1 (note that  $H_\infty(0) = 0$ ) is that  $m_\infty(b) < 0$ . Thus  $m_\infty(a) < m_\infty(b) < 0$ . Therefore, the condition (2.9) is equivalent to the condition

$$[a, b] \subset [z_\infty(-1/\sigma_{i_p+1}^{(p)}), z_\infty(-1/\sigma_{i_p}^{(p)})]. \quad (4.8)$$

We will consider four different choices of  $[a, b]$ . First fix  $1 \leq j \leq M_0$ . Take

$$[a, b] = \left[\alpha_j + \frac{c\alpha_j}{\alpha_j - 1} + \epsilon, \alpha_{j-1} + \frac{c\alpha_{j-1}}{\alpha_{j-1} - 1} - \epsilon\right] \quad (4.9)$$

for an arbitrary fixed  $\epsilon > 0$ . (Here  $\alpha_0 := +\infty$ .) From (4.5), we see that

$$[a, b] \subset (z_{j,+}^{(n)}, z_{j-1,-}^{(n)}) \quad (4.10)$$

for all large  $p$ , and hence condition (f) is satisfied using Proposition 3.5. Set

$$i_p := k_1 + \cdots + k_{j-1}. \quad (4.11)$$

(When  $j = 1$ ,  $i_p := 0$ .) For  $T_p$  given by (1.8),

$$\sigma_{i_p}^{(p)} = \alpha_{j-1}, \quad \sigma_{i_p+1}^{(p)} = \alpha_j. \quad (4.12)$$

But

$$z_\infty(-1/\alpha_j) = \alpha_j + \frac{c\alpha_j}{\alpha_j - 1} \quad (4.13)$$

and hence the condition (4.8) is satisfied. Therefore  $i_p$  is defined to satisfy the condition (2.9). Proposition 3.5 now implies that

$$\mathbb{P}\left(s_{k_1+\cdots+k_{j-1}}^{(p)} > \alpha_{j-1} + \frac{c\alpha_{j-1}}{\alpha_{j-1} - 1} - \epsilon \text{ and } s_{k_1+\cdots+k_{j-1}+1}^{(p)} < \alpha_j + \frac{c\alpha_j}{\alpha_j - 1} + \epsilon \text{ for all large } p\right) = 1. \quad (4.14)$$

This yields that,  $1 \leq j \leq M_0 - 1$ ,

$$\mathbb{P}\left(\alpha_j + \frac{c\alpha_j}{\alpha_j - 1} - \epsilon < s_{k_1 + \dots + k_{j-1} + k_j}^{(p)} \leq \dots \leq s_{k_1 + \dots + k_{j-1} + 1}^{(p)} < \alpha_j + \frac{c\alpha_j}{\alpha_j - 1} + \epsilon \text{ for all large } p\right) = 1, \quad (4.15)$$

which implies (1.10) for  $1 \leq j \leq M_0 - 1$ , and

$$\mathbb{P}\left(s_{k_1 + \dots + k_{M_0-1} + 1}^{(p)} < \alpha_{M_0} + \frac{c\alpha_{M_0}}{\alpha_{M_0} - 1} + \epsilon \text{ for all large } p\right) = 1. \quad (4.16)$$

For the second choice of  $[a, b]$ , set

$$[a, b] = [(1 + \sqrt{c})^2 + \epsilon, \alpha_{M_0} + \frac{c\alpha_{M_0}}{\alpha_{M_0} - 1} - \epsilon] \quad (4.17)$$

for an arbitrary fixed  $\epsilon > 0$ . Noting that

$$z_+^{(n)} \rightarrow (1 + \sqrt{c})^2 \quad (4.18)$$

and setting  $i_p := k_1 + \dots + k_{M_0}$ , a calculation similar to the above yields that

$$\mathbb{P}\left(s_{k_1 + \dots + k_{M_0}}^{(p)} > \alpha_{M_0} + \frac{c\alpha_{M_0}}{\alpha_{M_0} - 1} - \epsilon \text{ and } s_{k_1 + \dots + k_{M_0} + 1}^{(p)} < (1 + \sqrt{c})^2 + \epsilon \text{ for all large } p\right) = 1. \quad (4.19)$$

Thus, together with (4.16), we obtain (1.10) for  $j = M_0$ . Also as (4.2) and discussions around (2.6) implies that the support of the limiting spectral distribution of  $B_p$  is  $[(1 - \sqrt{c})^2, (1 + \sqrt{c})^2]$ , we obtain (1.11).

As the third and fourth choices of  $[a, b]$ , we set

$$[a, b] = [\alpha_{M_1+1} + \frac{c\alpha_{M_1+1}}{\alpha_{M_1+1} - 1} + \epsilon, (1 - \sqrt{c})^2 - \epsilon] \quad (4.20)$$

and

$$[a, b] = [\alpha_{j+1} + \frac{c\alpha_{j+1}}{\alpha_{j+1} - 1} + \epsilon, \alpha_j + \frac{c\alpha_j}{\alpha_j - 1} - \epsilon] \quad (4.21)$$

for some  $M_1 + 1 \leq j \leq M$  ( $\alpha_{M+1} := 0$ ), respectively. Arguments as above imply the remaining part of Theorem 1.1.

We now consider the case when an  $\alpha_j$  is equal to  $1 \pm \sqrt{c}$ . We first observe certain monotonicity of the eigenvalues  $s_j^{(p)}$  on  $\alpha_j$ 's. Note that the matrix  $\underline{B}_p := \frac{1}{n} Z_p' T_p Z_p$  has the same set of eigenvalues as  $B_p$  except for  $|p - n|$  zero eigenvalues. Consider a set of parameters  $\beta_j$ ,  $1 \leq j \leq M$ , such that  $\alpha_j \geq \beta_j$ . Let  $\hat{T}_p$  be the matrix  $T_p$  with  $\alpha_j$ 's replaced by  $\beta_j$ 's, and set  $\hat{B}_p = \frac{1}{n} \hat{T}_p^{1/2} Z_p Z_p' \hat{T}_p^{1/2}$  and  $\underline{\hat{B}}_p = \frac{1}{n} Z_p' \hat{T}_p Z_p$ . Then clearly,  $\underline{B}_p$  and  $\underline{\hat{B}}_p$  are Hermitian, and  $\underline{B}_p \geq \underline{\hat{B}}_p$ . Hence from the min-max principle (see e.g. [9]), we find that

$$s_j^{(p)} \geq \hat{s}_j^{(p)} \quad (4.22)$$

for all non-zero eigenvalues, where  $\hat{s}_j^{(p)}$  denotes the eigenvalues of  $\hat{B}_p$ .

Suppose that

$$\alpha_1 > \dots > \alpha_{M_0} > 1 + \sqrt{c} = \alpha_{M_0+1} > \dots > \alpha_{M-M_1} > 1 - \sqrt{c} > \alpha_{M-M_1+1} > \dots > \alpha_M. \quad (4.23)$$

Replacing in (4.23)  $\alpha_{M_0+1}$  by  $(1 \pm \epsilon)\alpha_{M_0+1} = (1 \pm \epsilon)(1 + \sqrt{c})$  for sufficiently small  $\epsilon > 0$ , the above monotonicity argument implies the following:

(i) For each  $1 \leq j \leq M_0$ ,

$$\alpha_j + \frac{c\alpha_j}{\alpha_j - 1} \leq \liminf s_{k_1+\dots+k_{j-1}+i}^{(p)} \leq \limsup s_{k_1+\dots+k_{j-1}+i}^{(p)} \leq \alpha_j + \frac{c\alpha_j}{\alpha_j - 1}, \quad 1 \leq i \leq k_j. \quad (4.24)$$

almost surely.

(ii)

$$(1 + \sqrt{c})^2 \leq \liminf s_{k_1+\dots+k_{M_0}+1}^{(p)} \leq \limsup s_{k_1+\dots+k_{M_0}+1}^{(p)} \leq (1 + \epsilon)\alpha_{M_0+1} + \frac{c(1 + \epsilon)\alpha_{M_0+1}}{(1 + \epsilon)\alpha_{M_0+1} - 1} \quad (4.25)$$

almost surely.

(iii)

$$(1 - \sqrt{c})^2 \leq \liminf s_{p-r+k_1+\dots+k_{M_1}}^{(p)} \leq \limsup s_{p-r+k_1+\dots+k_{M_1}}^{(p)} \leq (1 - \sqrt{c})^2 \quad (4.26)$$

almost surely.

(iv) For each  $M_1 + 1 \leq j \leq M$ ,

$$\begin{aligned} \alpha_j + \frac{c\alpha_j}{\alpha_j - 1} &\leq \liminf s_{p-r+k_1+\dots+k_{j-1}+i}^{(p)} \rightarrow \alpha_j + \frac{c\alpha_j}{\alpha_j - 1} \\ &\leq \limsup s_{p-r+k_1+\dots+k_{j-1}+i}^{(p)} \rightarrow \alpha_j + \frac{c\alpha_j}{\alpha_j - 1} \leq \alpha_j + \frac{c\alpha_j}{\alpha_j - 1} \quad 1 \leq i \leq k_j \end{aligned} \quad (4.27)$$

almost surely.

Since

$$\lim_{\epsilon \downarrow 0} (1 + \epsilon)\alpha_{M_0+1} + \frac{c(1 + \epsilon)\alpha_{M_0+1}}{(1 + \epsilon)\alpha_{M_0+1} - 1} = (1 + \sqrt{c})^2 \quad (4.28)$$

and the above result is true for arbitrary sufficiently small  $\epsilon > 0$ , Theorem 1.1 follows for the case when the parameters are given by (4.23). For the case when  $\alpha_{M-M_1} = 1 - \sqrt{c}$ , the argument is almost identical, and we skip the details.

## 4.2 When $c > 1$

From (4.3), when  $c > 1$ , the smallest value in the support of  $F_\infty$  is

$$x_0 = (1 - \sqrt{c})^2 > 0. \quad (4.29)$$

Hence Proposition 2.1 (i) implies that

$$s_n^{(p)} \rightarrow (1 - \sqrt{c})^2. \quad (4.30)$$

Since when  $p > n$ , at least  $p - n$  eigenvalues  $s_j^{(p)}$  are equal to 0, we conclude that

$$s_{n+1}^{(p)} = \dots = s_p^{(p)} = 0. \quad (4.31)$$

Therefore, (1.19) and (1.20) are obtained.

The proof of (1.17) and (1.18) is similar to the case when  $c < 1$  by using Proposition 3.6 and noting that an interval  $[a, b]$  satisfying condition (f) of Proposition 2.1 is contained in

$$(-\infty, (1 - \sqrt{c})^2) \cup \left( (1 + \sqrt{c})^2, \alpha_{M_0} + \frac{c\alpha_{M_0}}{\alpha_{M_0} - 1} \right) \cup \cdots \cup \left( \alpha_2 + \frac{c\alpha_2}{\alpha_2 - 1}, \alpha_1 + \frac{c\alpha_1}{\alpha_1 - 1} \right) \cup \left( \alpha_1 + \frac{c\alpha_1}{\alpha_1 - 1}, \infty \right), \quad (4.32)$$

which is a subset of

$$\text{supp}(F_\infty)^c = (-\infty, (1 - \sqrt{c})^2) \cup ((1 + \sqrt{c})^2, \infty). \quad (4.33)$$

### 4.3 When $c = 1$

Since the limiting distribution (1.2) for  $c = 1$  has a continuous density on the interval  $(0, 4)$ , it is easy to see (1.24).

We first observe a monotonicity of  $s_j^{(p)}$  in  $n$ . Let  $\hat{Z}_p = (Z_{ij})$ ,  $1 \leq i \leq p, 1 \leq j \leq \hat{n}$  and let  $\hat{B}_p := \frac{1}{\hat{n}} T_p^{1/2} \hat{Z}_p \hat{Z}_p' T_p^{1/2}$ . When  $\hat{n} > n$ , it is clear that

$$\hat{n} \hat{B}_p \geq n B_p. \quad (4.34)$$

Therefore, if the ordered eigenvalues of  $\hat{B}_p$  are denoted by  $\hat{s}_j^{(p)}$ , the min-max principle implies that

$$\hat{n} \hat{s}_j^{(p)} \geq n s_j^{(p)} \quad (4.35)$$

for all  $1 \leq j \leq p$ .

Take  $\hat{n} = \lceil \frac{n}{1+\epsilon} \rceil$  for  $\epsilon > 0$  where  $\lceil x \rceil$  denotes the largest integer  $\leq x$ . Then for sufficiently small  $\epsilon > 0$ ,

$$\alpha_1 > \cdots > \alpha_{M_0} > 1 + \sqrt{1+\epsilon} > \alpha_{M_0+1} > \cdots > \alpha_M. \quad (4.36)$$

By applying Theorem 1.2 and using (4.35), we obtain the following:

- For each  $1 \leq j \leq M_0$ ,

$$\liminf s_{k_1+\cdots+k_{j-1}+i}^{(p)} \geq \frac{1}{1+\epsilon} \left( \alpha_j + \frac{(1+\epsilon)\alpha_j}{\alpha_j - 1} \right), \quad 1 \leq i \leq k_j. \quad (4.37)$$

almost surely.

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$$\liminf s_{k_1+\cdots+k_{M_0}+1}^{(p)} \geq \frac{1}{1+\epsilon} (1 + \sqrt{1+\epsilon})^2 \quad (4.38)$$

almost surely.

On the other hand, take  $\hat{n} = \lceil \frac{n}{1-\epsilon} \rceil$  for  $\epsilon > 0$ . We first assume  $2 > \alpha_{M_0+1}$ . Then as  $\alpha_M > 0$ , for sufficiently small  $\epsilon > 0$ ,

$$\alpha_1 > \cdots > \alpha_{M_0} > 1 + \sqrt{1-\epsilon} > \alpha_{M_0+1} > \cdots > \alpha_M > 1 - \sqrt{1-\epsilon}, \quad (4.39)$$

and hence  $M_1 = M$ . By applying Theorem 1.1 and using (4.35), we obtain the following:

- For each  $1 \leq j \leq M_0$ ,

$$\limsup s_{k_1+\dots+k_{j-1}+i}^{(p)} \leq \frac{1}{1-\epsilon} \left( \alpha_j + \frac{(1-\epsilon)\alpha_j}{\alpha_j-1} \right), \quad 1 \leq i \leq k_j. \quad (4.40)$$

almost surely.

- 

$$\limsup s_{k_1+\dots+k_{M_0}+1}^{(p)} \leq \frac{1}{1-\epsilon} (1 + \sqrt{1-\epsilon})^2 \quad (4.41)$$

almost surely.

If  $\alpha_{M_0+1} = 2$ , then for sufficiently small  $\epsilon > 0$ ,

$$\alpha_1 > \dots > \alpha_{M_0} > \alpha_{M_0+1} > 1 + \sqrt{1-\epsilon} > \alpha_{M_0+2} > \dots > \alpha_M > 1 - \sqrt{1-\epsilon}. \quad (4.42)$$

Hence Theorem 1.1 implies (4.40) but (4.41) becomes

$$\limsup s_{k_1+\dots+k_{M_0}+1}^{(p)} \leq \frac{1}{1-\epsilon} \left( \alpha_{M_0+1} + \frac{(1-\epsilon)\alpha_{M_0+1}}{\alpha_{M_0+1}-1} \right) \quad (4.43)$$

almost surely.

Therefore (4.37) and (4.40) yield (1.22), and (4.38), (4.41) and (4.43) yield (1.23).

## References

- [1] Z. D. Bai. Methodologies in spectral analysis of large-dimensional random matrices, a review. *Statist. Sinica*, 9:611–677, 1999.
- [2] Z. D. Bai and J. W. Silverstein. No eigenvalues outside the support of the limiting spectral distribution of large-dimensional sample covariance matrices. *Ann. Probab.*, 26(1):316–345, 1998.
- [3] Z. D. Bai and J. W. Silverstein. Exact separation of eigenvalues of large dimensional sample covariance matrices. *Ann. Probab.*, 27(3):1536–1555, 1999.
- [4] J. Baik. Painlevé formulas of the limiting distributions for non-null complex sample covariance matrices. in preparation.
- [5] J. Baik, G. Ben Arous, and S. Péché. Phase transition of the largest eigenvalue for non-null complex sample covariance matrices. <http://xxx.lanl.gov/abs/math.PR/0403022>.
- [6] A. Buja, T. Hastie, and R. Tibshirani. Penalized discriminant analysis. *Ann. Statist.*, 23:73–102, 1995.
- [7] P.J. Forrester. The spectrum edge of random matrix ensembles. *Nuclear Physics B*, 402:709–728, 1993.
- [8] S. Geman. A limit theorem for the norm of random matrices. *Ann. Probab.*, 8(2):252–261, 1980.
- [9] R. Horn and C. Johnson. *Matrix Analysis*. Cambridge Press, New York, 1985.

- [10] D. Hoyle and M. Rattay. Limiting form of the sample covariance eigenspectrum in PCA and kernel PCA. to appear in proceedings of Neural Information Processing Systems 2003.
- [11] K. Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math. (2)*, 153:259–296, 2001.
- [12] I. Johnstone. On the distribution of the largest Principal Component. *Ann. Statist.*, 29:295–327, 2001.
- [13] L. Laloux, P. Cizeau, M. Potters, and J. Bouchaud. Random matrix theory and financial correlations. *Intern. J. Theor. Appl. Finance*, 3(3):391–397, 2000.
- [14] Y. Malevergne and D. Sornette. Collective origin of the coexistence of apparent RMT noise and factors in large sample correlation matrices. <http://xxx.lanl.gov/abs/cond-mat/0210115>.
- [15] V.A. Marcenko and L.A. Pastur. Distribution of eigenvalues for some sets of random matrices. *Math. USSR-Sbornik*, 1:457–486, 1967.
- [16] D. Paul. Asymptotic behaviour of the largest eigenvalues of sample covariance matrix when true covariance is a finite perturbation of identity. in preparation.
- [17] S. Péché. Universality of local eigenvalue statistics for random sample covariance matrices. *Ph.D. Thesis*, Ecole Polytechnique Fédérale de Lausanne, 2003.
- [18] V. Plerous, P. Gopikrishnan, B. Rosenow, L. Amaral, T. Guhr, and H. Stanley. Random matrix approach to cross correlations in financial data. *Phys. Rev. E*, 65(6):066126, 2002.
- [19] R. Sear and J. Cuesta. Instabilities in complex mixtures with a large number of components. *Phys. Rev. Lett.*, 91(24):245701, 2003.
- [20] J. W. Silverstein. The smallest eigenvalue of a large dimensional wishart matrix. *Annals of Probability*, 13(4):1364–1368, 1985.
- [21] J. W. Silverstein. Strong convergence of the empirical distribution of eigenvalues of large dimensional random matrices. *J. Multivariate Anal.*, 55(2):331–339, 1995.
- [22] J. W. Silverstein and Z. D. Bai. On the empirical distribution of eigenvalues of a class of large dimensional random matrices. *J. Multivariate Anal.*, 54(2):175–192, 1995.
- [23] J. W. Silverstein and S. Choi. Analysis of the limiting spectral distribution of large dimensional random matrices. *J. Multivariate Anal.*, 54(2):295–309, 1995.
- [24] A. Soshnikov. A note on universality of the distribution of the largest eigenvalues in certain sample covariance matrices. *J. Statist. Phys.*, 108(5-6), 2001.
- [25] E. Telatar. Capacity of multi-antenna Gaussian channels. *European transactions on Telecommunications*, 10(6):585–595, 1999.

- [26] C. Tracy and H. Widom. Level spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159:33–72, 1994.
- [27] C. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.*, 177:727–754, 1996.