

A GUE CENTRAL LIMIT THEOREM AND UNIVERSALITY OF DIRECTED FIRST AND LAST PASSAGE SITE PERCOLATION

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ABSTRACT. We prove a GUE central limit theorem for random variables with finite fourth moment. We apply this theorem to prove that the directed first and last passage percolation problems in thin rectangles exhibit universal fluctuations given by the Tracy-Widom law.

1. INTRODUCTION

In the last few years, it has become clear that random matrix theory is intimately related to a variety of questions arising in physics, statistics, combinatorics, representation theory, number theory, and probability theory (see e.g. [12, 5, 10, 19]). It is the fluctuations of random matrix ensembles which are common to the specific problems from each of these areas. In the hope of understanding the universal nature of the random matrix distributions, it is natural to search for central limit theorems for which these distributions are the limiting objects.

The following is a Gaussian Unitary Ensemble (GUE) central limit theorem.

Theorem 1 (GUECLT). *Suppose that $\{X_i^j\}_{i,j=1}^\infty$ is a family of independent identically distributed random variables such that $\mathbb{E}X_i^j = 0$, $\mathbb{E}|X_i^j|^2 = 1$, and $\mathbb{E}|X_i^j|^4 < \infty$. Let*

$$(1) \quad L(N, k) = \sup_{0=i_0 \leq i_1 \leq \dots \leq i_k=N} \sum_{j=1}^k \left\{ \sum_{i=i_{j-1}+1}^{i_j} X_i^j \right\},$$

$$(2) \quad R(N, k) = \inf_{0=i_0 \leq i_1 \leq \dots \leq i_k=N} \sum_{j=1}^k \left\{ \sum_{i=i_{j-1}+1}^{i_j} X_i^j \right\}.$$

If $k, N \rightarrow \infty$ such that $k = o(N^\alpha)$, $\alpha < \frac{3}{14}$, then

$$(3) \quad \left(\frac{L(N, k)}{N^{\frac{1}{2}}} - 2\sqrt{k} \right) k^{\frac{1}{6}} \implies F_{GUE},$$

$$(4) \quad \left(-\frac{R(N, k)}{N^{\frac{1}{2}}} - 2\sqrt{k} \right) k^{\frac{1}{6}} \implies F_{GUE},$$

where F_{GUE} is the GUE Tracy-Widom distribution. If $\{X_i^j\}_{i,j=1}^\infty$ are independent identically distributed Gaussian random variables satisfying the above conditions, then (3) and (4) hold if $k = o(N^\alpha)$, $\alpha < \frac{3}{7}$.

The GUE Tracy-Widom distribution function [18] is given by

$$(5) \quad F_{GUE}(x) = \exp \left\{ - \int_x^\infty (s-x) q^2(s) ds \right\},$$

where $q(x)$ solves the Painlevé II equation,

$$(6) \quad q'' = 2q^3 + xq$$

subject to the condition that $q(x) \sim Ai(x)$ as $x \rightarrow +\infty$; $Ai(x)$ denotes the Airy function. The function F_{GUE} is the limiting distribution function for the largest eigenvalue of the Gaussian Unitary Ensemble as the dimension of the matrices grows to infinity (see (18)).

Theorem 1 is intimately related to the directed first and last passage percolation problems. Consider the $\mathbb{N} \times \mathbb{N}$ lattice and a set of associated independent identically distributed random variables $\{X_i^j\}_{i,j=1}^\infty$ satisfying $\mathbb{E}X_i^j = \mu$, $\mathbb{E}|X_i^j|^2 - \mu^2 = \sigma^2$, and $\mathbb{E}|X_i^j|^4 < \infty$. An *up/right path* π from the site $(1, 1)$ to the site (N, k) is a collection of sites $\{(i_k, j_k)\}_{k=1}^{N+k-1}$ satisfying $(i_1, j_1) = (1, 1)$, $(i_{N+k-1}, j_{N+k-1}) = (N, k)$ and $(i_{k+1}, j_{k+1}) - (i_k, j_k)$ is either $(1, 0)$ or $(0, 1)$. Let $(1, 1) \nearrow (N, k)$ denote the set of such up/right paths. The directed first and last passage times to $(N, k) \in \mathbb{N} \times \mathbb{N}$, denoted by $L^f(N, k)$ and $L^l(N, k)$, respectively, are defined as

$$(7) \quad L^f(N, k) = \min_{\pi \in (1,1) \nearrow (N,k)} \sum_{(i,j) \in \pi} X_i^j,$$

$$(8) \quad L^l(N, k) = \max_{\pi \in (1,1) \nearrow (N,k)} \sum_{(i,j) \in \pi} X_i^j.$$

If X_i^j is interpreted as the time to pass the site (i, j) , $L^f(N, k)$ and $L^l(N, k)$ represent the minimal and the maximal time to travel from the site $(1, 1)$ to (N, k) along an admissible path. Since the directed last passage percolation time can be viewed as the departure time in queuing theory (see e.g. [7]), the result below also applies to queuing theory. In addition, Corollary 1.1 also applies to the flux of particles at a given site in the totally asymmetric simple exclusion process (see e.g. [17]). Theorem 1 implies the following.

Corollary 1.1. *Suppose that $\{X_i^j\}_{i,j=1}^\infty$ is a family of independent identically distributed random variables such that $\mathbb{E}X_i^j = \mu$, $\mathbb{E}|X_i^j|^2 - \mu^2 = \sigma^2$, and $\mathbb{E}|X_i^j|^4 < \infty$. For any $s \in \mathbb{R}$,*

$$(9) \quad \lim_{N,k \rightarrow \infty} \mathbb{P} \left(\frac{L^l(N, k) - \mu(N+k-1) - 2\sigma\sqrt{Nk}}{\sigma k^{-1/6} N^{\frac{1}{2}}} \leq s \right) = F_{GUE}(s),$$

$$(10) \quad \lim_{N,k \rightarrow \infty} \mathbb{P} \left(\frac{L^f(N, k) - \mu(N+k-1) + 2\sigma\sqrt{Nk}}{\sigma k^{-1/6} N^{\frac{1}{2}}} \leq s \right) = 1 - F_{GUE}(-s),$$

where $k = o(N^\alpha)$ and $\alpha < \frac{3}{14}$. If $\{X_i^j\}_{i,j=1}^\infty$ are independent identically distributed Gaussian random variables satisfying the above conditions, then (9) and (10) hold when $k = o(N^\alpha)$, $\alpha < \frac{3}{7}$.

The restriction $k = o(N^\alpha)$, $\alpha < \frac{3}{14}$, in the above corollary seems to be a technical matter. It is believed that such a central limit theorem (after possible changes in scaling) should hold as $N, k \rightarrow \infty$ with no restrictions on their relative rate of growth. When N and k are of the same order, such a limit law for L^l was obtained for geometric and exponential random variables by Johansson [9]¹ using Young tableaux theory in combinatorics and techniques from random matrix theory. However, analyzing the last passage percolation time for general random variables in arbitrary scaling regimes is still an open question.

Even the determination of the ‘time constants’ as $N, k \rightarrow \infty$ for general random variables is a challenge. Glynn and Whitt [7] proved that for $k = o(N)$,

$$(11) \quad \lim_{N,k \rightarrow \infty} \frac{L^l(N, k) - \mu N}{\sqrt{Nk}} = \alpha,$$

\mathbb{P} -almost surely for some α , where α was proved to be equal to 2σ by Seppäläinen [17] (see also [11]). Note that the definition of the last passage percolation, $L^l(N, k)$, is symmetric in N and k . Hence, (11) can not be true when N and k are of the same order. For exponential random variables of mean 1, an interpretation of a theorem of Rost [16] yields that

$$(12) \quad \lim_{n \rightarrow \infty} \frac{L^l([xn], [yn])}{n} = (\sqrt{x} + \sqrt{y})^2.$$

¹For slightly different choices of admissible paths, a few more explicit random variables are shown to have the same type of limit law (see e.g. [1, 8])

Johansson [9] (see equation (1.4)) proved that for geometric random variables of parameter q ,

$$(13) \quad \lim_{n \rightarrow \infty} \frac{L^l([xn], [yn])}{n} = \frac{q(x+y) + 2\sqrt{qxy}}{1-q}.$$

These two cases seem to be the only explicitly computed examples when N and k are of the same order.

Note that the directed first and last passage times can also be written as

$$(14) \quad L^f(N, k) = \inf_{1=i_0 \leq i_1 \leq \dots \leq i_k=N} \sum_{j=1}^k \left\{ \sum_{i=i_{j-1}}^{i_j} X_i^j \right\},$$

$$(15) \quad L^l(N, k) = \sup_{1=i_0 \leq i_1 \leq \dots \leq i_k=N} \sum_{j=1}^k \left\{ \sum_{i=i_{j-1}}^{i_j} X_i^j \right\}.$$

When k is *fixed* and $N \rightarrow \infty$, the last passage time is obtained along a path that lies on the first level, $j = 1$, for a certain ‘time’, $[t_1 N]$, then jumps to the second level, $j = 2$, and stays there until the ‘time’ $[t_2 N]$, and so on such that the total sum is maximized. As the sum of random variables on each level is ‘asymptotically equal’ to a Brownian motion after proper centering and scaling, by the Donsker’s theorem one can imagine that

$$(16) \quad \frac{L^l(N, k) - \mu N}{\sigma\sqrt{N}} \implies \hat{D}_k(1) := \sup_{0=t_0 \leq t_1 \leq \dots \leq t_k=1} \sum_{j=1}^k (B_j(t_j) - B_j(t_{j-1})),$$

where $B_j(t)$, $j = 1, \dots, k$, are independent Brownian motions. This was proved in [7] in the context of queuing theory.

The distribution function of $\hat{D}_k(1)$ is difficult to compute since $\hat{D}_k(1)$ is a complicated and unusual functional of k Brownian motions. However, as mentioned earlier, for some special random variables, Young tableaux theory in combinatorics can be applied to explicitly compute the distribution of L^l . By studying a special random variable, Baryshnikov [2] and Gravner, Tracy and Widom [8] (see also [14, 13]) showed that

$$(17) \quad \mathbb{P}(\hat{D}_k(1) \leq s) = \frac{1}{Z_k} \int_{-\infty}^s \dots \int_{-\infty}^s \prod_{1 \leq i \leq j \leq k} |\xi_i - \xi_j|^2 \prod_{j=1}^k e^{-\frac{1}{2}\xi_j^2} d\xi_j,$$

where $Z_k = 1!2! \dots k!(2\pi)^{k/2}$. As the integrand on the right-hand-side is the density of the eigenvalues ξ_1, \dots, ξ_k of $k \times k$ GUE random matrix, this identity implies that $\hat{D}_k(1)$ has precisely the same distribution as the largest eigenvalue, $\xi_{\max}(k)$, of $k \times k$ GUE random matrix. It is well-known in random matrix theory that (see e.g. [6, 18])

$$(18) \quad \lim_{k \rightarrow \infty} \mathbb{P}((\xi_{\max}(k) - 2\sqrt{k})k^{1/6} \leq s) = F_{GUE}(s).$$

Therefore,

$$(19) \quad \lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{P}\left(\left\{ \frac{L^l(N, k) - \mu N}{\sigma\sqrt{N}} - 2\sqrt{k} \right\} k^{1/6} \leq s\right) = F_{GUE}(s).$$

The content of Corollary 1.1 is that the two limits can be taken simultaneously as long as $k = o(N^\alpha)$, $\alpha < \frac{3}{14}$. Note that in the centering $\mu(N+k-1)$ of (9), the term $\mu(k-1)$ accounts for the ‘up movements’ of the path while μN comes from the ‘right movements’ of the path. The term $\mu(k-1)$ is negligible when $k = o(N^{\frac{3}{7}})$, but it is retained in the formula in order to emphasize the symmetry of N, k .

Theorem 1 is a central limit theorem whose limit is the Tracy-Widom GUE top and bottom eigenvalue distributions. The joint distribution of the top and bottom n eigenvalues is also a natural limit of a central limit theorem which depends on the O’Connell and Yor representation of the k -dimensional Dyson process [14,

13]. Following [14, 13], let $C_0([0, 1], \mathbb{R})$ be the space of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ with $f(0) = 0$. For $f, g \in C_0([0, 1], \mathbb{R})$, define

$$(20) \quad f \otimes g(t) = \inf_{0 \leq s \leq t} [f(s) + g(t) - g(s)],$$

$$(21) \quad f \odot g(t) = \sup_{0 \leq s \leq t} [f(s) + g(t) - g(s)].$$

The order of operations is from left to right. Define a sequence of mappings $\Gamma_k : C_0([0, 1], \mathbb{R})^k \rightarrow C_0([0, 1], \mathbb{R})^k$ by

$$(22) \quad \Gamma_2(f, g) = (f \otimes g, g \odot f),$$

and for $k > 2$,

$$(23) \quad \begin{aligned} \Gamma_k(f_1, \dots, f_k) &= (f_1 \otimes f_2 \otimes f_3 \otimes \dots \otimes f_k, \\ &\quad \Gamma_{k-1}(f_2 \odot f_1, f_3 \odot (f_1 \otimes f_2), \dots, f_k \odot (f_1 \otimes \dots \otimes f_{k-1}))). \end{aligned}$$

O'Connell and Yor proved that if B_1, \dots, B_k are independent standard one-dimensional Brownian motions, $\Gamma_k(B_1, \dots, B_k)$ has the same distribution on path space, $C_0([0, 1], \mathbb{R})^k$, as k one-dimensional Brownian motions starting from the origin, conditioned (in the sense of Doob) never to collide. It is well known that this process can be interpreted as the k -dimensional GUE Dyson eigenvalue process. The first coordinate of Γ_k is the smallest eigenvalue, the second coordinate of Γ_k is the second smallest eigenvalue, and so on. The proof of Theorem 1 immediately implies the following theorem once one notes that the first n coordinates of Γ_k are Lipschitz functions with a fixed Lipschitz constant independent of k . We omit the details.

Theorem 2. *Suppose that $\{X_i^j\}_{i,j=1}^\infty$ is a family of independent identically distributed random variables such that $\mathbb{E}X_i^j = 0$, $\mathbb{E}|X_i^j|^2 = 1$, and $\mathbb{E}|X_i^j|^4 < \infty$. Let $s_N^j(i) = \frac{1}{\sqrt{N}} \sum_{l=1}^i X_l^j$. As $N, k \rightarrow \infty$ such that $k = O(N^\alpha)$, $\alpha < \frac{3}{14}$,*

$$(24) \quad (\Gamma_k(s_N^1, \dots, s_N^k)_1(N), \dots, \Gamma_k(s_N^1, \dots, s_N^k)_n(N)) \implies F_{GUE}^n,$$

where F_{GUE}^n denotes the limiting joint probability distribution of the bottom n eigenvalues of the GUE. By symmetry, a similar statement holds for the top n eigenvalues.

Proving a GUE central limit theorem and analyzing certain scaling regimes for the directed first and last passage percolation problems is the main focus of this paper. It is interesting to seek central limit type theorems for the Gaussian Orthogonal Ensemble (GOE) and the Gaussian Symplectic Ensemble (GSE) Tracy-Widom distributions. The arguments of this paper can be adapted to prove such theorems. The authors will address this topic in several forthcoming papers. The authors will also describe the fluctuations of restricted Plancherel measures from this point of view.

Note. While completing this paper, the authors learned that T. Bodineau and J. Martin had announced a similar result [4]. Both [4] and this paper use the idea of approximating random walks by Brownian motion in order to analyze the last passage time. As opposed to [4], the authors use the Skorohod embedding theorem to couple random walks to Brownian motion. Bodineau and Martin use the Komlós, Major and Tusnády approximation theorem (which produces tighter bounds) to achieve such a coupling. The authors are grateful to Rongfeng Sun for bringing [4] to their attention.

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2. PROOFS

Theorem 1 and Corollary 1.1 are proven in this section.

Proof of Theorem 1. The proof is based on embedding general random walks in Brownian motions by means of the classical Skorohod embedding theorem [3].

Theorem 3 (Skorohod Embedding Theorem). *Let $\{B_t\}_{t \in \mathbb{R}^+}$ be a one-dimensional standard Brownian motion and X be a real valued random variable satisfying $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and $\mathbb{E}X^4 < \infty$. Then, there exists a Brownian stopping time, τ , such that B_τ is distributed as X , $\mathbb{E}\tau = \mathbb{E}X^2 = 1$, and $\mathbb{E}\tau^2 \leq 4\mathbb{E}X^4$.*

Let $\{X_i\}_{i=1}^\infty$ be a sequence of independent identically distributed random variables satisfying the assumptions of Theorem 3. There exists a sequence of independent identically distributed positive random variables $\{\tau_i\}_{i=1}^\infty$ such that $B_{\tau_1+\dots+\tau_n}$ is distributed as $X_1 + \dots + X_n$ and $\{B_{\tau_1+\dots+\tau_{n+1}} - B_{\tau_1+\dots+\tau_n}\}_{n=0}^\infty$ is a sequence of independent identically distributed random variables with distribution X_1 . This follows from Theorem 3 and the strong Markov property of Brownian motion.

To prove uniform estimates in time for the difference of embedded random walks and the scaled Brownian motion in which the walks are embedded, the following two real valued processes are useful:

$$(25) \quad S_N(t) = \frac{B_{\tau_1+\dots+\tau_{\lfloor Nt \rfloor}} + (Nt - \lfloor Nt \rfloor)(B_{\tau_{\lfloor Nt \rfloor}} - B_{\tau_{\lfloor Nt \rfloor}})}{\sqrt{N}},$$

$$(26) \quad \tilde{B}_N(t) = \frac{B_{Nt}}{\sqrt{N}}.$$

Let $C([0, 1], \mathbb{R}^k)$ be the space of continuous k -vector valued functions on the unit interval equipped with the sup norm, $\|\cdot\|_k$: for $f \in C([0, 1], \mathbb{R}^k)$, $\|f\|_k = \sum_{j=1}^k \sup_{t \in [0, 1]} |f_j(t)|$. Theorem 1 is proven in two steps. The first step involves establishing probabilistic estimates on differences of the above processes. The second step involves a Lipschitz bound on the relevant Brownian concatenation operations and the application of the quantitative estimates established in the first step.

Two estimates will be used in the proof of Theorem 1. Let $Y_i = \tau_i - 1$ and $Z_i^N = \frac{1}{N} \sum_{n=1}^i Y_n$. Note that Z_i^N and $(Z_i^N)^2$ are submartingales. By the Doob martingale inequalities [15],

$$(27) \quad \begin{aligned} \mathbb{P}\left(\sup_{0 \leq i \leq N} |Z_i^N| > \beta_N\right) &\leq \frac{\mathbb{E}(Z_N^N)^2}{\beta_N^2} \\ &= \frac{\mathbb{E}(\sum_{j=1}^N Y_j)^2}{N^2 \beta_N^2} = \frac{\mathbb{E}\tau_1^2 - (\mathbb{E}\tau_1)^2}{N \beta_N^2} \\ &\leq \frac{\text{Var}(\tau_1)}{N \beta_N^2}. \end{aligned}$$

For $\beta_N = \frac{c}{N^\lambda}$ where $c > 0$ and $\lambda < \frac{1}{2}$, this inequality implies

$$(28) \quad \mathbb{P}\left(\sup_{1 \leq i \leq N} \frac{|\sum_{j=1}^i \tau_j - i|}{N} > \frac{c}{N^\lambda}\right) \leq \frac{\text{Var}(\tau_1)}{N^{1-2\lambda} c^2}.$$

This is the first estimate.

The second estimate concerns the modulus of continuity for Brownian motion. If $\rho < \frac{1}{2}$, then for some appropriately chosen constants $A, \nu > 0$,

$$(29) \quad \mathbb{P}\left(\sup_{t \in [0, 1], |t-s| < cN^{-\lambda}} |\tilde{B}_N(t) - \tilde{B}_N(s)| > \frac{c}{N^\rho}\right) \leq Ae^{-\nu c}$$

uniformly in N , $c > 0$. This estimate is a consequence of Brownian scaling and standard estimates for the maximum of Brownian motion. The first part of the proof of Theorem 1 is complete.

Consider the functions $\overline{G}_k : C([0, 1], \mathbb{R}^k) \rightarrow \mathbb{R}$ and $\underline{G}_k : C([0, 1], \mathbb{R}^k) \rightarrow \mathbb{R}$ defined by

$$(30) \quad \underline{G}_k(f) = \inf_{0=t_0 \leq t_1 \leq \dots \leq t_k=1} \sum_{j=1}^k (f_j(t_j) - f_j(t_{j-1})),$$

$$(31) \quad \overline{G}_k(f) = \sup_{0=t_0 \leq t_1 \leq \dots \leq t_k=1} \sum_{j=1}^k (f_j(t_j) - f_j(t_{j-1})).$$

Lemma 2.1. *Both \overline{G}_k and \underline{G}_k are Lipschitz with Lipschitz constant 2:*

$$(32) \quad |\overline{G}_k(f) - \overline{G}_k(g)| \leq 2\|f - g\|_k \quad \text{and} \quad |\underline{G}_k(f) - \underline{G}_k(g)| \leq 2\|f - g\|_k,$$

for all $f, g \in C([0, 1], \mathbb{R}^k)$.

proof: Let $f, g \in C([0, 1], \mathbb{R}^k)$ satisfy $\|f - g\|_k = \epsilon$, $\|f_i - g_i\|_1 = \epsilon_i$, and $\sum_{i=1}^k \epsilon_i = \epsilon$ where $f = (f_1, \dots, f_k)$, $g = (g_1, \dots, g_k)$. $|\overline{G}_k(f) - \overline{G}_k(g)| \leq \sum_{i=1}^k |\overline{G}_k(f_1, \dots, f_{i-1}, g_i, \dots, g_k) - \overline{G}_k(f_1, \dots, f_i, g_{i+1}, \dots, g_k)|$ by the triangle inequality. $|\overline{G}_k(f_1, \dots, f_{i-1}, g_i, \dots, g_k) - \overline{G}_k(f_1, \dots, f_i, g_{i+1}, \dots, g_k)| \leq 2\epsilon_i$. Thus, $|\overline{G}_k(f) - \overline{G}_k(g)| \leq 2\|f - g\|_k$. Since the same argument holds for $|\underline{G}_k(f) - \underline{G}_k(g)|$, the proof of the lemma is complete. \diamond

By Skorohod embedding, for each $N > 0$ construct a sequence of pairs of random walks embedded in standard Brownian motions, $\{(S_N^j(t), \tilde{B}_N^j(t))\}_{j=1}^\infty$, as in (25) and (26). Lemma 2.1 and the estimates on the differences of the Skorohod coupling imply that if $N, k \rightarrow \infty$ and $k = o(N^\alpha)$, $\alpha < \frac{3}{14}$, then

$$(33) \quad k^{\frac{1}{6}} (\overline{G}_k(S_N^1, \dots, S_N^k) - \overline{G}_k(\tilde{B}_N^1, \dots, \tilde{B}_N^k)) \rightarrow 0,$$

$$(34) \quad k^{\frac{1}{6}} (\underline{G}_k(S_N^1, \dots, S_N^k) - \underline{G}_k(\tilde{B}_N^1, \dots, \tilde{B}_N^k)) \rightarrow 0,$$

in distribution. More explicitly, if $k = \lfloor N^\alpha \rfloor$, then

$$(35) \quad \begin{aligned} \mathbb{P}(k^{\frac{1}{6}} (\overline{G}_k(S_N^1, \dots, S_N^k) - \overline{G}_k(\tilde{B}_N^1, \dots, \tilde{B}_N^k)) > \epsilon) &\leq \mathbb{P}\left(\sum_{i=1}^{\lfloor N^\alpha \rfloor} \|S_N^i - \tilde{B}_N^i\|_1 > \frac{\epsilon}{2N^{\frac{1}{6}}}\right) \\ &\leq \frac{N^{\frac{7\alpha}{6}}}{\epsilon} \mathbb{E}\|S_N^1 - \tilde{B}_N^1\|_1. \end{aligned}$$

Note that

$$(36) \quad \begin{aligned} \mathbb{E}\|S_N^1 - \tilde{B}_N^1\|_1 &= \int_0^\infty \mathbb{P}(\|S_N^1 - \tilde{B}_N^1\|_1 > s) ds = \frac{1}{N^\rho} \int_0^\infty \mathbb{P}\left(\|S_N^1 - \tilde{B}_N^1\|_1 > \frac{u}{N^\rho}\right) du \\ &= \frac{1}{N^\rho} \int_0^\infty \mathbb{P}\left(\|S_N^1 - \tilde{B}_N^1\|_1 > \frac{u}{N^\rho}; \sup_{1 \leq i \leq N} \left| \sum_{j=1}^i \tau_j - i \right| > uN^{1-\lambda}\right) du \\ &\quad + \frac{1}{N^\rho} \int_0^\infty \mathbb{P}\left(\|S_N^1 - \tilde{B}_N^1\|_1 > \frac{u}{N^\rho}; \sup_{1 \leq i \leq N} \left| \sum_{j=1}^i \tau_j - i \right| < uN^{1-\lambda}\right) du. \end{aligned}$$

If $\lambda < \frac{1}{2}$, then (28) implies that the integrand of the first integral on the right hand side of (36) has an integrable tail. If the $\{X_i^j\}_{i,j=1}^\infty$ are Gaussian, this term is identically 0 since $\tau_i = 1$ \mathbb{P} -almost surely; in this case, there is no constraint on λ . The integrand of the second integral of (36) is dominated by

$$(37) \quad \mathbb{P}\left(\sup_{t \in [0,1], |t-s| < uN^{-\lambda}} |\tilde{B}_N(t) - \tilde{B}_N(s)| > \frac{u}{N^\rho}\right).$$

(29) implies that if $\rho < \frac{\lambda}{2}$, the second integrand has exponentially decaying tails uniformly in N . A necessary condition for the right hand side of (35) to vanish as $N \rightarrow \infty$ is $\alpha < \frac{6}{7}\rho$. If $\{X_i^j\}_{i,j=1}^\infty$ are Gaussian, then $\rho < \frac{1}{2}$ is sufficient to make the second integral of (36) finite; thus, if $\alpha < \frac{3}{7}$, the right hand side of (35) vanishes as $N \rightarrow \infty$. If $\{X_i^j\}_{i,j=1}^\infty$ are not Gaussian, then $\alpha < \frac{6}{7}\rho$ and $2\rho < \lambda < \frac{1}{2}$ imply that $\alpha < \frac{3}{14}$ is sufficient to make (36) vanish as $N \rightarrow \infty$.

As mentioned in the introduction, the theorems of Baryshnikov [2] and Gravener, Tracy, and Widom [8] state that $\overline{G}_k(\tilde{B}_N^1, \dots, \tilde{B}_N^k)$ is distributed as the largest eigenvalue of the GUE of $k \times k$ matrices. By (18) and (33),

$$(38) \quad k^{\frac{1}{6}}(\overline{G}_k(S_N^1, \dots, S_N^k) - 2\sqrt{k}) \implies F_{GUE},$$

in distribution. By (34) and the symmetry with respect to multiplication by -1 of Brownian motion and the Dyson process

$$(39) \quad k^{\frac{1}{6}}(-\underline{G}_k(S_N^1, \dots, S_N^k) - 2\sqrt{k}) \implies F_{GUE}.$$

Since $\overline{G}_k(S_N^1, \dots, S_N^k)$ and $\underline{G}_k(S_N^1, \dots, S_N^k)$ are distributed as $L(N, k)$ and $R(N, k)$, respectively, the proof of Theorem 1 is complete. \square

It is clear from the proof of Theorem 1 that when $k \gg o(N^{\frac{3}{7}})$ the ‘up movements’ of the paths contribute to $L^l(N, k)$. Hence, one needs to take those terms into account in order to prove a central limit theorem for general N, k scaling. The arguments in this paper do not seem to extend to the general case.

The proof of Corollary 1.1 remains.

proof of Corollary 1.1. With no loss of generality, assume that $\mathbb{E}X_i^j = 0$ and $\mathbb{E}|X_i^j|^2 = 1$. The definitions of $L^l(N, k)$ and $L(N, k)$ imply the following deterministic estimate:

$$(40) \quad k^{\frac{1}{6}} \inf_{1 \leq i_1 \leq \dots \leq i_{k-1} \leq N} \left\{ \frac{1}{\sqrt{N}} \sum_{j=1}^{k-1} X_{i_j}^{j+1} \right\} \leq k^{\frac{1}{6}} \frac{L^l(N, k) - L(N, k)}{\sqrt{N}} \leq k^{\frac{1}{6}} \sup_{1 \leq i_1 \leq \dots \leq i_{k-1} \leq N} \left\{ \frac{1}{\sqrt{N}} \sum_{j=1}^{k-1} X_{i_j}^{j+1} \right\}.$$

Note that

$$(41) \quad \begin{aligned} \mathbb{P}\left(k^{\frac{1}{6}} \sup_{1 \leq i_1 \leq \dots \leq i_{k-1} \leq N} \frac{1}{\sqrt{N}} \sum_{j=1}^{k-1} X_{i_j}^{j+1} > \epsilon\right) &\leq \frac{k^{\frac{1}{6}}}{\epsilon\sqrt{N}} \mathbb{E}\left(\sup_{1 \leq i_1 \leq \dots \leq i_{k-1} \leq N} \sum_{j=1}^{k-1} X_{i_j}^{j+1}\right) \\ &\leq \frac{k^{\frac{7}{6}}}{\epsilon\sqrt{N}} \mathbb{E}\left(\sup_{i \in \{1, \dots, N\}} |X_i^1|\right) \\ &\leq \frac{k^{\frac{7}{6}}}{\epsilon\sqrt{N}} \int_0^\infty \left\{1 - \mathbb{P}\left(|X_1^1| < s\right)^N\right\} ds \\ &\leq \frac{k^{\frac{7}{6}}}{\epsilon\sqrt{N}} \int_0^\infty \left\{1 - \left[1 - \mathbb{P}\left(|X_1^1| > s\right)\right]^N\right\} ds \\ &= \frac{k^{\frac{7}{6}}}{\epsilon N^{\frac{1}{4}}} \int_0^\infty \left\{1 - \left[1 - \mathbb{P}\left(|X_1^1| > sN^{\frac{1}{4}}\right)\right]^N\right\} ds. \end{aligned}$$

Now split the last integral into two pieces; one over the interval $[0, 1]$ and the other over $(1, \infty)$. Using the basic estimate $0 \leq \mathbb{P}\left(|X_1^1| > sN^{\frac{1}{4}}\right) \leq 1$ for the first integral and the Chebyshev inequality for the second integral, (41) is less than or equal to

$$(42) \quad \frac{k^{\frac{7}{6}}}{\epsilon N^{\frac{1}{4}}} \int_0^1 ds + \frac{k^{\frac{7}{6}}}{N^{\frac{1}{4}}} \int_1^\infty \left\{1 - \left[1 - \frac{\mathbb{E}|X_1^1|^4}{s^4 N}\right]^N\right\} ds \leq \frac{k^{\frac{7}{6}}}{\epsilon N^{\frac{1}{4}}} + \frac{k^{\frac{7}{6}}}{\epsilon N^{\frac{1}{4}}} \int_1^\infty \left\{1 - e^{-2\frac{\mathbb{E}|X_1^1|^4}{s^4}}\right\} ds.$$

If $N, k \rightarrow \infty$ such that $k = o(N^\alpha)$, $\alpha < \frac{3}{14}$, then the right hand side of (42) tends to 0. The inf can be treated similarly. Theorem 1 and (41), (42) imply Corollary 1.1 for the non-Gaussian case. If $\{X_i^j\}_{i,j=1}^\infty$ are Gaussian, then the arguments from the proof of Theorem 1 apply. \square

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