The asymptotics of monotone subsequences of involutions

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Abstract

We compute the limiting distributions of the lengths of the longest increasing subsequences of random (signed) involutions with or without constraint on the number of fixed points (and negated points) as the sizes of the involutions tend to infinity. The resulting distributions are (1) the Tracy-Widom distributions for the largest eigenvalues of random GOE, GUE, GSE matrices, (2) the normal distributions, or (3) new classes of distributions which interpolate between pairs of the Tracy-Widom distributions, depending on the number of the fixed points. We also compute the limiting distributions of the lengths of the second rows of corresponding Young disgrams and also prove the convergence of moments in each case. The proof is based on the algebraic work of the authors in [7] which establishes a connection between the statistics of random involutions and a family of orthogonal polynomials, and an asympotitic analysis of the orthogonal polynomials which is obtained by extending the Riemann-Hilbert analysis for the orthogonal polynomials by Deift, Johansson and the first author in [4].

1 Introduction

Recently it has been observed by many authors that there are certain connections between random permutations and/or Young tableaux, and random matrices. One of the earliest clues to this relationship appeared in the work of Regev [40] in 1981. A Young diagram, or equivalently a partition $\lambda = (\lambda_1, \lambda_2, \cdots) \vdash n$ ($\lambda_1 \geq \lambda_2 \geq \cdots$, $\sum \lambda_j = n$) is an array of n boxes with top and left adjusted as in the first picture of Figure 1, which represents the example $\lambda = (4,3,1) \vdash 8$. A standard Young tableau Q is a filling of the diagram by numbers $1, 2, \cdots, n$ such that numbers are increasing along each row and along each column. We say that the tableau Q has the shape λ. The second picture in Figure 1 is an example of a standard Young tableau with shape $\lambda = (4,3,1)$. Let d_λ denote the number of standard Young tableaux of shape λ. A result of [40] is that for fixed $\beta > 0$ and

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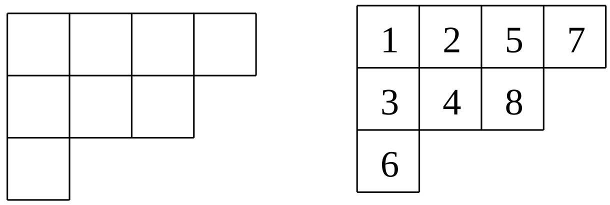

Figure 1: Young diagram and standard Young tableau

fixed l, as $n \to \infty$,

$$\sum_{\substack{\lambda \vdash n \\ \lambda_1 \le l}} (d_\lambda)^\beta \sim \left[\frac{l^{l^2/2} l^n}{(\sqrt{2\pi})^{(l-1)/2} n^{(l-1)(l+2)/4}} \right]^\beta \frac{n^{(l-1)/2}}{l!} \int_{\mathbb{R}^l} e^{-\frac{1}{2}\beta l \sum_j x_j^2} \prod_{j<k} |x_j - x_k|^\beta d^l x. \tag{1.1}$$

The multiple integral on the right hand side is called the Selberg integral, which can be computed exactly for each β in terms of the Gamma function. In particular, when $\beta = 1, 2, 4$, this integral is the normalization constant of the probability density of the eigenvalues in the Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE), Gaussian symplectic ensemble (GSE), respectively (see e.g. [37]) in the random matrix theory. Motivated by this result, we define the *β-Plancherel measure* on the set Y_n of Young diagrams (or partitions) of size n by

$$M_n^\beta(\lambda) := \frac{d_\lambda^\beta}{\sum_{\mu \vdash n} d_\mu^\beta}, \qquad \lambda \in Y_n. \tag{1.2}$$

A natural question is the limiting statistics of a random $\lambda \in Y_n$ under M_n^β, as $n \to \infty$.

The case when $\beta = 2$ is quite well studied. In this case, M_n^2 is called the *Plancherel measure* which arises in the representation theory of symmetric group S_n. Denote by $L_n^{(k)}$ the random variable λ_k under the Plancherel measure M_n^2, and denote $L_n = L_n^{(1)}$. There are many known results on the asymptotic statistics of λ under this Plancherel measure. In 1977, the limiting expected shape of λ under M_n^2 is obtained in [49], also independently in [36] for the so-called Poissonized Plancherel measure. In particular, it is shown that

$$\lim_{n\to\infty} \frac{\mathrm{E}(L_n)}{\sqrt{n}} = 2. \tag{1.3}$$

A central limit theorem is then obtained by [4] :

$$\lim_{n\to\infty} \Pr\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \right) = F_2(x), \tag{1.4}$$

where F_2 is the so-called Tracy-Widom distribution function, which is expressed in terms of the Painlevé II solution (see Definition 2 in Section 2 for the definition). The connection to the random matrix theory comes from this function F_2 : in 1994, Tracy and Widom [47] proved that under proper centering and scaling (which is different from the scaling for L_n in (1.4)), the largest eigenvalue of a random matrix taken from the Gaussian unitary ensemble (GUE) has the same limiting distribution given by F_2. In other words, after proper centering and scaling, the first row of a random Young diagram under the Plancherel measure behaves statistically for large n like the largest eigenvalue of a random GUE matrix. Then in the same paper [4], it was conjectured that $L_n^{(k)}$ of a random $\lambda \in Y_n$ under M_n^2 have the same limiting distribution as the k^{th} largest eigenvalue of a random GUE matrix for each k. This conjecture was supported by numerical simulations of Odlyzko and the second author, and was proved to be true for the second row $L_n^{(2)}$ in [5]. The full conjecture for the general row $L_n^{(k)}$ was proved by [39], [11] and [30], independently. The authors proved the convergence in joint distribution for general rows, and also in [11] and [30], the authors obtained the discrete sine kernel representations for the so-called bulk scaling limit of correlation functions, which is an analogue of the sine kernel appeared in the GUE matrix case. In [4] and [5], in addition to the convergence in distribution, the authors also proved the convergence of moments for $L_n^{(1)}$ and $L_n^{(2)}$, respectively. The convergence of (joint) moments for the general rows is obtained recently by [6].

One of the main topics in this paper is the case when $\beta = 1$. From (1.1) and the results for the case when $\beta = 2$, we expect that the limiting statistics of a random $\lambda \in Y_n$ under M_n^1 is same as the limiting statistics of the eigenvalues of GOE (Gaussian orthogonal ensemble) matrices. We establish this fact for the first two rows in this paper. More precisely, we have (see Theorems 3.4 and 3.6), denoting by $\tilde{L}_n^{(k)}$ the random variable λ_k of a random $\lambda \in Y_n$ under M_n^1,

$$\lim_{n\to\infty} \Pr\left(\frac{\tilde{L}_n^{(k)} - 2\sqrt{n}}{n^{1/6}} \le x \right) = F_1^{(k)}(x), \qquad k = 1, 2, \tag{1.5}$$

where $F_1^{(k)}(x)$ is the limiting distribution function [48] for the (scaled) k^{th} largest eigenvalue of a random matrix taken from GOE. We also prove the convergence of moments. As in the case of $\beta = 2$, we expect that the above result should extend to the general rows $k \ge 3$, and also to the joint distributions. For general values of $\beta > 0$, again from (1.1), we expect that in the large n limit, the k^{th} row of a random Young diagram under M_n^β correspond to the Coulomb charges on the real line with the quadratic potential at the inverse temperature β, which specializes to GOE, GUE, GSE eigenvalue distributions for the cases $\beta = 1, 2, 4$, respectively. This conjecture seems natural from the perspective of the discrete Coulomb gas interpretation for the Plancherel measure case M_n^2 given by Johansson [31, 30].

The Plancherel measure M_n^2 has a nice combinatorial interpretation. The well-known Robinson-Schensted correspondence [41] establishes a bijection between the permutations π of size n and the pairs of standard Young tableaux (P, Q) where the shape of P and the shape of Q are the same and the shape of P (or Q), denoted by $\lambda(\pi)$, is a partition of n. Thus the Plancherel measure M_n^2 on Y_n is the push forward of the uniform probability measure on S_n. Moreover, under this correspondence, $\lambda_1(\pi)$ is equal to the length of the longest increasing subsequence of π. More generally, a theorem of Greene [26] says that $\lambda_1(\pi)+\cdots+\lambda_k(\pi)$ is equal to the length of the longest so-called k-increasing subsequence of π. Thus the difference of the lengths of the longest k-increasing subsequence and the longest $k-1$-increasing subsequence of $\pi \in S_n$ under the uniform probability measure is equal to λ_k of $\lambda \in Y_n$ under the Plancherel measure M_n^2 in the sense of joint distributions. Thus for example, (1.3) and (1.4) can be restated for the results on the longest increasing subsequence of a random permutation. On the other hand, the sum of the lengths of the first k columns of λ is equal to the length of the longest *k-decreasing* subsequence of corresponding π. But since the transpose λ^t have the same statistics as λ under M_n^2, the results (1.3) and (1.4) also hold for the longest decreasing subsequence of a random permutation.

The measure M_n^1 also has a combinatorial interpretation. Under the Robinson-Schensted correspondence, if π corresponds to (P,Q), then π^{-1} corresponds to (Q,P) (see e.g. Section 5.1.4 of [34]). Therefore, the involution $\pi = \pi^{-1} \in S_n$ is in bijection to the set of *single* standard Young tableaux whose shape is a partition of n. Consequently, the uniform probability measure on the set of involutions $\tilde{S}_n = \{\pi \in S_n : \pi = \pi^{-1}\}$ is push-forwarded to the 1-Plancherel measure M_n^1 on Y_n. The result (1.5) with $k=1$ implies that asymptotically the length of the longest increasing (also decreasing) subsequence of a random involution behaves statistically like the largest eigenvalue of a random GOE matrix.

Any involution $\pi \in \tilde{S}_n$ consists of 1 cycles and 2 cycles. It turns out that if we put certain constrains on the number of 1 cycles, or fixed points, of π, the limiting distribution is different. Introduce a new ensemble

$$S_{n,m} = \{\pi \in \tilde{S}_{2n+m} : |\{x : \pi(x) = x\}| = m\}. \tag{1.6}$$

The corresponding tableaux ensemble is obtained by again the Robinson-Schensted correspondence. Under the map between the involutions π and the standard Young tableaux with shape λ, the number of fixed points of π is equal to the number of odd parts of λ^t, the transpose of λ (see [34]). Equivalently, the number of fixed points of π is equal to $\lambda_1 - \lambda_2 + \lambda_3 - \cdots$. Thus the set $S_{n,m}$ is in bijection to the set

$$Y_{n,m} = \{\lambda = (\lambda_1, \lambda_2, \cdots) \in Y_{2n+m} : \sum_j (-1)^{j-1}\lambda_j\}, \tag{1.7}$$

and the uniform probability measure on $S_{n,m}$ is push-forwarded to the measure

$$\frac{d_\lambda}{\sum_{\mu \in Y_{n,m}} d_\mu}, \qquad \lambda \in Y_{n,m}. \tag{1.8}$$

Note that for $\lambda \in Y_{n,m}$, the statistics of rows and columns are now different. We denote by $L_{n,m}^{O,(k)}$ and $L_{n,m}^{S,(k)}$ the random variables given by the lengths of the k^{th} row and k^{th} column of a random $\lambda \in Y_{n,m}$ under the measure (1.8), respectively. We also denote by $L_{n,m}^O = L_{n,m}^{O,(k)}$ and $L_{n,m}^S = L^{S,(k)}$ the length of the longest increasing and decreasing subsequences of a random $\pi \in S_{n,m}$ under the uniform probability measure.

In this paper, we obtain the convergence in distribution and the convergence of moments of $L_{n,m}^{O,(k)}$ and $L_{n,m}^{S,(k)}$ for $k = 1,2$ as $n, m \to \infty$ in a certain rate. Set

$$\alpha = \frac{m}{\sqrt{2n}}. \tag{1.9}$$

It turns out that the limiting distribution of L_n^O differs depending on α. The importance of the number of fixed points can be seen from the following point selecting picture. Consider a unit square $[0,1] \times [0,1]$ in the plane and denote by $\delta = \{(x,x) : 0 \le x \le 1\}$, the diagonal. Suppose we select n points at random in the lower triangle $0 \le x < y \le 1$ and take the mirror image of the points about the diagonal δ. And we also select m points at random on the diagonal δ. Hence there is a total of $2n+m$ points. As illustrated in Figure ..., one such choice of points gives rise a permutation π satisfying $\pi^2 = 1$ with m fixed points, i.e., $\pi \in S_{n,m}$. The length $L_n^O(\pi)$ of the longest increasing subsequence of π is then equal to the 'length' of the longest (piecewise linear) up/right path in the square from $(0,0)$ to $(1,1)$. Here the 'length' of the up/right path is defined by the number of points on the path. It is easy to see that the length of the longest up/right path in the above point selecting process has the same distribution as L_n^O. Then Note that the longest increasing subsequence is longer than the number of fixed points m : $L_n^O \geq \alpha\sqrt{2n}$. When α is large, there are many fixed points and it is likely that . (Theorem 3.1) :

On the other hand, in [36, 49, 50], the limiting shape of a typical Young diagram under Plancherel measure was obtained. In [33], it was shown that this limiting shape can be viewed as Wigner's semicircle law. A conceptual background for this connection between Y_n with the Plancherel measure and GUE matrices was provided by Okounkov [39] using two different points of view on topological surfaces, namely gluing polygons and ramified coverings.

In another direction, there are results on the limiting distribution of the length of the longest increasing subsequences of various type of permutations. Signed permutations were considered in [46], colored permutations in [10], a certain statistical model related to generalized permutations in [31], and random words in [45] and [30]. The limiting distribution for the first three cases is expressed in terms of the Tracy-Widom distribution $F_2(x)$, while random word is related to trace-free GUE matrix. There have been more recent work on random words in [28], [35], [44]. By way of comparison with the results of this paper, we mention the result in [46, 10] that the distribution function of the length of the longest increasing subsequence of a random signed permutation converges to $F_2(x)^2$ (while that of a random permutation converges to $F_2(x)$).

We refer the readers to [1, 38] for a survey and history of the longest increasing subsequence problem in random permutations, and to [37] for general reference on random matrix theory (see also the recent book [12]).

In this paper, we consider the case when $\beta = 1$. One of the main results is that the lengths of the first and the second rows of a random Young diagram under M_n^1, behave statistically like the largest and the second largest eigenvalues of a random GOE matrix in the limit $n \to \infty$. As in the permutation case, the length of the k^{th} row of a random Young diagram under M_n^1 is believed to correspond to the k^{th} largest eigenvalue of a random GOE matrix.

From the combinatorial point of view, the subject of this paper is involutions. We introduce the ensemble of involutions :

$$\tilde{S}_n = \{\pi \in S_n : \pi = \pi^{-1}\}. \tag{1.10}$$

The Robinson-Schensted correspondence establishes a bijection from this set to the set of standard Young tableaux of size n, $\pi \leftrightarrow Q(\pi)$ (see e.g. Section 5.1.4 of [34]), and hence the probability measure M_n^1 on Y_n is just the push forward of the uniform measure on $\tilde{S}_n$. We define $\tilde{L}_n^{(k)}(\pi)$ to be the length of the k^{th} row of $Q(\pi)$ for $k \geq 1$, and we call it simply *the length of the* k^{th} *row of* π. For brevity, in the case $k = 1$ we write $\tilde{L}_n(\pi)$ for the length of the first row, or equivalently the length of the longest increasing subsequence of π.

We also consider the ensemble of signed involutions of $2n$ letters, $\tilde{S}_n^u$, consisting of the bijections π from $\{-n, \cdots, -2, -1, 1, 2, \cdots, n\}$ onto itself satisfying $\pi = \pi^{-1}$ and $\pi(x) = -\pi(-x)$. From the tableaux point of view, this set is in one-to-one correspondence with the set of self-dual Young tableaux, or domino tableaux (see [7]). Similarly we define $\tilde{L}_n^{u,(k)}(\pi)$ to be the length of the k^{th} row of $\pi \in \tilde{S}_n^u$ and denote by $\tilde{L}_n^u(\pi)$ the length of the longest increasing subsequence of π. We equip this ensemble with the uniform distribution and consider the limiting distribution of $\tilde{L}_n^{u,(k)}$ as $n \to \infty$. The asymptotic results in the cases $k = 1, 2$ for both ensembles are stated in Theorems 3.4 and 3.6 in Section 3. Here we note that the lengths of the k^{th} row and k^{th} column of π have the same statistics.

In cyclic notation for permutations, an involution has only 1 cycles and 2 cycles. It turns out that the asymptotics differ depending on the number of 1 cycles, i.e. fixed points. We introduce additional ensembles :

$$S_{n,m} = \{\pi \in \tilde{S}_{2n+m} : \ |\{x : \pi(x) = x\}| = m\}, \tag{1.11}$$

$$S^u_{n,m_+,m_-} = \{\pi \in \tilde{S}^u_{2n+m_++m_-} : \ |\{x : \pi(x) = x\}| = 2m_+, \ |\{x : \pi(x) = -x\}| = 2m_-\}. \tag{1.12}$$

As in [7], if $\pi(x) = -x$, we say that the point x is negated. In contrast to $\tilde{S}_n$, if we impose constraints on the number of fixed points, the statistics of the rows are not identical to the statistics of the columns. We define $L^O_{n,m}(\pi)$ and $L^S_{n,m}(\pi)$ to be the length of the longest *increasing* and *decreasing* subsequences of $\pi \in S_{n,m}$, respectively. On the other hand, in S^u_{n,m_+,m_-}, we define $L^u_{n,m_+,m_-}(\pi)$ to be the length of the longest increasing subsequence of $\pi \in S^u_{n,m_+,m_-}$. We can give similar definitions for the k^{th} rows in each case. Again equip each ensemble with the uniform probability distribution, and we consider the limiting distributions as $n \to \infty$.

For convenience of future reference, we summarize the above definitions.

Definition 1. Let S_n be the symmetric group of n letters and let S^u_n be the set of the bijections from $\{-n, \cdots, -2, -1, 1, 2, \cdots, n\}$ onto itself satisfying $\pi(x) = -\pi(-x)$. We define

$$\begin{aligned}
\tilde{S}_n &= \{\pi \in S_n : \pi = \pi^{-1}\}, && (1.13)\\
\tilde{S}^u_n &= \{\pi \in S^u_n : \pi = \pi^{-1}\}, && (1.14)\\
S_{n,m} &= \{\pi \in \tilde{S}_{2n+m} : \ |\{x : \pi(x) = x\}| = m\}, && (1.15)\\
S^u_{n,m_+,m_-} &= \{\pi \in \tilde{S}^u_{2n+m_++m_-} : \ |\{x : \pi(x) = x\}| = 2m_+, \ |\{x : \pi(x) = -x\}| = 2m_-\}, && (1.16)\\
\tilde{L}^{(k)}_n(\pi) &= \text{the length of the } k^{th} \text{ row of } \pi \in \tilde{S}_n, \qquad \tilde{L}_n = \tilde{L}^{(1)}_n, && (1.17)\\
\tilde{L}^{u,(k)}_n(\pi) &= \text{the length of the } k^{th} \text{ row of } \pi \in \tilde{S}^u_n, \qquad \tilde{L}^u_n = \tilde{L}^{u,(1)}_n, && (1.18)\\
L^{O,(k)}_{n,m}(\pi) &= \text{the length of the } k^{th} \text{ row of } \pi \in S^O_{n,m}, \qquad L^O_{n,m} = L^{O,(1)}_{n,m}, && (1.19)\\
L^{S,(k)}_{n,m}(\pi) &= \text{the length of the } k^{th} \text{ column of } \pi \in S^S_{n,m}, \qquad L^S_{n,m} = L^{S,(1)}_{n,m}, && (1.20)\\
L^{u,(k)}_{n,m_+,m_-}(\pi) &= \text{the length of the } k^{th} \text{ row of } \pi \in S^u_{n,m_+,m_-}, \qquad L^u_{n,m_+,m_-} = L^{u,(1)}_{n,m_+,m_-}. && (1.21)
\end{aligned}$$

Main results are the followings. The asymptotic results for $L^O_{n,m}$ depend on the ratio $\alpha = m/\sqrt{2n}$. After suitable centering and scaling (see Theorem 3.1), the limiting distribution function is $F_4(x)$ when $\alpha < 1$ and $F_1(x)$ when $\alpha = 1$, where F_1 and F_4 are the Tracy-Widom distribution functions for GOE and GSE matrices (again see Section 2 for definition). This result suggests that there is a certain transition between GSE and GOE. In order to investigate this transition in detail, we introduce a new scaling : $m = \sqrt{2n} - 2w(2n)^{1/3}$, $w \in \mathbb{R}$. Then for fixed w, as $n \to \infty$, the distribution function converges to $F^O(x; w)$ which we will define in Section 2. This new class of distributions is expressed in terms of the solution of the Riemann-Hilbert problem for the Painlevé II equation. There is a similar phenomena, now between $F_2(x)$ and $F_1(x)^2$ for L^u_{n,m_+,m_-}, but there is no transition for $L^S_{n,m}$ (see Section 3). We also obtain the results for $L^{S,(2)}_{n,m}$ and $L^{u,(2)}_{n,m_+,m_-}$. Precise statement of the results are given in Section 3.

The result in this paper was announced in [8]. After we have completed this paper, there have been two unexpected applications. One is random vicious walker models [24, 8, 3], and the other is polynuclear growth model [43, 42, 9]. There are bijections of the above models into one of various ensembles considered in this paper. Therefore one can obtain asymptotic statistics using the results of this paper.

The proofs use the Poissonization and de-Poissonization scheme of [32, 4]. We define the Poisson generating function for $L^O_{n,m}$ by

$$Q^O_l(\lambda_1, \lambda_2) := e^{-\lambda_1-\lambda_2} \sum_{n_1,n_2 \geq 0} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{n_1!n_2!} \Pr\big(L^O_{n_2,n_1} \leq l\big). \tag{1.22}$$

A generalization of the de-Poissonization lemma due to Johansson [32] yields that $\Pr\big(L^O_{n_2,n_1} \leq l\big) \sim Q^O_l(n_1, n_2)$ as $n_1, n_2 \to \infty$; see Section 6 for precise statement. Thus if we have the asymptotics for the generating function, the asymptotics of the coefficients will follow. We also define similar Poisson generating functions for $L^S_{n,m}$ and L^u_{n,m_+,m_-}. The point of the scheme is that the generating functions have determinantal forms of Toeplitz or Hankel type, which in turn, using general theory, can be re-expressed in terms of quantities related to the monic polynomial $\pi_k(z;t) = z^k + \cdots$ on the unit circle which is orthogonal with respect to the weight $e^{t\cos\theta}d\theta/(2\pi)$. We note that only one class of orthogonal polynomials π_k is enough for all the cases of O, S, u. This orthogonal polynomial is precisely the same orthogonal polynomial used in [4] to analyze random permutation problem. Therefore as in [4], we can use the Deift-Zhou method for Riemann-Hilbert problems (RHP's) to obtain the asymptotics. However there are two main differences. The first one is that in [4], we need only the π_k, but now we need $\pi_k(-\alpha; t)$ for *all* $\alpha \geq 0$. Secondly, when we apply Deift-Zhou method, the asymptotic results are in terms of a so-called g-function. We need considerable further analysis to simply the results expressed in terms of g-function to our problem. Therefore the main part of the proofs in this paper is the RHP analysis and the estimation of the g-function. Once we have results for the constrained involutions, the general involution case of $\tilde{L}_n$ can be obtained by noting $\tilde{S}_n = \cup_{2k+m=n} S_{k,m}$. Similarly $\tilde{L}^u_n$ can be analyzed from the results on the constrained signed involutions.

This paper is organized as follows. Section 2 defines the Tracy-Widom distribution functions as well as new classes of distribution functions which will be used to state the main results. The main results are stated in Section 3. Determinantal formulae and orthogonal polynomial expressions for the Poisson generating functions are taken from [7] and summarized in Section 4. In Section 5, we state the main estimations on the relevant quantities of orthogonal polynomials which is the key to the proofs. The de-Poissonization lemmas are stated in Section 6. Proofs of the main theorems are given in Section 7 for involutions under fixed points constraints (Theorems 3.1, 3.2, 3.3 and 3.5), and in Section 8 for general involutions (Theorems 3.4 and 3.6), respectively. The case when $\alpha > 1$ is considered in Section 9 (see Remark to Theorem 3.3). Finally the RHP analysis and g-function analysis is given in Section 10, which proves the propositions in Section 5.

Notational remarks. The ensemble $\tilde{S}_n$ in the present paper is identical to $\tilde{S}^O_n$ in [7]. In [7], $\tilde{S}^S_n$ is introduced to denote the ensemble of neginvolutions and the longest increasing subsequence of $\pi \in \tilde{S}^S_n$ is considered. But there is a bijection between $\tilde{S}^S_n$ and $\tilde{S}^O_n$ and the longest increasing subsequence of $\pi \in \tilde{S}^S_n$ corresponds to the longest decreasing subsequence of the image of π. In the present paper, we choose the view point of considering both the increasing and decreasing subsequences of involutions of the same ensemble rather than considering only the increasing subsequences of involutions of the different ensembles.

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2 Limiting distribution functions

Let $u(x)$ be the solution of the Painlevé II (PII) equation,

$$u_{xx} = 2u^3 + xu, \tag{2.1}$$

with the boundary condition

$$u(x) \sim -Ai(x) \quad \text{as} \quad x \to +\infty, \tag{2.2}$$

where Ai is the Airy function. The proof of the (global) existence and the uniqueness of the solution was first established in [27] : the asymptotics as $x \to -\infty$ are (see e.g. [27, 19])

$$u(x) = -Ai(x) + O\bigg(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\bigg), \qquad \text{as } x \to +\infty, \tag{2.3}$$

$$u(x) = -\sqrt{\frac{-x}{2}}\bigg(1 + O\big(\frac{1}{x^2}\big)\bigg), \qquad \text{as } x \to -\infty. \tag{2.4}$$

Recall that $Ai(x) \sim \frac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}}$ as $x \to +\infty$. Define

$$v(x) := \int_\infty^x (u(s))^2 ds, \tag{2.5}$$

so that $v'(x) = (u(x))^2$.

We introduce the Tracy-Widom (TW) distributions. (Note that $q := -u$, which Tracy and Widom used in their papers, solves the same differential equation with the boundary condition $q(x) \sim +Ai(x)$ as $x \to \infty$.)

Definition 2 (TW distribution functions). Set

$$F(x) := \exp\bigg(\frac{1}{2}\int_x^\infty v(s)ds\bigg) = \exp\bigg(-\frac{1}{2}\int_x^\infty (s-x)(u(s))^2 ds\bigg), \tag{2.6}$$

$$E(x) := \exp\bigg(\frac{1}{2}\int_x^\infty u(s)ds\bigg), \tag{2.7}$$

and set

$$F_2(x) := F(x)^2 = \exp\bigg(-\int_x^\infty (s-x)(u(s))^2 ds\bigg), \tag{2.8}$$

$$F_1(x) := F(x)E(x) = \big(F_2(x)\big)^{1/2} e^{\frac{1}{2}\int_x^\infty u(s)ds}, \tag{2.9}$$

$$F_4(x) := F(x)\big[E(x)^{-1} + E(x)\big]/2 = \big(F_2(x)\big)^{1/2}\bigg[e^{-\frac{1}{2}\int_x^\infty u(s)ds} + e^{\frac{1}{2}\int_x^\infty u(s)ds}\bigg]/2. \tag{2.10}$$

In [47] and [48], Tracy and Widom proved that under proper centering and scaling, the distribution of the largest eigenvalue of a random GUE/GOE/GSE matrix converges to $F_2(x)$ / $F_1(x)$ / $F_4(x)$ as the size of the matrix becomes large. We note that from the asymptotics (2.3) and (2.4), for some positive constant c,

$$F(x) = 1 + O\big(e^{-cx^{3/2}}\big) \qquad \text{as } x \to +\infty, \tag{2.11}$$

$$E(x) = 1 + O\big(e^{-cx^{3/2}}\big) \qquad \text{as } x \to +\infty, \tag{2.12}$$

$$F(x) = O\big(e^{-c|x|^3}\big) \qquad \text{as } x \to -\infty, \tag{2.13}$$

$$E(x) = O\big(e^{-c|x|^{3/2}}\big) \qquad \text{as } x \to -\infty. \tag{2.14}$$

Hence in particular, $\lim_{x\to+\infty} F_\beta(x) = 1$ and $\lim_{x\to-\infty} F_\beta(x) = 0$, $\beta = 1, 2, 4$. Monotonicity of $F_\beta(x)$ follows from the fact that $F_\beta(x)$ is the limit of a sequence of distribution functions. Therefore $F_\beta(x)$ is indeed a distribution function.

Definition 3. Define χ_{GOE}, χ_{GUE} and χ_{GSE} to be random variables whose distribution functions are given by $F_1(x)$, $F_2(x)$ and $F_4(x)$, respectively. Define χ_{GOE^2} to be a random variable with the distribution function $F_1(x)^2$.

As indicated in Introduction, we need new classes of distribution functions to describe the phase transitions from χ_{GSE} to χ_{GOE} and from χ_{GUE} to χ_{GOE^2}. First we consider the Riemann-Hilbert problem (RHP) for the Painlevé II equation [20, 29]. Let Γ be the real line $\mathbb{R}$, oriented from $+\infty$ to $-\infty$. Let $m(\cdot\,;x)$ be the solution of the following RHP :

$$\begin{cases} m(z;x) \qquad \text{is analytic in } z \in \mathbb{C}\setminus\Gamma, \\ m_+(z;x) = m_-(z;x)\begin{pmatrix} 1 & -e^{-2i(\frac{4}{3}z^3+xz)} \\ e^{2i(\frac{4}{3}z^3+xz)} & 0 \end{pmatrix} \quad \text{for } z\in\Gamma, \\ m(z;x) = I + O(\frac{1}{z}) \qquad \text{as } z\to\infty. \end{cases} \tag{2.15}$$

Here $m_+(z;x)$ (resp,. m_-) is the limit of $m(z';x)$ as $z' \to z$ from the left (resp., right) of the contour Γ : $m_\pm(z;x) = \lim_{\epsilon\downarrow 0} m(z \mp i\epsilon; x)$. Relation (2.15) corresponds to the RHP for the PII equation with the special monodromy data $p = -q = 1, r = 0$ (see [20, 29], also [22, 19]). In particular if the solution is expanded at $z = \infty$,

$$m(z;x) = I + \frac{m_1(x)}{z} + O\big(\frac{1}{z^2}\big), \qquad \text{as } z\to\infty, \tag{2.16}$$

we have

$$2i(m_1(x))_{12} = -2i(m_1(x))_{21} = u(x), \tag{2.17}$$

$$2i(m_1(x))_{22} = -2i(m_1(x))_{11} = v(x), \tag{2.18}$$

where $u(x)$ and $v(x)$ are defined in (2.1)-(2.5).

Definition 4. Let $m(z;x)$ be the solution of RHP (2.15) and denote by $m_{jk}(z;x)$ the (jk)-entry of $m(z;x)$. For $w > 0$, define

$$F^O(x;w) := F(x)\Big\{\big[m_{22}(-iw;x) - m_{12}(-iw;x)\big]E(x)^{-1} + \big[m_{22}(-iw;x) + m_{12}(-iw;x)\big]E(x)\Big\}/2, \tag{2.19}$$

and for $w < 0$, define

$$\begin{aligned} F^O(x;w) &:= e^{\frac{8}{3}w^3 - 2xw} F(x) \\ &\quad \times \Big\{\big[-m_{21}(-iw;x) + m_{11}(-iw;x)\big]E(x)^{-1} - \big[m_{21}(-iw;x) + m_{11}(-iw;x)\big]E(x)\Big\}/2. \end{aligned} \tag{2.20}$$

Also define

$$F^u(x;w) := m_{22}(-iw;x)F_2(x), \qquad w > 0, \tag{2.21}$$

$$F^u(x;w) := -e^{\frac{8}{3}w^3 - 2xw} m_{21}(-iw;x)F_2(x), \qquad w < 0. \tag{2.22}$$

First $F^O(x;w)$ and $F^u(x;w)$ are real from Lemma 2.1 (i) below. Note that $F^O(x;w)$ and $F^u(x;w)$ are continuous at $w = 0$ since at $z = 0$, the jump condition of the RHP (2.15) implies $(m_{12})_+(0;x) = -(m_{11})_-(0;x)$ and $(m_{22})_+(0;x) = -(m_{21})_-(0;x)$. In fact, $F^O(x;w)$ and $F^u(x;w)$ are entire in $w \in \mathbb{C}$ from the RHP (2.15).

From (2.11)-(2.14) and (2.24)-(2.27) below, we see that

$$\lim_{x\to+\infty} F^O(x;w), F^u(x;w) = 1, \qquad \lim_{x\to-\infty} F^O(x;w), F^u(x;w) = 0 \tag{2.23}$$

for any fixed $w \in \mathbb{R}$. Also Theorem 3.2 below shows that $F^O(x;w)$ and $F^u(x;w)$ are limits of distribution functions, implying that they are monotone in x. Therefore, $F^O(x;w)$ and $F^u(x;w)$ are indeed distribution functions for each $w \in \mathbb{R}$.

Definition 5. Define χ^O_w and χ^u_w to be random variables with distribution functions $F^O(x;w)$ and $F^u(x;w)$, respectively.

We close this section summarizing some properties of $m(-iw;x)$ in the following lemma. In particular the lemma implies that $F^O(x;w)$ interpolates between $F_4(x)$ and $F_1(x)$, and $F^u(x;w)$ interpolates between $F_2(x)$ and $F_1(x)^2$ (see Corollary 2.2).

Lemma 2.1. *Let* $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, *and set* $[a,b] = ab - ba$.

(i). For real w, $m(-iw;x)$ *is real.*

(ii). For fixed $w \in \mathbb{R}$, *we have*

$$m(-iw;x) = \big(I + (e^{-cx^{3/2}})\big)\begin{pmatrix} 1 & -e^{\frac{8}{3}w^3 - 2xw} \\ 0 & 1 \end{pmatrix}, \qquad w > 0,\ x \to +\infty, \tag{2.24}$$

$$m(-iw;x) = \big(I + (e^{-cx^{3/2}})\big)\begin{pmatrix} 1 & 0 \\ -e^{-\frac{8}{3}w^3 + 2xw} & 1 \end{pmatrix}, \qquad w < 0,\ x \to +\infty, \tag{2.25}$$

$$m(-iw;x) \sim \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^{(-\frac{4}{3}w^3 + xw)\sigma_3} e^{(\frac{\sqrt{2}}{3}(-x)^{3/2} + \sqrt{2}w^2(-x)^{1/2})\sigma_3}, \quad w > 0,\ x \to -\infty, \tag{2.26}$$

$$m(-iw;x) \sim \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{(-\frac{4}{3}w^3 + xw)\sigma_3} e^{(-\frac{\sqrt{2}}{3}(-x)^{3/2} - \sqrt{2}w^2(-x)^{1/2})\sigma_3}, \quad w < 0,\ x \to -\infty. \tag{2.27}$$

(iii). For any x, we have

$$\lim_{w\to 0^+} m(-iw;x) = \lim_{w\to 0^-} \sigma_1 m(-iw;x)\sigma_1 = \begin{pmatrix} \frac{1}{2}\big(E(x)^2+E(x)^{-2}\big) & -E(x)^2 \\ \frac{1}{2}\big(-E(x)^2+E(x)^{-2}\big) & E(x)^2 \end{pmatrix}. \tag{2.28}$$

(iv). For fixed $w \in \mathbb{R}\setminus\{0\}$, $m(-iw;x)$ solves the differential equation

$$\frac{d}{dx}m = w[m,\sigma_3] + u(x)\sigma_1 m, \tag{2.29}$$

where $u(x)$ is the solution of the PII equation (2.1), (2.2).

(v). For fixed x, $m(-iw;x)$ solves

$$\frac{\partial}{\partial w}m = (-4w^2+x)[m,\sigma_3] - 4wu(x)\sigma_1 m - 2\begin{pmatrix} u^2 & -u' \\ u' & -u^2 \end{pmatrix} m. \tag{2.30}$$

(vi). For any x, we have

$$m(z;x) = \sigma_1 m(-z;x)\sigma_1. \tag{2.31}$$

Corollary 2.2. *We have*

$$F^O(x;0) = F_1(x), \tag{2.32}$$

$$\lim_{w\to\infty} F^O(x;w) = F_4(x), \tag{2.33}$$

$$\lim_{w\to-\infty} F^O(x;w) = 0, \tag{2.34}$$

$$F^u(x;0) = F_1(x)^2, \tag{2.35}$$

$$\lim_{w\to\infty} F^u(x;w) = F_2(x), \tag{2.36}$$

$$\lim_{w\to-\infty} F^u(x;w) = 0. \tag{2.37}$$

Proof. The values at $w = 0$ follow from (2.28). For $w \to \pm\infty$, note that from the RHP (2.15), we have $\lim_{z\to\infty} m(z;x) = I$. □

Proof of Lemma 2.1. Let $v(z) = v(z;x)$ denote the jump matrix of the RHP (2.15). Since $\overline{v(-z)} = v(z)$ for $z \in \mathbb{R}$, $M(z) := \overline{m(-\bar{z};x)}$ also solves the same RHP. By the uniqueness of the solution of the RHP (2.15), we have

$$\overline{m(-\bar{z};x)} = m(z;x), \qquad z \in \mathbb{C}\setminus\mathbb{R}. \tag{2.38}$$

Thus, $m(-iw;x)$ is real for $w \in \mathbb{R}$ proving (i).

By a symmetry of the jump matrix, $\sigma_1 v(-z)^{-1}\sigma_1 = v(z)$, as above, we obtain

$$\sigma_1 m(-z;x)\sigma_1 = m(z;x), \tag{2.39}$$

which is (vi).

The asymptotics results (ii) as $x \to \pm\infty$ follow from the calculations in Section 6, pp.329-333 of [19].

For the proof of (iv), define a new matrix function

$$f(z;x) := m(z;x)e^{-i\theta\sigma_3}, \qquad \theta := \frac{4}{3}z^3 + xz. \tag{2.40}$$

Then $f(\cdot\,;x)$ satisfies the jump condition $f_+(z;x) = f_-(z;x)\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ for $z \in \mathbb{R}$, and $f(z;x)e^{i\theta\sigma_3} \to I$ as $z \to \infty$. Since the jump matrix for $f(z;x)$ is independent of x, $f'(z;x)$, the derivative with respect to x, satisfies $f'_+(z;x) = f'_-(z;x)\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $f'e^{i\theta\sigma_3} + i\theta' f\sigma_3 e^{i\theta\sigma_3} \to 0$ as $z \to \infty$. Hence $f'f^{-1}$ has no jump across $\mathbb{R}$, and satisfies $f'f^{-1} + i\theta' f\sigma_3 f^{-1} \to 0$ as $z \to \infty$. If we write $m(z;x) = I + \frac{m_1(x)}{z} + O(1/z^2)$ as $z \to \infty$, we have $i\theta' f\sigma_3 f^{-1} = iz\sigma_3 + i[m_1,\sigma_3] + O(z^{-1})$ as $z \to \infty$. Thus $f'f^{-1}$ is entire and as $z \to \infty$, $f'f^{-1} \sim -iz\sigma_3 - i[m_1,\sigma_3]$. Therefore by Liouville's theorem, we obtain

$$f'(z;x)(f(z;x))^{-1} = -iz\sigma_3 - i[m_1,\sigma_3]. \tag{2.41}$$

Recalling that $u(x) = 2i(m_1(x))_{12} = -2i(m_1(x))_{21}$ in (2.17), we have $[m_1,\sigma_3] = iu(x)\sigma_1$. Changing f to m from (2.40), (2.41) is

$$\frac{d}{dx}m(z;x) = iz[m(z;x),\sigma_3] + u(x)\sigma_1 m(z;x). \tag{2.42}$$

This is (2.29) when $z = -iw$.

The proof of (v) is very similar to that of (iv), and the detail is left to the reader. We only note that in the derivation of (v), we need the identity

$$\frac{d}{dx}m_1 = i[m_2,\sigma_3] - i[m_1,\sigma_3]m_1, \tag{2.43}$$

which can be obtained from (2.42) by setting $m(z;x) = I + \frac{m_1(x)}{z} + \frac{m_2(x)}{z^2} + O(1/z^3)$ as $z \to \infty$.

Finally we prove (iii). Note that $\lim_{w\to 0^\pm} m(-iw;x) = m_\pm(0;x)$. From the jump condition at $z = 0$, we have

$$m_+(0;x) = m_-(0;x)\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.44}$$

Letting $z \to 0$, $Imz > 0$, in (vi), we have $\sigma_1 m_+(0;x)\sigma_1 = m_-(0;x)$, which together with (2.44) implies that $m_+(0;x) = \sigma_1 m_+(0;x)\sigma_1\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus we have

$$m_+(0;x) = \begin{pmatrix} a(x) & b(x) \\ a(x)+b(x) & -b(x) \end{pmatrix} \tag{2.45}$$

for some $a(x)$, $b(x)$. Also the condition $\det v(z) = 1$ for all $z \in \mathbb{R}$ implies that $\det m(z;x) = 1$ for all $z \in \mathbb{C}\setminus\mathbb{R}$, and hence we have

$$b^2 + 2ab + 1 = 0. \tag{2.46}$$

Now letting $z \to 0$, $Imz < 0$ in (2.42), we obtain

$$m'_+(0;x)(m_+(0;x))^{-1} = \begin{pmatrix} 0 & u(x) \\ u(x) & 0 \end{pmatrix}. \tag{2.47}$$

Thus from (2.45) and (2.46), $b'/b = -u$, which has the solution

$$b(x) = b(y)e^{-\int_y^x u(s)ds}. \tag{2.48}$$

From (2.24) with $w = 0^+$, we have

$$b(x) = (m_{12})_+(0;x) \to -1, \qquad \text{as } x \to +\infty. \tag{2.49}$$

Therefore, $b(x) = -e^{\int_x^\infty u(s)ds}$, which is $-E(x)^2$ from (2.7). Now (2.46) gives $a(x) = \frac{1}{2}(E(x)^2 + E(x)^{-2})$, proving (2.28). □

3 Statement of results

3.1 Involutions with constraint on the number of fixed points

Recall the ensembles $\tilde{S}_{n,m}$, $\tilde{S}^u_{n,m_+,m_-}$ of (signed) involutions with constraint on the number of fixed (and negated) points. We scale the random variables,

$$\chi^O_{n,m} := \frac{L^O_{n,m} - 2\sqrt{2n+m}}{(2n+m)^{1/6}}, \tag{3.1}$$

$$\chi^S_{n,m} := \frac{L^S_{n,m} - 2\sqrt{2n+m}}{(2n+m)^{1/6}}, \tag{3.2}$$

$$\chi^u_{n,m_+,m_-} := \frac{L^u_{n,m_+,m_-} - 2\sqrt{4n+2m_+ + 2m_-}}{2^{2/3}(4n+2m_+ + 2m_-)^{1/6}}. \tag{3.3}$$

The first result concerns the case when α and β are fixed.

Theorem 3.1. *For fixed α and β, we have the following asymptotic results.*

$$\lim_{n\to\infty} \Pr\big(\chi^O_{n,[\sqrt{2n}\alpha]} \le x\big) = F_4(x), \qquad 0 \le \alpha < 1, \tag{3.4}$$

$$\lim_{n\to\infty} \Pr\big(\chi^O_{n,[\sqrt{2n}]} \le x\big) = F_1(x), \tag{3.5}$$

$$\lim_{n\to\infty} \Pr\big(\chi^O_{n,[\sqrt{2n}\alpha]} \le x\big) = 0, \qquad \alpha > 1, \tag{3.6}$$

$$\lim_{n\to\infty} \Pr\big(\chi^S_{n,[\sqrt{2n}\beta]} \le x\big) = F_1(x), \qquad 0 \le \beta, \tag{3.7}$$

$$\lim_{n\to\infty} \Pr\big(\chi^u_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]} \le x\big) = F_2(x), \qquad 0 \le \alpha < 1, \beta \ge 0, \tag{3.8}$$

$$\lim_{n\to\infty} \Pr\big(\chi^u_{n,[\sqrt{n}],[\sqrt{n}\beta]} \le x\big) = F_1(x)^2, \qquad \beta \ge 0, \tag{3.9}$$

$$\lim_{n\to\infty} \Pr\big(\chi^u_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]} \le x\big) = 0, \qquad \alpha > 1, \beta \ge 0. \tag{3.10}$$

As indicated earlier in the introduction, as $\alpha \to 1$ at a certain rate, we see smooth transitions.

Theorem 3.2. *For fixed $w \in \mathbb{R}$ and $\beta \ge 0$, we have*

$$\lim_{n\to\infty} \Pr\big(\chi^O_{n,m} \le x\big) = F^O(x;w), \qquad m = [\sqrt{2n} - 2w(2n)^{1/3}], \tag{3.11}$$

$$\lim_{n\to\infty} \Pr\big(\chi^u_{n,m_+,m_-} \le x\big) = F^u(x;w), \qquad m_+ = [\sqrt{n} - 2wn^{1/3}], \; m_- = [\sqrt{n}\beta]. \tag{3.12}$$

From Corollary 2.2, this result is consistent with Theorem 3.1. We also have convergence of moments.

Theorem 3.3. *For any $p = 1, 2, 3, \cdots$, the followings hold. For fixed α and β,*

$$\lim_{n\to\infty} \mathrm{E}\big((\chi^O_{n,[\sqrt{2n}\alpha]})^p\big) = \mathrm{E}\big((\chi_{GSE})^p\big), \qquad 0 \le \alpha < 1, \tag{3.13}$$

$$\lim_{n\to\infty} \mathrm{E}\big((\chi^O_{n,[\sqrt{2n}]})^p\big) = \mathrm{E}\big((\chi_{GOE})^p\big), \tag{3.14}$$

$$\lim_{n\to\infty} \mathrm{E}\big((\chi^S_{n,[\sqrt{2n}\beta]})^p\big) = \mathrm{E}\big((\chi_{GOE})^p\big), \qquad 0 \le \beta, \tag{3.15}$$

$$\lim_{n\to\infty} \mathrm{E}\big((\chi^u_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]})^p\big) = \mathrm{E}\big((\chi_{GUE})^p\big), \qquad 0 \le \alpha < 1, \beta \ge 0, \tag{3.16}$$

$$\lim_{n\to\infty} \mathrm{E}\big((\chi^u_{n,[\sqrt{n}],[\sqrt{n}\beta]})^p\big) = \mathrm{E}\big((\chi_{GOE^2})^p\big), \qquad \beta \ge 0. \tag{3.17}$$

Also for fixed $w \in \mathbb{R}$ and $\beta \ge 0$,

$$\lim_{n\to\infty} \mathrm{E}\big((\chi^O_{n,m})^p\big) = \mathrm{E}\big((\chi^O_w)^p\big), \qquad m = [\sqrt{2n} - 2w(2n)^{1/3}], \tag{3.18}$$

$$\lim_{n\to\infty} \mathrm{E}\big((\chi^u_{n,m_+,m_-})^p\big) = \mathrm{E}\big((\chi^u_w)^p\big), \qquad m_+ = [\sqrt{n} - 2wn^{1/3}],\ m_- = [\sqrt{n}\beta]. \tag{3.19}$$

Remark. Theorem 3.1 shows that when $\alpha > 1$ is fixed, we have not used the correct scaling. When properly scaled, the resulting limiting distribution is Gaussian. See Section 9 for the statement and the proof.

Suppose we select N points at random in a square, say $R = [0,1] \times [0,1]$. With probability 1, a configuration of random points induces a random permutation : order x-coordinate, then use the relative order of the y-coordinate. We define the length of a up/right path from the corner $(0,0)$ to the corner $(1,1)$ to be the number of points in the path. Under this setting, the statistics of the length of the longest up/right path in random n point selection process is equal to that of the length of the longest increasing subsequence of a random permutation in S_n. This process can be regarded as a version of (directed site) percolation in 2-dimension from (lower left) corner to (upper right) corner . The result of [4] is that in the limit, the expected length of the longest up/right path is of order $\sqrt{n}$ and the fluctuation is of order $n^{1/6}$ with the distribution function $F_2(x)$.

On the other hand, one can consider the corner to line percolation. It is believed by the people working in the field that the corner to corner percolation and the corner to line percolation should have the same expected value and fluctuation order. For the special case of $(0,0)$ to the anti-diagonal line $\{(t, 1-t) : 0 \le t \le 1\}$, if we take the mirror image of the n selected points in the low-left triangle to the anti-diagonal line, we are in the case of S. The length of the longest up/right path from $(0,0)$ to the anti-diagonal line is the half of $L^S_{n,0}$. The above result shows that the expected value and the fluctuation has the same order as the corner to corner percolation, but the limiting distribution functions is difference (F_1 in this case) ! One can also state the similar results for Poisson process (see 7.3) and also certain directed site percolation considered by [31] (see [8]). This observation came from discussions of one of us (J.B.) with Charles Newman to whom we are specially grateful.

3.2 General involutions

Now we consider general involutions and signed involutions without constraints on the number of fixed or negated points.

Theorem 3.4. *For any fixed $x \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} \Pr\biggl(\tilde{\chi}_n := \frac{\tilde{L}_n - 2\sqrt{n}}{n^{1/6}} \le x \biggr) = F_1(x), \tag{3.20}$$

$$\lim_{n\to\infty} \Pr\biggl(\tilde{\chi}^u_n := \frac{\tilde{L}^u_n - 2\sqrt{2n}}{2^{2/3}(2n)^{1/6}} \le x \biggr) = F_1(x)^2. \tag{3.21}$$

Also for any $p = 1, 2, 3, \cdots$,

$$\lim_{n\to\infty} \mathrm{E}\bigl((\tilde{\chi}_n)^p\bigr) = \mathrm{E}\bigl((\chi_{GOE})^p\bigr), \tag{3.22}$$

$$\lim_{n\to\infty} \mathrm{E}\bigl((\tilde{\chi}^u_n)^p\bigr) = \mathrm{E}\bigl((\chi_{GOE^2})^p\bigr). \tag{3.23}$$

As indicated in the introduction, from the Robinson-Schensted correspondence, this result proves that the first row of a random Young diagram under the 1-Plancherel measure M^1_n behaves statistically like the the largest eigenvalue of a random GOE matrix as $n \to \infty$.

3.3 Second rows

For the second row, we scale in the same way as in (3.1)-(3.3), and denote the scaled random variables by $\chi^{O,(2)}_{n,m}$, $\chi^{S,(2)}_{n,m}$ and $\chi^{u,(2)}_{n,m_+,m_-}$, respectively.

Theorem 3.5. *Let $\alpha, \beta \ge 0$ be fixed. Then*

$$\lim_{n\to\infty} \Pr\bigl(\chi^{O,(2)}_{n,\sqrt{2n}\alpha}\bigr) = F_4(x), \tag{3.24}$$

$$\lim_{n\to\infty} \Pr\bigl(\chi^{S,(2)}_{n,[\sqrt{2n}\beta]}\bigr) = F_4(x), \tag{3.25}$$

$$\lim_{n\to\infty} \Pr\bigl(\chi^{u,(2)}_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]}\bigr) = F_2(x), \tag{3.26}$$

and for any $p = 1, 2, 3, \cdots$,

$$\lim_{n\to\infty} \mathrm{E}\bigl((\chi^{O,(2)}_{n,\sqrt{2n}\alpha})^p\bigr) = \mathrm{E}\bigl((\chi_{GSE})^p\bigr), \tag{3.27}$$

$$\lim_{n\to\infty} \mathrm{E}\bigl((\chi^{S,(2)}_{n,[\sqrt{2n}\beta]})^p\bigr) = \mathrm{E}\bigl((\chi_{GSE})^p\bigr), \tag{3.28}$$

$$\lim_{n\to\infty} \mathrm{E}\bigl((\chi^{u,(2)}_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]})^p\bigr) = \mathrm{E}\bigl((\chi_{GUE})^p\bigr). \tag{3.29}$$

As in the first row, these results yield the following theorem.

Theorem 3.6. *For any fixed $x \in \mathbb{R}$, we have*

$$\lim_{n\to\infty} \Pr\biggl(\tilde{\chi}^{(2)}_n := \frac{\tilde{L}^{(2)}_n - 2\sqrt{n}}{n^{1/6}} \le x \biggr) = F_4(x), \tag{3.30}$$

$$\lim_{n\to\infty} \Pr\biggl(\tilde{\chi}^{u,(2)}_n := \frac{\tilde{L}^{u,(2)}_n - 2\sqrt{2n}}{2^{2/3}(2n)^{1/6}} \le x \biggr) = F_2(x). \tag{3.31}$$

Also for any $p = 1, 2, 3, \cdots$,

$$\lim_{n\to\infty} \mathrm{E}\bigl((\tilde{\chi}^{(2)}_n)^p\bigr) = \mathrm{E}\bigl((\chi_{GSE})^p\bigr), \tag{3.32}$$

$$\lim_{n\to\infty} \mathrm{E}\bigl((\tilde{\chi}^{u,(2)}_n)^p\bigr) = \mathrm{E}\bigl((\chi_{GUE})^p\bigr). \tag{3.33}$$

We conclude this section with some remarks on GOE and GSE. If the conjecture given in the introduction that the k^{th} row of a random involution corresponds to the k^{th} largest eigenvalue of a random GOE matrix were true, the result (3.30) suggests that the second largest eigenvalues of a random GOE matrix and those of a random GSE matrix have the same limiting distribution (GSE matrix has double eigenvalues, hence the largest and the second largest eigenvalues are the same). Indeed according to Theorem 10.6.1 of [37], the statistical properties of N alternate angles of the eigenvalues of a random $2N \times 2N$ matrix taken from the circular orthogonal ensemble (COE) are identical to those of the N angles of the eigenvalues of a random $N \times N$ matrix taken from the circular symplectic ensemble (CSE). Also for Laguerre ensembles, we proved that the joint distributions of the second, fourth, sixth, etc. largest eigenvalues of the Laguerre orthogonal ensemble (LOE) and the Laguerre symplectic ensemble (LSE) are identical (see Remark 1 to Corollary 7.6 of [7]). Recently the same results for GOE and GSE cases are proved to be true in [23] and [25]. In those papers, one can find many other related results including that the second eigenvalue of superimposition of two random GOE matrices has the same distribution with the first eigenvalue of a random GUE matrix.

4 Poisson generating functions

We review the results from [7] which we will need in the sequel. As in [7], throughout the paper, the notation G indicates an arbitrary member of the set $\{O, S, u\}$.

Definition 6. We define the Poisson generating functions for the probabilities introduced above as follows.

$$Q_l^O(\lambda_1, \lambda_2) := e^{-\lambda_1-\lambda_2} \sum_{n_1,n_2\geq 0} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{n_1!n_2!} \Pr\big(L^O_{n_2,n_1} \leq l\big), \tag{4.1}$$

$$Q_l^S(\lambda_1, \lambda_2) := e^{-\lambda_1-\lambda_2} \sum_{n_1,n_2\geq 0} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{n_1!n_2!} \Pr\big(L^S_{n_2,n_1} \leq l\big), \tag{4.2}$$

$$Q_l^u(\lambda_1, \lambda_2, \lambda_3) := e^{-\lambda_1-\lambda_2-\lambda_3} \sum_{n_1,n_2,n_3\geq 0} \frac{\lambda_1^{n_1}\lambda_2^{n_2}\lambda_3^{n_3}}{n_1!n_2!n_3!} \Pr\big(L^u_{n_3,n_1,n_2} \leq l\big). \tag{4.3}$$

As in [7], let $\tilde{f}^O_{nml}$ (resp., $\tilde{f}^S_{nml}$) be the number of involutions on n numbers with m fixed points with no increasing (resp., decreasing) subsequence of length greater than l. Thus $\tilde{f}^O_{(2n_2+n_1)n_1 l} = \Pr(L^O_{n_2,n_1} \leq l) \cdot |S_{n_2,n_1}|$, etc. Also let $\tilde{f}^u_{nm_+m_-l}$ be the number of signed involutions on $2n$ letters with $2m_+$ fixed points and $2m_-$ negated points with no increasing subsequence of length greater than l. We also define

$$P_l^O(t; \alpha) := e^{-\alpha t - t^2/2} \sum_{0\leq n} \frac{t^n}{n!} \sum_{0\leq m} \alpha^m \tilde{f}^O_{nml}, \tag{4.4}$$

$$P_l^S(t; \beta) := e^{-\beta t - t^2/2} \sum_{0\leq n} \frac{t^n}{n!} \sum_{0\leq m} \beta^m \tilde{f}^S_{nml}, \tag{4.5}$$

$$P_l^u(t; \alpha, \beta) := e^{-\alpha t - \beta t - t^2} \sum_{0\leq n} \frac{t^n}{n!} \sum_{0\leq m_+,m_-} \alpha^{m_+}\beta^{m_-} \tilde{f}^u_{nm_+m_-l}. \tag{4.6}$$

One can easily check that

$$\begin{aligned} P_l^O(t;\alpha) &= Q_l^O(\alpha t, t^2/2), && (4.7)\\ P_l^S(t;\beta) &= Q_l^S(\beta t, t^2/2), && (4.8)\\ P_l^u(t;\alpha,\beta) &= Q_l^u(\alpha t, \beta t, t^2). && (4.9) \end{aligned}$$

It turns out that the P-formulae in (4.4)-(4.6) are useful for algebraic manipulations (see [7]), while the Q-formulae (4.1)-(4.3) are well adapted to asymptotic analysis.

The following results from [7] provide the stating point for our analysis in this paper. For a nonnegative integer k, define $\pi_k(z;t) = z^k + \cdots$ to be the monic orthogonal polynomial of degree k with respect to the weight function $\exp(t(z+1/z))dz/(2\pi i)$ on the unit circle. Let the norm of $\pi_k(z;t)$ be $N_k(t)$:

$$\int_\Sigma \pi_n(z;t)\overline{\pi_m(z;t)} e^{t(z+1/z)} \frac{dz}{2\pi i z} = N_n(t)\delta_{nm}. \tag{4.10}$$

We note that all the coefficients of $\pi_n(z;t)$ are real. Define

$$\pi_n^*(z;t) := z^n \pi_n(z^{-1};t). \tag{4.11}$$

Then

Theorem 4.1. *For $\alpha, \beta \geq 0$,*

$$P_{2l}^O(t;\alpha) = e^{-\alpha t - t^2/2}\frac{1}{2}\Big\{\big[\pi_{2l-1}^*(-\alpha;t) - \alpha\pi_{2l-1}(-\alpha;t)\big]D_l^{--}(t) + \big[\pi_{2l-1}^*(-\alpha;t) + \alpha\pi_{2l-1}(-\alpha;t)\big]D_{l-1}^{++}(t)\Big\}, \tag{4.12}$$

$$P_{2l+1}^O(t;\alpha) = e^{-\alpha t - t^2/2}\frac{1}{2}\Big\{\big[\pi_{2l}^*(-\alpha;t) + \alpha\pi_{2l}(-\alpha;t)\big]e^t D_l^{+-}(t) + \big[\pi_{2l}^*(-\alpha;t) - \alpha\pi_{2l}(-\alpha;t)\big]e^{-t} D_l^{-+}(t)\Big\}, \tag{4.13}$$

$$P_{2l+1}^S(t;\beta) = e^{-t^2/2} D_l^{++}(t), \tag{4.14}$$

$$P_{2l+1}^u(t;\alpha,\beta) = e^{-\alpha t - t^2}\pi_l^*(-\alpha;t) D_l(t), \tag{4.15}$$

where for any real $t \geq 0$, $D_l(t)$ and $D_l^{\pm\pm}(t)$ are certain Toeplitz and Hankel determinants which in turn can be written as

$$\begin{aligned} e^{-t^2} D_l(t) &= \prod_{j\geq l} N_j(t)^{-1}, && (4.16)\\ e^{-t^2/2} D_l^{--}(t) &= \prod_{j\geq l} N_{2j+2}(t)^{-1}(1+\pi_{2j+2}(0;t)), && (4.17)\\ e^{-t^2/2} D_l^{++}(t) &= \prod_{j\geq l} N_{2j+2}(t)^{-1}(1-\pi_{2j+2}(0;t)), && (4.18)\\ e^{-t^2/2+t} D_l^{+-}(t) &= \prod_{j\geq l} N_{2j+1}(t)^{-1}(1-\pi_{2j+1}(0;t)), && (4.19)\\ e^{-t^2/2-t} D_l^{-+}(t) &= \prod_{j\geq l} N_{2j+1}(t)^{-1}(1+\pi_{2j+1}(0;t)). && (4.20) \end{aligned}$$

As a special case:

Theorem 4.2. *For $l \geq 0$, we have the following formulae:*

$$P^O_{2l+2}(t;0) = e^{-t^2/2}\left[D^{--}_{l+1}(t) + D^{++}_{l}(t)\right]/2, \tag{4.21}$$
$$P^O_{2l+1}(t;0) = e^{-t^2/2}\left[e^{t}D^{+-}_{l}(t) + e^{-t}D^{-+}_{l}(t)\right]/2, \tag{4.22}$$
$$P^S_{2l}(t;0) = e^{-t^2/2}D^{++}_{l}(t), \tag{4.23}$$
$$P^O_{2l}(t;1) = P^S_{2l}(t;1) = e^{-t-t^2/2}D^{-+}_{l}(t), \tag{4.24}$$
$$P^O_{2l+1}(t;1) = P^S_{2l+1}(t;1) = e^{-t^2/2}D^{++}_{l}(t), \tag{4.25}$$
$$P^u_{2l}(t;0,0) = e^{-t^2}D_l(t), \tag{4.26}$$
$$P^u_{4l+1}(t;1,\beta) = e^{-t-t^2}D^{++}_{l}(t)D^{-+}_{l}(t), \tag{4.27}$$
$$P^u_{4l+3}(t;1,\beta) = e^{-t-t^2}D^{++}_{l}(t)D^{-+}_{l+1}(t). \tag{4.28}$$

Also $P^O_0(t;0) = e^{-t^2/2}D^{--}_0(t) = e^{-t^2/2}$.

For the second row, we define the Poisson generating functions in a similar manner. Then we have

Theorem 4.3. *For $\alpha, \beta \geq 0$,*

$$P^{O,(2)}_{l}(t;\alpha) = P^O_l(t;0), \tag{4.29}$$
$$P^{S,(2)}_{2l+1}(t;\beta) = P^O_{2l}(t;0), \tag{4.30}$$
$$P^{u,(2)}_{2l+1}(t;\alpha,\beta) = P^u_{2l}(t;0,0). \tag{4.31}$$

5 Asymptotics of orthogonal polynomials

Let Σ be the unit circle in the complex plane, $\Sigma = \{z \in \mathbb{C} : |z| = 1\}$, oriented counterclockwise. Set

$$\psi(z;t) := e^{t(z+z^{-1})}. \tag{5.1}$$

As before, let $\pi_n(z;t) = z^n + \cdots$ be the n-th monic orthogonal polynomial with respect to the measure $\psi(z;t)dz/(2\pi i z)$ on the unit circle.

Define the 2×2 matrix-valued function of z in $\mathbb{C}\setminus\Sigma$ by

$$Y(z;k;t) := \begin{pmatrix} \pi_k(z;t) & \int_\Sigma \frac{\pi_k(s;t)}{s-z}\frac{\psi(s;t)ds}{2\pi i s^k} \\ -N_{k-1}(t)^{-1}\pi^*_{k-1}(z;t) & -N_{k-1}(t)^{-1}\int_\Sigma \frac{\pi^*_{k-1}(s;t)}{s-z}\frac{\psi(s;t)ds}{2\pi i s^k} \end{pmatrix}, \qquad k \geq 1. \tag{5.2}$$

Then $Y(\cdot\,;k;t)$ solves the following Riemann-Hilbert problem (RHP) (see Lemma 4.1 in [4]) :

$$\begin{cases} Y(z;k;t) \quad \text{is analytic in} \quad z \in \mathbb{C}\setminus\Sigma, \\ Y_+(z;k;t) = Y_-(z;k;t)\begin{pmatrix} 1 & \frac{1}{z^k}\psi(z;t) \\ 0 & 1 \end{pmatrix}, \quad \text{on} \quad z \in \Sigma, \\ Y(z;k;t)\begin{pmatrix} z^{-k} & 0 \\ 0 & z^k \end{pmatrix} = I + O(\frac{1}{z}) \quad \text{as} \quad z \to \infty. \end{cases} \tag{5.3}$$

Here the notation $Y_+(z;k)$ (resp., Y_-) denotes the limiting value $\lim_{z'\to z} Y(z';k)$ with $|z'|<1$ (resp., $|z'|>1$). Note that k and t play the role of external parameters in the above RHP ; in particular, the term $O(\frac{1}{z})$ does not imply a uniform bound in k and t. One can easily show that the solution of the above RHP is unique, hence (5.2) is the unique solution of the above RHP. This RHP formulation of orthogonal polynomials on the unit circle is an adaptation of a result of Fokas, Its and Kitaev in [21] where they considered orthogonal polynomials on the real line.

From Theorems 4.1 and 4.2, in order to obtain the asymptotics of the Poisson generating functions, we need the asymptotics, as $k,t\to\infty$, of

$$N_{k-1}(t), \qquad \pi_k(z;t), \qquad \pi^*_{k-1}(z;t). \tag{5.4}$$

Since

$$\begin{aligned} N_{k-1}(t)^{-1} &= -Y_{21}(0;k;t), && (5.5)\\ \pi_k(z;t) &= Y_{11}(z;k;t), && (5.6)\\ \pi^*_k(z;t) &= z^k Y_{11}(z^{-1};k;t) = Y_{21}(z;k+1;t)(Y_{21}(0;k+1;t))^{-1}, && (5.7) \end{aligned}$$

these asymptotics can be obtained by a steepest descent type analysis of the above RHP. In [4], the authors obtained the asymptotics of $N_k(t)$ ($=\kappa_k^{-2}$ in [4]) as $k,t\to\infty$ by this steepest descent type analysis for the 21 component of $Y(z;k;t)$ at $z=0$. Precisely the same analysis applies to the other components of $Y(z;k;t)$ of general z. In this section, we summarize the results needed in this paper (cf. Lemma 5.1 and Lemma 6.3 in [4]). Proofs are provided in Section 10 below.

Proposition 5.1. *There exists $M_0>0$ such that as $k,t\to\infty$, we have the following asymptotic results for $N_{k-1}(t)$ and $\pi_k(0;t)$ in each different region of k and t.*

(i). If $0\le 2t\le ak$ with $0<a<1$, then

$$\big|N_{k-1}(t)^{-1}-1\big|, \quad \big|\pi_k(0;t)\big| \le Ce^{-ck}, \tag{5.8}$$

for some constants C,c.

(ii). If $ak\le 2t\le k-Mk^{1/3}$ with some $M>M_0$ and $0<a<1$, then

$$\big|N_{k-1}(t)^{-1}-1\big|, \quad \big|\pi_k(0;t)\big| \le \frac{C}{k^{1/3}}e^{-\frac{2\sqrt{2}}{3}k\left(1-\frac{2t}{k}\right)^{3/2}}, \tag{5.9}$$

for some constant C.

(iii). If $2t=k-\frac{x}{2^{1/3}}k^{1/3}$ with $-M\le x\le M$ for some constant $M>0$, then

$$\left|N_{k-1}(t)^{-1}-1-\frac{2^{1/3}}{k^{1/3}}v(x)\right|, \quad \left|\pi_k(0;t)+(-1)^k\frac{2^{1/3}}{k^{1/3}}u(x)\right| \le \frac{C}{k^{2/3}}, \tag{5.10}$$

for some constant C, where $u(x)$ and $v(x)$ are defined in (2.1) *and* (2.5) *respectively.*

(iv). If $k + Mk^{1/3} \le 2t \le ak$ with some $M > M_0$ and $a > 1$, then

$$\left| \sqrt{\frac{2t}{k}} e^{k(2t/k - \log(2t/k) - 1)} N_{k-1}(t)^{-1} - 1 \right|, \quad \left| (-1)^k \sqrt{\frac{2t}{2t-k}} \pi_k(0;t) - 1 \right| \le \frac{C}{2t-k}, \tag{5.11}$$

for some constant C.

(v). If $ak \le 2t \le bk$ with $1 < a < b$, then

$$\left| \sqrt{\frac{2t}{k}} e^{k(2t/k - \log(2t/k) - 1)} N_{k-1}(t)^{-1} - 1 \right|, \quad \left| (-1)^k \sqrt{\frac{2t}{2t-k}} \pi_k(0;t) - 1 \right| \le \frac{C}{k}, \tag{5.12}$$

for some constant C.

Proof. Proofs for the case (i), (ii), (iv), (v) are given in Subsubsection 10.1.1, 10.1.2, 10.2.2, 10.2.1, respectively. The case (iii) with $x \ge 0$ is given in 10.1.3, and (iii) with $x < 0$ is given in 10.2.3. ☐

Remark. For the other entries of Y, one can check directly from (5.2) that $Y_{12}(0;k;t) = N_k(t)$, $Y_{22}(0;k;t) = \pi_k(0;t)$.

Proposition 5.2. *For $2t = k - x(k/2)^{1/3}$, x fixed, and for each fixed $z \in \mathbb{C} \setminus \Sigma$, we have*

$$\lim_{k\to\infty} e^{tz}\pi_k(z;t) = 0, \qquad \lim_{k\to\infty} e^{tz}\pi_k^*(z;t) = 1, \qquad |z| < 1, \tag{5.13}$$

$$\lim_{k\to\infty} z^{-k} e^{tz^{-1}}\pi_k(z;t) = 1, \quad \lim_{k\to\infty} z^{-k} e^{tz^{-1}}\pi_k^*(z;t) = 0, \qquad |z| > 1. \tag{5.14}$$

Proof. See Subsubsection 10.1.3, especially the paragraph between (10.71) and (10.72), for the case $x \ge 0$, and Subsubsection 10.2.3, especially (10.129) - (10.137), for the case $x < 0$. ☐

Corollary 5.3. *For $2t = k - x(k/2)^{1/3}$, x fixed, we have for $\alpha > 1$ fixed,*

$$\lim_{k\to\infty} e^{-\alpha t}\pi_k(-\alpha;t) = 0, \qquad \lim_{k\to\infty} e^{-\alpha t}\pi_k^*(-\alpha;t) = 0. \tag{5.15}$$

Proof. Write

$$\begin{aligned} e^{-\alpha t}\pi_k(-\alpha;t) &= \alpha^k e^{t(-\alpha+\alpha^{-1})}\alpha^{-k} e^{-t\alpha^{-1}}\pi_k(-\alpha;t) \\ &= e^{kf(\alpha;2t/k)}\alpha^{-k} e^{-t\alpha^{-1}}\pi_k(-\alpha;t), \end{aligned} \tag{5.16}$$

where

$$f(\alpha;\gamma) = \frac{\gamma}{2}(-\alpha + \alpha^{-1}) + \log\alpha. \tag{5.17}$$

The function $f(\alpha;1)$ is strictly decreasing for $\alpha > 0$, and $f(1;1) = 0$. Hence $f(\alpha;1) < 0$ for $\alpha > 1$. Note that

$$f(\alpha;\gamma) = f(\alpha;1) + \frac{\gamma - 1}{2}(-\alpha + \alpha^{-1}). \tag{5.18}$$

When $x \le 0$, $2t/k \ge 1$, hence $f(\alpha;2t/k) \le f(\alpha;1)$. On the other hand, when $x > 0$, since $2t/k - 1 = -x/(2^{1/3}k^{2/3})$, if $k > \left(\frac{2^{2/3}x(-\alpha+\alpha^{-1})}{f(\alpha;1)}\right)^{3/2}$, $f(\alpha;2t/k) \le \frac{1}{2}f(\alpha;1)$. Therefore (5.14) implies that

$$\left|e^{-\alpha t}\pi_k(-\alpha;t)\right| \le e^{\frac{k}{2}f(\alpha;1)}\left|(-\alpha)^{-k}e^{-t\alpha^{-1}}\pi_k(-\alpha;t)\right| \to 0, \tag{5.19}$$

as $k \to \infty$. Similar calculations give the desired result for π_k^*. ☐

Recall from Section 2 that $m(\cdot\,,x)$ solves the RHP for the PII equation (2.15).

Proposition 5.4. *Let $2t = k - x(k/2)^{1/3}$ where x is a fixed number. Set*

$$\alpha = 1 - \frac{2^{4/3}w}{k^{1/3}}. \tag{5.20}$$

We have for $w > 0$ fixed,

$$\lim_{k\to\infty} (-1)^k e^{-t\alpha}\pi_k(-\alpha;t) = -m_{12}(-iw;x), \tag{5.21}$$
$$\lim_{k\to\infty} e^{-t\alpha}\pi_k^*(-\alpha;t) = m_{22}(-iw;x), \tag{5.22}$$

and for $w < 0$ fixed,

$$\lim_{k\to\infty} (-\alpha)^{-k} e^{-t\alpha^{-1}}\pi_k(-\alpha;t) = m_{11}(-iw;x), \tag{5.23}$$
$$\lim_{k\to\infty} \alpha^{-k} e^{-t\alpha^{-1}}\pi_k^*(-\alpha;t) = -m_{21}(-iw;x). \tag{5.24}$$

Proof. See Subsubsection 10.1.3 (10.72) - (10.82) for the case $x \geq 0$, and Subsubsection 10.2.3 (10.138) - (10.148) for the case $x < 0$. □

Corollary 5.5. *For $w < 0$, under the same condition as the above proposition, we have*

$$\lim_{k\to\infty} (-1)^{-k} e^{-t\alpha}\pi_k(-\alpha;t) = m_{11}(-iw;x)e^{(8/3)w^3-2xw}, \tag{5.25}$$
$$\lim_{k\to\infty} e^{-t\alpha}\pi_k^*(-\alpha;t) = -m_{21}(-iw;x)e^{(8/3)w^3-2xw}. \tag{5.26}$$

Proof. Note that under the stated conditions, we have

$$e^{t(\alpha^{-1}-\alpha)}\alpha^k = e^{(8/3)w^3-2xw+O(k^{-1/3})}. \tag{5.27}$$

□

Remark. As noted in 2, it follows from the RHP (2.15) that $(m_{12})_+(0;x) = -(m_{11})_-(0;x)$ and $(m_{22})_+(0;x) = -(m_{21})_-(0;x)$, and hence by the above corollary, the limits in (5.21)-(5.22) are in fact continuous across $w = 0$.

Proposition 5.6. *Define x through the relation*

$$\frac{2t}{k} = 1 - \frac{x}{2^{1/3}k^{2/3}}. \tag{5.28}$$

Then there exists M_0 such that the following holds for any fixed $M > M_0$, Let $0 < b < 1$ and $0 < L < 2^{-3/2}\sqrt{M}$ be fixed. Then as $k, t \to \infty$, we have for $x \geq M$,

$$\left|e^{tz}\pi_k(z;t)\right| \leq Ce^{-c|x|^{3/2}}, \qquad |z| \leq b, \tag{5.29}$$
$$\left|e^{tz^{-1}}z^{-k}\pi_k(z;t) - 1\right| \leq Ce^{-c|x|^{3/2}}, \qquad |z| \geq b^{-1}, \tag{5.30}$$
$$\left|e^{-t\alpha}\pi_k(-\alpha;t)\right| \leq Ce^{c|x|}, \qquad \alpha = 1 - 2^{4/3}k^{-1/3}w, \quad -L \leq w \leq L, \tag{5.31}$$
$$\left|e^{-t\alpha^{-1}}(-\alpha)^{-k}\pi_k(-\alpha;t) - 1\right| \leq Ce^{-c|x|^{3/2}}, \qquad \alpha = 1 - 2^{4/3}k^{-1/3}w, \quad -L \leq w \leq L, \tag{5.32}$$

and for $x \le -M$,

$$\left|e^{-t\alpha}\pi_k(-\alpha;t)\right| \le C, \qquad 0 < \alpha \le 1, \tag{5.33}$$

$$\left|e^{-t\alpha^{-1}}(-\alpha)^{-k}\pi_k(-\alpha;t)\right| \le C, \qquad \alpha \ge 1. \tag{5.34}$$

Proof. See Subsubsection 10.1.1 and 10.1.2 for the case $x \ge M$, and Subsubsection 10.2.1 and 10.2.2 for the case $x \le -M$. □

Corollary 5.7. *Let $\alpha = 1 - 2^{4/3}wk^{-1/3}$ and $-L \le w \le L$ for fixed $L > 0$. Under the assumption of the above proposition, for $x \le -M$, we have*

$$\left|e^{-t\alpha}\pi_k(-\alpha;t)\right| \quad \le \quad Ce^{c|x|}, \tag{5.35}$$

$$\left|e^{-t\alpha^{-1}}(-\alpha)^{-k}\pi_k(-\alpha;t)\right| \quad \le \quad Ce^{c|x|}. \tag{5.36}$$

for some positive constants C, c.

Proof. We have

$$|e^{-t(\alpha-\alpha^{-1})}\alpha^k| \quad = \quad e^{2xw+\frac{8}{3}w^3+O(k^{-1})}, \tag{5.37}$$

$$|e^{t(\alpha-\alpha^{-1})}\alpha^{-k}| \quad = \quad e^{-2xw-\frac{8}{3}w^3+O(k^{-1})}. \tag{5.38}$$

Above proposition shows that (5.35) is true for $w \ge 0$. For $w < 0$, write

$$e^{-t\alpha}\pi_k(-\alpha;t) = \left[e^{-t(\alpha-\alpha^{-1})}(-\alpha)^k\right]\left[e^{-t\alpha^{-1}}(-\alpha)^{-k}\pi_k(-\alpha;t)\right]. \tag{5.39}$$

Now (5.35) follows from (5.34) and (5.37). The estimation (5.38) is proved similarly. □

Proposition 5.8. *Let $\alpha > 1$ be fixed. When*

$$\frac{t}{k} = \frac{\alpha}{\alpha^2+1} - \frac{\alpha(\alpha^2-1)^{1/2}}{(\alpha^2+1)^{3/2}} \cdot \frac{x}{\sqrt{k}}, \qquad x \text{ fixed}, \tag{5.40}$$

we have

$$\lim_{k\to\infty} e^{-\alpha t}(-\alpha)^k \pi_k(-\alpha^{-1};t) = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy. \tag{5.41}$$

Proof. See Subsection 10.3. □

6 De-Poissonization lemmas

In this section, we describe a series of Tauberian type de-Poissonization lemmas, which enable us to extract the asymptotics of the coefficient from the knowledge of the asymptotics of its generating function. Lemma 6.1 below is due to Johansson [32]. The following two lemmas are taken from Section 8 in [4]. Lemma 6.1 and 6.3 are enough for both convergence in distribution and the convergence of moments, but for convenience, we use Lemma 6.2 and 6.4 for the convergence of moments in the subsequent sections.

For a sequence $q = \{q_n\}_{n\geq 0}$, we define its Poisson generating function by

$$\phi(\lambda) = e^{-\lambda} \sum_{0\leq n} q_n \frac{\lambda^n}{n!}. \tag{6.1}$$

Lemma 6.1. *For any fixed real number $d > 0$, set*

$$\mu_n^{(d)} = n + (2\sqrt{d+1}+1)\sqrt{n\log n}, \tag{6.2}$$

$$\nu_n^{(d)} = n - (2\sqrt{d+1}+1)\sqrt{n\log n}. \tag{6.3}$$

Then there are constants C and n_0 such that for any sequence $q = \{q_n\}_{n\geq 0}$ satisfying (i) $q_n \geq q_{n+1}$ (ii) $0 \leq q_n \leq 1$, for all $n \geq 0$,

$$\phi(\mu_n^{(d)}) - Cn^{-d} \leq q_n \leq \phi(\nu_n^{(d)}) + Cn^{-d} \tag{6.4}$$

for all $n \geq n_0$.

Lemma 6.2. *For any fixed real number $d > 0$, there exist constants C and n_0 such that for any sequence $q = \{q_n\}_{n\geq 0}$ satisfying (i) and (ii) above,*

$$q_n \leq C\phi(n - d\sqrt{n}), \tag{6.5}$$

$$1 - q_n \leq C(1 - \phi(n + d\sqrt{n})) \tag{6.6}$$

for all $n \geq n_0$.

For multi-indexed sequences, there are similar results. For $q = \{q_{n_1,n_2}\}_{n_1,n_2\geq 0}$, define

$$\phi(\lambda_1, \lambda_2) = e^{-\lambda_1-\lambda_2} \sum_{n_1,n_2\geq 0} q_{n_1n_2} \frac{\lambda_1^{n_1}\lambda_2^{n_2}}{n_1!n_2!}. \tag{6.7}$$

From the above two lemmas, we easily obtain the following lemmas.

Lemma 6.3. *For any real number $d > 0$, define $\mu_n^{(d)}$ and $\nu_n^{(d)}$ as in Lemma 6.1. Then there exist constants C and n_0 such that for any $q = \{q_{n_1,n_2}\}_{n_1,n_2\geq 0}$ satisfying (i) $q_{n_1,n_2} \geq q_{n_1+1,n_2}$, $q_{n_1,n_2} \geq q_{n_1.n_2+1}$ (ii) $0 \leq q_{n_1,n_2} \leq 1$, for all $n_1, n_2 \geq 0$,*

$$\phi(\mu_{n_1}^{(d)}, \mu_{n_2}^{(d)}) - C(n_1^{-d} + n_2^{-d}) \leq q_{n_1n_2} \leq \phi(\nu_{n_1}^{(d)}, \nu_{n_2}^{(d)}) + C(n_1^{-d} + n_2^{-d}) \tag{6.8}$$

for all $n_1, n_2 \geq n_0$.

Similarly,

Lemma 6.4. *For any fixed real number $d > 0$, there exist constants C and n_0 such that for any $q = \{q_{n_1,n_2}\}_{n_1,n_2\geq 0}$ satisfying two condition in Lemma 6.3,*

$$q_{n_1n_2} \leq C\phi(n_1 - d\sqrt{n_1}, n_2 - d\sqrt{n_2}), \tag{6.9}$$

$$1 - q_{n_1n_2} \leq C(1 - \phi(n_1 - d\sqrt{n_1}, n_2 + d\sqrt{n_2})) \tag{6.10}$$

for $n_1, n_2 \geq n_0$.

Remark. Similar lemmas hold true for sequences of arbitrarily many number of indices.

7 Proofs of Theorems 3.1, 3.2, 3.3 and 3.5

From Proposition 5.1 above, by taking products (equivalently, sums for their logarithms), we obtain the following results. A detailed derivation is given in [4] for (7.1), and the other cases are similar.

Corollary 7.1. *Let $M > M_0$, where M_0 is given in Proposition 5.6. Then there exist positive constants C and c which are independent of M, and a positive constant $C(M)$ which may depend on M, such that the following results hold for large l.*

(i) Define x by $2t = l - x(l/2)^{1/3}$. For $-M < x < M$,

$$\Big|\sum_{j\ge l} \log N_j(t)^{-1} - 2\log F(x)\Big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}. \tag{7.1}$$

(ii) Define x by $t = l - (x/2)l^{1/3}$. For $-M < x < M$,

$$\Big|\sum_{j\ge l} \log N_{2j+2}(t)^{-1} - \log F(x)\Big|, \quad \Big|\sum_{j\ge l} \log N_{2j+1}(t)^{-1} - \log F(x)\Big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}, \tag{7.2}$$

$$\Big|\sum_{j\ge l} \log(1-\pi_{2j+2}(0;t)) - \log E(x)\Big|, \quad \Big|\sum_{j\ge l} \log(1+\pi_{2j+1}(0;t)) - \log E(x)\Big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}, \tag{7.3}$$

$$\Big|\sum_{j\ge l} \log(1+\pi_{2j+2}(0;t)) + \log E(x)\Big|, \quad \Big|\sum_{j\ge l} \log(1-\pi_{2j+1}(0;t)) + \log E(x)\Big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}. \tag{7.4}$$

These results yield the asymptotics of the determinants in Theorem 4.1.

Corollary 7.2. *There exits M_1 such that for $M > M_1$, there exist positive constants C and c which are independent of M, and a positive constant $C(M)$ which may depend on M such that the following results hold for large l.*

(i) Define x by $2t = l - x(l/2)^{1/3}$. For $-M < x < M$,

$$\big|e^{-t^2} D_l(t) - F(x)^2\big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}. \tag{7.5}$$

(ii) Define x by $t = l - (x/2)l^{1/3}$. For $-M < x < M$,

$$\big|e^{-t^2/2} D_l^{--}(t) - F(x)E(x)^{-1}\big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}, \tag{7.6}$$

$$\big|e^{-t^2/2} D_{l-1}^{++}(t) - F(x)E(x)\big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}, \tag{7.7}$$

$$\big|e^{-t^2/2+t} D_l^{+-}(t) - F(x)E(x)^{-1}\big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}, \tag{7.8}$$

$$\big|e^{-t^2/2-t} D_l^{-+}(t) - F(x)E(x)\big| \le \frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}}. \tag{7.9}$$

Proof. For C and c in the above corollary, take $M_1 > M_0$ such that $Ce^{-cM_1^{3/2}} \le \frac{1}{2}$. Once we fix $M > M_1$ and for l is large, $\frac{C(M)}{l^{1/3}} + Ce^{-cM^{3/2}} < 1$, hence by (7.1) above, $\big|\sum_{j\ge l} \log N_j(t)^{-1} - 2\log F(x)\big| \le 1$. Using

$|e^x - 1| \le (e-1)|x|$ for $|x| \le 1$,

$$\begin{aligned} \left|e^{-t^2} D_l(t) - F(x)^2\right| &= F(x)^2 \left|e^{\left(\sum_{j\ge l} \log N_j(t)^{-1} - 2\log F(x)\right)} - 1\right| \\ &\le (e-1)F(x)^2 \left|\sum_{j\ge l} \log N_j(t)^{-1} - 2\log F(x)\right|. \end{aligned} \tag{7.10}$$

But from (2.11) and (2.13), $F(x)$ is bounded for $x \in \mathbb{R}$. Hence using (7.1), we obtain the result for (i) with new constants C, c and $C(M)$. For (ii), we note that $F(x)E(x)$ and $F(x)E(x)^{-1}$ are bounded for $x \in \mathbb{R}$ from (2.11)-(2.14). □

From Proposition 5.2, Corollary 5.3 and Theorems 4.1, 4.2, this Corollary immediately yields the following asymptotics for Poisson generating functions.

Proposition 7.3. *Let $2t = l - x(l/2)^{1/3}$ where x is fixed. As $l \to \infty$, for each fixed α, β,*

$$P_l^O(t;\alpha) \to F_4(x), \qquad 0 \le \alpha < 1, \tag{7.11}$$
$$P_l^O(t;1) \to F_1(x), \tag{7.12}$$
$$P_l^O(t;\alpha) \to 0, \qquad \alpha > 1, \tag{7.13}$$
$$P_l^S(t;\beta) \to F_1(x), \qquad \beta \ge 0. \tag{7.14}$$

Let $4t = l - x(2l)^{1/3}$ where x is fixed. As $l \to \infty$, for each fixed α, β,

$$P_l^u(t;\alpha,\beta) \to F_2(x), \qquad 0 \le \alpha < 1,\ \beta \ge 0, \tag{7.15}$$
$$P_l^u(t;1,\beta) \to F_1(x)^2, \qquad \beta \ge 0, \tag{7.16}$$
$$P_l^u(t;\alpha,\beta) \to 0, \qquad \alpha > 1, \quad \beta \ge 0. \tag{7.17}$$

Similarly, using Proposition 5.4 and Corollary 5.5 and Theorem 4.1, we have :

Theorem 7.4. *Let $2t = l - x(l/2)^{1/3}$ where x is fixed. As $l \to \infty$, we have for any fixed $w \in \mathbb{R}$,*

$$P_l^O(t;\alpha) \to F^O(x;w), \qquad \alpha = 1 - \frac{2^{4/3}w}{l^{1/3}}. \tag{7.18}$$

Let $4t = l - x(2l)^{1/3}$ where x is fixed. As $l \to \infty$, we have for each fixed β and $w \in \mathbb{R}$,

$$P_l^u(t;\alpha,\beta) \to F^u(x;w), \qquad \alpha = 1 - \frac{2^{5/3}w}{l^{1/3}},\ \beta \ge 0. \tag{7.19}$$

Recall the relation between $Q_l^G(\lambda)$ and $P_l^G(t)$ in (4.7)-(4.9). We can now use the de-Poissonization Lemma 6.3 to obtain the asymptotic results of Theorem 3.1 and 3.2. In order to apply the de-Poissonization Lemma, we need the following monotonicity results.

Lemma 7.5 (Monotonicity). *For any l, $\Pr(L_{k,m}^O \le l)$, $\Pr(L_{k,m}^S \le l)$ and $\Pr(L_{k,m_+,m_-}^u \le l)$ are monotone decreasing in k, m, m_+ and m_-.*

Proof. We first consider $\Pr(L^O_{k,m} \le l)$. Let $f_{km} := \Pr(L^O_{k,m} \le l) \cdot |S_{k,m}|$ be the number of elements in $S_{k,m}$ with no increasing subsequence greater than l. Consider the map $h : S_{k,m-1} \times \{1, 2, \cdots, 2k+m\} \to S_{k,m}$ defined as follows : for $\pi \in S_{k,m-1}$ and $1 \le j \le 2k+m$, set $h((\pi, j))(x) = \pi(x)$ for $1 \le x < j-1$, $h((\pi, j))(j) = j$, and $h((\pi, j))(x) = \pi(x-1)$ for $j < x \le 2k+m$. Then it is easy to see that $h^{-1}(\sigma)$ consists of m elements, hence $(2k+m)|S_{k,m-1}| = m|S_{k,m}|$. Moreover if $\pi \in S_{k,m-1}$ has an increasing subsequence of length greater than l, then $h((\pi, j)$ has an increasing subsequence of length greater than l. Thus $(2k+m)f_{k(m-1)} \ge m f_{km}$. But since $|S_{k,m}| = \frac{(2k+m)!}{2^k k! m!}$, we obtain $\Pr(L^O_{k,m-1} \le l) \ge \Pr(L^O_{k,m} \le l)$.

Similar argument works for the other cases. Note that $|S^u_{k,m_+,m_-}| = \frac{(2k+m_+,m_-)!}{k! m_+! m_-!}$. □

Now we can apply Lemma (6.3) to obtain the asymptotics results in Theorems 3.1 and 3.2. The proofs are similar to that in Section 9 of [4].

Now we consider the convergence of moments. For this, we first obtain the following estimates which follow from Proposition 5.1 (i),(ii),(iv),(v) above. The proof is very similar to that of Lemma 7.1 (i),(ii),(iv),(v) of [4]. Compare the results with (2.11)-(2.14) noting Corollary 7.1.

Corollary 7.6. *Set*

$$t = l - \frac{x}{2} l^{1/3}. \tag{7.20}$$

There exits M_2 such that for a fixed $M > M_2$, there are positive constants $C = C(M)$ and $c = c(M)$ such that the following results hold.

(i). For $x \ge M$,

$$1 - \prod_{j \ge l} N_{2j+2}(t)^{-1}, \quad 1 - \prod_{j \ge l} N_{2j+1}(t)^{-1} \quad \le \quad C e^{-c|x|^{3/2}}, \tag{7.21}$$

$$1 - \prod_{j \ge l} (1 - \pi_{2j+2}(0; t)), \quad 1 - \prod_{j \ge l} (1 + \pi_{2j+1}(0; t)) \quad \le \quad C e^{-c|x|^{3/2}}, \tag{7.22}$$

$$1 - \prod_{j \ge l} (1 + \pi_{2j+2}(0; t)), \quad 1 - \prod_{j \ge l} (1 - \pi_{2j+1}(0; t)) \quad \le \quad C e^{-c|x|^{3/2}}, \tag{7.23}$$

(ii). For $x \le -M$,

$$\prod_{j \ge l} N_{2j+2}(t)^{-1}, \quad \prod_{j \ge l} N_{2j+1}(t)^{-1} \quad \le \quad C e^{-c|x|^3}, \tag{7.24}$$

$$\prod_{j \ge l} (1 - \pi_{2j+2}(0; t)), \quad \prod_{j \ge l} (1 + \pi_{2j+1}(0; t)) \quad \le \quad C e^{-c|x|^{3/2}}, \tag{7.25}$$

$$\prod_{j \ge l} (1 + \pi_{2j+2}(0; t)), \quad \prod_{j \ge l} (1 - \pi_{2j+1}(0; t)) \quad \le \quad C e^{+c|x|^{3/2}}. \tag{7.26}$$

Remark. From the definitions of $P^G_l(t)$ and the equalities of Theorem 4.1, we know that all the infinite products above are between 0 and 1.

Now as in Section 9 of [4], using Lemma 6.4 and Theorems 3.1, 3.2, this implies Theorem 3.3.

Theorem 3.5 follows from Theorem 4.3.

8 Proofs of Theorems 3.4 and 3.6

In this section, we prove Theorem 3.4 by summing up the asymptotic results of Theorems 3.1, 3.2 and 3.3. Theorem 3.6 can be proved in a similar way from Theorem 4.3.

Proof of (3.20)**.** Note that we have a disjoint union

$$\tilde{S}_n = \bigcup_{2k+m=n} S_{k,m}. \tag{8.1}$$

Set $p^S_{kml} = \Pr(L^S_{km} \le l)$, the probability that the length of the longest decreasing subsequence of $\pi \in S_{k,m}$ is less than or equal to l. As the first row and the first column of π in $\tilde{S}_n$ have the same statistics, we have

$$\Pr(\tilde{L}_n \le l) = \frac{1}{|\tilde{S}_n|} \sum_{2k+m=n} p^S_{kml} |S_{k,m}|. \tag{8.2}$$

Note that

$$|S_{k,m}| = \binom{2k+m}{2k} \frac{(2k)!}{2^k k!}. \tag{8.3}$$

As $n \to \infty$ (see pp.66-67 of [34]), we have

$$|\tilde{S}_n| = \sum_{2k+m=n} |S_{k,m}| = \frac{1}{\sqrt{2}} n^{n/2} e^{-n/2+\sqrt{n}-1/4} \Big(1 + \frac{7}{24} n^{-1/2} + O\big(n^{-3/4}\big)\Big), \tag{8.4}$$

and the main contribution to the sum comes from $\sqrt{n} - n^{\epsilon+1/4} \le m \le \sqrt{n} + n^{\epsilon+1/4}$.

Fix $0 < a < 1 < b$. We split the sum in (8.2) into two pieces :

$$\Pr(\tilde{L}_n \le l) = \frac{1}{|\tilde{S}_n|} \Big[\sum_{(*)} p^S_{kml} |S_{k,m}| + \sum_{(**)} p^S_{kml} |S_{k,m}| \Big], \tag{8.5}$$

where $(*)$ is the region $a\sqrt{n} \le m \le b\sqrt{n}$ and $(**)$ is the rest.

For $2k + m = n$, the quantity $|S_{k,m}| = \binom{n}{2k} \frac{(2k)!}{2^k k!}$ is unimodal for $0 \le k \le n$, and the maximum is achieved when $k \sim (n - \sqrt{n})/2$ as $n \to \infty$. Hence

$$\sum_{(**)} p^S_{kml} |S_{k,m}| \le n \cdot \max\big(|S_{k,[a\sqrt{n}]}|, \quad |S_{k,[b\sqrt{n}]}|\big). \tag{8.6}$$

Using Stirling's formula for (8.3), for any fixed c, when $2k + [c\sqrt{n}] = n$,

$$|S_{k,[c\sqrt{n}]}| \sim n^{n/2} e^{-n/2+\sqrt{n}(c-c\log c)} \frac{e^{-1/2+c^2/4}}{\sqrt{\pi c} n^{1/4}}. \tag{8.7}$$

Hence using (8.4), we have

$$\frac{1}{|\tilde{S}_n|} \sum_{(**)} p^S_{kml} |S_{k,m}| \le C n^{3/4} \cdot \max\big(e^{\sqrt{n}(a-1-a\log a)}, \quad e^{\sqrt{n}(b-1-b\log b)}\big). \tag{8.8}$$

But $f(x) = x - 1 - x\log x$ is increasing in $0 < x < 1$, is decreasing in $x > 1$, and $f(1) = 0$. Therefore there are positive constants C and c such that for large n,

$$\frac{1}{|\tilde{S}_n|} \sum_{(**)} p^S_{kml} |S_{k,m}| \le C e^{-c\sqrt{n}}. \tag{8.9}$$

On the other hand, Lemma 6.3 says that (recall (4.8)) for any fixed real number $d > 0$, there is a constant C such that for $a\sqrt{n} \le m \le b\sqrt{n}$,

$$\begin{aligned} P_l^S\Big((2\mu_k^{(d)})^{1/2}; \mu_m^{(d)}(2\mu_k^{(d)})^{-1/2}\Big) - Cn^{-d/2} \\ \le p_{kml}^S \le P_l^S\Big((2\nu_k^{(d)})^{1/2}; \nu_m^{(d)}(2\nu_k^{(d)})^{-1/2}\Big) + Cn^{-d/2}, \end{aligned} \tag{8.10}$$

for sufficiently large n. Since $P_l^S(t;\beta) \le P_{l+1}^S(t;\beta)$, Theorem 4.1 for $P_{2l+1}^S(t;\beta)$ yields

$$e^{-\mu_k^{(d)}} D_{[(l-1)/2]}^{++}((2\mu_k^{(d)})^{1/2}) - Cn^{-d/2} \le p_{nml}^S \le e^{-\nu_k^{(d)}} D_{[l/2]}^{++}((2\nu_k^{(d)})^{1/2}) + Cn^{-d/2}. \tag{8.11}$$

Let $l = [2\sqrt{n} + xn^{1/6}]$. For $a\sqrt{n} \le m \le b\sqrt{n}$, hence for $\frac{n-b\sqrt{n}}{2} \le k \le \frac{n-a\sqrt{n}}{2}$,

$$\big(l/2 - (2\mu_k^{(d)})^{1/2}\big)2(l/2)^{-1/3}, \quad \big(l/2 - (2\nu_k^{(d)})^{1/2}\big)2(l/2)^{-1/3} = x + O\big(n^{-1/6}\sqrt{\log n}\big). \tag{8.12}$$

Also note that from the asymptotics (2.3), (2.4) and (2.11)-(2.14),

$$(F(x)E(x))' = -\frac{1}{2}(v(x) + u(x))F(x)E(x) \tag{8.13}$$

is bounded for $x \in \mathbb{R}$. Hence using (7.7) in Corollary 7.2, (8.12) and (8.13), we obtain

$$\begin{aligned} &\big|e^{-\nu_k^{(d)}} D_{[l/2]}^{++}((2\nu_k^{(d)})^{1/2}) - (FE)(x)\big| \\ &\quad \le \big|e^{-\nu_k^{(d)}} D_{[l/2]}^{++}((2\nu_k^{(d)})^{1/2}) - (FE)(\big(l/2 - (2\nu_k^{(d)})^{1/2}\big)2(l/2)^{-1/3})\big| \\ &\qquad + \big|(FE)(\big(l/2 - (2\nu_k^{(d)})^{1/2}\big)2(l/2)^{-1/3}) - (FE)(x)\big| \\ &\quad \le C(M)n^{-1/6} + Ce^{-cM^{3/2}} + Cn^{-1/6}\sqrt{\log n}. \end{aligned} \tag{8.14}$$

Therefore we have

$$\sum_{(*)} p_{nml}^S |S_{n,m}| \le \Big(F(x)E(x) + C(M)n^{-1/6} + Ce^{-cM^{3/2}} + Cn^{-1/6}\sqrt{\log n}\Big) \sum_{(*)} |S_{n,m}|. \tag{8.15}$$

Similarly,

$$\sum_{(*)} p_{nml}^S |S_{n,m}| \ge \Big(F(x)E(x) - C(M)n^{-1/6} - Ce^{-cM^{3/2}} - Cn^{-1/6}\sqrt{\log n}\Big) \sum_{(*)} |S_{n,m}|. \tag{8.16}$$

But from (8.9),

$$\frac{1}{|\tilde{S}_n|} \sum_{(*)} |S_{n,m}| = 1 - \frac{1}{|\tilde{S}_n|} \sum_{(**)} |S_{n,m}| = 1 + O(e^{-c\sqrt{n}}). \tag{8.17}$$

Thus using (8.5), (8.9), (8.15), (8.16) and (8.17), we obtain (3.20). □

Proof of (3.22)**.** As in Section 9 in [4], integrating by parts,

$$\mathrm{E}\big((\tilde{\chi}_n)^p\big) = \int_{-\infty}^{\infty} x^p dF_n(x) = -\int_{-\infty}^{0} px^{p-1}F_n(x)dx + \int_0^{\infty} px^{p-1}(1 - F_n(x))dx, \tag{8.18}$$

where $F_n(x) := \Pr(\tilde{\chi}_n \le x) = \Pr(\tilde{L}_n \le 2\sqrt{n} + xn^{1/6})$. From Theorem 4.1 and Corollary 7.6, we have

$$\begin{aligned} 1 - e^{-t^2/2} D_l^{++}(t) &\le Ce^{-c|x|^{3/2}}, \qquad x \ge M, && (8.19)\\ e^{-t^2/2} D_l^{++}(t) &\le Ce^{-c|x|^{3}}, \qquad x \le -M, && (8.20)\end{aligned}$$

for a fixed $M > M_2$ where $t = l - (x/2)l^{1/3}$. Noting that $P_{2l+1}^S(t;\beta) = e^{-t^2/2}D_l^{++}(t)$ for all $\beta \ge 0$, from (8.2), Lemma 6.4 and (8.19), (8.20), we obtain

$$\begin{aligned} 1 - F_n(x) &\le Ce^{-c|x|^{3/2}}, \qquad x \ge M, && (8.21)\\ F_n(x) &\le Ce^{-c|x|^{3}}, \qquad x \le -M. && (8.22)\end{aligned}$$

Now using convergence in distribution, dominated convergence theorem gives (3.22). □

Remark. We could proceed using

$$\Pr(\tilde{L}_n \le l) = \frac{1}{|\tilde{S}_n|} \sum_{2k+m=n} p_{kml}^O |S_{k,m}|. \tag{8.23}$$

The main contribution to the sum from $|S_{k,m}|$ comes from the region $|m - \sqrt{n}| \le n^{1/4+\epsilon}$. On the other hand, from Theorem 3.2, when $m = \sqrt{n} - 2wn^{1/3}$, the quantity p_{kml}^O converges to $F(x;w)$. But the region $m = \sqrt{n} + cn^{1/4+\epsilon}$ is much narrower than the region $m = \sqrt{n} + cn^{1/3}$, hence the main contribution to the sum comes from when $w = 0$, implying

$$\Pr(\tilde{L}_n \le l) \sim \frac{1}{|\tilde{S}_n|} \sum_{m=0}^{n} F(x;0)|S_{n,m}| = F_1(x). \tag{8.24}$$

In the following proof for signed involutions, we make this arguments rigorously.

Proof of (3.21)**.** We have a disjoint union

$$\tilde{S}_n^u = \bigcup_{2k+m_++m_-=n} S_{k,m_+,m_-}^u. \tag{8.25}$$

Hence again

$$\Pr(\tilde{L}_n^u \le l) = \frac{1}{|\tilde{S}_n^u|} \sum_{2k+m_++m_-=n} p_{km_+m_-l}^u \left|S_{k,m_+,m_-}^u\right|. \tag{8.26}$$

One can check that

$$\left|S_{k,m_+,m_-}^u\right| = \frac{(2k+m_++m_-)!}{k!m_+!m_-!}. \tag{8.27}$$

Hence, we have

$$\left|\tilde{S}_n^u\right| = \sum_{2k+m_++m_-=n} \left|S_{k,m_+,m_-}^u\right| = \sum_{0\le k\le [\frac{n}{2}]} \sum_{0\le m_+\le n-2k} f(m_+,k) \tag{8.28}$$

where

$$f(m_+,k) := \frac{n!}{m_+!(n-m_+-2k)!k!}. \tag{8.29}$$

For fixed $0 \le k \le [\frac{n}{2}]$, $f(m_+,k)$ is unimodal in m_+ and achieves its maximum when $m_+ \sim n/2-k$. And $f(n/2-k,k)$ is unimodal in k and the maximum is attained when $k \sim n/2 - \sqrt{n/2}$. Hence $f(m_+,k)$ has its maximum when $(m_+,k) \sim (\sqrt{\frac{n}{2}}, \frac{n}{2} - \sqrt{\frac{n}{2}})$. Consider the disc D of radius $n^{1/4+\epsilon}$ centered at $(\sqrt{\frac{n}{2}}, \frac{n}{2} - \sqrt{\frac{n}{2}})$. We will show that the main contribution to the sum in (8.28) comes from D. Set

$$m_+ = \sqrt{\frac{n}{2}} + x, \quad k = \frac{n}{2} - \sqrt{\frac{n}{2}} + y, \qquad |x|, |y| \le n^{1/4+\epsilon}, \tag{8.30}$$

By Stirling's formula,

$$f(m_+,k) = \frac{1}{\sqrt{en\pi}}(2n)^{n/2} e^{-n/2+\sqrt{2n}} e^{-\frac{x^2+(x+2y)^2}{\sqrt{2n}}} \big(1 + O(n^{-1/4+3\epsilon/2})\big). \tag{8.31}$$

Hence from the unimodality discussed above,

$$\sum_{(m_+,k)\notin D} f(m_+,k) \le n^2 \max_{(m_+,k)\in\partial D} f(m_+,k) \le \frac{n^2}{\sqrt{en\pi}}(2n)^{n/2} e^{-n/2+\sqrt{2n}} e^{-5\sqrt{2}n^{2\epsilon}}, \tag{8.32}$$

and by summing up using (8.31),

$$\sum_{(m_+,k)\in D} f(m_+,k) = \frac{1}{\sqrt{2e}}(2n)^{n/2} e^{-n/2+\sqrt{2n}} \big(1 + O(n^{-1/4+3\epsilon/2})\big). \tag{8.33}$$

Hence we have

$$\big|\tilde{S}^u_n\big| = \frac{1}{\sqrt{2e}}(2n)^{n/2} e^{-n/2+\sqrt{2n}} \big(1 + O(n^{-1/4+3\epsilon/2})\big), \tag{8.34}$$

and the main contribution to the sum in (8.28) comes from D.

As in Theorem 3.4, we write

$$\Pr(\tilde{L}^u_n \le l) = \frac{1}{\big|\tilde{S}^u_n\big|}\Big[\sum_D p_{km_+m_-l}\big|\tilde{S}^u_{k,m_+,m_-}\big| + \sum_{D^c} p_{km_+m_-l}\big|\tilde{S}^u_{k,m_+,m_-}\big|\Big]. \tag{8.35}$$

From (8.32) and (8.34),

$$\frac{1}{\big|\tilde{S}^u_n\big|}\sum_{D^c} p_{nm_+m_-l}\big|\tilde{S}^u_{n,m_+,m_-}\big| \le e^{-10n^{2\epsilon}}. \tag{8.36}$$

On the other hand, by the remark to Lemma 6.3 and Theorem 4.1, (recall (4.9))

$$p_{km_+m_-l} \le e^{-\nu^{(d)}_{m_+} - \nu^{(d)}_k} \pi^*_{[l/2]}\Big(-\frac{\nu^{(d)}_{m_+}}{(\nu^{(d)}_k)^{1/2}}; (\nu^{(d)}_k)^{1/2}\Big) D_{[\frac{l}{2}]}((\nu^{(d)}_k)^{1/2}) + Cn^{-d/2}, \tag{8.37}$$

for large n. We have a similar inequality of the other direction with ν, l and $+Cn^{-d/2}$ replaced by μ, $l-1$ and $-Cn^{-d/2}$.

Let $l = [2\sqrt{2n} + x2^{2/3}(2n)^{1/6}]$. In the region D,

$$(l/2 - 4(\nu^{(d)}_k)^{1/2})(l/4)^{-1/3} = x + O(n^{-1/6}\sqrt{\log n}), \tag{8.38}$$

and

$$(1 - \nu^{(d)}_{m_+}/(\nu^{(d)}_k)^{1/2})2^{-4/3}(l/2)^{1/3} = O(n^{-1/12+\epsilon}). \tag{8.39}$$

Hence as in (8.14), using Corollary 7.2 (7.5), Proposition 5.4 and Corollary 5.5

$$\begin{aligned}&\Big|e^{-\nu_{m_+}^{(d)}-\nu_k^{(d)}}\pi^*_{[l/2]}\Big(-\frac{\nu_{m_+}^{(d)}}{(\nu_k^{(d)})^{1/2}};(\nu_k^{(d)})^{1/2}\Big)D_{[\frac{l}{2}]}((\nu_k^{(d)})^{1/2})-F^u(x;0)\Big|\\&\qquad\le Cn^{-1/6}\sqrt{\log n}+Cn^{-1/12+\epsilon}+C(M)n^{-1/6}+Ce^{-cM^{3/2}}\end{aligned}\tag{8.40}$$

for a constant $C(M)$ which may depend on M and constants C and c which is independent of M. Thus for large n,

$$\Pr\Big(\frac{\tilde{L}_n^u-2\sqrt{2n}}{2^{2/3}(2n)^{1/6}}\le x\Big)\le F^u(x;0)+e(n,M)\tag{8.41}$$

with some error $e(n,M)$ such that $\lim_{M\to\infty}\lim_{n\to\infty}e(n,M)=0$. Similarly we have an inequality for the other direction. Recalling $F^u(x;0)=F_1(x)^2$ from (2.35), we obtain (3.21). □

Proof of (3.23)**.** Integrating by parts,

$$\mathrm{E}\big((\tilde{\chi}_n^u)^p\big)=\int_{-\infty}^{\infty}x^p dF_n(x)=-\int_{-\infty}^{0}px^{p-1}F_n(x)dx+\int_0^{\infty}px^{p-1}(1-F_n(x))dx,\tag{8.42}$$

where $F_n(x):=\Pr(\tilde{\chi}_n^u\le x)=\Pr(\tilde{L}_n^u\le 2\sqrt{2n}+x2^{2/3}(2n)^{1/6})$. Note that when $x<-(4n)^{1/3}$, $F_n(x)=0$, and when $x>2^{1/6}n^{5/6}-(4n)^{1/3}$, $F_n(x)=1$.

Let $M>M_0$ fixed. Consider the case when $-(4n)^{1/3}\le x\le -M$. From (8.35) and (8.36),

$$F_n(x)\le\frac{1}{|\tilde{S}_n^u|}\sum_D p_{km_+m_-l}\big|\tilde{S}^u_{k,m_+,m_-}\big|+Ce^{-10n^{2\epsilon}},\tag{8.43}$$

where $l=[2\sqrt{2n}+x2^{2/3}(2n)^{1/6}]$. We apply Lemma 6.4, Corollary 7.6 and Corollary 5.7 (5.36). Note that we are in the region $\alpha\to 1$ faster than $k^{-1/3}$, hence w is bounded, say $-1\le w\le 1$. So we can apply Corollary 5.7 (5.36). Then we obtain

$$F_n(x)\le Ce^{-c|x|^3}+Ce^{-10n^{2\epsilon}}.\tag{8.44}$$

Since $-(4n)^{1/3}\le x\le -M$, we have

$$e^{-10n^{2\epsilon}}\le e^{-\frac{10}{2^{4\epsilon}}|x|^{6\epsilon}},\tag{8.45}$$

thus,

$$F_n(x)\le Ce^{-c|x|^3}+Ce^{-\frac{10}{2^{4\epsilon}}|x|^{6\epsilon}}.\tag{8.46}$$

On the other hand, when $M\le x\le 2^{1/6}n^{5/6}-(4n)^{1/3}$, similarly we have

$$1-F_n(x)\le\frac{1}{|\tilde{S}_n^u|}\sum_D(1-p_{km_+m_-l})\big|\tilde{S}^u_{k,m_+,m_-}\big|+Ce^{-10n^{2\epsilon}},\tag{8.47}$$

Using Lemma 6.4, Corollary 7.6 and Proposition 5.6 (5.32), we obtain

$$1-F_n(x)\le Ce^{-c|x|^{3/2}}+Ce^{-10n^{2\epsilon}}.\tag{8.48}$$

Since $M\le x\le 2^{1/6}n^{5/6}-(4n)^{1/3}$,

$$e^{-10n^{2\epsilon}}\le e^{-\frac{10}{2^{2\epsilon/5}}|x|^{12\epsilon/5}},\tag{8.49}$$

thus

$$1 - F_n(x) \le Ce^{-c|x|^{3/2}} + Ce^{-\frac{10}{2^{2\epsilon/5}}|x|^{12\epsilon/5}}. \tag{8.50}$$

Therefore, using dominated convergence theorem, we obtain (3.23). □

9 Asymptotics for $\alpha > 1$

As we remarked following Theorem 3.1, when $\alpha > 1$, we must use a different scaling to obtain useful results.

Let $L^O(t;\alpha)$ and $L^u(t;\alpha,\beta)$ be random variables with the distribution functions given by $\Pr(L^O(t;\alpha) \le l) = P_l^O(t;\alpha)$ and $\Pr(L^u(t;\alpha,\beta) \le l) = P_l^u(t;\alpha,\beta)$, respectively. Under appropriate scalings, we obtain the Gaussian distribution in the limit.

Theorem 9.1. *For $\alpha > 1$ and $\beta \ge 0$ fixed,*

$$\lim_{t\to\infty} \Pr\left(\frac{L^O(t;\alpha) - (\alpha+\alpha^{-1})t}{\sqrt{(\alpha-\alpha^{-1})t}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy, \tag{9.1}$$

$$\lim_{t\to\infty} \Pr\left(\frac{L^u(t;\alpha,\beta) - 2(\alpha+\alpha^{-1})t}{\sqrt{2(\alpha-\alpha^{-1})t}} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy. \tag{9.2}$$

Proof. Let $l = (\alpha+\alpha^{-1})t + \sqrt{(\alpha-\alpha^{-1})t}$ for L^O. For large t, $2t/l \le c < 1$ for some $c > 0$. From Proposition 5.1 (i), using Theorem 4.1, it is easy to see that $e^{-t^2/2}D_l^{\pm\pm}(t), e^{-t^2 \pm t}D_l^{\pm\mp}(t) \to 1$ exponentially as $l \to \infty$. Now Theorem 4.1, (5.29) and Proposition 5.8 imply (9.1). For L^u, let $l = 2(\alpha+\alpha^{-1})t + \sqrt{2(\alpha-\alpha^{-1})t}$. Similarly, $e^{-t^2}D_l(t) \to 1$ exponentially as $l \to \infty$, and we obtain (9.2). □

Unfortunately, we can no longer apply the de-Poissonization technique; the difficulty is that $(\alpha+\alpha^{-1})t$ depends too strongly on small perturbations in α. Indeed, as we shall see, the asymptotics of the non-Poisson process are different.

Consider the case of involutions with $[\alpha t]$ fixed points and $[t^2/2]$ 2-cycles; the case of signed involutions is analogous. By symmetry, this is the same as the largest increasing subset distribution in the triangle $0 \le y \le x \le 1$ with $[t^2/2]$ generic points and $[\alpha t]$ diagonal points. As was observed in Remark 2 to Corollary 7.6 of [7], it is equivalent to consider weakly increasing subsets where the extra points are added to the line $y = 0$ instead of to the diagonal.

As in (3.1), let

$$\chi^O_{[t^2/2],[\alpha t]} = \frac{L^O_{[t^2/2],[\alpha t]} - (\alpha + 1/\alpha)t}{\sqrt{(1/\alpha - 1/\alpha^3)t}}. \tag{9.3}$$

Theorem 9.2. *As $t \to \infty$, the variable $\chi^O_{[t^2/2],[\alpha t]}$ converges in distribution and moments to $N(0,1)$.*

Proof. Let $S(t)$ be the set of points at time t, and let I be a largest increasing subset of $S(t)$. Then there will exist some number $0 \le s^+ \le 1$ (not unique) such that

$$(S(t) \cap \{y = 0, 0 \le x \le s^+\}) \subset I, \tag{9.4}$$

and such that every other point of I has $x > s^+$ and $y > 0$. For any $0 \le s \le 1$, we thus have

$$f_1(s) + f_2(s) \le |I|, \tag{9.5}$$

where $f_1(s)$ is the number of points of $S(t)$ with $y = 0$ and $0 \le x \le s$, and where $f_2(s)$ is the largest increasing subset of $S(t)$ lying entirely in the (part-open) trapezoid with $x \ge s$, $y > 0$.

Since $f_1(s)$ is binomial with parameters $[\alpha t]$ and s,

Lemma 9.3. *Let $M > 0$ be sufficiently large and fixed. For all $0 \le s < 1$, there exist positive constants C, c independent of s such that for $w \ge M$,*

$$\Pr(f_1(s) > \alpha st + wt^{1/2}) \le Ce^{-c|w|^2}, \tag{9.6}$$

while for $w \le -M$,

$$\Pr(f_1(s) < \alpha st + wt^{1/2}) \le Ce^{-c|w|^2}. \tag{9.7}$$

For f_2, we have:

Lemma 9.4. *Let $M > 0$ be sufficiently large and fixed. For $0 \le s < 1$, there exist positive constants C and c independent of s such that for all $w \ge M$,*

$$\Pr(f_2(s) > 2\sqrt{(1-s)}t + wt^{1/3}) \le Ce^{-c|w|^{3/2}}, \tag{9.8}$$

and for all $w \le -M$,

$$\Pr(f_2(s) < 2\sqrt{(1-s)}t + wt^{1/3}) \le Ce^{-c|w|^3}. \tag{9.9}$$

Proof. We first show the corresponding large-deviation result for the Poissonization. Define $f_2'(s,t)$ to be the length of the longest increasing subsequence when the number of points in the trapezoid is Poisson with parameter $t^2(1-s^2)/2$. Then $f_2'(s,t)$ is bounded between the corresponding processes for the rectangle $s \le x \le 1$, $0 \le y \le 1$ and for the triangle $0 \le (x-s)/(1-s) \le y \le 1$. In particular, if $f_2'(s,t)$ deviates significantly from $\sqrt{1-s}t$, so must the appropriate bounding process; the result follows immediately from the corresponding results for rectangles and triangles.

The corresponding large-deviation result when the number of points is fixed then follows from Lemma 6.2. In our case, the number of points in the trapezoid is binomial with parameters $t^2/2$ and $(1-s^2)$; the lemma follows via essentially the same argument used to prove Lemma 6.2. □

As we will see, the value $s = 1 - \alpha^{-2}$ deserves special attention:

Lemma 9.5. *Let $M > 0$ be sufficiently large and fixed. There exist positive constants C, c such that for $w \ge M$,*

$$\Pr(f_1(1-\alpha^{-2}) + f_2(1-\alpha^{-2}) > (\alpha + 1/\alpha)t + wt^{1/2}) \le Ce^{-c\min(|w|^2, |w|^{3/2}t^{1/4})}, \tag{9.10}$$

and for $w \le -M$,

$$\Pr(f_1(1-\alpha^{-2}) + f_2(1-\alpha^{-2}) < (\alpha + 1/\alpha)t + wt^{1/2}) \le Ce^{-c|w|^2}. \tag{9.11}$$

Moreover, if we define

$$\chi_0(t) = \frac{f_1(1-\alpha^{-2}) + f_2(1-\alpha^{-2}) - (\alpha+1/\alpha)t}{\sqrt{(1/\alpha - 1/\alpha^3)t}}, \tag{9.12}$$

then $\chi_0(t)$ converges to a standard normal distribution, both in distribution and in moments.

Proof. That $\chi_0(t)$ converges as stated follows from the fact that if we write $\chi_0(t) = \chi_1(t) + \chi_2(t)$, with

$$\chi_1(t) = \frac{f_1(1-\alpha^{-2}) - (\alpha - 1/\alpha)t}{\sqrt{(1/\alpha - 1/\alpha^3)t}} \tag{9.13}$$

$$\chi_2(t) = \frac{f_2(1-\alpha^{-2}) - (2/\alpha)t}{\sqrt{(1/\alpha - 1/\alpha^3)t}}, \tag{9.14}$$

then $\chi_1(t)$ converges in distribution and moments to a standard normal distribution, and $\chi_2(t)$ converges in distribution and moments to 0.

For the large deviation bounds, we note that if $x + y > z + w$, then either $x > z$ or $y > w$. Thus for any $0 \leq b \leq 1$, we have

$$\Pr(f_1(1-\alpha^{-2}) + f_2(1-\alpha^{-2}) > (\alpha + 1/\alpha)t + wt^{1/2}) \tag{9.15}$$

$$\leq \Pr(f_1(1-\alpha^{-2}) > (\alpha - 1/\alpha)t + bwt^{1/2}) + \Pr(f_2(1-\alpha^{-2}) > (2/\alpha)t + (1-b)wt^{1/2}) \tag{9.16}$$

$$\leq Ce^{-c|bw|^2} + Ce^{-c|(1-b)w|^{3/2}t^{1/4}}; \tag{9.17}$$

the result follows by balancing the two terms. In the other case, the $Ce^{-c|w|^2}$ term always dominates. ☐

Lemma 9.6. *For any sufficiently small $\epsilon > 0$, there exist positive constants C, c such that*

$$\Pr(s^+ - (1 - 1/\alpha^2) > t^{\epsilon/3 - 1/3}) < Ce^{-ct^\epsilon} \tag{9.18}$$

and

$$\Pr((1 - 1/\alpha^2) - s^+ > t^{\epsilon/2 - 1/2}) < Ce^{-ct^\epsilon} \tag{9.19}$$

for all sufficiently large t.

Proof. Define a sequence s_i by taking

$$s_i = 1 - (1 - 2/(i+2))^2/\alpha^2 \tag{9.20}$$

for all $i \geq 0$. Similarly define a sequence s'_i by

$$s'_i = \max(t_i, 0), \tag{9.21}$$

with

$$t_i = 1 - (1 + 2e^{2^{-1-i}})^2/\alpha^2 \tag{9.22}$$

for $i < 0$ and

$$t_i = 1 - (1 + 4/(i+1))^2/\alpha^2 \tag{9.23}$$

for $i \geq 0$. Note that s_i is strictly decreasing and t_i is strictly increasing.

Lemma 9.7. *For all $i \geq 0$,*

$$\alpha s_i + 2\sqrt{1 - s_{i+1}} < \alpha + 1/\alpha. \tag{9.24}$$

For all i,

$$\alpha s'_{i+1} + 2\sqrt{1 - s'_i} < \alpha + 1/\alpha. \tag{9.25}$$

Proof. In the first case, we have

$$\alpha + 1/\alpha - (\alpha s_i + 2\sqrt{1 - s_{i+1}}) = \frac{4}{(i+2)^2(i+3)\alpha}. \tag{9.26}$$

In the second case, it suffices to verify the formula with s' replaced by t. For $i < -1$,

$$\alpha + 1/\alpha - (\alpha t_{i+1} + 2\sqrt{1 - t_i}) = 4e^{2^{-i}/4}/\alpha. \tag{9.27}$$

For $i = -1$,

$$\alpha + 1/\alpha - (\alpha t_{i+1} + 2\sqrt{1 - t_i}) = (24 - 4e)/\alpha \tag{9.28}$$

Finally, for $i \geq 0$,

$$\alpha + 1/\alpha - (\alpha t_{i+1} + 2\sqrt{1 - t_i}) = \frac{8i}{(i+1)(i+2)^2\alpha}. \tag{9.29}$$

□

Let $i_1 = t^{1/6-\epsilon/6}$, $i_2 = t^{1/4-\epsilon/4}$. Then there exist constants C, c such that for $0 \leq i \leq i_1$,

$$\Pr(f_1(s_i) + f_2(s_{i+1}) > f_1(1 - \alpha^{-2}) + f_2(1 - \alpha^{-2})) < Ce^{-ct^\epsilon}. \tag{9.30}$$

Since $f_1(s_i) + f_2(s_{i+1})$ is an upper bound on $f_1(s) + f_2(s)$ with $s_{i+1} \leq s \leq s_i$, it follows that

$$\Pr(s^+ \in [s_{i+1}, s_i]) < Ce^{-ct^\epsilon}. \tag{9.31}$$

Similarly, for $i \leq i_2$,

$$\Pr(f_1(s'_{i+1}) + f_2(s'_i) > f_1(1 - \alpha^{-2}) + f_2(1 - \alpha^{-2})) < Ce^{-ct^\epsilon}, \tag{9.32}$$

and thus

$$\Pr(s^+ \in [s'_i, s'_{i+1}]) < Ce^{-ct^\epsilon}. \tag{9.33}$$

Since there are only $i_1 + i_2 + \log\log\alpha$ such events to consider, the result follows. □

In particular, with probability $1 - Ce^{-ct^\epsilon}$, we have

$$f_1(1-\alpha^{-2}-t^{\epsilon/2-1/2})+f_2(1-\alpha^{-2}+t^{\epsilon/3-1/3}) \leq L(t) \leq f_1(1-\alpha^{-2}+t^{\epsilon/3-1/3})+f_2(1-\alpha^{-2}-t^{\epsilon/2-1/2}). \tag{9.34}$$

But then using the fact that

$$f_1(1 - \alpha^{-2}) - f_1(1 - \alpha^{-2} - t^{\epsilon/2-1/2}) \tag{9.35}$$

and

$$f_1(1 - \alpha^{-2} - t^{\epsilon/2-1/2}) - f_1(1 - \alpha^{-2}) \tag{9.36}$$

are Poisson, and using the large deviation behavior of $f_2(s)$, we find that

$$\Pr(L(t) - (f_1(1-\alpha^{-2} - t^{\epsilon/2-1/2}) + f_2(1-\alpha^{-2} + t^{\epsilon/3-1/3})) \geq t^{1/2-\epsilon'}) \leq Ce^{-ct^{\epsilon}}, \tag{9.37}$$

and

$$\Pr((f_1(1-\alpha^{-2} - t^{\epsilon/3-1/3}) + f_2(1-\alpha^{-2} + t^{\epsilon/2-1/2})) - L(t) \geq t^{1/2-\epsilon'}) \leq Ce^{-ct^{\epsilon}}. \tag{9.38}$$

So $\chi(t) - \chi_0(t)$ converges to 0 in a fairly strong sense; in particular, they must have the same limiting distribution and limiting moments. □

Remark. The above proof could equally well be applied to the Poisson process; the beta distribution would then be replaced by a Poisson distribution.

For the signed involution case, with $[2\alpha t]$ fixed points, $[2\beta t]$ negated points, and $[2t^2]$ 2-cycles, again we let

$$\chi^u_{[t^2],[2\alpha t],[2\beta t]} = \frac{L^u_{[t^2/2],[\alpha t],[\beta t]} - 2(\alpha + 1/\alpha)t}{\sqrt{2(1/\alpha - 1/\alpha^3)t}}. \tag{9.39}$$

Then the analogous argument proves that $\chi'(t)$ also converges in moments and distribution to a standard normal.

10 Steepest descent type analysis for Riemann-Hilbert problems

In this section, we prove the asymptotics of orthogonal polynomials stated in Section 5 by applying the steepest descent type method to the Riemann-Hilbert problem (RHP) (5.3). In [4], the RHP (5.3) is analyzed asymptotically as $k, t \to \infty$, and a variety of estimates for the specific quantity of interest, $Y_{21}(0; k; t)$, are obtained. In the present paper, as mentioned in Introduction, additional quantities related to (5.3) are needed, and we use and refine the analysis in [4] to obtain the asymptotics of the new quantities. Also we simplify the g-function expression to obtain desired estimation. When the analysis overlaps to that of [4], we only sketch the method, and instead we focus on new features to indicate how to prove the propositions in Section 5.

The steepest descent type method for RHP's, the Deift-Zhou method, was introduced by Deift and Zhou in [18] and is developed further in [19], [16], and finally placed in a systematic form by Deift, Venakides and Zhou in [17]. The analysis of the RHP (5.3) has many similarities with [13, 14, 15] where the asymptotics of orthogonal polynomial on the real line with respect to a general weight is obtained, leading to a proof of universality conjectures in random matrix theory. The main difference of analysis of [4] and this paper with [13, 14, 15] is that now we need full range of asymptotics as $k, t \to \infty$ with good enough bound on error terms so that the summation process in the proofs of Collorary 7.1 and etc. in sections 7 and 8. Also since the main contribution to the sum comes from a narrow region when $(2t)/k \sim 1$, we need finer analysis around this region.

We say that a RHP normalized at ∞ if the solution m satisfies the condition $m \to I$ as $z \to \infty$. Thus for instance, the RHP's (2.15), (10.1) are normalized at ∞, while the RHP (5.3) is not.

It turns out that The asymptotic analysis differs critically when $(2t)/k \leq 1$ and $(2t)/k > 1$, due to the difference of (support of) the associated equilibrium measure (see Lemma 4.3 [4]). Hence we discuss those two cases separately in Subsections 10.1 and 10.2. In Subsection 10.3, we prove Proposition 5.8.

10.1 When $(2t)/k \le 1$.

Define

$$
\begin{aligned}
m^{(1)}(z;k;t) &:= Y(z;k;t)\begin{pmatrix}(-1)^k e^{tz} & 0\\ 0 & (-1)^k e^{-tz}\end{pmatrix}, \qquad |z|<1,\\
m^{(1)}(z;k;t) &:= Y(z;k;t)\begin{pmatrix}z^{-k} e^{tz^{-1}} & 0\\ 0 & z^k e^{-tz^{-1}}\end{pmatrix}, \qquad |z|>1.
\end{aligned}
\tag{10.1}
$$

Then $m^{(1)}$ solves a new RHP which is equivalent to the RHP (5.3) in the sense that a solution of one RHP yields algebraically a solution of the other RHP, and vice versa :

$$
\begin{cases}
m^{(1)}(z;k;t) \qquad \text{is analytic in } \mathbb{C}\setminus\Sigma,\\
m^{(1)}_+(z;k;t) = m^{(1)}_-(z;k;t)\begin{pmatrix}(-1)^k z^k e^{t(z-z^{-1})} & (-1)^k\\ 0 & (-1)^k z^{-k} e^{-t(z-z^{-1})}\end{pmatrix} \quad \text{on } \Sigma,\\
m^{(1)}(z;k;t) = I + O(\frac{1}{z}) \qquad \text{as } z\to\infty,
\end{cases}
\tag{10.2}
$$

where Σ is the unit circle oriented counterclockwise as before. Here and in the below, $m_+(z)$ (resp. $m_-(z)$) is understood as the limit from the left (reps. right) side of the contour as one goes along the orientation of the contour. Now we define $m^{(2)}(z;k;t)$ in terms of $m^{(1)}(z;k;t)$ as follows :

$$
\begin{aligned}
&\text{for even } k,\\
&\begin{cases}
m^{(2)} \equiv m^{(1)} & |z|>1,\\
m^{(2)} \equiv m^{(1)}\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) & |z|<1.
\end{cases}\\
&\text{for odd } k,\\
&\begin{cases}
m^{(2)} \equiv \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) m^{(1)}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) & |z|>1,\\
m^{(2)} \equiv \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) m^{(1)}\left(\begin{smallmatrix}0 & -1\\ -1 & 0\end{smallmatrix}\right) & |z|<1.
\end{cases}
\end{aligned}
\tag{10.3}
$$

Then $m^{(2)}(\cdot;k;t)$ solves another RHP

$$
\begin{cases}
m^{(2)}_+(z;k;t) = m^{(2)}_-(z;k;t)v^{(2)}(z;k;t) \quad \text{on} \quad \Sigma,\\
m^{(2)}(z;k;t) = I + O(\frac{1}{z}) \ \text{ as } \ z\to\infty
\end{cases}
\tag{10.4}
$$

where

$$
v^{(2)}(z;k;t) = \begin{pmatrix}1 & -(-1)^k z^k e^{t(z-z^{-1})}\\ (-1)^k z^{-k} e^{-t(z-z^{-1})} & 0\end{pmatrix}.
\tag{10.5}
$$

The jump matrix has the following factorization,

$$
v^{(2)} = \begin{pmatrix}1 & 0\\ (-1)^k z^{-k} e^{-t(z-z^{-1})} & 1\end{pmatrix}\begin{pmatrix}1 & -(-1)^k z^k e^{t(z-z^{-1})}\\ 0 & 1\end{pmatrix} =: (b^{(2)}_-)^{-1} b^{(2)}_+,
\tag{10.6}
$$

We note that through the changes $Y \to m^{(1)} \to m^{(2)}$, we have

$$Y_{11}(z;k;t) = -(-1)^k e^{-tz} m_{12}^{(2)}(z;k;t), \qquad |z| < 1, \tag{10.7}$$

$$Y_{21}(z;k;t) = -e^{-tz} m_{22}^{(2)}(z;k;t), \qquad |z| < 1, \tag{10.8}$$

$$Y_{11}(z;k;t) = z^k e^{-tz^{-1}} m_{11}^{(2)}(z;k;t), \qquad |z| > 1, \tag{10.9}$$

$$Y_{21}(z;k;t) = (-z)^k e^{-tz^{-1}} m_{21}^{(2)}(z;k;t), \qquad |z| > 1. \tag{10.10}$$

The absolute value of the (12)-entry of the jump matrix $v^{(2)}$ is $e^{kF(\rho,\theta;\frac{2t}{k})}$ where

$$F(z;\gamma) = F(\rho e^{i\theta};\gamma) := \frac{\gamma}{2}(\rho - \rho^{-1})\cos\theta + \log\rho, \quad z = \rho e^{i\theta}. \tag{10.11}$$

The absolute value of the (21)-entry of $v^{(2)}$ is $e^{-kF(\rho e^{i\theta};\frac{2t}{k})}$. Note that

$$F(\rho,\theta;\gamma) = -F(\rho^{-1},\theta;\gamma). \tag{10.12}$$

Figure 2 shows the curves $F(z;\gamma) = 0$. In $\Omega_1^{sig} \cup \Omega_3^{sig}$, $F > 0$, and in $\Omega_2^{sig} \cup \Omega_4^{sig}$, $F < 0$. The region Ω_2^{sig}

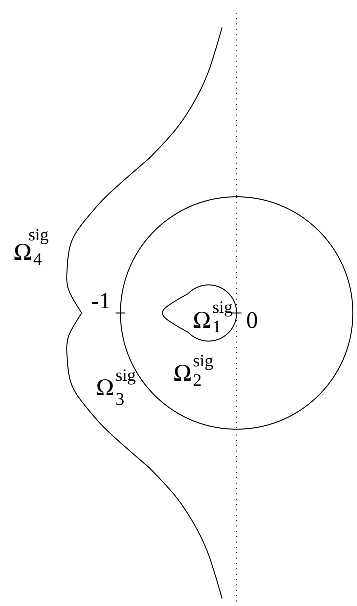

Figure 2: curves of $F(z;\gamma) = 0$ when $0 < \gamma < 1$

becomes smaller as γ increases, and when $\gamma = 1$, the curve $F(z;\gamma) = 0$ contacts the unit circle Σ at $z = -1$ with the angle $\pi/3$.

10.1.1 Case $0 \le 2t \le ak$ for some $0 < a < 1$.

It is possible to fix $\rho_a < 1$ such that the circle $\{z : |z| = \rho_a\}$ is in the region Ω_2^{sig} for all such t and k. Define $m^{(3)}(z;k;t)$ by

$$\begin{cases} m^{(3)} = m^{(2)}(b_+^{(2)})^{-1}, & \rho_a < |z| < 1, \\ m^{(3)} = m^{(2)}(b_-^{(2)})^{-1}, & 1 < |z| < \rho_a^{-1}, \\ m^{(3)} = m^{(2)} & |z| < \rho_a, \quad |z| > \rho_a^{-1}. \end{cases} \tag{10.13}$$

Then $m^{(3)}$ satisfies a new jump condition $m_+^{(3)} = m_-^{(3)} v^{(3)}$ on $\Sigma^{(3)} := \{z : |z| = \rho_a, \rho_a^{-1}\}$, where $v^{(3)} = b_+^{(2)}$, $|z| = \rho_a$ and $v^{(3)} = (b_-^{(2)})^{-1}$, $|z| = \rho_a^{-1}$. From the choice of ρ_a, we see that

$$|v^{(3)}(z;k;t) - I| \le e^{-ck} \qquad \text{for all } z \in \Sigma^{(3)}, \tag{10.14}$$

which implies that $I - C_{w^{(3)}}$ is invertible and the norm of the inverse is uniformly bounded, where $w^{(3)} := v^{(3)} - I$ and $C_{w^{(3)}}(f) := C_-(fw^{(3)})$ on $L^2(\Sigma^{(3)}, |dz|)$, $C_\pm$ being Cauchy operators (see (2.5)-(2.9) in [4] and references in it). From the general theory of RHP, we have

$$m^{(3)}(z) = I + \frac{1}{2\pi i}\int_{\Sigma^{(3)}} \frac{((I - C_{w^{(3)}})^{-1}I)(s)w^{(3)}(s)}{s-z} ds, \quad z \notin \Sigma^{(3)}. \tag{10.15}$$

This implies the estimates

$$|m_{22}^{(3)}(0;k;t) - 1|, \quad |m_{12}^{(3)}(0;k;t)| \le Ce^{-ck}, \tag{10.16}$$

which using (10.13), (10.7), (10.8) and (5.5), (5.6), yield Proposition 5.1 (i).

From (10.13), (10.7), (10.9) and (5.6), we have

$$\begin{aligned} \pi_k(z;t) &= -(-1)^k e^{-tz} m_{12}^{(3)}(z;k;t), & |z| < \rho_a, \quad (10.17)\\ \pi_k(z;t) &= z^k e^{-tz^{-1}} m_{11}^{(3)}(z;k;t), & |z| > \rho_a^{-1}, \quad (10.18)\\ \pi_k(z;t) &= z^k e^{-tz^{-1}} m_{11}^{(3)}(z;k;t) - (-1)^k e^{-tz} m_{12}^{(3)}(z;k;t), & \rho_a < |z| < \rho_a^{-1}. \quad (10.19) \end{aligned}$$

Let $0 < b < 1$ be a fixed number. We could have chosen ρ_a such that $\rho_a > b$. When $|z| \le b$ and $|z| \ge b^{-1}$, we have $\mathrm{dist}(z, \Sigma^{(3)}) \ge c > 0$. Hence using (10.15), (10.17) and (10.18) imply that

$$\begin{aligned} |e^{tz}\pi_k(z;t)| &\le Ce^{-ck}, & |z| \le b, \quad (10.20)\\ |e^{tz^{-1}} z^{-k}\pi_k(z;t)| &\le Ce^{-ck}, & |z| \ge b^{-1}. \quad (10.21) \end{aligned}$$

These are (5.29), (5.30) in Proposition 5.6 of the special case $x \ge 2^{1/3}(1-a)k^{2/3}$.

On the other hand, let $L > 0$ be a fixed number. Set $\alpha = 1 - 2^{4/3}k^{-1/3}w$ with $-L \le w \le L$ as in Proposition 5.6. Since ρ_a is fixed, when k is large, $\mathrm{dist}(-\alpha, \Sigma^{(3)}) \ge c > 0$. Then from (10.15),

$$|m_{11}^{(3)}(-\alpha;k;t) - 1|, \quad |m_{12}^{(3)}(-\alpha;k;t)| \le Ce^{-ck}. \tag{10.22}$$

Note that

$$\begin{aligned} \frac{1}{2}(s-s^{-1}) &\le s-1, \quad s>0, && (10.23)\\ -\frac{1}{2}(s-s^{-1})+\log s &\le \frac{2}{3}(1-s)^3, \quad \frac{1}{2}\le s\le 1, && (10.24)\\ -\frac{1}{2}(s-s^{-1})+\log s &\le 0, \quad s\ge 1. && (10.25)\end{aligned}$$

Thus for $\gamma \le 1$, $s \ge \frac{1}{2}$,

$$\begin{aligned} F(-s;\gamma) &= -\frac{\gamma}{2}(s-s^{-1})+\log s = \frac{1-\gamma}{2}(s-s^{-1}) - \frac{1}{2}(s-s^{-1})+\log s \\ &\le (1-\gamma)(s-1)+\frac{2}{3}|s-1|^3. \end{aligned} \tag{10.26}$$

For large k, $\alpha \ge \frac{1}{2}$ for all $-L \le w \le L$, and hence

$$\left|(-\alpha)^k e^{-t(\alpha-\alpha^{-1})}\right| = e^{kF(-\alpha;\frac{2t}{k})} \le e^{\frac{32}{2}|w|^3 - k(1-\frac{2t}{k})\frac{2^{4/3}w}{k^{1/3}}} \le e^{\frac{32}{2}L^3} e^{2^{4/3}Lk^{2/3}} = Ce^{ck^{2/3}}. \tag{10.27}$$

Similarly, since $\alpha^{-1} \ge \frac{1}{2}$ for large k,

$$\left|(-\alpha)^{-k} e^{t(\alpha-\alpha^{-1})}\right| = e^{kF(-\alpha^{-1};\frac{2t}{k})} \le e^{k(1-\frac{2t}{k})\frac{2^{4/3}w}{k^{1/3}\alpha} + \frac{32}{2}|w|^3\alpha^{-3}} \le Ce^{ck^{2/3}}. \tag{10.28}$$

Therefore from (10.19) and (10.22),

$$\begin{aligned} \left|e^{-t\alpha}\pi_k(-\alpha;t)\right| = \left|(-\alpha)^k e^{-t(\alpha-\alpha^{-1})} m_{11}^{(3)}(-\alpha;k;t) - (-1)^k m_{12}^{(3)}(-\alpha;k;t)\right| &\le Ce^{ck^{2/3}}, && (10.29)\\ \left|e^{-t\alpha^{-1}}(-\alpha)^{-k}\pi_k(-\alpha;t)-1\right| = \left|m_{11}^{(3)}(-\alpha;k;t) - 1 - \alpha^{-k}e^{t(\alpha-\alpha^{-1})} m_{12}^{(3)}(-\alpha;k;t)\right| &\le Ce^{-ck}. && (10.30)\end{aligned}$$

Noting $x \sim k^{2/3}$, these are (5.31), (5.32) in Proposition 5.6 of the special case $x \ge 2^{1/3}(1-a)k^{2/3}$.

10.1.2 Case $ak \le 2t \le k - M2^{-1/3}k^{1/3}$ for some $0 < a < 1$ and $M > M_0$.

In the previous section, the contour $\Omega^{(3)}$ was not the best choice. We could have chosen the steepest descent curve for $F(z;\gamma)$. For the previous case, it was not necessary to use the steepest descent curve to obtain the desired results, but for the case at hand and in the future calculations, we need to use the steepest descent curve.

One can check that (see (5.5) in [4]) the minimum of $F(\rho,\theta;\gamma)$ for $0<\rho\le 1$ is attained for fixed $\pi/2 \le \theta \le (3\pi)/2$, $\gamma \le 1$ at

$$\rho = \rho_\theta := \frac{1-\sqrt{1-\gamma^2\cos^2\theta}}{-\gamma\cos\theta} \tag{10.31}$$

and $F(\rho_\theta,\theta;\gamma)<0$. For $0\le\theta<\pi/2$ and $(3\pi)/2<\theta<2\pi$, $F(\rho,\theta;\gamma)$ is always negative for $0<\rho<1$ and as $\rho\downarrow 0$, it decreases to infinity. It is straightforward to check that

$$F(\rho_\theta,\theta;\gamma) = \sqrt{1-\gamma^2\cos^2\theta} + \log\left[\frac{1-\sqrt{1-\gamma^2\cos^2\theta}}{-\gamma\cos\theta}\right] \le -\frac{2\sqrt{2}}{3}(1+\gamma\cos\theta)^{3/2}, \tag{10.32}$$

for all $0\le\gamma\le 1$, $\pi/2\le\theta\le 3\pi/2$. Also $F(\rho_\theta,\theta;\gamma)$ is increasing in $\pi/2\le\theta\le\pi$ and is decreasing in $\pi\le\theta\le 3\pi/2$. In fact, the saddle points for $\frac{\gamma}{2}(z-z^{-1})+\log z$ are $z=-\rho_\pi$ and $z=-\rho_\pi^{-1}$.

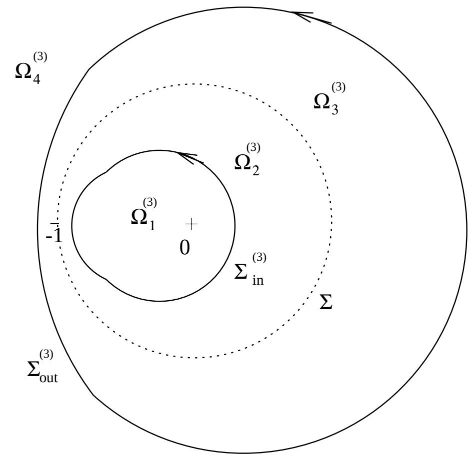

Figure 3: $\Sigma^{(3)}$ and $\Omega^{(3)}$ when $gamma < 1$

In this time, define $\Sigma^{(3)} := \Sigma^{(3)}_{in} \cup \Sigma^{(3)}_{out}$ by

$$\begin{aligned}\Sigma^{(3)}_{in} &= \{\rho_\theta e^{i\theta} : 3\pi/4 \le \theta \le 5\pi/4\} \cup \{\rho_{3\pi/4} e^{i\theta} : 0 \le \theta \le 3\pi/4, 5\pi/4 \le \theta < 2\pi\},\\ \Sigma^{(3)}_{out} &= \{\rho_\theta^{-1} e^{i\theta} : 3\pi/4 \le \theta \le 5\pi/4\} \cup \{\rho_{3\pi/4}^{-1} e^{i\theta} : 0 \le \theta \le 3\pi/4, 5\pi/4 \le \theta < 2\pi\},\end{aligned} \tag{10.33}$$

where ρ_θ is defined in (10.31) with $\gamma = (2t)/k$. Orient $\Sigma^{(3)}$ as in Figure 3. Note that $\Sigma^{(3)}$ lies in Ω_2^{sig} and for $3\pi/4 \le \theta \le 5\pi/4$, it is the steepest descent curve. The reason to choose part of a circle for the rest of angles is just to make the contour closed to ensure the uniform boundedness of Cauchy operators (see Section 5 of [4].)

Define the regions $\Omega^{(3)}_j$, $j = 1, \cdots, 4$ as in Figure 3. Define $m^{(3)}(z;k;t)$ by

$$\begin{cases} m^{(3)} = m^{(2)}(b_+^{(2)})^{-1}, & \text{in } \Omega_2^{(3)},\\ m^{(3)} = m^{(2)}(b_-^{(2)})^{-1}, & \text{in } \Omega_3^{(3)},\\ m^{(3)} = m^{(2)} & \text{in } \Omega_1^{(3)}, \Omega_4^{(3)}, \end{cases} \tag{10.34}$$

where $b_\pm^{(2)}$ are defined in (10.6). Then $m^{(3)}$ solves a new RHP with the jump matrix $v^{(3)}(z;k;t)$ where

$$\begin{cases} v^{(3)} = \begin{pmatrix} 1 & -(-1)^k z^k e^{t(z-z^{-1})} \\ 0 & 1 \end{pmatrix} & \text{on} \quad \Sigma^{(3)}_{in},\\ v^{(3)} = \begin{pmatrix} 1 & 0 \\ (-1)^k z^{-k} e^{-t(z-z^{-1})} & 1 \end{pmatrix} & \text{on} \quad \Sigma^{(3)}_{out}. \end{cases} \tag{10.35}$$

Set $w^{(3)} := v^{(3)} - I$. For $z \in \Sigma^{(3)}_{in}$, from the choice of $\Sigma^{(3)}$ and (10.32), the (12)-entry of the jump matrix satisfies for $3\pi/4 \le \arg z \le 5\pi/4$,

$$|z^k e^{t(z-z^{-1})}| = e^{kF(\rho_\theta, \theta; (2t)/k)} \le e^{-\frac{2\sqrt{2}}{3}k(1+\frac{2t}{k}\cos\theta)^{3/2}} \le e^{-\frac{2\sqrt{2}}{3}k(1-\frac{2t}{k})^{3/2}} \le e^{-\frac{2}{3}M^{3/2}}, \tag{10.36}$$

and for $0 \le \arg z \le 3\pi/4$ or $5\pi/4 \le \arg z < 2\pi$,

$$|z^k e^{t(z-z^{-1})}| = e^{kF(\rho_{3\pi/4}, \theta; \frac{2t}{k})} \le e^{kF(\rho_{3\pi/4}, \frac{3\pi}{4}; \frac{2t}{k}} \le e^{-\frac{2\sqrt{2}}{3}k(1+\frac{2t}{k}\cos\frac{3\pi}{4})^{3/2}} \le e^{-\frac{1}{24}k}. \tag{10.37}$$

From (10.12), similar estimates hold for $z^{-k}e^{-t(z-z^{-1})}$, $z \in \Sigma^{(3)}_{out}$. Therefore there exists M_0 such that for $M > M_0$, $\|C_{w^{(3)}}\|_{L^2(\Sigma^{(3)})\to L^2(\Sigma^{(3)})} \le c_1 < 1$, and hence (10.15) holds.

We are also interested in L^1 norm of $w^{(3)}$. Note that $|dz| \le C|d\theta|$ on $\Sigma^{(3)}$. Hence using estimates in (10.36) and (10.37),

$$\int_{\Sigma^{(3)}_{in}} \left|z^k e^{t(z-z^{-1})}\right||dz| \le C\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} e^{-\frac{2\sqrt{2}}{3}k(1+\frac{2t}{k}\cos\theta)^{3/2}}d\theta + C\int_{[0,2\pi)\setminus[3\pi/4,5\pi/4]} e^{-\frac{1}{24}k}d\theta. \tag{10.38}$$

The second integral is less than $Ce^{-\frac{1}{24}k}$. Recall the inequality $(x+y)^a \ge x^a + y^a$ for $x, y > 0$, $a \ge 1$. Using the inequality $1+\cos\theta \ge \frac{1}{2\sqrt{2}}(\theta-\pi)^2$ for $\theta \in [\frac{3\pi}{4}, \frac{5\pi}{4}]$, and the condition $ak \le 2t$, the first integral is less than or equal to

$$\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} e^{-\frac{2\sqrt{2}}{3}k\left[\left(1-\frac{2t}{k}\right)^{3/2}+\left(\frac{2t}{k}(1+\cos\theta)\right)^{3/2}\right]}d\theta \le e^{-\frac{2\sqrt{2}}{3}k\left(1-\frac{2t}{k}\right)^{3/2}}\int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} e^{-\frac{a^{3/2}}{3\cdot 2^{3/4}}k|\theta-\pi|^3}d\theta. \tag{10.39}$$

The last inequality is less than or equal to $Ck^{-1/3}$ for some constant $C > 0$. Therefore adjusting constants, we obtain

$$\int_{\Sigma^{(3)}_{in}} \left|z^k e^{t(z-z^{-1})}\right||dz| \le \frac{C}{k^{1/3}}e^{-\frac{2\sqrt{2}}{3}k\left(1-\frac{2t}{k}\right)^{3/2}}. \tag{10.40}$$

Similarly,

$$\int_{\Sigma^{(3)}_{out}} \left|z^{-k} e^{-t(z-z^{-1})}\right||dz| \le \frac{C}{k^{1/3}}e^{-\frac{2\sqrt{2}}{3}k\left(1-\frac{2t}{k}\right)^{3/2}}. \tag{10.41}$$

These results are refinement of (5.23) in [4]. From (10.15), we have

$$m^{(3)}(z) = I + \frac{1}{2\pi i}\int_{\Sigma^{(3)}} \frac{w^{(3)}(s)}{s-z}ds + \frac{1}{2\pi i}\int_{\Sigma^{(3)}} \frac{[(I-C_{w^{(3)}})^{-1}C_{w^{(3)}}I](s)w^{(3)}(s)}{s-z}ds, \quad z \notin \Sigma^{(3)}. \tag{10.42}$$

The first integral is less than or equal to

$$\frac{C}{\mathrm{dist}(z,\Sigma^{(3)})k^{1/3}}e^{-\frac{2\sqrt{2}}{3}k\left(1-\frac{2t}{k}\right)^{3/2}}, \tag{10.43}$$

while the second integral is less than or equal to

$$\begin{aligned}
&\frac{1}{2\pi\,\mathrm{dist}(z,\Sigma^{(3)})}\|(I-C_{w^{(3)}})^{-1}C_{w^{(3)}}I\|_{L^2}\|w^{(3)}\|_{L^2} \\
&\le \frac{1}{2\pi\,\mathrm{dist}(z,\Sigma^{(3)})}\|(I-C_{w^{(3)}})^{-1}\|_{L^2\to L^2}\|C_{w^{(3)}}I\|_{L^2}\|w^{(3)}\|_{L^2} \\
&\le \frac{C}{\mathrm{dist}(z,\Sigma^{(3)})}\|w^{(3)}\|^2_{L^2} \\
&\le \frac{C}{\mathrm{dist}(z,\Sigma^{(3)})}\|w^{(3)}\|_{L^\infty}\|w^{(3)}\|_{L^1} \\
&\le \frac{C}{\mathrm{dist}(z,\Sigma^{(3)})k^{1/3}}e^{-\frac{4\sqrt{2}}{3}k\left(1-\frac{2t}{k}\right)^{3/2}}.
\end{aligned} \tag{10.44}$$

When $z = 0$, $\mathrm{dist}(z,\Sigma^{(3)}) \ge c_1 > 0$, hence using (10.34), (10.7), (10.8) and (5.5), (5.6), we obtain Proposition 5.1 (ii).

As in (10.17)-(10.19), from (10.13), (10.7), (10.9) and (5.6), we have

$$\begin{aligned} \pi_k(z;t) &= -(-1)^k e^{-tz} m^{(3)}_{12}(z;k;t), && z \in \Omega^{(3)}_1, && (10.45)\\ \pi_k(z;t) &= z^k e^{-tz^{-1}} m^{(3)}_{11}(z;k;t), && z \in \Omega^{(3)}_4, && (10.46)\\ \pi_k(z;t) &= z^k e^{-tz^{-1}} m^{(3)}_{11}(z;k;t) - (-1)^k e^{-tz} m^{(3)}_{12}(z;k;t), && z \in \Omega^{(3)}_2 \cup \Omega^{(3)}_3. && (10.47)\end{aligned}$$

Define x by

$$\frac{2t}{k} = 1 - \frac{x}{2^{1/3}k^{2/3}} \tag{10.48}$$

as in Proposition 5.6. Let $0 < b < 1$ be a fixed number. Given b, from the beginning, we could have chosen $0 < a < 1$ such that

$$\rho_{\theta_b} = \frac{1-\sqrt{1-a^2\cos^2\theta_b}}{-a\cos\theta_b} \tag{10.49}$$

is strictly greater than b for some $\pi/2 \le \theta_b < \pi$. Note that in (10.33) defining $\Sigma^{(3)}$, the choice of $3\pi/4$ and $5\pi/4$ was arbitrary. Instead of $3\pi/4$ and $5\pi/4$, we use θ_b and $2\pi - \theta_b$ in this time, and carry the whole following calculations. Then we obtain the same estimates of (10.43) and (10.44) with different constants C. Now $|z| \le b$ lies in $\Omega^{(3)}_1$ and $|z| \ge b^{-1}$ lies in $\Omega^{(3)}_4$. Since the distance $\mathrm{dist}(z, \Sigma^{(3)}) \ge c_2 > 0$, using (10.42), (10.43), (10.44), (10.45) and (10.46), we obtain (5.29) and (5.30) in Proposition 5.6 for $M \le x \le (1-a)2^{1/3}k^{2/3}$. Since in (10.20) and (10.21) in Subsubsection 10.1.1, the choice of $0 < a < 1$ was arbitrary, we obtain (5.29) and (5.30) in Proposition 5.6 for all $x \ge M$.

On the other hand, let $0 < L < 2^{-3/2}\sqrt{M}$ be a fixed number. Set $\alpha = 1 - 2^{4/3}k^{-1/3}w$ with $-L \le w \le L$ as in Proposition 5.6. From the inequality $\frac{1-\sqrt{1-\gamma^2}}{\gamma} \le 1 - \sqrt{1-\gamma}$ for all $0 \le \gamma \le 1$, we have

$$\rho_\pi = \frac{1-\sqrt{1-\left(\frac{2t}{k}\right)^2}}{\frac{2t}{k}} \le 1 - \sqrt{1-\frac{2t}{k}} \le 1 - \frac{\sqrt{M}}{2^{1/6}k^{1/3}}. \tag{10.50}$$

But

$$\alpha = 1 - \frac{2^{4/3}w}{k^{1/3}} \ge 1 - \frac{2^{4/3}L}{k^{1/3}}. \tag{10.51}$$

Hence $\mathrm{dist}(-\alpha, \Sigma^{(3)}) \ge Ck^{-1/3}$. Thus from (10.42), (10.43) and (10.44), we have

$$|m^{(3)}(-\alpha;k;t) - I| \le Ce^{-\frac{2}{3}x^{3/2}}, \tag{10.52}$$

which together with (10.47) (note that $-\alpha \in \Omega^{(3)}_2 \cup \Omega^{(3)}_3$), implies that

$$\begin{aligned} \left|e^{-t\alpha}\pi_k(-\alpha;t)\right| &\le \left|(-\alpha)^k e^{-t(\alpha-\alpha^{-1})}\right|\left(1 + Ce^{-c|x|^{3/2}}\right) + Ce^{-c|x|^{3/2}}, && (10.53)\\ \left|e^{-t\alpha^{-1}}(-\alpha)^{-k}\pi_k(-\alpha;t) - 1\right| &\le Ce^{-c|x|^{3/2}} + \left|(-\alpha)^{-k}e^{t(\alpha-\alpha^{-1})}\right|Ce^{-c|x|^{3/2}}. && (10.54)\end{aligned}$$

For large k, $\alpha \ge \frac{1}{2}$ for all $-L \le w \le L$, hence using (10.26), we obtain as in (10.27) and (10.28),

$$\left|(-\alpha)^k e^{-t(\alpha-\alpha^{-1})}\right| = e^{kF(-\alpha;\frac{2t}{k})} \le e^{-2wx+\frac{32}{2}|w|^3} \le Ce^{c|x|}, \tag{10.55}$$

and

$$\left|(-\alpha)^{-k}e^{t(\alpha-\alpha^{-1})}\right| = e^{kF(-\alpha^{-1};\frac{2t}{k})} \le Ce^{c|x|}. \tag{10.56}$$

Thus from (10.53) and (10.54), we obtain (5.29), (5.30) in Proposition 5.6.

10.1.3 Case $k - M2^{-1/3}k^{1/3} \leq 2t \leq k$ for some $M > 0$.

In this case, as $k \to \infty$, the point $z = -\rho_\pi$ on the deformed contour $\Sigma^{(3)}$ defined in (10.33) is approaching to $z = -1$ rapidly and we need a special attention to the neighborhood of $z = -1$. More precisely, we need so called parametrix for the RHP, which is an approximate local solution.

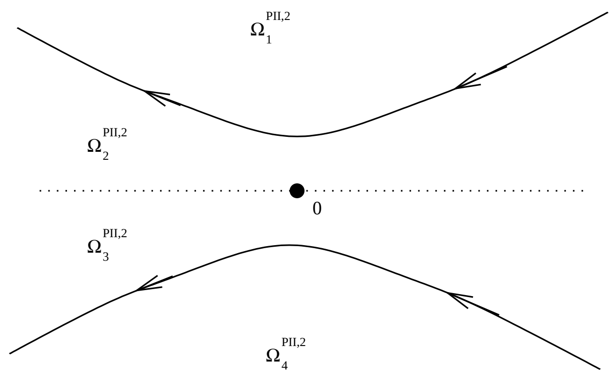

Figure 4: $\Sigma^{PII,2}$ and $\Omega_j^{PII,2}$

Let $\Sigma^{PII,2} = \Sigma_1^{PII,2} \cup \Sigma_2^{PII,2}$ be a contour of the general shape indicated in Figure 4. Asymptotically for large z, the curves are straight lines of angle less than $\pi/3$. For more precise discussions, see the paragraph after (2.18) of [4]. We will define the exact shape of $\Sigma^{PII,2}$ below. Define $m^{PII,2}(z;x)$ by

$$\begin{cases} m^{PII,2}(z,x) = m(z;x) \begin{pmatrix} 1 & e^{-2i(\frac{4}{3}z^3+xz)} \\ 0 & 1 \end{pmatrix} & \text{in } \Omega_2^{PII,2}, \\ m^{PII,2}(z,x) = m(z;x) \begin{pmatrix} 1 & 0 \\ e^{2i(\frac{4}{3}z^3+xz)} & 1 \end{pmatrix} & \text{in } \Omega_3^{PII,2}, \\ m^{PII,2}(z,x) = m(z;x) & \text{in } \Omega_1^{PII,2}, \Omega_4^{PII,2}, \end{cases} \tag{10.57}$$

where $m(z;x)$ is the solution of RHP for PII equation given in (2.15). Then $m^{PII,2}$ solves the RHP,

$$\begin{cases} m^{PII,2} & \text{is analytic in } \mathbb{C} \setminus \Sigma^{PII,2}, \\ m_+^{PII,2} = m_-^{PII,2} \begin{pmatrix} 1 & -e^{-2i(\frac{4}{3}z^3+xz)} \\ 0 & 1 \end{pmatrix} & \text{in } \Sigma_1^{PII,2}, \\ m_+^{PII,2} = m_-^{PII,2} \begin{pmatrix} 1 & 0 \\ e^{2i(\frac{4}{3}z^3+xz)} & 1 \end{pmatrix} & \text{in } \Sigma_2^{PII,2}, \\ m^{PII,2} = I + O\left(\frac{1}{z}\right) & \text{as } z \to \infty. \end{cases} \tag{10.58}$$

Also $m_1^{PII,2}(x)$ defined by $m^{PII,2}(z;x) = I + \frac{m_1^{PII,2}(x)}{z} + O(z^{-2})$ satisfies

$$m_1^{PII,2}(x) = m_1(x) \tag{10.59}$$

Set x by

$$\frac{2t}{k} = 1 - \frac{x}{2^{1/3}k^{2/3}}. \tag{10.60}$$

We define $\Sigma^{(3)}$ and $m^{(3)}$ as in (10.33) and (10.34). Let $\mathcal{O}$ be the the ball of radius ϵ around $z = -1$, where $\epsilon > 0$ is a small fixed number. Define the map

$$\lambda(z) := -i2^{-4/3}k^{1/3}\frac{1}{2}(z - z^{-1}) \tag{10.61}$$

in $\mathcal{O}$. Define $\Sigma^{PII,2}$ by $\Sigma^{PII,2} \cap \lambda(\mathcal{O}) := \lambda(\Sigma^{(3)} \cap \mathcal{O})$ and extend smoothly outside $\lambda(\mathcal{O})$ as indicated in (5.29) of [4]. And define $m^{PII,2}$ as above using this contour. Now we define the parametrix by

$$\begin{cases} m_p(z;k;t) = m^{PII,2}(\lambda(z), x) & \text{in } \mathcal{O} \setminus \Sigma^{(3)}, \\ m_p(z;k;t) = I & \text{in } \bar{\mathcal{O}}^c \setminus \Sigma^{(3)}. \end{cases} \tag{10.62}$$

It is proved in (5.25)-(5.34) in [4] that if we take ϵ small enough but fix it, then the ratio

$$R(z;k;t) := m^{(3)}m_p^{-1} \tag{10.63}$$

solves a new RHP

$$\begin{cases} R(z;k;t) & \text{is analytic in } \mathbb{C} \setminus \Sigma_R, \\ R_+(z;k;t) = R_-(z;k;t)v_R(z;k;t), & \text{on } \Sigma_R, \\ R(z;k;t) = I + O(\frac{1}{z}), & \text{as } z \to \infty, \end{cases} \tag{10.64}$$

where $\Sigma_R := \partial\mathcal{O} \cup \Sigma^{(3)}$, and the jump matrix satisfies

$$\begin{cases} \|v_R - I\|_{L^\infty} \leq \frac{C}{k^{2/3}} & \text{on } \mathcal{O} \cap \Sigma^{(3)}, \\ \|v_R - I\|_{L^\infty} \leq Ce^{-ck} & \text{on } \mathcal{O}^c \cap \Sigma^{(3)}, \\ \|v_R - I + \frac{m_1^{PII,2}(x)}{\lambda(z)}\|_{L^\infty} \leq \frac{C}{k^{2/3}} & \text{on } \partial\mathcal{O}, \text{ as } k \to \infty, \end{cases} \tag{10.65}$$

with some constants C and c which may depend on M. Set $w_R := v_R - I$. Using (10.15) which holds generally, we have (see (5.35) and the preceding calculations in [4])

$$\begin{aligned} R(z;k;t) &= I + \frac{1}{2\pi i}\int_{\Sigma_R} \frac{((I - C_{w_R})^{-1}I)(s)(v_R(s) - I)}{s - z}ds \\ &= I + \frac{1}{2\pi i}\int_{\Sigma_R} \frac{v_R(s) - I}{s - z}ds + \frac{1}{2\pi i}\int_{\Sigma_R} \frac{[(I - C_{w_R})^{-1}C_{w_R}I](s)w_R(s)}{s - z}ds \end{aligned} \tag{10.66}$$

Since $\|(I - C_{w_R})^{-1}\|_{L^2(\Sigma_R)\to L^2(\Sigma_R)} \leq C$, and $|\partial\mathcal{O}| \leq \epsilon C$, $|\mathcal{O} \cap \Sigma^{(3)}| \leq \epsilon C$ and $|\mathcal{O}^c \cap \Sigma^{(3)}| \leq C$, using (10.65), the absolute value of the second integral is less than or equal to

$$\begin{aligned} &\frac{C}{\operatorname{dist}(z, \Sigma_R)}\|(I - C_{w_R})^{-1}\|_{L^2(\Sigma_R)\to L^2(\Sigma_R)}\|C_{w_R}I\|_{L^2(\Sigma_R)}\|w_R\|_{L^2(\Sigma_R)} \\ &\leq \frac{C}{\operatorname{dist}(z, \Sigma_R)}\|w_R\|^2_{L^2(\Sigma_R)} \\ &\leq \frac{\epsilon C}{\operatorname{dist}(z, \Sigma_R)k^{2/3}}. \end{aligned} \tag{10.67}$$

Similarly, the first integral satisfies

$$\left|\frac{1}{2\pi i}\int_{\Sigma_R}\frac{v_R(s)-I}{s-z}ds+\frac{m_1^{PII,2}(x)}{2\pi i}\int_{\partial\mathcal{O}}\frac{1}{\lambda(s)(s-z)}ds\right|\le\frac{\epsilon C}{\mathrm{dist}(z,\Sigma_R)k^{2/3}}. \tag{10.68}$$

Hence

$$\left|m^{(3)}(z;k;t)\big(m_p(z;k;t)\big)^{-1}-I+\frac{m_1^{PII,2}(x)}{2\pi i}\int_{\partial\mathcal{O}}\frac{1}{\lambda(s)(s-z)}ds\right|\le\frac{\epsilon C}{\mathrm{dist}(z,\Sigma_R)k^{2/3}}. \tag{10.69}$$

For $z=0$, from (10.62) and (10.63), $R(0)=m^{(3)}(0)$, and $\mathrm{dist}(0,\Sigma_R)\ge c_1>0$. Note that $\lambda(s)$ is analytic in $\mathcal{O}$ except at $s=-1$, and

$$\lambda(s)=-i2^{-4/3}k^{1/3}[(s+1)+\frac{1}{2}(s+1)^2+\cdots],\quad s\sim-1. \tag{10.70}$$

By a residue calculation, we have

$$m^{(3)}(0;k;t)=I+\frac{i2^{4/3}m_1^{PII,2}(x)}{k^{1/3}}+O(\frac{1}{k^{2/3}}). \tag{10.71}$$

Thus using (2.17) and (2.18), from (5.5), (5.6), (10.7), (10.8) and (10.34), we obtain Proposition 5.1 (iii) of the case when $0\le x\le M$.

We now prove Proposition 5.2 when $x\ge0$. Since the choice of M was arbitrary in our calculations, for fixed x, we choose M large enough so that $0\le x<M$. Let $z\in\mathbb{C}\setminus\Sigma$ be fixed. We first assume $|z|<1$. By modifying the contour $\Sigma^{(3)}$, if necessary, as in (10.49) and the following paragraph, we have $z\in\Omega_1^{(3)}$ and $\mathrm{dist}(z,\Sigma^{(3)})\ge c_1>0$. Thus from (10.69), $|R(z)-I|\le Ck^{-1/3}$ with some constant C depending on x. Thus from (5.6), (10.7), (10.34) and (10.63), we obtain the first limit of (5.13). Similar calculation applies to the case $|z|>1$, and we obtain the first limit of (5.14). The second limits of (5.13) and (5.14) follow from the first limits of (5.14) and (5.13), respectively.

Finally, we prove Proposition 5.4 when $x>0$. Set

$$\alpha=1-\frac{2^{4/3}w}{k^{1/3}},\qquad w\text{ fixed}, \tag{10.72}$$

and

$$\frac{2t}{k}=1-\frac{x}{2^{1/3}k^{2/3}},\qquad x>0\text{ fixed}. \tag{10.73}$$

In this case, $-\alpha\in\mathcal{O}$. By a residue calculation again, for w not equal to 0,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\mathcal{O}}\frac{1}{\lambda(s)(s+\alpha)}ds&=\frac{i2^{4/3}}{(-1+\alpha)k^{1/3}}+\frac{1}{\lambda(-\alpha)}\\&=-\frac{i}{w}+\frac{i}{w+\frac{2^{1/3}}{k^{1/3}}w^2+\cdots}=O(\frac{1}{k^{1/3}}).\end{aligned} \tag{10.74}$$

When $w=0$, similarly we have the same order. On the other hand, since

$$\rho_\pi=\frac{1-\sqrt{1-\left(\frac{2t}{k}\right)^2}}{\frac{2t}{k}}=1-\frac{2^{1/3}\sqrt{x}}{k^{1/3}}+\frac{x}{2^{1/3}k^{2/3}}+O(\frac{1}{k}), \tag{10.75}$$

we have $\mathrm{dist}(-\alpha, \Sigma_R) \geq Ck^{-2/3}$. Thus we have

$$|R(-\alpha;k;t) - I| \leq \epsilon C. \tag{10.76}$$

Using $\lambda(-\alpha) \sim -iw$, from (10.62) and (10.63), we have

$$\lim_{k\to\infty} m^{(3)}(-\alpha;k;t) = m^{PII,2}(-iw,x) \tag{10.77}$$

since ϵ is arbitrarily small. From the conditions on t and α, we have

$$\lim_{k\to\infty} \alpha^k e^{-t(\alpha-\alpha^{-1})} = e^{\frac{8}{3}w^3-2xw}. \tag{10.78}$$

Using (10.34), we have

$$\lim_{k\to\infty} m^{(2)}(-\alpha;k;t) = m^{PII,2}(-iw,x), \qquad -\alpha \in \Omega_1^{(3)}, \Omega_4^{(3)}, \tag{10.79}$$

$$\lim_{k\to\infty} m^{(2)}(-\alpha;k;t) = m^{PII,2}(-iw,x)\begin{pmatrix}1 & -e^{\frac{8}{3}w^3-2xw}\\ 0 & 1\end{pmatrix}, \quad -\alpha \in \Omega_2^{(3)}, \tag{10.80}$$

$$\lim_{k\to\infty} m^{(2)}(-\alpha;k;t) = m^{PII,2}(-iw,x)\begin{pmatrix}1 & 0\\ -e^{-\frac{8}{3}w^3+2xw} & 1\end{pmatrix}, \quad -\alpha \in \Omega_3^{(3)}. \tag{10.81}$$

Thus, using (10.57), for each fixed w and x, we have

$$\lim_{k\to\infty} m^{(2)}(-\alpha;k;t) = m(-iw;x). \tag{10.82}$$

From (10.7)-(10.10) and (10.78), we obtain Proposition 5.4 of the case when $x > 0$.

10.2 When $(2t)/k > 1$.

Through this subsection, we set

$$\gamma := \frac{2t}{k} > 1. \tag{10.83}$$

We need some definition. Define $0 < \theta_c < \pi$ by $\sin^2 \frac{\theta_c}{2} = \frac{1}{\gamma}$. Define a probability measure on an arc,

$$d\mu(\theta) := \frac{\gamma}{\pi}\cos(\frac{\theta}{2})\sqrt{\frac{1}{\gamma} - \sin^2(\frac{\theta}{2})}d\theta, \qquad -\theta_c \leq \theta \leq \theta_c, \tag{10.84}$$

and define a constant

$$l := -\gamma + \log\gamma + 1. \tag{10.85}$$

Now we define so-called g-function,

$$g(z;k;t) := \int_{-\theta_c}^{\theta_c} \log(z - e^{i\theta})d\mu(\theta), \qquad z \in \mathbb{C} \setminus \Sigma \cup (-\infty, -1]. \tag{10.86}$$

The measure $d\mu(\theta)$ is the equilibrium measure of a certain variational problem and the constant l is a related constant (see (4.6) and Lemma 4.3 in [4]). For each $|\theta| \leq \theta_c$, the branch is chosen such that $\log(z - e^{i\theta})$ is analytic in $\mathbb{C}\setminus(-\infty,-1]\cup\{e^{i\phi}:-\pi\le\phi\le\theta\}$ and behaves like $\log z$ as $z\in\mathbb{R}\to+\infty$. Basic properties of $g(z)$ is summarized in Lemma 4.2 of [4]. In general, the role of g-function in RHP, first introduced in [19] and then generalized in [17], is to replace exponentially growing terms in the jump matrix by oscillating or exponentially decaying terms. In [13], the authors introduced g-function of the form similar to (10.86) to analyze RHP associated to orthogonal polynomials on the real line. The above g-function (10.86) is an adaptation of their work to the circle case. When $0\le\gamma\le 1$, the related equilibrium measure is

$$d\mu(\theta)=\frac{1}{2\pi}(1+\gamma\cos\theta)d\theta,\qquad -\pi\le\theta<\pi, \tag{10.87}$$

with the related constant $l=0$, and hence

$$g(z)=\begin{cases}\log z-\frac{\gamma}{2z}, & |z|>1, z\notin(-\infty,-1)\\ -\frac{\gamma}{2}z+\pi i, & |z|<1.\end{cases} \tag{10.88}$$

Since $g(z)$ is explicit in this case, we did not introduce it of the form (10.86) in the previous subsection.

Write $\Sigma=C_1\cup\overline{C_2}$ where $C_2:=\{e^{i\theta}:-\theta_c<\theta<\theta_c\}$ and $C_1:=\Sigma\setminus\overline{C_2}$. Define $m^{(1)}(z;k;t)$ by

$$m^{(1)}(z;k;t):=e^{\frac{kl}{2}\sigma_3}Y(z;k;t)e^{-kg(z;k;t)\sigma_3}e^{-\frac{kl}{2}\sigma_3}, \tag{10.89}$$

where $\sigma_3=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Then $m^{(1)}$ solves (see (6.1) in [4]) a new RHP

$$\begin{cases}m^{(1)}(z;k;t)\qquad\text{is analytic in}\quad\mathbb{C}\setminus\Sigma,\\ m^{(1)}_+(z;k;t)=m^{(1)}_-(z;k;t)\begin{pmatrix}e^{-2k\widetilde{\alpha}(z;k;t)} & (-1)^k\\ 0 & e^{2k\widetilde{\alpha}(z;k;t)}\end{pmatrix}\quad\text{on}\quad C_2,\\ m^{(1)}_+(z;k;t)=m^{(1)}_-(z;k;t)\begin{pmatrix}1 & (-1)^k e^{-2k\widetilde{\alpha}_-(z;k;t)}\\ 0 & 1\end{pmatrix}\quad\text{on}\quad C_1,\\ m^{(1)}(z;k;t)=I+O\left(\frac{1}{z}\right)\quad\text{as}\quad z\to\infty,\end{cases} \tag{10.90}$$

where $\widetilde{\alpha}(z;k;t)$ is defined by (see Lemma 6.1 in [4])

$$\widetilde{\alpha}(z;k;t):=-\frac{\gamma}{4}\int_{e^{i\theta_c}}^{z}\frac{s+1}{s^2}\sqrt{(s-e^{i\theta_c})(s-e^{-i\theta_c})}ds,\qquad \xi:=e^{i\theta_c}. \tag{10.91}$$

(Notation : We use $\widetilde{\alpha}$ here instead of α in [4] to avoid confusion with α in (10.138) below.) The branch is chosen such that $\sqrt{(s-e^{i\theta_c})(s-e^{-i\theta_c})}$ is analytic in $\mathbb{C}\setminus\overline{C_1}$ and behaves like s as $s\in\mathbb{R}\to+\infty$.

Define $m^{(2)}(z;k;t)$ as in (10.3). Then $m^{(2)}$ solves a new RHP, normalized as $z\to\infty$, with the jump matrices

$$\begin{cases}v^{(2)}(z;k;t)=\begin{pmatrix}1 & -e^{-2k\widetilde{\alpha}(z;k;t)}\\ e^{2k\widetilde{\alpha}(z;k;t)} & 0\end{pmatrix}\qquad\text{on}\quad C_2,\\ v^{(2)}(z;k;t)=\begin{pmatrix}e^{-2k\widetilde{\alpha}_-(z;k;t)} & -1\\ 1 & 0\end{pmatrix}\qquad\text{on}\quad C_1.\end{cases} \tag{10.92}$$

Through the changes $Y \to m^{(1)} \to m^{(2)}$, we have

$$\mathrm{Y}_{11}(z;k;t) = -e^{kg(z;k;t)} m_{12}^{(2)}(z;k;t), \qquad |z| < 1, \tag{10.93}$$

$$\mathrm{Y}_{21}(z;k;t) = -(-1)^k e^{kg(z;k;t)+kl} m_{22}^{(2)}(z;k;t), \qquad |z| < 1, \tag{10.94}$$

$$\mathrm{Y}_{11}(z;k;t) = e^{kg(z;k;t)} m_{11}^{(2)}(z;k;t), \qquad |z| > 1, \tag{10.95}$$

$$\mathrm{Y}_{21}(z;k;t) = (-1)^k e^{kg(z;k;t)+kl} m_{21}^{(2)}(z;k;t), \qquad |z| > 1. \tag{10.96}$$

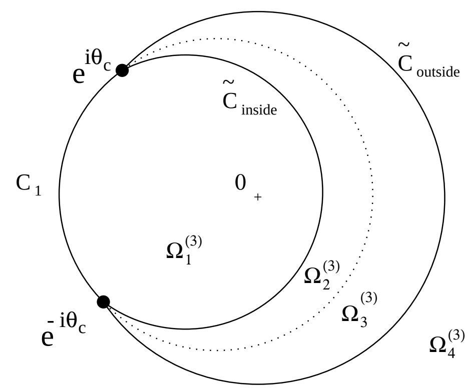

Figure 5: $\Sigma^{(3)}$ and $\Omega^{(3)}$when $\gamma > 1$

Set $\Sigma^{(3)} := \overline{C_1} \cup \widetilde{C}_{inside} \cup \widetilde{C}_{outside}$ as in the Figure 5, which divides $\mathbb{C}$ into four regions, $\Omega_j^{(3)}$, $j = 1, \cdots, 4$. Again, there is a certain freedom choosing the shape of $\widetilde{C}_{inside}$ and $\widetilde{C}_{outside}$. For example, $\widetilde{C}_{inside}$ (resp., $\widetilde{C}_{outside}$) can be any smooth curve lying in $\Omega_2^{(3)}$ (resp. $\Omega_3^{(3)}$) connecting $e^{i\theta_c}$ and $e^{-i\theta_c}$. Precise requirement is given in [4] (see also (10.99)-(10.100) below). Define $m^{(3)}(z;k;t)$ by

$$\begin{cases} m^{(3)} = m^{(2)} \begin{pmatrix} 1 & -e^{-2k\widetilde{\alpha}(z;k;t)} \\ 0 & 1 \end{pmatrix}^{-1} & \text{in} \quad \Omega_2^{(3)}, \\ m^{(3)} = m^{(2)} \begin{pmatrix} 1 & 0 \\ e^{2k\widetilde{\alpha}(z;k;t)} & 1 \end{pmatrix} & \text{in} \quad \Omega_3^{(3)}, \\ m^{(3)} = m^{(2)} & \text{in} \quad \Omega_1^{(3)}, \Omega_4^{(3)}. \end{cases} \tag{10.97}$$

Then $m^{(3)}$ solves a RHP, normalized as $z \to \infty$, with the jump matrix given by

$$v^{(3)}(z;k;t) = \begin{cases} \begin{pmatrix} 1 & -e^{-2k\widetilde{\alpha}(z;k;t)} \\ 0 & 1 \end{pmatrix} & \text{on} \quad \widetilde{C}_{inside}, \\ \begin{pmatrix} 1 & 0 \\ e^{2k\widetilde{\alpha}(z;k;t)} & 1 \end{pmatrix} & \text{on} \quad \widetilde{C}_{outside}, \\ \begin{pmatrix} e^{-2k\widetilde{\alpha}-(z;k;t)} & -1 \\ 1 & 0 \end{pmatrix} & \text{on} \quad C_1. \end{cases} \tag{10.98}$$

From the properties of $g(z)$, it is proved in [4] that

$$e^{-k\widetilde{\alpha}-(z;k;t)} \to 0 \quad \text{as} \quad k\to\infty, \qquad z\in C_1, e^{-k\widetilde{\alpha}(z;k;t)} \to 0 \quad \text{as} \quad k\to\infty, \qquad z\in \widetilde{C}_{inside}, \tag{10.99}$$

$$e^{k\widetilde{\alpha}(z;k;t)} \to 0 \quad \text{as} \quad k\to\infty, \qquad z\in \widetilde{C}_{outside}. \tag{10.100}$$

The choice of $\widetilde{C}_{inside}$ and $\widetilde{C}_{outside}$ is precisely for these properties. Here the convergence is uniform for any compact part of the each contour *away* from the end points $e^{i\theta_c}$ and $e^{-i\theta_c}$, but is not uniform on the while contour. This gives rise the technical difficulty which will be overcome below using the idea of parametrix. Formally $v^{(3)} \to v^\infty$ as $k\to\infty$ where

$$\begin{cases} v^\infty(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{on} \quad \widetilde{C}_{inside}\cup\widetilde{C}_{outside}, \\ v^\infty(z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{on} \quad C_1. \end{cases} \tag{10.101}$$

Thus we expect that $m^{(3)}$ converges to m^∞, the solution of the RHP $m^\infty_+ = m^\infty_- v^\infty$ with $m^\infty \to I$ as $z\to\infty$. The solution m^∞ is easily given by (see Lemma 6.2 in [4])

$$m^\infty(z) = \begin{pmatrix} \frac{1}{2}(\beta+\beta^{-1}) & \frac{1}{2i}(\beta-\beta^{-1}) \\ -\frac{1}{2i}(\beta-\beta^{-1}) & \frac{1}{2}(\beta+\beta^{-1}) \end{pmatrix}, \tag{10.102}$$

where $\beta(z) := \big(\frac{z-e^{i\theta_c}}{z-e^{-i\theta_c}}\big)^{1/4}$, which is analytic $\mathbb{C}\setminus\overline{C}_1$ and $\beta\sim +1$ as $z\in\mathbb{R}\to+\infty$

10.2.1 Case $2t \geq ak$ for some $a > 1$.

We introduce parametrix $m_p(z)$ as follows. In the neighborhood,$\mathcal{O}$, of size ϵ around the points $e^{i\theta_c}$ and $e^{-i\theta_c}$, $m_p(z)$ is constructed using Airy function such that $(m_p(z))_+ = (m_p(z))_- v^{(3)}(z)$ for $z\in\Sigma^{(3)}\cap\mathcal{O}$, and $\|m_p(m^\infty)^{-1} - I\|_{L^\infty(\partial\mathcal{O})} = O(k^{-1})$. In $\mathbb{C}\setminus\mathcal{O}$, we set $m_p(z) := m^\infty(z)$. Then the ratio $R(z) := m^{(3)}(z)m_p^{-1}(z)$ has no jump on $\Sigma^{(3)}\cap\mathcal{O}$, and has a jump $v_R := w_R + I$ converging to I uniformly of order $O(k^{-1})$ on $\partial\mathcal{O}$, and of order $O(e^{-ck})$ on $\Sigma^{(3)}\cap\mathcal{O}^c$ as $k\to\infty$. This implies that $R(z) = I + O(k^{-1})$ for any $z\in\mathbb{C}\setminus\Sigma_p$, $\Sigma_p := (\Sigma^{(3)}\cap\mathcal{O}^c)\cup\partial\mathcal{O}$. Moreover following the arguments in Section 8 of [14], the error is uniform up to the boundary in each open regions in $\mathbb{C}\setminus\Sigma_p$. In particular, for $z\in\overline{\Omega_1^{(3)}}\cup\overline{\Omega_4^{(3)}}$,

$$m^{(3)}(z) = \big(I + O(k^{-1})\big)m^\infty(z). \tag{10.103}$$

Here the error is uniform for $ak \leq 2t \leq bk$ for some $0 < a < b$. For $(2t)/k\to\infty$ case, by shrinking the size of $\mathcal{O}$ properly, we again obtain the unform error (see [2]). Therefore for any $a > 0$, we obtain the uniform error for (10.103) for $ak \leq 2t$.

When $z = 0$, $\beta(0) = -ie^{i\theta_c/2}$ and $g(0) = \pi i$ (Lemma 4.2 (vi) of [4]), Also (10.97) says $m^{(3)}(0) = m^{(2)}(0)$. Thus Proposition 5.1 (v) follows from (10.93) and (10.94).

Now we consider Proposition 5.6 of the case whe $x \leq -2^{1/3}(a-1)k^{2/3}$. For $z = -\alpha$ real, $|\beta(-\alpha)| = 1$, so $m^\infty(-\alpha)$ is bounded. Hence from (10.97) and (10.93),(10.95), we have for $\alpha \geq 1$,

$$|\pi_k(-\alpha;k;t)| \leq C|e^{kg(-\alpha;k;t)}|. \tag{10.104}$$

The rest of the proof is the same as in the following subsubsetion.

10.2.2 Case $k + M2^{-1/3}k^{1/3} \le 2t \le ak$ for some $a > 1$ and $M > M_0$.

In this case, the points $e^{i\theta_c}$ and $e^{-i\theta_c}$ is allowed to approach -1, but the rate is restricted :

$$|e^{i\theta_c} + 1| = 2\left(1 - \frac{k}{2t}\right)^{1/2} \ge \frac{\sqrt{M}}{k^{1/3}}, \tag{10.105}$$

for large k. We now take neighborhood $\mathcal{O}$ of size $\epsilon\sqrt{\frac{2t}{k} - 1}$ around $e^{i\theta_c}$ and $e^{-i\theta_c}$. From (10.105), $\mathcal{O}$ consists of two disjoint disks and their boundary do not touch the real axis. We introduce the same parametrix m_p as in the previous subsubsection. Then we have a similar results : there is $M_0 > 0$ such that for $M > M_0$,

$$m^{(3)}(z) = \left(I + O\left(\frac{1}{k(\frac{2t}{k} - 1)}\right)\right)m^{\infty}(z), \tag{10.106}$$

for $z \in \overline{\Omega_1^{(3)}} \cup \overline{\Omega_4^{(3)}}$.

When $z = 0$, as in the previous subsubsection, we obtain Proposition 5.1 (iv).

Now we prove Proposition 5.6 when $x \le -M$. As in the previous subsubsection, we have

$$|\pi_k(-\alpha; k; t)| \le C|e^{kg(-\alpha;k;t)}|. \tag{10.107}$$

But note that

$$\begin{aligned} \mathrm{Re}\big(g(-\alpha)\big) &= \frac{1}{2}\int_{-\theta_c}^{\theta_c} \log(1 + \alpha^2 + 2\alpha\cos\theta)d\mu(\theta) \\ &= \log 2 + \frac{1}{2}\log\frac{\alpha}{\gamma} + I(s), \end{aligned} \tag{10.108}$$

where

$$I(s) = \frac{1}{\pi}\int_{-1}^{1} \log(s^2 - x^2)\sqrt{1 - x^2}dx, \qquad s := \frac{\sqrt{\gamma}(1 + \alpha)}{2\sqrt{\alpha}} > 1. \tag{10.109}$$

The inequality $s > 1$ follows from arithmetic-geometric mean inequality and the assumption $\gamma > 1$. Residue calculation gives us

$$I'(y) = \frac{1}{\pi}\int_{-1}^{1} \frac{2y}{y^2 - x^2}\sqrt{1 - x^2}dx = 2y - 2\sqrt{y^2 - 1}, \qquad y > 1. \tag{10.110}$$

Integrating from 1 to $s > 1$, we have

$$I(s) = s^2 - 1 - 2\int_1^s \sqrt{y^2 - 1}dy + I(1). \tag{10.111}$$

The constant $I(1)$ can be evaluated (cf. Lemma 4.3 (ii)-(a) in [4]) :

$$I(1) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(\sin^2\theta)\sin^2\theta d\theta = \frac{1}{2} - \log 2. \tag{10.112}$$

Thus we have

$$\mathrm{Re}\big(g(-\alpha)\big) = -\frac{1}{2} + \frac{1}{2}\log\frac{\alpha}{\gamma} + s^2 - s\sqrt{s^2-1} + \log(s+\sqrt{s^2-1}). \tag{10.113}$$

Assume $0<\alpha\le 1$. We change the variables γ, α into s, ξ where s is define in (10.109) and

$$\xi := \left(\frac{\gamma}{\alpha}\right)^{1/2} > 1. \tag{10.114}$$

Then

$$F(\xi) := g(-\alpha) - \frac{\gamma}{2}\alpha = -\frac{1}{2} - \log\xi - \frac{1}{2}\xi^2 + 2s\xi - s^2 - s\sqrt{s^2-1} + \log(s+\sqrt{s^2-1}). \tag{10.115}$$

Differentiating with respect to ξ,

$$F'(\xi) = -\frac{1}{\xi} - \xi + 2s. \tag{10.116}$$

Thus the maximum of F occurs at $\xi = s+\sqrt{s^2-1}$. But $F(s+\sqrt{s^2-1})=0$, hence $F(\xi)\le 0$. Thus we obtain

$$|e^{-t\alpha}\pi_k(-\alpha;k)| \le Ce^{k\,\mathrm{Re}\left(g(-\alpha;k;t)-\frac{\gamma}{2}\alpha\right)} \le C, \qquad 0<\alpha\le 1. \tag{10.117}$$

For $\alpha\ge 1$, note that

$$\mathrm{Re}(g(-\alpha)) = \log\alpha + \mathrm{Re}(g(-\alpha^{-1})). \tag{10.118}$$

Thus using (10.117), we have

$$|e^{-t\alpha^{-1}}(-\alpha)^{-k}\pi_k(-\alpha;k)| \le Ce^{k\,\mathrm{Re}\left(g(-\alpha;k;t)-\frac{\gamma}{2}\alpha^{-1}-\log\alpha\right)} = Ce^{k\,\mathrm{Re}\left(g(-\alpha^{-1};k;t)-\frac{\gamma}{2}\alpha^{-1}\right)} \le C. \tag{10.119}$$

10.2.3 Case $k<2t\le k+M2^{-1/3}k^{1/3}$ for some $M>0$.

Set

$$g^{PII}(z) := \frac{4}{3}\big(z^2+\frac{x}{2}\big)^{3/2}, \tag{10.120}$$

which is analytic in $\mathbb{C}\setminus\big[-\sqrt{\frac{-x}{2}},\sqrt{\frac{-x}{2}}\big]$ and behaves like $\frac{4}{3}z^3+xz+\frac{x^2}{8z}+O(z^{-3}) =: \theta_{PII}(z)+Oz^{-1}$ as $z\to+\infty$. Let $\Sigma^{PII,3} := \cup_{j=1}^5\Sigma_j^{PII,3}$ as shown in Figure 6. The angles of the rays with the real line are between 0 and

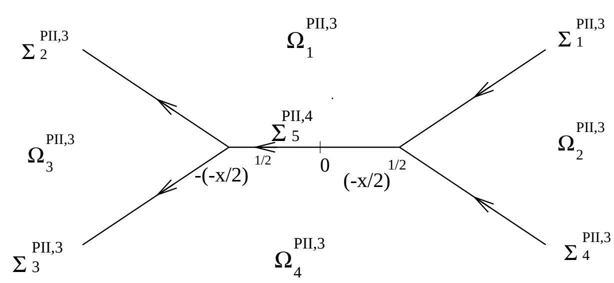

Figure 6: $\Sigma^{PII,3}$ and $\Omega_j^{PII,3}$

$\pi/3$. Recall that $m(z;x)$ solves (2.15) the RHP for PII equation. Define $m^{PII,3}(z;x)$ by

$$\begin{cases} m^{PII,3} = m(z;x)e^{i(g^{PII}-\theta_{PII})\sigma_3}, & z \in \Omega_1^{PII,3}, \Omega_4^{PII,3}, \\ m^{PII,3} = m(z;x)\begin{pmatrix} 1 & e^{-2i\theta_{PII}} \\ 0 & 1 \end{pmatrix} e^{i(g^{PII}-\theta_{PII})\sigma_3}, & z \in (\Omega_2^{PII,3} \cup \Omega_3^{PII,3}) \cap \mathbb{C}_-, \\ m^{PII,3} = m(z;x)\begin{pmatrix} 1 & 0 \\ e^{2i\theta_{PII}} & 1 \end{pmatrix} e^{i(g^{PII}-\theta_{PII})\sigma_3}, & z \in (\Omega_2^{PII,3} \cup \Omega_3^{PII,3}) \cap \mathbb{C}_+. \end{cases} \tag{10.121}$$

Then $m^{(3)}$ solves the RHP (see (2.25) in [4]) normalized at ∞ with the jump matrix

$$v^{(3)}(z;k;t) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2ig^{PII}} & 1 \end{pmatrix} & \text{on} \quad \Sigma_1^{PII,3}, \Sigma_2^{PII,3} \\ \begin{pmatrix} 1 & -e^{-2ig^{PII}} \\ 0 & 1 \end{pmatrix} & \text{on} \quad \Sigma_3^{PII,3}, \Sigma_4^{PII,3} \\ \begin{pmatrix} e^{-2ig_-^{PII}} & -1 \\ 1 & 0 \end{pmatrix} & \text{on} \quad \Sigma_5^{PII,3}. \end{cases} \tag{10.122}$$

Also we have

$$m_1(x) = m_1^{PII,3}(x) - (\frac{ix^2}{8})\sigma_3. \tag{10.123}$$

where $m^{PII,3}(z;x) = I + \frac{m_1^{PII,3}(x)}{z} + O(z^{-2})$ as $z \to \infty$.

Set x by

$$\frac{2t}{k} = 1 - \frac{x}{2^{1/3}k^{2/3}}. \tag{10.124}$$

Hence we have $-M \le x < 0$ in this subsubsection. Define the parametrix

$$\begin{cases} m_p(z;k;t) = m^{PII,3}\big(\lambda(z), x\big) & \text{in} \quad \mathcal{O} \setminus \Sigma^{(3)}, \\ m_p(z;k;t) = I & \text{in} \quad \bar{\mathcal{O}}^c \setminus \Sigma^{(3)}, \end{cases} \tag{10.125}$$

where $\lambda(z)$ is defined in (10.61) and $\mathcal{O}$ is a small neighborhood of size $\epsilon > 0$ around $z = -1$ (see [4] case (iii) of Section 6 for details). As in Subsubsection 10.1.3, the ratio $R(z;k;t) := m^{(3)}m_p^{-1}$ satisfies a new RHP, normalized at ∞, with the jump matrix v_R which satisfies the estimation (10.65) with now $m_1^{PII,2}(x)$ replaced by $m_1^{PII,3}(x)$. Hence we have

$$\left| m^{(3)}(z;k;t)\big(m_p(z;k;t)\big)^{-1} - I + \frac{m_1^{PII,3}(x)}{2\pi i}\int_{\partial\mathcal{O}} \frac{1}{\lambda(s)(s-z)}ds \right| \le \frac{\epsilon C}{\operatorname{dist}(z,\Sigma_R)k^{2/3}}. \tag{10.126}$$

As in (10.71), we have

$$m^{(3)}(0;k;t) = I + \frac{i2^{4/3}m_1^{PII,3}(x)}{k^{1/3}} + O(\frac{1}{k^{2/3}}). \tag{10.127}$$

From (10.97) and (10.123),

$$m^{(2)}(0;k;t) = I + \frac{i2^{4/3}m_1(x) - 2^{-5/3}x^2\sigma_3}{k^{1/3}} + O(\frac{1}{k^{2/3}}). \tag{10.128}$$

Hence using $e^{kl} = 1 - \frac{x^2}{2^{5/3}k^{1/3}} + O(k^{-2/3})$ and $g(0) = \pi i$, (10.93) and (10.94) yield Proposition 5.1 (iii) of the case when $-M \le x < 0$.

For the proof of Proposition 5.2, note that for each fixed $z \in \mathbb{C} \setminus \Sigma$, we can use the freedom of the shape of $\Sigma^{(3)}$ (and Σ_R) so that $z \in \Omega_1^{(3)} \cup \Omega_4^{(3)}$, and $\operatorname{dist}(z, \Sigma_R) \ge c_1 > 0$. Thus we obtain

$$\lim_{k\to\infty} m^{(2)}(z;k;t) = I, \qquad z \in \mathbb{C} \setminus \Sigma \quad \textit{fixed}. \tag{10.129}$$

From (10.93) and (10.95), we have

$$\lim_{k\to\infty} e^{-kg(z;k;t)}\pi_k(z) = 0, \qquad |z| < 1, \tag{10.130}$$

$$\lim_{k\to\infty} e^{-kg(z;k;t)}\pi_k(z) = 1, \qquad |z| > 1. \tag{10.131}$$

To prove Proposition 5.2, since $\gamma = (2t)/k$, it is enough to show that

$$\lim_{k\to\infty} (-1)^k e^{k[g(z;k;t) + \frac{\gamma}{2}z]} = 1, \qquad |z| < 1, \tag{10.132}$$

$$\lim_{k\to\infty} e^{k[g(z;k;t) + \frac{\gamma}{2}z^{-1} - \log z]} = 1, \qquad |z| > 1. \tag{10.133}$$

But proof of Lemma 4.3 (ii) of [4] says that for $|z| > 1$, $z \notin (-\infty, -1)$,

$$g(z) = \frac{1}{2}\log z - \frac{\gamma}{4}(z + z^{-1}) + \frac{\gamma}{2} + \frac{\gamma}{4}\int_{1_{+0}}^{z} \frac{s+1}{s^2}\sqrt{(s - e^{i\theta_c})(s - e^{-i\theta_c})}ds + g_-(1), \tag{10.134}$$

where the integral is taken over a curve from 1_{+0} to z lying in $\{z \in \mathbb{C} : |z| > 1, z \notin (-\infty, -1)\}$. Here $\sqrt{(s - e^{i\theta_c})(s - e^{-i\theta_c})}$ is analytic in $\mathbb{C} \setminus C_1$ and behaves like s as $s \in \mathbb{R} \to +\infty$, and $\log z$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and is real for $z\mathbb{R}_+$. Calculations in the same proof, together with Lemma 4.2 (viii) of [4], gives us $g_-(1) = -\frac{1}{2} - \frac{1}{2}\log\gamma$. Also using $\sin^2\frac{\theta_c}{2} = \frac{1}{\gamma}$, for $|s| > 1$, $s \notin (-\infty, -1)$,

$$\sqrt{(s - e^{i\theta_c})(s - e^{-i\theta_c})} = (s+1) - \frac{2s}{s+1}(\gamma - 1) + O((\gamma-1)^2). \tag{10.135}$$

Thus expanding in $\gamma - 1$, we have

$$g(z) + \frac{\gamma}{2}z^{-1} - \log z = O((\gamma-1)^2) = O(k^{-4/3}), \tag{10.136}$$

which implies (10.133). Similar calculations using for $|z| < 1$, $z \notin (-1, 0]$,

$$g(z) = \frac{1}{2}\log z - \frac{\gamma}{4}(z + z^{-1}) + \frac{\gamma}{2} + \frac{\gamma}{4}\int_{1_{+0}}^{z} \frac{s+1}{s^2}\sqrt{(s - e^{i\theta_c})(s - e^{-i\theta_c})}ds + g_+(1), \tag{10.137}$$

and $g_+(1) = -\frac{1}{2} - \frac{1}{2}\log z + \pi i$ yield (10.132).

For the proof of Proposition 5.4 when $x < 0$, set

$$\alpha = 1 - \frac{2^{4/3}w}{k^{1/3}}. \tag{10.138}$$

When w and x are fixed, again as in (10.76), we have $\lim_{k\to\infty} R(-\alpha;k;t) = I$, which implies that

$$\lim_{k\to\infty} m^{(3)}(-\alpha;k;t) = m^{PII,3}(-iw, x). \tag{10.139}$$

From (6.8) of [4], we have

$$\lim_{k\to\infty} k\widetilde{\alpha}(-\alpha;k;t) = i\frac{4}{3}\big((-iw)^2+\frac{x}{2}\big)^{3/2} = ig^{PII}(-iw), \tag{10.140}$$

which from (10.97) and (10.121) implies

$$\lim_{k\to\infty} m^{(2)}(-\alpha;k;t) = m(-iw,x)e^{i(g^{PII}(-iw)-\theta_{PII}(-iw))\sigma_3}. \tag{10.141}$$

Now we compute the large k limit of $kg(-\alpha;k;t)-t\alpha$ when $w>0$, and of $kg(-\alpha;k;t)-t\alpha^{-1}-\log\alpha$ when $w<0$. For $-\pi<\theta<\pi$, $\lim_{\epsilon\downarrow 0}\arg(-\alpha+i\epsilon-e^{i\theta}) = \pi+\tan^{-1}(\frac{\sin\theta}{\alpha+\cos\theta})$ where $-\pi<\tan^{-1}\phi<\pi$. Since $\tan^{-1}(\frac{\sin\theta}{\alpha+\cos\theta})$ is odd in θ, we have from (10.113),

$$\begin{aligned}\lim_{\epsilon\downarrow 0} g(-\alpha+i\epsilon) &= \lim_{\epsilon\downarrow 0}\operatorname{Re} g(-\alpha+i\epsilon)+\pi i\\ &= -\frac{1}{2}+\frac{1}{2}\log\frac{\alpha}{\gamma}+s^2-s\sqrt{s^2-1}+\log(s+\sqrt{s^2-1})+\pi i\end{aligned} \tag{10.142}$$

where $s=\frac{\sqrt{\gamma}(1+\alpha)}{2\sqrt{\alpha}}>1$. Under the conditions of γ and α, as $k\to\infty$,

$$\frac{\sqrt{\gamma}(1+\alpha)}{2\sqrt{\alpha}} = 1+\left(\frac{w^2}{2^{1/3}}-\frac{x}{2^{4/3}}\right)\cdot\frac{1}{k^{2/3}}+\frac{2w^3}{k}+O\big(\frac{1}{k^{-4/3}}\big), \tag{10.143}$$

$$\frac{\alpha}{\gamma} = 1-\frac{2^{4/3}w}{k^{1/3}}+\frac{x}{2^{1/3}k^{2/3}}-\frac{2xw}{k}+O\big(\frac{1}{k^{-4/3}}\big). \tag{10.144}$$

Note that $\lim_{\epsilon\downarrow 0} g(-\alpha+i\epsilon)-\lim_{\epsilon\downarrow 0} g(-\alpha-i\epsilon)$ is $2\pi i$ for $\alpha>1$, and is 0 for $0<\alpha<1$. Therefore we obtain

$$\lim_{k\to\infty}(-1)^k e^{kg(-\alpha)-t\alpha} = e^{\frac{4}{3}w^3-xw-\frac{4}{3}(w^2-\frac{x}{2})^{3/2}}, \qquad w>0, \tag{10.145}$$

$$\lim_{k\to\infty}(-1)^k e^{kg(-\alpha)-t\alpha^{-1}-k\log\alpha} = e^{-\frac{4}{3}w^3+xw-\frac{4}{3}(w^2-\frac{x}{2})^{3/2}}, \qquad w<0. \tag{10.146}$$

Also being careful of branch, we have

$$i(g^{PII}(-iw)-\theta_{PII}(-iw)) = \frac{4}{3}w^3-xw-\frac{4}{3}\left(w^2-\frac{x}{2}\right)^{3/2}, \qquad w>0, \tag{10.147}$$

$$i(g^{PII}(-iw)-\theta_{PII}(-iw)) = \frac{4}{3}w^3-xw+\frac{4}{3}\left(w^2-\frac{x}{2}\right)^{3/2}, \qquad w<0. \tag{10.148}$$

Since $\lim_{k\to\infty} e^{kl}=1$, using (10.93)-(10.96) and (10.141), this implies (5.21)-(5.24) when $x<0$.

10.3 Proof of Proposition 5.8

Let $\alpha>1$ be fixed. Set

$$\frac{t}{k} = \frac{\alpha}{\alpha^2+1}-\frac{\alpha(\alpha^2-1)^{1/2}}{(\alpha^2+1)^{3/2}}\cdot\frac{x}{\sqrt{k}}, \qquad x\in\mathbb{R}\setminus\{0\}\text{ is fixed.} \tag{10.149}$$

We are interested in the asymptotics of $e^{-\alpha t}(-\alpha)^l\pi_k(-\alpha^{-1};t)$. Since $\frac{2\alpha}{\alpha^2+1}<1$ and x is fixed, we are in the case of Subsubsection 10.1.1 and/or 10.1.2. We define $m^{(1)}$ and $m^{(2)}$ as in (10.1) and (10.3). Recall that we have certain freedom of the choice of $\Sigma^{(3)}$. We choose a contour passing through saddle points of

$$f(z) := \frac{t}{k}(z-z^{-1})+\log(-z), \tag{10.150}$$

the exponent of the (12) entry of $v^{(2)}$ divided by k. The saddle points (see (10.31) and the following discussions) are $-\rho_\pi$ and $-\rho_\pi^{-1}$, where

$$\rho_\pi = \frac{1-\sqrt{1-(2t/k)^2}}{2t/k} = \frac{1}{\alpha} - \frac{(\alpha^2+1)^{1/2}}{\alpha(\alpha^2-1)^{1/2}} \cdot \frac{x}{\sqrt{k}} + O\big(\frac{1}{k}\big) =: \rho_c + O\big(\frac{1}{k}\big). \tag{10.151}$$

Take $\delta > 0$ and $\epsilon > 0$ small such that $\Sigma_c := \{-\rho_c + is : -k^{\delta-1/2} \le s \le \epsilon k^{\delta-1/2}\}$ lies inside the open unit disc for all $k \ge 1$. Define (see Figure 7) $\Sigma^{(3)} := \Sigma^{(3)}_{in} \cup \Sigma^{(3)}_{out}$ by $\Sigma^{(3)}_{in} := \Sigma_c \cup \Sigma_r$ and $\Sigma^{(3)}_{out} := \{r^{-1}e^{i\phi} : re^{i\phi} \in \Sigma^{(3)}_{in}\}$,

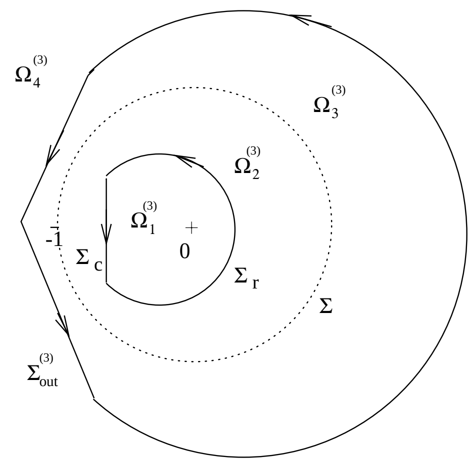

Figure 7: $\Sigma^{(3)}$ and $\Omega^{(3)}$

where $\Sigma_r := \{|-\rho_c + i\epsilon k^{\delta-1/2}|e^{i\theta} : |\theta| < \theta_0\}$, $-\rho_c + i\epsilon k^{\delta-1/2} = |-\rho_c + i\epsilon k^{\delta-1/2}|e^{i\theta_0}$. Let $\Omega^{(3)}_j$ be as in Figure 7 and define $m^{(3)}$ as in (10.34). As in (10.45)-(10.47), the quantities we are interested in are

$$e^{-\alpha t}(-\alpha)^k \pi_k(-\alpha^{-1};t) = -\alpha^k e^{-t(\alpha-\alpha^{-1})} m^{(3)}_{12}(-\alpha^{-1};k;t), \qquad x<0, \tag{10.152}$$

$$e^{-\alpha t}(-\alpha)^k \pi_k(-\alpha^{-1};t) = -\alpha^k e^{-t(\alpha-\alpha^{-1})} m^{(3)}_{12}(-\alpha^{-1};k;t) + m^{(3)}_{11}(-\alpha^{-1};k;t), \quad x>0. \tag{10.153}$$

For the estimates of $w^{(3)} := v^{(3)} - I$, note that for any fixed $0 < \rho < 1$, $\operatorname{Re} f(\rho e^{i\theta}) = F(\rho e^{i\theta}; \frac{2t}{k})$ (recall (10.11)) is increasing in $0 < \theta < \pi$, and is decreasing in $\pi < \theta < 2\pi$, hence $\|e^{kf(z)}\|_{L^\infty(\Sigma_r)} = e^{k \operatorname{Re} f(-\rho_c + \epsilon k^{\delta-1/2})}$. But we have

$$f(-\rho_c + ia) = \frac{t}{k}(\alpha - \alpha^{-1}) - \log\alpha - \frac{1}{2}\bigg(\frac{\alpha^2(\alpha^2-1)}{\alpha^2+1}a^2 + \frac{x^2}{k}\bigg) + O(k^{-\frac{3}{2}+2\delta}), \qquad |a| \le \epsilon k^{\delta-1/2}. \tag{10.154}$$

Thus,

$$\|\alpha^k e^{-t(\alpha-\alpha^{-1})} e^{kf(z)}\|_{L^\infty(\Sigma_r)} \le Ce^{-ck^{2\delta}}, \tag{10.155}$$

and also using

$$\alpha^{-k} e^{t(\alpha-\alpha^{-1})} = e^{k\left[\log\alpha - \frac{\alpha^2-1}{\alpha^2+1} + O\left(\frac{1}{\sqrt{k}}\right)\right]}, \qquad \log\alpha - \frac{\alpha^2-1}{\alpha^2+1} > 0, \ \text{ for } \alpha > 1, \tag{10.156}$$

we have

$$\|e^{kf(z)}\|_{L^\infty(\Sigma_r)} \le Ce^{-ck}. \tag{10.157}$$

On the other hand, one can directly check that $\operatorname{Re} f(-\rho_c + a)$ has its maximum at $a = 0$ for $-\epsilon k^{\delta-1/2} \le a \le \epsilon k^{\delta-1/2}$, hence $\|e^{kf(z)}\|_{L^\infty(\Sigma_c)} = e^{k \operatorname{Re} f(-\rho_c)}$. Again (10.154) and (10.156) yield

$$\|e^{kf(z)}\|_{L^\infty(\Sigma_c)} \le e^{-ck}. \tag{10.158}$$

Similarly, we have $\|e^{-kf(z)}\|_{L^\infty(\Sigma^{(3)}_{out})} \le e^{-ck}$. Now calculations as in Subsubsection 10.1.2 give us the result (10.42). Hence using (10.156) and (10.158) and noting $\operatorname{dist}(-\alpha^{-1}, \Sigma^{(3)}) = \frac{(\alpha^2+1)^{1/2}}{\alpha(\alpha^2-1)^{1/2}} \cdot \frac{x}{\sqrt{k}}$, we have

$$m^{(3)}_{11}(-\alpha^{-1}; k; t) = 1 + O(\sqrt{k} e^{-2(1-\epsilon_1)c_1 k}), \tag{10.159}$$

$$\alpha^k e^{-t(\alpha-\alpha^{-1})} m^{(3)}_{12}(-\alpha^{-1}; k; t) = \alpha^k e^{-t(\alpha-\alpha^{-1})} \int_{\Sigma_c} \frac{(-s)^k e^{t(s-s^{-1})}}{s+\alpha^{-1}} \frac{ds}{2\pi i} + O(\sqrt{k} e^{-ck^{2\delta}}). \tag{10.160}$$

To evaluate the integral asymptotically, first we change the variable by $s = -\rho_c - \frac{i(\alpha^2+1)^{1/2}}{\alpha(\alpha^2-1)} \cdot \frac{y}{\sqrt{k}}$. Then from (10.154), the numerator of the integrand becomes

$$\alpha^{-k} e^{t(\alpha-\alpha^{-1})} e^{-\frac{1}{2}(y^2+x^2) + O(k^{-\frac{1}{2}+2\delta})}. \tag{10.161}$$

Hence setting $A := \frac{\epsilon\alpha(\alpha^2-1)^{1/2}}{(\alpha^2+1)^{1/2}}$,

$$\alpha^k e^{-t(\alpha-\alpha^{-1})} \frac{1}{2\pi i} \int_{\Sigma_c} \frac{(-s)^k e^{t(s-s^{-1})}}{s+\alpha^{-1}} ds = \frac{1}{2\pi i} \int_{-Ak^\delta}^{Ak^\delta} \frac{e^{-\frac{1}{2}(y^2+x^2)}}{y+ix} dy \big(1 + O(e^{k^{-\frac{1}{2}+2\delta}})\big) + O(e^{-ck^{2\delta}}). \tag{10.162}$$

Thus from (10.152) and (10.153), we obtain

$$\lim_{k\to\infty} e^{-\alpha t}(-\alpha)^k \pi_k(-\alpha^{-1}; t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(y^2+x^2)}}{y+ix} dy, \qquad x < 0, \tag{10.163}$$

$$\lim_{k\to\infty} e^{-\alpha t}(-\alpha)^k \pi_k(-\alpha^{-1}; t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(y^2+x^2)}}{y+ix} dy + 1, \qquad x > 0. \tag{10.164}$$

The function $h(x) := \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(y^2+x^2)}}{y+ix} dy$ is smooth in $x > 0$ and $x < 0$. The derivative is $h'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. As $x \to \pm\infty$, $h(x) \to 0$. Therefore we see that

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(y^2+x^2)}}{y+ix} dy = \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy, & x < 0, \\ \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{-\frac{1}{2}y^2} dy, & x > 0. \end{cases} \tag{10.165}$$

Thus we proved Proposition 5.8.

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