

# Riemann-Hilbert problems for last passage percolation

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## Abstract

Last three years have seen new developments in the theory of last passage percolation, which has variety applications to random permutations, random growth and random vicious walks. It turns out that a few class of models have determinant formulas for the probability distribution, which can be analyzed asymptotically. One of the tools for the asymptotic analysis has been the Riemann-Hilbert method. In this paper, we survey the use of Riemann-Hilbert method in the last passage percolation problems.

## 1 introduction

Let  $\Sigma$  be the unit circle in  $\mathbb{C}$ , oriented counterclockwise, and let  $\Omega_+ = \{z \in \mathbb{C} : |z| < 1\}$ ,  $\Omega_- = \{z \in \mathbb{C} : |z| > 1\}$ . Set

$$\varphi(z) = e^{t(z+z^{-1})}. \quad (1.1)$$

Consider the following Riemann-Hilbert problem :  $Y(z)$  is the  $2 \times 2$  matrix-valued function satisfying

$$\begin{cases} Y(z) \text{ is analytic in } z \in \Omega_{\pm}, \text{ and is continous in } \overline{\Omega_{\pm}}, \\ Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-k}\varphi(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma, \\ Y(z) = (I + O(z^{-1}))z^{k\sigma_3}, \quad \text{as } z \rightarrow \infty. \end{cases} \quad (1.2)$$

Here  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , and  $Y_+(z)$  (resp.,  $Y_-(z)$ ) is the limit from the inside (resp., outside) of  $\Sigma$ .

This Riemann-Hilbert problem appears in a class of problems which have many different interpretations like longest increasing subsequence in a random permutation, last passage percolation, polynuclear growth model, and random vicious walks. In this paper, we survey a list of problems related to the above Riemann-Hilbert problems and discuss various aspects of the RHP (1.2).

Before we state the problems, we first consider the solution to (1.2). From the work of Fokas, Its and Kitaev [20], the RHP (1.2) is related to the orthogonal polynomials on the unit circle. Indeed (see e.g. [4]), if we let  $\pi_k(z)$  be the  $k^{\text{th}}$  monic (leading coefficient=1) orthogonal polynomial with respect to the measure  $\varphi(z)dz/(2\pi iz)$  on  $\Sigma$ ,

$$\int_{|z|=1} \pi_k(z) \overline{z^j} \varphi(z) \frac{dz}{2\pi iz} = N_k \delta_{jk} \quad 0 \leq j \leq k, \quad (1.3)$$

the function

$$Y(z) = \begin{pmatrix} \pi_k(z) & (C\pi_k)(z) \\ -N_{k-1}^{-1} \pi_{k-1}^*(z) & -N_{k-1}^{-1} (C\pi_{k-1}^*)(z) \end{pmatrix} \quad (1.4)$$

is a solution to (1.2), where  $\pi_n^*(z) = z^n \pi_n(1/z)$ , and  $(Cf)(z)$  is the Cauchy transform of  $f$  :

$$(Cf)(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(s)}{s-z} ds. \quad (1.5)$$

Also it is a standard argument to show that the solution is unique. Here  $\varphi$  can be a general function, not necessarily of the form (1.1). Thus for instance,

$$\pi_k(z) = Y_{11}(z; k), \quad N_{k-1}^{-1} = -Y_{21}(0; k). \quad (1.6)$$

In this paper, we discuss three different uses of this RHP formulation for orthogonal polynomials  $\pi_k$  : (1) obtain asymptotic result using the Deift-Zhou steepest-descent method (2) obtain differential (or difference) equations (3) make a connection to so-called integrable operators. These three topics are discussed in Section 4, Section 5 and Section 6, respectively. The rest of the paper is organized as follows. In Section 2, we state the last passage percolation problems whose solution has Toeplitz/Hankel determinant formulas, which can be expressed in terms of the solution to the RHP (1.2). These percolation problems have various different interpretations, and Section 3 discusses some of these applications. There are a few other last passage percolation problems which can also be solved in terms of the RHP of the type (1.2), but with a different function  $\varphi$ . These problem are discussed in the final section 7. Most of the material in this paper have appeared somewhere. We indicate the references in each section. The difference equation, Theorem 5.1 is the only new result.

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## 2 Last passage percolation

### Square case

Consider a Poisson process of rate 1 in the plane. An up/right path is, by definition, a piece-wise linear curve with positive slope where defined. The ‘length’ of an up/right path is defined by the number of

Poisson points on the path. Let  $L(t)$  be the length of the ‘longest’ up/right path starting from  $(0, 0)$  ending at  $(t, t)$  (see Figure 1). See Section 3 below for other different interpretations. The basic formula

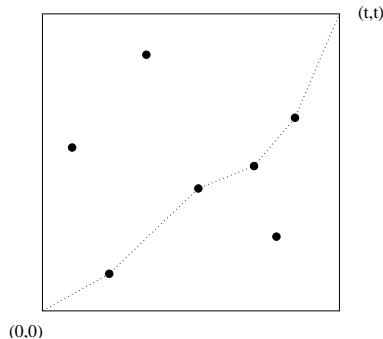


Figure 1: Poisson points in  $(0, 0) \times (t, t)$  and the longest up/right path

for us is the following, proved by [22] (see also [38]) :

$$\mathbb{P}(L(t) \leq \ell) = \frac{1}{Z} \mathbb{E}_{U \in U(\ell)} (\det(\varphi(U))) \quad (2.1)$$

where  $\varphi$  is given in (1.1) and  $Z = e^{t^2}$  is the normalization constant to make the left hand side 1 as  $\ell \rightarrow \infty$ .

As is well-known, using the Weyl’s integration formula for the unitary group, the above expected value is equal to a Toeplitz determinant :

$$\mathbb{E}_{U \in U(\ell)} \det(\varphi(U)) = \det(\varphi_{j-k})_{0 \leq j, k < \ell} =: D_\ell(t), \quad (2.2)$$

where  $\varphi_j$  is the  $j^{\text{th}}$  Fourier coefficient of  $\varphi$ . Now we will relate this Toeplitz determinant with the RHP (1.2).

As in Section 1, let  $\pi_k(z)$  be the  $k^{\text{th}}$  monic orthogonal polynomial with respect to  $\varphi(z)dz/(2\pi iz)$  on the unit circle and  $N_k$  be defined by (1.3). It is direct to check that the orthogonal polynomials have the determinant expression (see e.g., [43])

$$\pi_k(z) := \frac{1}{D_k(\varphi)} \det \begin{pmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_k \\ \varphi_{-1} & \varphi_0 & \cdots & \varphi_{k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{-k+1} & \varphi_{-k+2} & \cdots & \varphi_1 \\ 1 & z & \cdots & z^k \end{pmatrix}. \quad (2.3)$$

Thus we have

$$N_k = \frac{D_{k+1}}{D_k}. \quad (2.4)$$

Note from (2.1) and (2.2) that

$$e^{t^2} = Z = \lim_{n \rightarrow \infty} D_n, \quad (2.5)$$

which also can be seen from the strong Szegő limit theorem. Hence we have from (1.6)

$$\mathbb{P}(L(t) \leq \ell) = e^{-t^2} D_\ell = \lim_{n \rightarrow \infty} \frac{D_\ell}{D_n} = \prod_{k=\ell}^{\infty} N_k^{-1} = \prod_{k=\ell}^{\infty} (-Y_{21}(0; k+1)). \quad (2.6)$$

*Remark 2.1.* In addition to the Toeplitz determinant formula, there are two other determinant formulas for  $\mathbb{P}(L(t) \leq \ell)$ . Both of them are Fredholm determinants of operators, one acting on  $\Sigma$  (see [5]), and the other acting on the discrete set  $\{\ell, \ell+1, \dots\}$  (see [13, 29, 31]). The first Fredholm determinant has the RHP expression which is algebraically equivalent to (1.2). See Section 6 below for details. The second Fredholm determinant also has the RHP expression, but now the RHP has a jump condition on the discrete set. However, for the second formula, the kernel of the operator has integral representation, and hence the classical steepest-descent method is sufficient to obtain the asymptotics ([13, 29]). Thus the limit of  $\mathbb{P}(L(t) \leq \ell)$  can be obtained from the second algebraic formula, which avoids the RHP analysis in Section 4 below. But for some applications (see [28, 41]) one also needs uniform tail estimates of the ‘scaled’ random variable  $(L(t) - 2t)/t^{1/3}$  in  $t$  (see (4.17), (4.18) below). This problem is more difficult than the convergence in distribution, and the lower bound (4.18) has been obtained only from RHP analysis so far [4]. The identities between the Toeplitz determinants and the Fredholm determinants are discussed in [5, 12, 11, 6, 15, 14].

## Triangle case

Suppose now that we take a (2-dimensional) Poisson process of rate 1 in the half plane  $y < x$ , and take a (1-dimensional) Poisson process of rate  $\alpha \geq 0$  on the line  $y = x$  (see Figure 2). Let  $L_s(t; \alpha)$  be

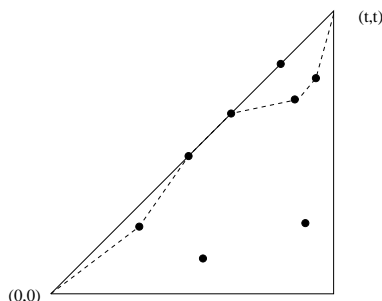


Figure 2: Triangle case

again the length of the longest up/right path from  $(0, 0)$  to  $(t, t)$ . The algebraic formula for this case is obtained in [38] for  $\alpha = 0$  and in [9] for general  $\alpha > 0$  :

$$\mathbb{P}(L_s(t; \alpha) \leq \ell) = \frac{1}{Z} \mathbb{E}_{U \in O(\ell)} \det((1 + \alpha U)e^{tU}), \quad (2.7)$$

where  $Z = e^{\alpha t + t^2/2}$ .

There are two components of  $O(\ell)$  depending on the sign of the determinant of the matrix. Using the Weyl’s integration formula, the expected value over each component is expressed in terms of one of

the Hankel determinants of the form (see e.g., Theorem 2.2 of [9])  $\tilde{H}_\ell = \det(h_{j+k})_{\ell \times \ell}$  where

$$h_j = \int_{-1}^1 x^j h(x) (1-x)^{\pm \frac{1}{2}} (1+x)^{\pm \frac{1}{2}} dx, \quad (2.8)$$

with

$$h(x) = (1 + \alpha^2 + 2\alpha x)e^{2tx} = (1 + \alpha z)(1 + \alpha/z)\varphi(z), \quad x = \frac{1}{2}(z + z^{-1}). \quad (2.9)$$

As an analogue of (2.3), these Hankel determinants are related to the orthogonal polynomials on the interval  $(-1, 1)$ . But the orthogonal polynomials on the unit circle with weight  $(1 + \alpha z)(1 + \alpha/z)\varphi(z)$  and the orthogonal polynomials on the interval  $(-1, 1)$  with weight  $h(x)(1-x)^{\pm \frac{1}{2}}(1+x)^{\pm \frac{1}{2}}$  are related in a simple way (see [43]). Thus the ration of Hankel determinants  $\tilde{H}_\ell/\tilde{H}_{\ell+1}$  can be expressed (see theorem 2.3 of [9]) in terms of orthogonal polynomials (norms and the values at  $z = 0$ ) on the unit circle  $\Sigma$  with respect to the measure

$$(1 + \alpha z)(1 + \alpha/z)\varphi(z) \frac{dz}{2\pi iz}. \quad (2.10)$$

Moreover, after some algebraic work (see Section 3 of [9]), the simple factor  $(1 + \alpha z)(1 + \alpha/z)$  in the above measure can be removed and  $\tilde{H}_\ell/\tilde{H}_{\ell+1}$  can be expressed purely in terms of orthogonal polynomials  $\pi_k$  for the measure  $\varphi(z)dz/(2\pi iz)$ . The result is [9]

$$\mathbb{P}(L_s(t; \alpha) \leq 2\ell + 1) = e^{-\alpha t} \frac{1}{2} \{ (\pi_{2\ell}^*(-\alpha) + \alpha\pi_{2\ell}(-\alpha))H_\ell^+ + (\pi_{2\ell}^*(-\alpha) - \alpha\pi_{2\ell}(-\alpha))H_\ell^- \} \quad (2.11)$$

where

$$H_\ell^\pm = \prod_{k=\ell}^{\infty} N_{2k+1}^{-1} (1 \mp \pi_{2k+1}(0)). \quad (2.12)$$

There is a similar formula for  $\mathbb{P}(L_t(t; \alpha) \leq 2\ell)$ . Thus again, the probability distribution for  $L_s(t; \alpha)$  can be expressed in terms of the solution of the RHP (1.2).

*Remark 2.2.* In [9, 10, 8], the authors considered five types of symmetry of the Poisson model. Reflection symmetry about the anti-diagonal, reflection symmetry about both diagonal and anti-diagonal, and rotation symmetry about the center are considered in addition to the square case and the triangle case. The distribution of the longest up/right path in each case has the determinantal expression (either Toeplitz or Hankel determinants with a simple change of the weight), and can be expressed in terms of the RHP (1.2).

*Remark 2.3.* As in square case (see Remark 2.1), there is also a different algebraic formula for the triangle case [37]. But the result is so-called Fredholm Pfaffian of an operator acting on a discrete set. The kernel of the operator is rather involved, and so far there has been no work for the asymptotic analysis from this Fredholm Pfaffian formula.

## External sources

Suppose that in the square model above, we have additional (1-dimensional) Poisson processes of rate  $\alpha_+$  and  $\alpha_-$  on the lines  $y = 0$  and  $x = 0$ , respectively. We assume that the corner  $(0, 0)$  has no point (see Figure 3). Let  $L_e(t; \alpha_+, \alpha_-)$  be the length of the longest up/right path from  $(0, 0)$  to  $(t, t)$ . This

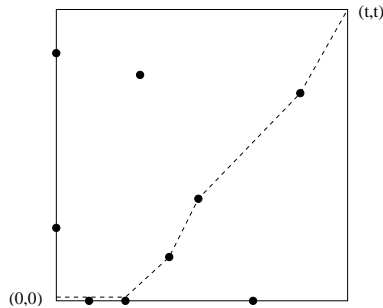


Figure 3: External sources case

problem arises from a polynuclear growth model with random initial data [34], [33].

The distribution in this case is given in [7] :

$$\mathbb{P}(L_e(t; \alpha_+, \alpha_-) = \ell) = \frac{1}{Z} (D'_\ell - \alpha_+ \alpha_- D'_{\ell-1}) \quad (2.13)$$

where

$$D'_\ell := \mathbb{E}_{U \in U(\ell)} \det((1 + \alpha_+ U)(1 + \alpha_- U^{-1}) \varphi(U)), \quad (2.14)$$

and  $Z = e^{\alpha_+ t + \alpha_- t + t^2}$ .

By a similar argument as in (2.10),  $D'_\ell$  can be written as (see Theorem 3.2 of [9])

$$D'_\ell = \frac{\pi_\ell^*(-\alpha_+) \pi_\ell^*(-\alpha_-) - \alpha_+ \alpha_- \pi_\ell(-\alpha_+) \pi_\ell(-\alpha_-)}{1 - \alpha_+ \alpha_-} D_\ell, \quad \alpha_+ \alpha_- \neq 1, \quad (2.15)$$

where  $D_\ell = D_\ell(t)$  is given in (2.2). For  $\alpha_+ \alpha_- = 1$ , we use the l'Hopital's rule in the above formula. Thus again the distribution for  $L_e(t; \alpha_+, \alpha_-)$  can be expressed in terms of the solution  $Y$  of the RHP (1.2).

*Remark 2.4.* In the triangular case above, we may put an additional (1-dimensional) Poisson process of rate  $\alpha_+$  on the line  $y = 0$ . Then there still is a determinant formula for the distribution of the longest up/right path, which again can be expressed in terms of the RHP (1.2).

## 3 Applications

In this section, we discuss various different interpretations of the last passage percolation problems in Section 2.

## Random permutations

Given a permutation  $\pi$ , a subsequence  $\{\pi(i_1), \dots, \pi(i_k)\}$  such that  $i_1 < \dots < i_k$  and  $\pi(i_1) < \dots < \pi(i_k)$  is called ‘increasing’. The length of such an increasing subsequence is defined to be  $k$ . Let  $L_N(\pi)$  be the length of the longest increasing subsequence of a permutation  $\pi \in S_N$ . Now take  $\pi$  randomly from  $S_N$ , and we ask the distribution of  $L_N$ . This problem is known as Ulam’s problem since early 1960’s (see e.g., [2], [4]).

If we take  $N$  itself as a Poisson random variable of rate  $t^2$ , the distribution of  $L_N$  is same as the distribution of  $L(t)$  above :

$$\mathbb{P}(L(t) \leq \ell) = \sum_{N=0}^{\infty} \frac{e^{-t^2} (t^2)^N}{N!} \mathbb{P}(L_N \leq \ell). \quad (3.1)$$

In other words,  $\mathbb{P}(L(t) \leq \ell)$  is the exponential generating function of  $\mathbb{P}(L_N \leq \ell)$ . This follows from the fact that a configuration of  $N$  points in a square can be regarded as a permutation  $\pi \in S_N$  : consider the relative  $x$ -orders, then relative  $y$ -orders. The longest up/right path in a points configuration is precisely the longest increasing subsequence of the corresponding permutation. Thus the asymptotics of  $L_N$  as  $N \rightarrow \infty$  can be obtained from the asymptotics of  $L(t)$  as  $t \rightarrow \infty$  (see [26]).

The limiting distribution of  $L(t)$  (see (4.16) below), and thus the limiting distribution of  $L_N$ , is first obtained in [4] by using the RHP (1.2). Later the same result was obtained by [32, 13, 29] using different methods which avoids the RHP analysis. The uniform tail estimates (see (4.17), (4.18)) and the convergence of moments of the scaled random variable were obtained in [4]. As mentioned in Remark 2.1, the (lower) tail estimate obtained in [4] was later used in other applications [28], [41].

Now we consider the corresponding permutation model for the triangle case. Given random points in the triangle model, take mirror image of the points in  $y < x$  about the line  $y = x$ . Hence we have points in the whole plane which is symmetric about  $y = x$ . It is each to see that in terms of permutation, the symmetry condition implies that  $\pi^2 = 1$ . Also the points on the line  $y = x$  corresponds to the fixed points  $\pi(i) = i$ . Thus it is natural to consider the set  $\tilde{S}_{m,n} = \{\pi \in S_{2m+n} : \pi^2 = 1, |\{x : \pi(x) = x\}| = n\}$ . We equip this set with the uniform probability measure and consider the longest increasing subsequence of a random  $\pi \in \tilde{S}_{m,n}$ . As before, the asymptotics of  $L_s(t; \alpha)$  implies the asymptotics of the longest increasing subsequence in this set. See [9, 10, 8] for details.

## Plancherel measure

Let  $Y_N$  be the set of partitions (Young diagrams)  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N)$ ,  $\lambda_j \in \mathbb{N} \cup \{0\}$ ,  $\sum_j \lambda_j = N$ . Given a partition  $\lambda$ , let  $d_\lambda$  be the number of standard Young tableaux of shape  $\lambda$  (see e.g. [42] for the definitions). It is a basic fact in the representation of symmetric group that  $\sum_{\lambda \vdash N} d_\lambda^2 = |S_N| = N!$ . Define the Plancherel measure on  $Y_N$  by

$$\mathbb{P}(\lambda) = \frac{d_\lambda^2}{N!}, \quad \lambda \in Y_N. \quad (3.2)$$

The famous Robinson-Schensted (RS) correspondence (see e.g. [39], [42]) implies that the first part  $\lambda_1$  from the Plancherel measure has the same distribution as the longest increasing subsequence of

random permutation  $\pi \in S_N$  from the uniform measure. Thus from the result of the longest increasing subsequence, the asymptotics of the first row  $\lambda_1$  can be obtained. The convergence in distribution and the convergence of moments for the second row  $\lambda_2$  was obtained in [5]. For the general rows  $\lambda_j$ ,  $j \geq 3$ , the convergence in distribution is obtained in [32, 13, 29], and the convergence of moments are obtained recently in [6].

Let  $\beta \geq 0$ . As a generalization, consider the  $\beta$ -Plancherel measure on  $Y_N$  defined by

$$\mathbb{P}(\lambda) = \frac{d_\lambda^\beta}{\sum_{\mu \in Y_N} d_\mu^\beta}, \quad \lambda \in Y_N. \quad (3.3)$$

One can regard  $\beta$  as the inverse temperature. The above Plancherel measure corresponds to the case when  $\beta = 2$ . The 1-Plancherel measure also has a simple permutation interpretation. Consider the set of involutions  $\tilde{S}_N := \{\pi \in S_N : \pi^2 = 1\}$ . Equip  $\tilde{S}_N$  with the uniform probability measure and consider the length of the longest increasing subsequence of a random  $\pi \in \tilde{S}_N$ . The RS correspondence implies that the first row  $\lambda_1$  of  $\lambda \in Y_N$  under the 1-Plancherel measure has the same distribution as the longest increasing subsequence of a random involution. In [10] (see also [8]), the limiting distribution of the first row of the 1-Plancherel measure was obtained. It would be interesting to obtain the limiting distributions for the general  $\beta$ .

## Polynuclear growth model

Consider a flat 1-dimensional substrate. There are random nucleation events happening on the substrate, which can be thought as a 2-dimensional Poisson process in space-time coordinates. An island of height 1 with width 0 is made at each nucleation position, and the island grows laterally with speed 1. When two islands meet, they just form one island and keep growing at the edges. Further nucleation events may happen on top of islands. This model is called Polynuclear growth (PNG) model. An interest is the height  $h(x, t)$  as time  $t$  tends to infinity.

It is an observation by Prähofer and Spohn that for certain choices of initial data, PNG models are in bijection to the last passage percolation models in Section 2. As the first case, suppose that initially there is an island of width 0 at the position  $x = 0$ . It grows with speed 1, and we assume that further nucleation events occur only on top of the base island and on top of islands on the base island. Then the height  $h(0, t)$  is equal, in the sense of distribution, to  $L(t/\sqrt{2})$  of the square case (see [35]). Secondly, we consider the half infinite case :  $x \in [0, \infty)$ . When an island hits the edge  $x = 0$ , it stops growing at that side. As before there is the base island at the position  $x = 0$  at time  $t = 0$ . In addition to the nucleation events of rate 1 in  $(0, \infty)$ , and we assume that there are extra nucleation events at the position  $x = 0$  of rate  $\alpha$ . Then the height  $h(0, t)$  is equal to  $L(t/\sqrt{2}; \alpha)$  of the triangle case (see [33]). If one considers flat case, i.e. no base island, the PNG model is in bijection to yet a different symmetry model of the Poisson process mentioned in Remark 2.2 above (see [35]). Finally the case of external sources in Section 2 has the PNG model which has random initial data. See [34, 33] for more details.



## Random turn vicious walks

At time  $t = 0$ , each site of the semi-infinite lattice  $\{\dots, -2, -1, 0\}$  is occupied by a walker. A walker is called ‘right-movable’ if its right-neighboring site is vacant. At each discrete time, we pick a random walker among all right-movable walkers (there are finitely many such walkers), and move it to its right-neighboring site. Hence at each time, one and only one walker moves to its right. We let the walkers walk by the above rule for  $N$  time steps. In the next  $N$  time steps, we let walkers move to their *left* according to the similar rule. We further impose the condition that after total  $2N$  time steps, all the walkers return to their original positions. The interest is the position  $x_1(N)$  of the right-most walker at time  $N$ . It is shown in [21] (see also [24]) that the distribution of  $x_1(N)$  is same as the distribution of  $L_N$ , the length of the longest increasing subsequence of a random permutation.

If we just let the walkers walk up to time  $N$  without assuming that they all return to their original positions,  $x_1(N)$  is, in the sense of distribution, equal to the length of the longest increasing subsequence of a random involution ( $\pi^2 = 1$ ). See [8] for the asymptotic results of this case.

## Queueing theory and Totally asymmetric simple exclusion process

Lattice versions of the last passage percolation problems are known to be related to the queueing theory and also so-called totally asymmetric simple exclusion processes (TASEP) (see e.g. [27]). See Section 7 below for lattice last passage percolation problems.

## 4 Asymptotics

In this section, we sketch asymptotic analysis of the RHP (1.2) and discuss the asymptotic results for the random variables  $L(t)$ ,  $L_s(t; \alpha)$  and  $L_e(t; \alpha_+, \alpha_-)$ .

The asymptotics of the RHP (1.2) as  $k, t \rightarrow \infty$  in all different regimes was first considered in [4], and later in a little more detail in [10]. A special interest is the case when  $k$  and  $t$  are related as

$$\frac{2t}{k} = 1 - \frac{x}{2^{1/3}k^{2/3}}. \quad (4.1)$$

For  $x$  in a compact set, the RHP (1.2) is localized around  $z = -1$  as  $k \rightarrow \infty$ , and the local RHP is, after scaling, the RHP for the Painlevé II solution. In the below, we will discuss, heuristically, how the Painlevé II solution arises. This heuristic argument is taken from Section 6 of [5].

First, we show why the point  $z = -1$  plays an important role in the asymptotic analysis. Since the RHP (1.2) is not normalized at  $\infty$ , we algebraically transform it to a normalized problem. Set

$$m(z) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} e^{tz} & 0 \\ 0 & e^{-tz} \end{pmatrix}, \quad |z| < 1, \quad (4.2)$$

and set

$$m(z) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} 0 & z^{-k} e^{tz^{-1}} \\ -z^k e^{-tz^{-1}} & 0 \end{pmatrix}, \quad |z| > 1. \quad (4.3)$$

Then  $m(z)$  solves the new Riemann-Hilbert problem

$$\begin{cases} m(z) & \text{is analytic in } z \in \Omega_{\pm}, \text{ and is continuous in } \overline{\Omega_{\pm}}, \\ m_+(z) = m_-(z)V(z), & z \in \Sigma, \\ m(z) = I + O(z^{-1}), & z \rightarrow \infty, \end{cases} \quad (4.4)$$

where

$$V(z) = \begin{pmatrix} 0 & -z^{-k}\psi(z)^{-1} \\ z^k\psi(z) & 1 \end{pmatrix}, \quad \psi(z) := e^{t(z-z^{-1})}. \quad (4.5)$$

Now the oscillatory factors in the jump matrix are  $V_{12}$  and  $V_{21} = -1/V_{12}$ . It is direct to check that the critical points of

$$\log V_{21}(z) = k\left(\frac{t}{k}(z - z^{-1}) + \log z\right) \quad (4.6)$$

are

$$z = -\frac{k}{2t} \pm \sqrt{\left(\frac{k}{2t}\right)^2 - 1}. \quad (4.7)$$

When  $2t < k$ , there are two real critical points. These two points collapses to one point  $z = -1$  when  $2t = k$ . And when  $2t > k$ , there are two complex critical points. Thus one can imagine that the nature of asymptotics changes at  $2t = k$ . This is indeed the case. It is well-known that in the (so-called Plancherel-Rotach type) asymptotics of the orthogonal polynomials, the equilibrium measure of an associated variational problem plays a key role (see e.g. [17, 18]). For the case at hand, the support of the equilibrium measure is the full circle when  $2t \leq k$ , and it is a part of the circle when  $2t > k$ . And the point  $z = -1$  is the place where the ‘gap’ of the support starts to open up at  $2t = k$ .

Thus it is natural to analyze the problem near  $z = -1$  and in the regime  $2t \sim k$ . The scaling (4.1) is chosen so that the local RHP around  $z = -1$  becomes non-trivial. Writing  $z = -1 + s$  for  $z$  near  $-1$ , we have

$$\log V_{12}(z) - \pi i = -\frac{x}{k^{1/3}}(k^{1/3}s) - \frac{x}{2^{4/3}k^{1/3}}(k^{1/3}s)^2 + \frac{(k^{1/3}s)^3}{6}\left(1 - \frac{3x}{2^{1/3}k^{2/3}}\right) + \dots \quad (4.8)$$

If we take  $w = 2^{-4/3}k^{1/3}s$ , then we have

$$\log V_{12}(z) - \pi i = -2xw + \frac{8}{3}w^3 + \dots, \quad (4.9)$$

and hence the jump matrix near  $z = -1$  becomes as  $k \rightarrow \infty$ ,

$$\tilde{V}(w) = \begin{pmatrix} 0 & -(-1)^k e^{-2(-xw + \frac{4}{3}w^3)} \\ (-1)^k e^{2(-xw + \frac{4}{3}w^3)} & 1 \end{pmatrix} \quad (4.10)$$

for  $w \in i\mathbb{R}$ . But after rotation by  $\pi/2$  and removing  $(-1)^k$  by a simple conjugation, this is precisely the RHP for Painlevé II equation with the choice of parameters  $p = -q = 1$ ,  $r = 0$  (see e.g. [19], [4]). Of

course, this is only a heuristics, and one needs to justify the convergence of the original RHP (1.2) to the above local RHP. As mentioned above, the justification proceeds differently for  $2t < k$  and  $2t > k$ , due to the change of the support of the equilibrium measure. See [4], [10] for details.

The asymptotic results we obtain are, for example, the following [4, 10] : There are numerical constants  $C, c, x_0 > 0$  such that for large  $k$  and  $t$ ,

(i). if  $x \geq x_0$ ,

$$\left| -Y_{21}(0; k) - 1 \right|, \quad \left| Y_{11}(0; k) \right| \leq \frac{C}{k^{1/3}} e^{-cx^{3/2}}, \quad (4.11)$$

(ii). if  $-x_0 \leq x \leq x_0$ ,

$$\left| -Y_{21}(0; k) - 1 - \frac{2^{1/3}}{k^{1/3}} v(x) \right|, \quad \left| Y_{11}(0; k) + (-1)^k \frac{2^{1/3}}{k^{1/3}} u(x) \right| \leq \frac{C}{k^{2/3}}, \quad (4.12)$$

(iii). if  $x \leq -x_0$ ,

$$\left| -\sqrt{\frac{2t}{k}} e^{k(2t/k - \log(2t/k) - 1)} Y_{21}(0; k) - 1 \right|, \quad \left| (-1)^k \frac{2t}{2t - k} Y_{11}(0; k) \right| \leq \frac{C}{2t - k}. \quad (4.13)$$

Here  $v(x)$ ,  $u(x)$  in case (ii) are given by the Painlevé II solution

$$\begin{cases} u'' = 2u^3 + xu, \\ u(x) \sim -\frac{1}{2\sqrt{\pi}x^{1/4}} e^{-2/3x^{3/2}}, \quad x \rightarrow +\infty. \end{cases} \quad (4.14)$$

and

$$v(x) = \int_{\infty}^x (u(s))^2 ds. \quad (4.15)$$

We also obtain the asymptotics of  $Y(z)$  for general points  $z \in \mathbb{C}$  [10].

Now using (4.11)-(4.13), (2.6) and (1.6) yield the convergence in distribution in the square case [4] : for fixed  $x \in \mathbb{R}$ ,

$$\lim_{t \rightarrow \infty} \mathbb{P} \left( \frac{L(t) - 2t}{t^{1/3}} \leq x \right) = \exp \left( - \int_x^{\infty} (y - x)(u(y))^2 dy \right) =: F_{GUE}(x). \quad (4.16)$$

The function  $F_{GUE}(x)$  in (4.16) is called the GUE Tracy-Widom distribution [44], which was first obtained as the limiting distribution for the fluctuation of the largest eigenvalue of a random matrix from the Gaussian unitary ensemble in the random matrix theory (see e.g. [30] for an introduction to random matrix theory). Also the uniform tail estimates can be obtained [4] : there are constants  $C, c, x_0 > 0$  such that for all  $t > 0$ ,

$$\mathbb{P} \left( \frac{L(t) - 2t}{t^{1/3}} \leq x \right) \leq C e^{-cx^{3/2}}, \quad x \geq x_0, \quad (4.17)$$

and

$$\mathbb{P} \left( \frac{L(t) - 2t}{t^{1/3}} \leq x \right) \leq C e^{-cx^3}, \quad x \leq -x_0. \quad (4.18)$$

The upper estimate (4.17) also follows from a large deviation result of [40]. But the lower estimate (4.18) is obtained only from the RHP analysis so far. These estimates imply the convergence of moments of the scaled random variable  $(L(t) - 2t)/t^{1/3}$  [4]. These estimates have also been crucial for the analysis of the transversal fluctuation of the longest up/right path [28] and of the perturbation of the equilibrium measure for a related dynamical system, called stick process [41].

The asymptotic analysis of the RHP (1.2) also yields similar results for the triangle case  $L_s(t; \alpha)$  [10] and the external sources case  $L_e(t; \alpha_+, \alpha_-)$  [7]. The results show more interesting feature. Depending on the values  $\alpha, \alpha_+, \alpha_-$ , the scaled random variables converge to different distributions. For example,  $(L_s(t; \alpha) - 2t)/t^{1/3}$  converges in distribution to the so-called GSE TW-function [45],

$$F_{GSE}(x) := \frac{1}{2} \left( e^{\frac{1}{2} \int_x^\infty u(y) dy} + e^{-\frac{1}{2} \int_x^\infty u(y) dy} \right) (F_{GUE}(x))^{1/2} \quad (4.19)$$

for fixed  $0 \leq \alpha < 1$ , and to the GOE TW-function [45],

$$F_{GOE}(x) := e^{\frac{1}{2} \int_x^\infty u(y) dy} (F_{GUE}(x))^{1/2} \quad (4.20)$$

for  $\alpha = 1$ . When  $\alpha > 1$  and fixed, a differently scaled random variable converges to the Gaussian distribution. Moreover if we scale  $\alpha = 1 - \frac{2w}{t^{1/3}}$  with fixed  $w \in \mathbb{R}$  and take  $t \rightarrow \infty$ , the above scaled random variable converges to yet another distribution function for each fixed  $w \in \mathbb{R}$ . This new one-parameter family of distributions interpolates  $F_{GSE}$  and  $F_{GOE}$  as  $w = \infty$  and  $w = 0$ . Similar feature appears for the analysis of  $L_e(t; \alpha_+, \alpha_-)$  where two-parameter family of new distributions are obtained [7].

## 5 Difference equation

In this section, we show that the RHP (1.2) yields difference equations for the entries of  $Y(0)$ . It is direct to check from (1.4) that  $Y_{11}(0) = -Y_{22}(0)$ . Also since the determinant of the jump matrix for (1.2) is 1, we have  $\det Y(0) = 1$ .

**Theorem 5.1.** *Let*

$$Y(0; k) = \begin{pmatrix} -b(k) & d(k) \\ a(k) & b(k) \end{pmatrix}. \quad (5.1)$$

*Then  $b$  satisfy the discrete Painlevé II equation*

$$\frac{k}{t} b(k) + (b(k-1) + b(k+1))(1 - b(k)^2) = 0, \quad (5.2)$$

*and  $a$  and  $d$  satisfy*

$$a(k) = (1 - b(k)^2)a(k+1), \quad (5.3)$$

$$d(k) = (1 - b(k)^2)d(k-1). \quad (5.4)$$

*Proof.* Consider (4.4). Note that the jump matrix has determinant 1. Thus  $\det(m_+) = \det(m_-)$ , and hence  $\det(m)$  is an entire function. Also  $\det(m) \rightarrow 1$  as  $z \rightarrow \infty$ . Thus by Liouville's theorem, we have  $\det(m(z)) \equiv 1$ , and in particular,  $(m(z))^{-1}$  is analytic in  $\Omega_{\pm}$ , and is continuous in  $\overline{\Omega_{\pm}}$ .

Set

$$m(z) = m(z; k) = I + \frac{m_1^{\infty}(k)}{z} + O(z^{-2}), \quad z \rightarrow \infty, \quad (5.5)$$

and set  $m(0; k) = B(k)$  and

$$m(z) = m(z; k) = B(k)(I + m_1^0(k)z + O(z^2)), \quad z \rightarrow 0. \quad (5.6)$$

Set

$$\Psi(z) = \Psi(z; k) := m(z) \begin{pmatrix} 1 & 0 \\ 0 & z^k \psi(z) \end{pmatrix}, \quad J_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.7)$$

Note that by (5.5) and (5.6),

$$\Psi(z; k) = \left( I + \frac{m_1^{\infty}(k)}{z} + \frac{m_2^{\infty}(k)}{z^2} + O(z^{-3}) \right) e^{tzJ_1 + k \log z J_1 - \frac{1}{z} J_1}, \quad z \rightarrow \infty, \quad (5.8)$$

$$\Psi(z; k) = B(k) \left( I + m_1^0(k)z + O(z^2) \right) e^{-\frac{1}{z} J_1 + k \log z J_1 - \frac{1}{z} J_1 + tzJ_1}, \quad z \rightarrow 0. \quad (5.9)$$

Now the jump matrix for  $\Psi(z)$  is a constant matrix :

$$\Psi_+(z) = \Psi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad z \in \Sigma. \quad (5.10)$$

**Equations :** By differentiating (5.10),  $\frac{\partial}{\partial z} \Psi$  satisfies the same jump condition. Hence  $\frac{\partial \Psi}{\partial z} \Psi^{-1}$  has no jump cross  $\Sigma$ . Also from (5.7),  $\frac{\partial \Psi}{\partial z} \Psi^{-1}$  has a double pole at  $z = 0$ . Therefore using (5.8), (5.9), we have

$$\frac{\partial \Psi(z; k)}{\partial z} \Psi(z; k)^{-1} = tJ_1 + \frac{A_1(k)}{z} + \frac{A_2(k)}{z^2} =: P(z; k), \quad (5.11)$$

for some constant matrices  $A_1, A_2$  which depend on  $k$ . On the other hand,  $\Psi(z; k+1)\Psi(z; k)^{-1}$  again has no jump cross  $\Sigma$ , and it is now entire. Thus using (5.8), (5.9), we have

$$\Psi(z; k+1)\Psi(z; k)^{-1} = zJ_1 + X_0(k) =: Q(z; k), \quad (5.12)$$

for some constant matrix  $X_0(k)$ .

Now we take 'cross differentiation' of (5.11) and (5.12) in two different ways :

$$\begin{aligned} \frac{\partial}{\partial z} \Psi(z; k+1) &= \frac{\partial}{\partial z} Q(z; k) \Psi(z; k) = \frac{\partial Q(z; k)}{\partial z} \Psi(z; k) + Q(z; k) \frac{\partial \Psi(z; k)}{\partial z} \\ &= \left\{ \frac{\partial Q(z; k)}{\partial z} + Q(z; k) P(z; k) \right\} \Psi(z; k). \end{aligned} \quad (5.13)$$

and

$$\frac{\partial \Psi(z; k)}{\partial z} = \frac{\partial \Psi(z; k)}{\partial z} \Big|_{k \rightarrow k+1} = P(z; k+1) \Psi(z; k+1) = P(z; k+1) Q(z; k) \Psi(z; k). \quad (5.14)$$

Thus we obtain an equation

$$\frac{\partial Q(z; k)}{\partial z} + Q(z; k)P(z; k) = P(z; k+1)Q(z; k). \quad (5.15)$$

By plugging in the formulas of  $P$  and  $Q$  and comparing the coefficients in  $z$ , we obtain the relations for  $A_1, A_2, X_0$  :

$$J_1 + J_1 A_1(k) - A_1(k+1)J_1 + t[X_0, J_1] = 0, \quad (5.16)$$

$$J_1 A_2(k) - A_2(k+1) + X_0(k)A_1(k) - A_1(k+1)X_0(x) = 0, \quad (5.17)$$

$$X_0(k)A_2(k) - A_2(k+1)X_0(k) = 0. \quad (5.18)$$

**Constant matrices  $A_1, A_2, X_0$  :** Now we express  $A_1, A_2, X_0$  in terms of  $m$ . We plug in (5.8) and (5.9) into (5.11) and (5.12). This determines the constant matrices  $A_1, A_2, X_0$ . Using (5.8) for (5.11),

$$O(z^{-1}) : A_1(k) = t[m_1^\infty(k), J_1] + kJ_1. \quad (5.19)$$

Using (5.9) for (5.11),

$$O(z^{-1}) : A_1(k) = B(k)\{t[m_1^0(k), J_1] + kJ_1\}B(k)^{-1}, \quad (5.20)$$

$$O(z^{-2}) : A_2(k) = tB(k)J_1B(k)^{-1}. \quad (5.21)$$

Using (5.8) for (5.12),

$$O(1) : X_0(k) = J_0 + m_1^\infty(k+1)J_1 - J_1m_1^\infty(k). \quad (5.22)$$

Using (5.9) for (5.12),

$$O(z) : J_1 = B(k+1)\{J_1 + m_1^0(k+1)J_0 - J_0m_1^0(k)\}B(k)^{-1}, \quad (5.23)$$

$$O(1) : X_0(k) = B(k+1)J_0B(k)^{-1}. \quad (5.24)$$

**Symmetry :** Note that the jump matrix  $v(z)$  for  $m$  satisfies

$$\sigma_3 v(z^{-1})^t \sigma_3 = v(z), \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.25)$$

Thus it is standard to show that  $m$  satisfies the symmetry condition

$$m(z) = \sigma_3 B^t (m(z^{-1})^t)^{-1} \sigma_3. \quad (5.26)$$

By taking  $z \rightarrow 0$ , and considering terms of order  $O(1)$  and  $O(z)$ , we obtain the symmetry relations

$$B = \sigma_3 B^t \sigma_3, \quad (5.27)$$

$$m_1^0 = -\sigma_3 (m_1^\infty)^t \sigma_3. \quad (5.28)$$

Since  $\det B = 1$  and  $\text{tr}(m_1^0) = 0$ , we can set

$$B(k) := \begin{pmatrix} a(k) & b(k) \\ -b(k) & d(k) \end{pmatrix}, \quad a(k)d(k) + b(k)^2 = 1, \quad (5.29)$$

and set

$$m_1^0(k) := \begin{pmatrix} \lambda(k) & \mu(k) \\ \nu(k) & -\lambda(k) \end{pmatrix}, \quad m_1^\infty(k) := \begin{pmatrix} -\lambda(k) & \nu(k) \\ \mu(k) & \lambda(k) \end{pmatrix}. \quad (5.30)$$

**Equations from (5.16)-(5.18) :** Now we re-write the equations (5.16)-(5.18) in terms  $a, b, d$  and  $\lambda, \mu, \nu$ . First, if we use (5.19) and (5.24), use (5.29) and (5.30), the equation (5.16) yields

$$\mu(k) = b(k+1)d(k), \quad (5.31)$$

$$\nu(k+1) = -a(k+1)b(k). \quad (5.32)$$

Second, if we use (5.20), (5.21) and (5.29), (5.30), the equation (5.17) yields the same equations (5.31) and (5.32). Finally, the equation (5.18) is trivial if we use (5.21) and (5.24).

**Equations from (5.19)-(5.23) :** We also have the condition that the two different formulas (5.19) and (5.20) for  $A_1(k)$  are equal. Similarly (5.22) and (5.24) for  $X_0(k)$  are equal. Also there is an additional condition (5.23). From (5.19) and (5.20), we have

$$(t[m_1^\infty(k), J_1] + kJ_1)B(k) = B(k)(t[m_1^\infty(k), J_1] + kJ_1). \quad (5.33)$$

Using (5.29), (5.30), this yields

$$t(\mu(k)a(k) - \nu(k)d(k)) + kb(k) = 0. \quad (5.34)$$

If we use equations (5.31) and (5.32), and the relation  $ad + b^2 = 1$ , this equation becomes

$$kb(k) + t(b(k-1) + b(k+1))(1 - b(k)^2) = 0. \quad (5.35)$$

From (5.22) and (5.24), we have

$$(J_0 + m_1^\infty(k+1)J_1 - J_1m_1^\infty(k))B(k) = B(k+1)J_0, \quad (5.36)$$

which yields

$$a(k+1) - a(k) + \nu(k+1)b(k) = 0, \quad (5.37)$$

$$b(k) + \nu(k+1)d(k) = 0, \quad (5.38)$$

$$b(k+1) - (\lambda(k+1) - \lambda(k))b(k) - \mu(k)a(k) = 0, \quad (5.39)$$

$$(\lambda(k+1) - \lambda(k))d(k) - \mu(k)b(k) = 0. \quad (5.40)$$

Thus using (5.32), (5.37) yields

$$a(k) = (1 - b(k)^2)a(k+1). \quad (5.41)$$

On the other hand, the equation (5.23) implies

$$J_1B(k) = B(k+1)(J_1 + m_1^0(k+1)J_0 - J_0m_1^0(k)), \quad (5.42)$$

which yields

$$(\lambda(k+1) - \lambda(k))a(k+1) - \nu(k+1)b(k+1) = 0, \quad (5.43)$$

$$b(k+1) - \mu(k)a(k+1) = 0, \quad (5.44)$$

$$b(k) - (\lambda(k+1) - \lambda(k))b(k+1) - \nu(k+1)d(k+1) = 0, \quad (5.45)$$

$$d(k+1) - a(k) + \mu(k+1)b(k) = 0. \quad (5.46)$$

Thus using (5.31), (5.46) yields

$$d(k) = (1 - b(k)^2)d(k-1). \quad (5.47)$$

□

*Remark 5.2.* The RHP (1.2) has also the parameter  $t$ . In addition to the above equation (5.2)-(5.4) obtained from the  $z$ -derivative and ‘ $n$ -derivative’ of  $Y$ , we can also obtain two more equations from  $z$ -derivative and  $t$ -derivative of  $Y$ , and from the  $t$ -derivative and  $n$ -derivative of  $Y$ . In this paper, we would not consider those equations. See Section 3 of [4] and [46, 47, 1] for those other equations.

## 6 Integrable operators

In this section, we related the RHP (1.2) to a so-called integrable operator on  $L^2(\Sigma)$ , and also discuss an identity between Toeplitz determinants and Fredholm determinants on  $L^2(\Sigma)$ .

It is shown in [5] that the RHP (1.2) for orthogonal polynomials is also the RHP for an integrable operator on  $\Sigma$  after a simple algebraic transformation. For this purpose, the particular choice  $\varphi(z) = e^{t(z+z^{-1})}$  in (1.2) is not important. In the below, we just assume that  $\varphi(z)$  is a continuous function on  $\Sigma$  which has the factorization  $\varphi(z) = \varphi_+(z)\varphi_-(z)$  where  $\varphi_+(z), \varphi_+(z)^{-1}$  are analytic in  $|z| < 1$ , continuous in  $|z| \leq 1$ ,  $\varphi_+(0) = 1$ , and  $\varphi_-(z), \varphi_-(z)^{-1}$  are analytic in  $|z| > 1$ , continuous in  $|z| \geq 1$ ,  $\lim_{z \rightarrow \infty} \varphi_-(z) = 1$ . We also assume that 1 is not in the spectrum of the operator  $K$  defined in (6.7) below.

Define (cf. (4.2), (4.3))

$$M(z) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} \frac{1}{2}\varphi_+(z) & 0 \\ 0 & 2\varphi_+(z)^{-1} \end{pmatrix}, \quad |z| < 1, \quad (6.1)$$

and set

$$M(z) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} 0 & z^{-k}\varphi_-(z) \\ -z^k\varphi_-(z)^{-1} & 0 \end{pmatrix}, \quad |z| > 1. \quad (6.2)$$

Then  $M(z)$  solves the new Riemann-Hilbert problem

$$\begin{cases} M(z) & \text{is analytic in } z \in \Omega_{\pm}, \text{ and is continuous in } \overline{\Omega_{\pm}}, \\ M_+(z) = M_-(z)V(z), & z \in \Sigma, \\ M(z) = I + O(z^{-1}), & z \rightarrow \infty, \end{cases} \quad (6.3)$$



where

$$V(z) = \begin{pmatrix} 0 & -2z^{-k}\psi(z)^{-1} \\ \frac{1}{2}z^k\psi(z) & 2 \end{pmatrix}, \quad \psi(z) := \varphi_+(z)\varphi_-(z)^{-1}. \quad (6.4)$$

Note that the jump matrix  $V$  is of the form

$$V(z) = I - 2\pi i f(z)g(z)^T, \quad (6.5)$$

where

$$f(z) = (f_1, f_2)^T = (z^{-k}, -\frac{1}{2}\psi(z))^T, \quad g(z) = (g_1, g_2)^T = \frac{1}{2\pi i}(z^k, 2\psi(z)^{-1})^T. \quad (6.6)$$

This is precisely the RHP canonically associated to the so-called integrable operator  $K = K_k$  on  $L^2(\Sigma)$  (see [25], [16]) whose kernel is defined by

$$K(z, w) = \frac{f(z)^T g(w)}{z - w} = \frac{z^{-k}w^k - \psi(z)\psi(w)^{-1}}{2\pi i(z - w)}, \quad (Kf)(z) = \int_{\Sigma} K(z, w)f(w)dw. \quad (6.7)$$

It is shown in [25] (see also [16]) that given an integrable operator  $K$  of the form (6.7) (with general column vectors  $f, g$  such that  $f(z)^T g(z) = 0$ ), the resolvent kernel of  $K$ , if exists, is again an integrable operator

$$\left( \frac{1}{1 - K} \right) (z, w) = \frac{F(z)^T G(w)}{z - w} \quad (6.8)$$

and moreover, the column vector  $F, G$  are expressed in terms of the solution  $M$  to the (normalized) RHP with the jump matrix (6.5) :

$$F(z) = \left( \frac{1}{1 - K} f \right) (z) = M_+(z)f(z), \quad (6.9)$$

$$G(z) = \left( \frac{1}{1 - K} g \right) (z) = (M_+(z)^T)^{-1}g(z). \quad (6.10)$$

From the structure of the integrable operators, one can also show (see Lemma 4 of [5]) that

$$M_{11}(0) = \frac{\det(1 - K_{k-1})}{\det(1 - K_k)}. \quad (6.11)$$

Thus, using the definition of  $M$ , (1.6) and (2.4), we obtain

$$\frac{D_{k-1}(\varphi)}{D_k(\varphi)} = 2 \frac{\det(1 - K_{k-1})}{\det(1 - K_k)}. \quad (6.12)$$

Moreover, when  $[1, \infty)$  has no intersection to the spectrum of  $K_k$  for all  $k \geq k_0$  for some  $k_0$ , it is shown in Lemma 5 of [5] that

$$\lim_{p \rightarrow \infty} 2^{-p} \det(1 - K_p) = 1. \quad (6.13)$$

Thus for this case, by taking infinite product in  $k$  of (6.12), we have

$$\frac{D_n(\varphi)}{D_\infty(\varphi)} = 2^{-n} \det(1 - K_n), \quad (6.14)$$

where

$$D_\infty(\varphi) = \lim_{p \rightarrow \infty} D_p(\varphi) = \exp\left(\sum_{j=1}^{\infty} j(\log \varphi)_j (\log \varphi)_{-j}\right) \quad (6.15)$$

is given by the strong Szegő theorem.

It would be of interest to find if the other entries  $M_{12}$ ,  $M_{21}$  have expressions involving  $K$ , analogous to (6.11). Such formulas, if exist, would yield similar formulas to (6.12), (6.14) for the Hankel determinants (2.12).

We note that as mentioned in Remark 2.1, there is yet another identity between the Toeplitz determinant and the Fredholm determinant of an integrable operator. But in this case, the integrable operator acts on a discrete set  $\{k, k+1, \dots\}$ . Hence there is another RHP with jump conditions on the discrete set. It is not clear yet if for example, there is an algebraic transformations between these two RHP's, one with jump on  $\Sigma$  and the other with jump on the discrete set.

## 7 Other models

There are other last passage percolation models which have the similar Toeplitz/Hankel determinant formulas of the form (2.1), (2.7) and (2.13). The only difference now is the symbol  $\varphi$  which depends on the model. In the below, we will describe the models and the corresponding symbols. Here we only consider the square case and the triangle case.

Since the RHP formulation for the Toeplitz/Hankel determinants in Section 2 is general, for all the models below there are associated RHP of the form (1.2) with new function  $\varphi(z)$ . Thus we can follow the procedures in Section 4, Section 5 and Section 6. The determinant identity in Section 6 is general and it includes all the cases below. The difference equations for the cases below can be worked out as in Section 5. It will appear in some other place. See [1] for some results for the difference/differential equations. The asymptotic analysis of Section 4 can be worked out in all cases below. However, as mentioned several places above, for the square case, the convergence in distribution result of the form (4.16) can be obtained from a Fredholm determinant formula on a discrete set, which does not require the RHP analysis. In the literature, many such results are obtained from the Fredholm determinant analysis. But for the triangle cases and also for the tail estimates, RHP analysis have been used (see [3, 8]). For the survey of asymptotic results of the cases below, see e.g. [36].

### 7.1 Square case

Consider a lattice version of the Poisson process. Let  $X(i, j)$ ,  $i, j \in \mathbb{N}$ , be a planar array of independent random variables. We consider a *weakly-up/weakly-right path*, which a sequence  $\{(i_k, j_k)\}_{k=1}^l$  such that

$i_1 \leq i_2 \leq \dots \leq i_l$  and  $j_1 \leq j_2 \leq \dots \leq j_l$ . Let  $(1, 1) \nearrow (M, N)$  denote the set of all weakly-up/weakly-right paths from  $(1, 1)$  to  $(M, N)$ . Now define

$$L(M, N) := \max_{\pi \in (1, 1) \nearrow (M, N)} \sum_{(i, j) \in \pi} X(i, j). \quad (7.1)$$

If one thinks of the random variable  $X(i, j)$  as the passage time at the site  $(i, j)$ , the sum  $\sum_{(i, j) \in \pi} X(i, j)$  is the total passage time to travel from  $(1, 1)$  to  $(M, N)$  along the particular path  $\pi$ . Thus  $L(M, N)$  can be regarded as the *last passage percolation time* from  $(1, 1)$  to  $(M, N)$ . As in the Poisson case of Section 2, this problem arises in many different fields. See for example, [27, 23, 33] for various applications.

In the definition of  $L(M, N)$  in (7.1), we could take different admissible up/right paths. For example, we could take weakly-up/strictly-right paths (sequences  $\{(i_k, j_k)\}_{k=1}^l$  such that  $i_1 \leq \dots \leq i_l$ ,  $j_1 < \dots < j_l$ ) or strictly-up/strictly-right paths (sequence  $\{(i_k, j_k)\}_{k=1}^l$  such that  $i_1 < \dots < i_l$ ,  $j_1 < \dots < j_l$ ). Also we can take continuum model instead of lattice model. In the below, we list the models which have the determinant formula for the distribution of  $L(M, N)$ . In each case, we have

$$\mathbb{P}(L(M, N) \leq \ell) = \frac{1}{Z_{M, N}} \mathbb{E}_{U \in U(\ell)} \det(\varphi(U)). \quad (7.2)$$

with a function  $\varphi(z)$  for  $|z| = 1$  which depends on the model. The constant  $Z_{M, N} = \lim_{\ell \rightarrow \infty} D_\ell(\varphi)$  is a finite number in each case.

We use the notation  $g(q)$  for a geometric random variable with parameter  $q \in (0, 1)$  :  $\mathbb{P}(g(q) = k) = (1 - q)q^k$ ,  $k = 0, 1, 2, \dots$ . We understand that  $g(0)$  means the identically 0 random variable. The notation  $b(q)$  is used for the Bernoulli random variable :  $\mathbb{P}(b(q) = 0) = \frac{q}{1+q}$ ,  $\mathbb{P}(b(q) = 1) = \frac{1}{1+q}$ .

**Lattice models :**

- (a) (weakly-up/weakly-right) Fix  $q_i, q_j \geq 0$  such that  $q_i q_j \in [0, 1)$ ,  $i = 1, \dots, M$ ,  $j = 1, \dots, N$ . Take  $X(i, j) \sim g(q_i q_j)$ . Then

$$\varphi(z) = \prod_{i=1}^M \prod_{j=1}^N (1 + q_i z)(1 + q_j / z), \quad Z_{M, N} = \prod_{i=1}^M \prod_{j=1}^N (1 - q_i q_j)^{-1}. \quad (7.3)$$

- (b) (weakly-up/strictly-right) Fix  $q_i, q_j \geq 0$ ,  $i = 1, \dots, M$ ,  $j = 1, \dots, N$ . Take  $X(i, j) \sim b(q_i q_j)$ . Then

$$\varphi(z) = \prod_{i=1}^M \prod_{j=1}^N (1 + q_i z)(1 - q_j / z)^{-1}, \quad Z_{M, N} = \prod_{i=1}^M \prod_{j=1}^N (1 + q_i q_j). \quad (7.4)$$

- (c) (strictly-up/strictly-right) Fix  $q_i, q_j \geq 0$  such that  $q_i q_j \in [0, 1)$ ,  $i = 1, \dots, M$ ,  $j = 1, \dots, N$ . Take  $X(i, j) \sim g(q_i q_j)$ . Then

$$\varphi(z) = \prod_{i=1}^M \prod_{j=1}^N (1 - q_i z)^{-1}(1 - q_j / z)^{-1}, \quad Z_{M, N} = \prod_{i=1}^M \prod_{j=1}^N (1 - q_i q_j)^{-1}. \quad (7.5)$$

**Lattice-Continuum models :**

- (d) (Poisson : weakly-right) Fix  $q_i \geq 0$ ,  $i = 1, \dots, N$ . Consider a Poisson process of rate  $q_i$  on  $\mathbb{R} \times \{i\} \subset \mathbb{R} \times \{1, \dots, N\}$ . Let  $L(t, N)$  be the length of the longest weakly-up/weakly-right path from  $(0, 1)$  to  $(t, N)$ . Then  $\mathbb{P}(L(t, N) \leq \ell) = \frac{1}{Z_{t,N}} \mathbb{E}_{U \in U(t)}(\varphi)$  with

$$\varphi(z) = e^{tz} \prod_{i=1}^N (1 + q_i/z), \quad Z_{t,N} = \prod_{i=1}^N e^{tq_i}. \quad (7.6)$$

- (e) (Poisson : strictly-right) In (f), replace the admissible path as weakly-up/strictly-right paths. Then

$$\varphi(z) = e^{tz} \prod_{i=1}^N (1 - q_i/z)^{-1}, \quad Z_{t,N} = \prod_{i=1}^N e^{tq_i}. \quad (7.7)$$

**Continuum model :**

- (f) (Poisson process) This is the model in the square case of Section 2. We have  $\mathbb{P}(L(t) \leq \ell) = \frac{1}{Z_t} \mathbb{E}_{U \in U(t)}(\varphi)$  with

$$\varphi(z) = e^{t(z+z^{-1})}, \quad Z_t = e^{-t^2}. \quad (7.8)$$

*Remark 7.1.* In (d) and (e), weakly-up condition can be changed to strictly-up, since in a Poisson process, the event of two points occurring at the same position has measure 0.

*Remark 7.2.* We could have mixture of the above models. Then the corresponding  $\varphi$  is simply multiplication of each symbol. See [36] for the most general setting.

## 7.2 Symmetrized models

For models (a), (c), (f), one can think of 4 different symmetrized models (see section 7 of [9]). We could impose that the model is symmetric under (1) rotation about the center, (2) reflection about the diagonal, (3) reflection about the anti-diagonal and (4) reflection about both the diagonal and the anti-diagonal. In each case, there are still determinant formulas for the distribution of the longest up/right path. However, for the symmetry types (2),(3),(4) for (a), (c), we only have formulas for the square model  $M = N$ . Depending on the symmetry type, instead of Toeplitz determinant above, we sometimes have Hankel determinant.

Here we will consider only the case (2). That is, for example in (a), we impose that  $X(j, i)$  should be equal to the value of  $X(i, j)$  for each  $i, j$ . In each case, the algebraic formula is of the form

$$\mathbb{P}(L \leq \ell) = \frac{1}{Z} \mathbb{E}_{U \in O(\ell)} \det(\psi(U)). \quad (7.9)$$

Note that now we have expectation over the orthogonal group. We will specify the function  $\psi(z)$ .

Let  $g'(\alpha, q)$  be the random variable given by

$$\mathbb{P}(g'(\alpha, q) = k) = \frac{1 - q^2}{1 + \alpha q} \alpha^{k \bmod 2} q^k. \quad (7.10)$$

(a-S) Fix  $q_i \geq 0$ ,  $i = 1, \dots, N$  and  $\alpha \geq 0$  such that  $q_i \in [0, 1)$  and  $\alpha q_i \in [0, 1)$ . Take  $X(i, j) = X(j, i) \sim g(q_i q_j)$  for  $i \neq j$  and  $X(i, i) \sim g(\alpha q_i)$ . Then the longest weakly-up/weakly-right  $L(N; \alpha)$  path from  $(1, 1)$  to  $(N, N)$  satisfies (7.9) with

$$\psi(z) = (1 + \alpha z) \prod_{i=1}^N (1 + q_i z), \quad Z = \prod_{i=1}^N (1 - \alpha q_i)^{-1} \prod_{1 \leq i < j \leq N} (1 - q_i q_j)^{-1}. \quad (7.11)$$

(c-S) Fix  $q_i \geq 0$ ,  $i = 1, \dots, N$  and  $\alpha \geq 0$  such that  $q_i \in [0, 1)$ . Take  $X(i, j) = X(j, i) \sim g(q_i q_j)$  for  $i \neq j$  and  $X(i, i) \sim g'(\alpha, q_i)$ . Then the longest strictly-up/strictly-right  $L(N; \alpha)$  path from  $(1, 1)$  to  $(N, N)$  satisfies (7.9) with

$$\psi(z) = (1 + \alpha z) \prod_{i=1}^N (1 - q_i z)^{-1}, \quad Z = \prod_{i=1}^N (1 + \alpha q_i)(1 - q_i^2)^{-1} \prod_{1 \leq i < j \leq N} (1 - q_i q_j)^{-1}. \quad (7.12)$$

(f-S) The Triangle case of Section 2. Take a Poisson process of rate 1 in the region  $x < y$  in the  $xy$ -plane. Then take the mirror image of the points about the line  $x = y$ . In addition, take a Poisson process of rate  $\alpha \geq 0$  on the line  $x = y$ . Then the longest up/right path  $L(t; \alpha)$  from  $(0, 0)$  to  $(t, t)$  satisfies (7.9) with

$$\psi(z) = (1 + \alpha z)e^{tz}, \quad Z = e^{\alpha t + t^2/2}. \quad (7.13)$$

## References

- [1] M. Adler and P. van Moerbeke. Integrals over classical groups, random permutations, Toda and Toeplitz lattices. *Comm. Pure Appl. Math.*, 54(2):153–205, 2001.
- [2] D. Aldous and P. Diaconis. Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem. *Bull. Amer. Math. Soc. (N.S.)*, 36(4):413–432, 1999.
- [3] J. Baik. Random vicious walks and random matrices. *Comm. Pure Appl. Math.*, 53(11):1185–1410, 2000.
- [4] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12(4):1119–1178, 1999.
- [5] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the second row of a Young diagram under Plancherel measure. *Geom. Funct. Anal.*, 10(4):702–731, 2000.
- [6] J. Baik, P. Deift, and E. Rains. A Fredholm determinant identity and the convergence of moments for random Young tableaux. math.CO/0012117; <http://xxx.lanl.gov/abs/>.
- [7] J. Baik and E. M. Rains. Limiting distributions for a polynuclear growth model with external sources. *J. Statist. Phys.*, 100:523–541, 2000.

- [8] J. Baik and E. M. Rains. Symmetrized random permutations. In P. Bleher and A. Its, editors, *Random Matrix Models and Their Applications*, volume 40 of *Mathematical Sciences Research Institute Publications*, pages 1–19, Cambridge, 2001. Cambridge.
- [9] J. Baik and E. M. Rains. Algebraic aspects of increasing subsequences. math.CO/9905083 ; <http://xxx.lanl.gov/abs/>, to appear in *Duke Math. J.*
- [10] J. Baik and E. M. Rains. The asymptotics of monotone subsequences of involutions. math.CO/9905084 ; <http://xxx.lanl.gov/abs/>, to appear in *Duke Math. J.*
- [11] E. Basor and H. Widom. On a Toeplitz determinant identity of Borodin and Okounkov. *Integral Equations Operator Theory*, 37(4):397–401, 2000.
- [12] A. Borodin and A. Okounkov. A Fredholm determinant formula for Toeplitz determinants. *Integral Equations Operator Theory*, 37(4):386–396, 2000.
- [13] A. Borodin, A. Okounkov, and G. Olshanski. On asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.*, 13(3):481–515, 2000.
- [14] A. Böttcher. On the determinant formulas by Borodin, Okounkov, Baik, Deift and Rains. math.FA/0101008 ; <http://xxx.lanl.gov/abs/>.
- [15] A. Böttcher. One more proof of the Borodin-Okounkov formula for Toeplitz determinants. math.FA/0012200 ; <http://xxx.lanl.gov/abs/>.
- [16] P. Deift. Integrable operators. *Amer. Math. Soc. Transl. Ser. 2*, 189:69–84, 1999.
- [17] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides, and X. Zhou. Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.*, 52(12):1491–1552, 1999.
- [18] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides, and X. Zhou. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.*, 52(11):1335–1425, 1999.
- [19] P. Deift and X. Zhou. Asymptotics for the Painlevé II equation. *Comm. Pure Appl. Math.*, 48:277–337, 1995.
- [20] A. Fokas, A. Its, and V. Kitaev. Discrete Painlevé equations and their appearance in quantum gravity. *Comm. Math. Phys.*, 142:313–344, 1991.
- [21] P. Forrester. Random walks and random permutations. math.CO/9907037 ; <http://xxx.lanl.gov/abs/>.
- [22] I. Gessel. Symmetric functions and P-recursiveness. *J. Combin. Theory Ser. A*, 53:257–285, 1990.

- [23] Y. Gravner, C. Tracy, and H. Widom. Limiting theorems for height fluctuations in a class of discrete space and time growth models. math.PR/0005133 ; <http://xxx.lanl.gov/abs/>.
- [24] A. Guttmann, A. Owczarek, and X. Viennot. Vicious walkers and Young tableaux I: Without walls. *J. Phys. A*, 31(40):8123–8135, 1998.
- [25] A. Its, A. Izergin, V. Korepin, and N. Slavnov. Differential equations for quantum correlation functions. *Internat. J. Modern Phys. B*, 4(5):1003–1037, 1990.
- [26] K. Johansson. The longest increasing subsequence in a random permutation and a unitary random matrix model. *Math. Res. Lett.*, 5(1-2):63–82, 1998.
- [27] K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, 209(2):437–476, 2000.
- [28] K. Johansson. Transversal fluctuations for increasing subsequences on the plane. *Prob. Theory Related Fields*, 116(4):445–456, 2000.
- [29] K. Johansson. Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math.*, 153(1):259–296, 2001.
- [30] M. Mehta. *Random matrices*. Academic press, San Diego, second edition, 1991.
- [31] A. Okounkov. Infinite wedge and random partitions. math.RT/9907127 ; <http://xxx.lanl.gov/abs/>.
- [32] A. Okounkov. Random matrices and random permutations. *Intern. Math. Res. Notices*, 20:1043–1095, 2000.
- [33] M. Prähofer and H. Spohn. Current fluctuations for the totally asymmetric simple exclusion process. cond-math/0101200 ; <http://xxx.lanl.gov/abs/>.
- [34] M. Prähofer and H. Spohn. Statistical self-similarity of one-dimensional growth processes. *Phys. Rev. Lett.*, 84:4882–4885, 2000.
- [35] M. Prähofer and H. Spohn. Universal distributions for growth processes in 1 + 1 dimensions and random matrices. *Physica A*, 279:342–352, 2000.
- [36] E. Rains. A mean identity for longest increasing subsequence problems. math.CO/0004082 ; <http://xxx.lanl.gov/abs/>.
- [37] E. M. Rains. Correlation functions for symmetrized increasing subsequences. math.CO/0006097 ; <http://xxx.lanl.gov/abs/>.
- [38] E. M. Rains. Increasing subsequences and the classical groups. *Electron. J. Combin.*, 5(1):R12, 1998.
- [39] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

- [40] T. Seppäläinen. Large deviations for increasing sequences on the plane. *Prob. Theory Related Fields*, 112:221–244, 1998.
- [41] T. Seppäläinen. Perturbation of the equilibrium measure for a totally asymmetric stick process in one dimension. *Ann. of Prob.*, 29(1):176–204, 2001.
- [42] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, Cambridge, United Kingdom, 1999.
- [43] G. Szegő. *Orthogonal Polynomials*, volume 23 of *American Mathematical Society, Colloquium Publications*. AMS, Providence, R.I., fourth edition, 1975.
- [44] C. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159:151–174, 1994.
- [45] C. Tracy and H. Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.*, 177:727–754, 1996.
- [46] C. Tracy and H. Widom. Random unitary matrices, permutations and Painlevé. *Comm. Math. Phys.*, 207(3):665–685, 1999.
- [47] C. Wang. Orthonormal polynomials on the unit circle and spatially discrete Painlevé II equation. *J. Phys. A*, 32(41):7207–7217, 1999.