Total integrals of global solutions to Painlevé II

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Abstract

We evaluate the total integral from negative infinity to positive infinity of all global solutions to the Painlevé II equation on the real line. The method is based on the interplay between one of the equations of the associated Lax pair and the corresponding Riemann-Hilbert problem. In addition, we evaluate the total integral of a function related to a special solution to the Painlevé V equation. As a corollary, we obtain short proofs of the computation of the constant terms of the limiting gap probabilities in the edge and the bulk of the Gaussian Orthogonal and Gaussian Symplectic Ensembles which were obtained recently in [4] and [18]. We also evaluate the total integrals of certain polynomials of the Painlevé functions and their derivatives. These polynomials are the densities of the first integrals of the modified Korteweg-de Vries equation. We discuss the relations of the formulae we have obtained to the classical trace formulae for the Dirac operator on the line.

1 Introduction

In this paper we compute the total integral, or integral from negative infinity to positive infinity, of all global solutions to the Painlevé II equation on the real line (modulo an additve factor of $2\pi i\mathbb{Z}$ for one case; see Theorem 3.2). If the solutions do not decay sufficiently fast as $x \to \pm\infty$ then appropriate terms from the asymptotic expansion of the solution are subtracted off to make the integral convergent. One of the motivations is to give a new, short proof of the constant terms (first computed in [4]) in the asymptotic expansions of the distributions of the largest eigenvalue of a GOE or GSE matrix in the edge scaling limit. This employs the total integral of the special Hastings-McLeod solution (see Theorem 2.2). In addition, in Section 5 we compute the total integral of a function related to a special solution of the Painlevé V equation. This allows us to give a short proof of the constant terms (first computed in [18]) in the asymptotic expansions of the limiting gap probabilities in the bulk for a GOE or GSE matrix. In the last two sections we evaluate the total integrals of the polynomials of the Painlevé functions and their derivatives that are produced by the densities of the first integrals of the modified Korteweg-de Vries equation. The evaluation of these integrals, although much simpler than the evaluation of the total integrals of the Painelvé functions themselves, allows us to introduce the Painlevé analogs of the classical trace formulae of the scattering theory (see equations (250) in Section 7).

The homogenous Painlevé II equation

$$u_{xx}(x) = 2u^3(x) + xu(x) \tag{1}$$

can be solved via a certain Riemann-Hilbert problem (see [22]; also see [20, 29, 21, 26] for the derivation and for the history of the subject). Define the six rays $\gamma_k := \{e^{i(2k-1)\pi/6}\mathbb{R}^+\},\ k = 1, \ldots, 6$ oriented outwards from 0 in the complex plane. On each γ_k define the jump matrix S_k as shown in figure 1. The complex constants s_1, s_2, and s_3 satisfy

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0. \tag{2}$$

Solving the Riemann-Hilbert problems means finding a 2×2 matrix valued function $\Psi(\lambda; x)$ such that

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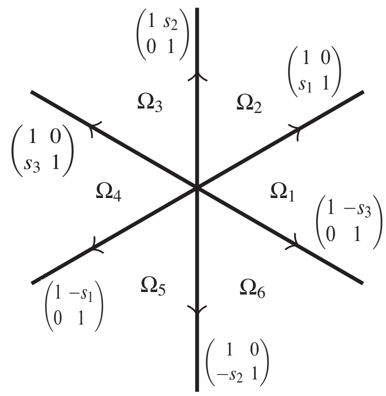

Figure 1: The Riemann-Hilbert problem for Painlevé II.

$$\begin{cases} \Psi(\lambda;x) \text{ is analytic for } \lambda \notin \gamma_k, \quad k=1,\ldots,6 \\ \Psi_+(\lambda;x) \text{ and } \Psi_-(\lambda;x) \text{ are continuous for } \lambda \in \gamma_k, \quad k=1,\ldots,6 \\ \Psi_+(\lambda;x) = \Psi_-(\lambda;x) S_k \text{ on } \gamma_k, \text{ with } S_k \text{ defined in figure 1} \\ \Psi(\lambda;x) e^{\theta(\lambda;x)\sigma_3} = I + O(\frac{1}{\lambda}) \text{ as } \lambda \to \infty. \end{cases} \tag{3}$$

Here $\Psi_+(\lambda;x)$ and $\Psi_-(\lambda;x)$ denote the nontangential limits of $\Psi(\lambda;x)$ from the left and right sides of the jump contour, respectively, and

$$\theta(\lambda;x) := i\left(\frac{4}{3}\lambda^3 + x\lambda\right). \tag{4}$$

If Ψ exists,

$$u(x) := 2\lim_{\lambda\to\infty}(\lambda\Psi_{12}(\lambda;x)e^{-\theta(\lambda;x)}) \tag{5}$$

is a solution of (1). Indeed, the Riemann-Hilbert problem is always uniquely solvable (the solution is a meromorphic function of x) and the map

$$\{s_1,s_2,s_3\} \to \{\text{set of all solutions of (1)}\}, \tag{6}$$

defined by formula (5), is a bijection ([22] Theorem 3.4, Theorem 4.2, and Corollary 4.4).

We define the Pauli matrices as

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

It is easy to check from the Riemann-Hilbert problem that $\Psi(\lambda;x)$ satisfies the Lax pair

$$\frac{\partial}{\partial\lambda}\Psi = (-i(4\lambda^2 + x + 2u^2)\sigma_3 - 4u\lambda\sigma_2 - 2v\sigma_1)\Psi \tag{8}$$

$$\frac{\partial}{\partial x}\Psi = (-i\lambda\sigma_3 - u\sigma_2)\Psi. \tag{9}$$

Here the function $v(x)$ satisfies $v(x) = u_x(x) \equiv du(x)/dx \equiv u'(x)$. The Painleve II equation (1) is indeed the compatability condition for this overdetermined system[1].

[1]Note that here we use the Lax pair from p. 174 of [22]. This Lax pair differs from the original Lax pair suggested in [20] and reproduced on page 161 of [22] by a matrix conjugation with the matrix $e^{i\frac{\pi}{4}\sigma_3}$.

Let $\Psi_k(\lambda;x)$ indicate the function $\Psi(\lambda;x)$ restricted to $\lambda \in \Omega_k$, where the regions Ω_k are defined in figure 1. The x differential equation (9) is particularly simple when $\lambda = 0$:

$$\frac{d}{dx}P(x) = -u(x)\sigma_2 P(x), \qquad P(x) := \lim_{\lambda\to 0} \Psi_k(\lambda;x) \tag{10}$$

for some k, where λ approaches 0 in Ω_k. This limit is well defined since $\Psi_k(\lambda;x)$ takes continuous boundary values. The general solution of (10) is

$$P(x) = e^{-U(a,x)\sigma_2}P(a) = \begin{pmatrix} \cosh U(a,x) & i\sinh U(a,x) \\ -i\sinh U(a,x) & \cosh U(a,x) \end{pmatrix} P(a), \tag{11}$$

where a is a constant and

$$U(a,x) := \int_a^x u(y)dy. \tag{12}$$

Hence

$$\begin{pmatrix} \cosh U(a,x) & i\sinh U(a,x) \\ -i\sinh U(a,x) & \cosh U(a,x) \end{pmatrix} = P(x)P(a)^{-1}. \tag{13}$$

If we take $x \to +\infty$ and $a \to -\infty$, this yields a relation between the total integral $\int_{-\infty}^{\infty} u(y)dy$ and the solution of the Riemann-Hilbert problem (3). Therefore analyzing the Riemann-Hilbert problem asymptotically as $x \to \pm\infty$ by using the Deift-Zhou steepest-descent method, we can compute the total integral.

The asymptotic analysis as $x \to \pm\infty$ for the Painlevé II Riemann-Hilbert problem has been worked out in [14] and [22][2]. Most of the asymptotic analysis we will need in this paper is carried out in these references with the exception of the $O(x^{-1})$ term in the generic purely imaginary global solutions, which we compute in Section 4. Nevertheless, the value of the solution at $z = 0$, $P(x) := \lim_{\lambda\to 0}\Psi_k$, has not been specifically addressed before, and in the subsequent sections we compute this term explicitly. We adopt the notation in [22] except when computing the total integral of the Hastings-McLeod solutions, when it is convenient to follow [14] where the original Riemann-Hilbert setting of [20] is used. We remark that the monodromy data (p,q,r), jump matrix V_{DZ}, and solution $m^{(1)}$ to the Riemann-Hilbert problem in [14] are related to those in [22] by

$$p = is_3, \quad q = is_1, \quad r = -is_2, \quad V_{\mathrm{DZ}} = e^{-i\pi\sigma_3/4}e^{-\theta\sigma_3}Ve^{\theta\sigma_3}e^{i\pi\sigma_3/4}, \quad m^{(1)} = e^{-i\pi\sigma_3/4}\Psi e^{\theta\sigma_3}e^{+i\pi\sigma_3/4}. \tag{14}$$

Note that the phase factor $e^{\theta\sigma_3}$ appears in the normalization condition in [22] and in the jump matrices in [14].

We conclude the introduction with the following useful observation that relates the solution corresponding to monodromy data (s_1,s_2,s_3) to the solution corresponding to monodromy data $(-s_1,-s_2,-s_3)$ (cf. [33] and Chapter 11 of [22]).

Lemma 1.1. *If $u(x;s_1,s_2,s_3)$ is the solution to* (1) *with monodromy data (s_1,s_2,s_3), then*

$$u(x;s_1,s_2,s_3) = -u(x;-s_1,-s_2,-s_3). \tag{15}$$

Proof. Define $\widetilde{\Psi}(\lambda;x) := e^{i\pi\sigma_3/2}\Psi(\lambda;x)e^{-i\pi\sigma_3/2}$. Then $\widetilde{\Psi}(\lambda;x)$ satisfies the Riemann-Hilbert problem (3) with the jump condition replaced by $\widetilde{\Psi}_+(\lambda;x) = \widetilde{\Psi}_-(\lambda;x)e^{i\pi\sigma_3/2}S_k e^{-i\pi\sigma_3/2}$ on γ_k. The only effect this conjugation has is to change (s_1,s_2,s_3) to $(-s_1,-s_2,-s_3)$ in the jump matrices. Therefore, if $\Psi(\lambda;x,s_1,s_2,s_3)$ is the solution to the Riemann-Hilbert problem (3) with monodromy data (s_1,s_2,s_3), then $\widetilde{\Psi}(\lambda;x,s_1,s_2,s_3)$ $= \Psi(\lambda;x,-s_1,-s_2,-s_3)$ by the existence and uniqueness of the solution of the Riemann-Hilbert problem. From (5),

$$\begin{aligned} u(x;s_1,s_2,s_3) &= 2\lim_{\lambda\to\infty}(\lambda\widetilde{\Psi}_{12}(\lambda;x,-s_1,-s_2,-s_3)e^{-\theta(\lambda;x)}) \\ &= -2\lim_{\lambda\to\infty}(\lambda\Psi_{12}(\lambda;x,-s_1,-s_2,-s_3)e^{-\theta(\lambda;x)}) \\ &= -u(x;-s_1,-s_2,-s_3), \end{aligned} \tag{16}$$

[2]We refer to the introduction of [22] for a detailed historic review on the asymptotic analysis of the Painlevé equations via the Riemann-Hilbert-isomonodromy method.

as desired. □

The paper is organized as follows. In Section 2, the total integrals of the purely real solutions of Painlevé II equation are evaluated. In particular, Theorem 2.2 gives a new short proof of the evaluation of the constant term of the asymptotics of the GOE and GSE Tracy-Widom distribution functions in random matrix theory obtained in [4]. The total integrals of the purely imaginary solutions are computed in Section 3. In Section 4, we compute the asymptotic expansion of the generic purely imaginary solution up to $O(x^{-3/2})$ as $x \to \infty$, whose total integral is studied in Theorem 3.2. In Section 5, the total integral of a special solution to Painlevé V equation is computed, and a new simple proof of the constant term in the asymptotics of the gap distribution of orthogonal and symplectic ensembles of random matrix theory is given. Finally, in the last two sections the total integrals of the densities of the mKdV conservation laws evaluated for the Painlevé functions are computed (Section 6), and the relations to the trace formulae of the scattering theory for the Dirac operator are discussed (Section 7).

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2 Purely real solutions

A solution of Painlevé II is real for all real x if and only if the monodromy data satisfy

$$s_3 = \overline{s_1}, \quad s_2 = \overline{s_2}. \tag{17}$$

See, for example, page 158 in [22]. The constraint (2) on the monodromy data shows that if $|s_1| = 1$ then s_1 must be $\pm i$ and s_2 can be any real number. If $|s_1| \neq 1$ then $s_2 = (s_1 + \overline{s_1})/(1 - |s_1|^2)$. If $s_2 \neq 0$, then $u(x)$ has infinitely many poles; specifically ([34]; see also page 349 in [22]),

$$u(x) \sim \pm\sqrt{x} \tan\left(\frac{\sqrt{2}}{3}x^{3/2} + O(\ln x)\right) \text{ as } x \to +\infty \text{ for purely real solutions with } s_2 \neq 0. \tag{18}$$

Since we want to integrate $u(x)$ we will assume $s_2 = 0$, and thus that s_1 is purely imaginary. If $|s_1| > 1$ then $u(x)$ again has infinitely many poles; specifically ([34]; see also page 349 in [22]),

$$u(x) \sim \pm\sqrt{-x} \Big/ \sin\left(\frac{2}{3}(-x)^{3/2} + O(\ln(-x))\right) \text{ as } x \to -\infty \text{ for purely real solutions with } |s_1| > 1. \tag{19}$$

There are two cases of global purely real solutions:

- The purely real Ablowitz-Segur solutions [40, 41] with monodromy data

$$-1 < is_1 < 1, \quad s_3 = \overline{s_1} = -s_1, \quad s_2 = 0 \tag{20}$$

  and asymptotics

$$u(x) = \frac{\sqrt{-2\beta}}{(-x)^{1/4}} \cos\left(\frac{2}{3}(-x)^{3/2} + \beta\log(8(-x)^{3/2}) + \phi\right) + O\left(\frac{\log(-x)}{(-x)^{5/4}}\right) \quad \text{as } x \to -\infty, \tag{21}$$

$$u(x) = is_1\mathrm{Ai}(x) + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right) \quad \text{as } x \to +\infty. \tag{22}$$

  Here

$$\beta := \frac{1}{2\pi}\log(1 - |s_1|^2) < 0, \quad \phi := -\frac{\pi}{4} - \arg\Gamma(i\beta) - \arg s_1, \tag{23}$$

  and $\mathrm{Ai}(x)$ is the standard Airy function. A representative solution with $s_1 = -\frac{i}{2}$ is shown in figure 2(a).

- The Hastings-McLeod solutions [25] with monodromy data

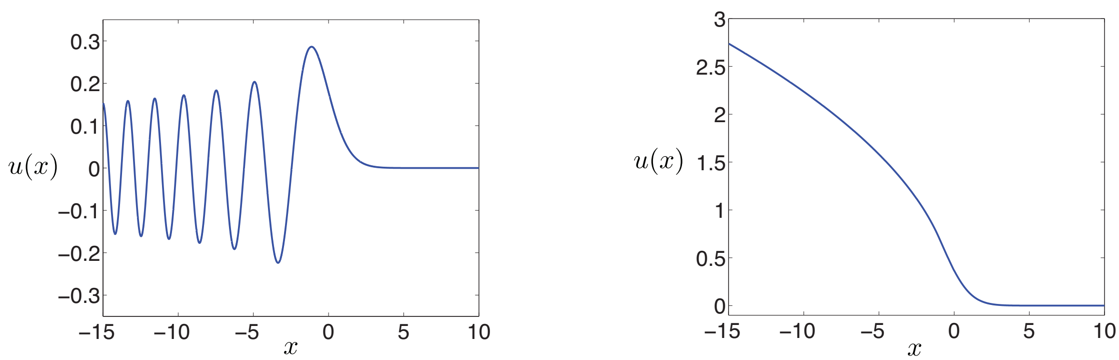
$$s_1 = \pm i, \quad s_3 = \mp i, \quad s_2 = 0. \tag{24}$$

and asymptotics

$$u(x) = is_1\sqrt{\frac{-x}{2}} + O((-x)^{-5/2}) \quad \text{as } x \to -\infty, \tag{25}$$

$$u(x) = is_1 \mathrm{Ai}(x) + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right) \quad \text{as } x \to +\infty. \tag{26}$$

The solution with $s_1 = -i$ is shown in figure 2(b).



(a) The real Ablowitz-Segur solution with $s_1 = -i/2$.

(b) The Hastings-McLeod solution with $s_1 = -i$.

Figure 2: Plots of purely real solutions to Painlevé II.

The error estimates above come from [14]. These solutions have no singularities for finite x [3, 25]. Both of these solutions look like the Airy function (up to a constant) as $x \to +\infty$. However, as $x \to -\infty$, their asymptotic behaviors differ dramatically: the Ablowitz-Segur solutions decay, whereas the Hastings-McLeodo solutions grow. We begin with the total integral for the Ablowitz-Segur solutions.

Theorem 2.1. [Purely real Ablowitz-Segur solutions] *Suppose that $u(x)$ is a solution to the Painlevé II equation* (1) *with monodromy data $-1 < is_1 < 1$, $s_3 = \overline{s_1} = -s_1$, $s_2 = 0$ (that is, with asymptotics given by* (21) *and* (22)*). Then*

$$\int_{-\infty}^{+\infty} u(y)dy = \frac{1}{2}\log\left(\frac{1+is_1}{1-is_1}\right). \tag{27}$$

Proof. Consider λ approaching 0 in the region Ω_2 (see figure 1). That is, set

$$P(x) := \Psi_2(0; x). \tag{28}$$

Since the Ablowitz-Segur solutions are integrable on the entire real line, we could choose $a = -\infty$ or $a = +\infty$ in (11); we pick $a = +\infty$. Then

$$\lim_{x\to+\infty} P(x) = \lim_{x\to+\infty} e^{U(+\infty,x)\sigma_2} C = C. \tag{29}$$

We therefore find C by analyzing the Riemann-Hilbert problem as $x \to +\infty$. This analysis is done in [22], Chapter 11, Section 6, so we merely provide a short sketch of the argument. Note that since $s_2 = 0$ the jump contour consists of only four rays. We use the scalings

$$z := \frac{\lambda}{x^{1/2}}, \quad t := x^{3/2}, \quad \Phi(z;t) := \Psi(\lambda(z);x), \quad \phi(z) := i\frac{4}{3}z^3 + iz. \tag{30}$$

Using standard contour deformations, this Riemann-Hilbert problem for $\Phi(z;t)$ may be transformed into the following Riemann-Hilbert problem for $\Phi^{\mathrm{def}}(z;t)$:

$$\begin{cases} \Phi^{\mathrm{def}}(z;t) \text{ is analytic in } \mathbb{C} \setminus \{\Im z = \pm\frac{1}{2}\} \\ \Phi^{\mathrm{def}}_+(z;t) = \Phi^{\mathrm{def}}_-(z;t) \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \qquad z \in \{\Im z = \frac{1}{2}\} \\ \Phi^{\mathrm{def}}_+(z;t) = \Phi^{\mathrm{def}}_-(z;t) \begin{pmatrix} 1 & s_1 \\ 0 & 1 \end{pmatrix}, \qquad z \in \{\Im z = -\frac{1}{2}\} \\ \Phi^{\mathrm{def}}(z;t) e^{t\phi(z)\sigma_3} = I + O(z^{-1}), \qquad z \to \infty. \end{cases} \tag{31}$$

Here the two jump contours are oriented from $-\infty$ to $+\infty$. Under this deformation,

$$\Phi(z;t) = \Phi^{\mathrm{def}}(z;t) \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \text{ for } z \in \Omega_2 \cap \left\{ 0 < \Im z < \frac{1}{2} \right\}. \tag{32}$$

Then, from a standard Riemann-Hilbert problem small norm argument [14],

$$\Phi^{\mathrm{def}}(z;t) e^{t\phi(z)\sigma_3} = I + O\left(\frac{e^{-2t/3}}{\sqrt{t}} \right) \text{ as } t \to +\infty. \tag{33}$$

Undoing the contour deformations gives

$$C = \lim_{x\to+\infty} P(x) = \lim_{t\to+\infty} \Psi^{\mathrm{def}}(0;t) \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}. \tag{34}$$

Thus

$$P(x) = \begin{pmatrix} \cosh U(+\infty,x) + is_1 \sinh U(+\infty,x) & i \sinh U(+\infty,x) \\ -i \sinh U(+\infty,x) + s_1 \cosh U(+\infty,x) & \cosh U(+\infty,x) \end{pmatrix}. \tag{35}$$

The analysis for x near $+\infty$ goes through even if $s_1 = \pm ia$, $a \geq 1$ (and $s_3 = -s_1$, $s_2 = 0$). We will use this fact when studying the Hastings-McLeod solution below. However, for the analysis at x near $-\infty$ the analysis is different for $s_1 = \pm i$ (since $u(x)$ is not integrable at that endpoint).

The analysis of $\Psi_1(0;x)$ as $x \to -\infty$ is identical for both the Ablowitz-Segur solutions and the generic purely imaginary solutions. This calculation is carried out below as part of the proof of Theorem 3.2. Specifically, for the Ablowitz-Segur solutions equation (85) holds with $s_3 = -s_1$. At $\lambda = 0$ the two functions $\Psi_1(\lambda;x)$ and $\Psi_2(\lambda;x)$ are related by a multiplicative jump:

$$\begin{aligned} \lim_{x\to-\infty} P(x) = \lim_{x\to-\infty} \Psi_2(0;x) = \lim_{x\to-\infty} \Psi_1(0;x) \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} &= \frac{1}{\sqrt{1-s_1s_3}} \begin{pmatrix} 1 & -s_3 \\ -s_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{1-s_1s_3} & \frac{-s_3}{\sqrt{1-s_1s_3}} \\ 0 & \frac{1}{\sqrt{1-s_1s_3}} \end{pmatrix}. \end{aligned} \tag{36}$$

Combining (35) and (36) and using $s_3 = -s_1$ shows

$$\begin{pmatrix} \cosh U(+\infty,-\infty) + is_1 \sinh U(+\infty,-\infty) & i \sinh U(+\infty,-\infty) \\ -i \sinh U(+\infty,-\infty) + s_1 \cosh U(+\infty,-\infty) & \cosh U(+\infty,-\infty) \end{pmatrix} = \begin{pmatrix} \sqrt{1+s_1^2} & \frac{s_1}{\sqrt{1+s_1^2}} \\ 0 & \frac{1}{\sqrt{1+s_1^2}} \end{pmatrix}. \tag{37}$$

The (21) entry gives

$$(-i+s_1)e^{U(+\infty,-\infty)} + (i+s_1)e^{-U(+\infty,-\infty)} = 0. \tag{38}$$

Solving for $U(+\infty,-\infty)$ gives

$$\int_{-\infty}^{+\infty} u(y)dy = \frac{1}{2} \log\left(\frac{1+is_1}{1-is_1} \right) + 2i\pi m \tag{39}$$

for some $m \in \mathbb{Z}$. Since $u(x)$ is purely real, $m = 0$, which gives equation (27). □

Next we compute the total integral of the Hastings-McLeod solutions. Since these functions are not integrable near $x = -\infty$ we will subtract off the nonintegrable part.

The integral of the Hastings-McLeod solution with $s_1 = -i$ appears in the Tracy-Widom distribution functions that arise in random matrix theory [43, 44]. The proof of Theorem 2.2 below is a new, shorter way to show a result that was obtained previously by the first three authors using the asymptotics of orthogonal polynomials in [4].

Theorem 2.2. [Hastings-McLeod solutions] *Suppose that $u(x)$ is a solution to the Painlevé II equation* (1) *with monodromy data $s_1 = \pm i$, $s_3 = \mp i$, $s_2 = 0$ (that is, with asymptotics given by* (25) *and* (26)*). Then, for any $c \in \mathbb{R}$,*

$$\int_c^{+\infty} u(y)dy + \int_{-\infty}^{c} \left(u(y) - is_1 \sqrt{\frac{|y|}{2}} \right) dy = -is_1 \frac{\sqrt{2}}{3} c|c|^{1/2} + is_1 \frac{1}{2} \log(2). \tag{40}$$

Proof. We set $s_1 = i$. The alternate case $s_1 = -i$ follows immediately from noting that if $u(x)$ is a solution to (1) then so is $-u(x)$.

Take $\lambda \in \Omega_2$ and define

$$P(x) := \Psi_2(0; x). \tag{41}$$

The Hastings-McLeod solution is integrable at $x = +\infty$, so set $a = +\infty$. The constant matrix C was computed above in (34) in the section on Ablowitz-Segur solutions. Explicitly,

$$C = \lim_{x \to +\infty} P(x) = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \tag{42}$$

and therefore

$$P(x) = \begin{pmatrix} e^{-U(+\infty,x)} & i \sinh U(+\infty, x) \\ ie^{-U(+\infty,x)} & \cosh U(+\infty, x) \end{pmatrix}. \tag{43}$$

Now we compute the asymptotics of $P(x)$ as $x \to -\infty$, taking into account the nonintegrable term using a g-function. This Riemann-Hilbert problem was analyzed in [14], and we give a sketch of the argument. Recall that the function $\Psi(\lambda; x)$ and $m^{(1)}(\lambda; x)$ used in [14] are related as in (14). Define

$$g(\lambda) := (\lambda^2 - 1)^{3/2} \tag{44}$$

with branch cut on $[-1, 1]$ and sheet chosen so $g(\lambda) \sim \lambda^3$ as $\lambda \to \infty$. Then set (see (6.17) in [14])

$$m^{g}(\lambda; x) := m^{(1)} \left(\sqrt{\frac{-x}{2}} \lambda; x \right) e^{it(g(\lambda) - (\lambda^3 - \frac{3}{2}\lambda))\sigma_3}, \quad t := \frac{\sqrt{2}}{3}(-x)^{3/2}. \tag{45}$$

By standard changes of variables we can transform $m^g(\lambda; x)$ to $m^{(23)}(\lambda; x)$, which solves the Riemann-Hilbert problem[3]

$$\begin{cases} m^{(23)}(\lambda; x) \text{ is analytic in } \mathbb{C} \setminus \Sigma \\ m_+^{(23)}(\lambda; x) = m_-^{(23)}(\lambda; x) e^{-i\pi\sigma_3/4} e^{-itg_-(\lambda)\sigma_3} V^{\mathrm{HM}} e^{itg_+(\lambda)\sigma_3} e^{i\pi\sigma_3/4}, & \lambda \in \Sigma \\ m^{(23)}(\lambda; x) = I + O(\lambda^{-1}), \qquad \lambda \to \infty, \end{cases} \tag{46}$$

with the contour Σ and the constant jump V^{HM} given in figure 3. For $\lambda \in \Omega_2$,

$$\Psi(\lambda; x) = e^{i\pi\sigma_3/4} m^{(23)}(\lambda; x) e^{-\theta(\lambda;x)\sigma_3} e^{-i\pi\sigma_3/4}. \tag{47}$$

In [14] the analysis includes $s_2 \neq 0$, but for us $S_2 = S_5 = I$. As $x \to -\infty$ (that is, $t \to +\infty$), formally the jump approaches the identity on all portions of the contour in figure 3 with the exception of the jump $S_4^{-1} S_3^{-1} = S_6 S_1$ on the interval $[-1, 1]$. Now $m^g(\lambda; x) = (I + O(t^{-1/2}))m^{\mathrm{mod}}(\lambda)$ [14], where $m^{\mathrm{mod}}(\lambda)$ solves

[3] We use the notation $m^{(23)}(\lambda; x)$ to correspond with reference [14].

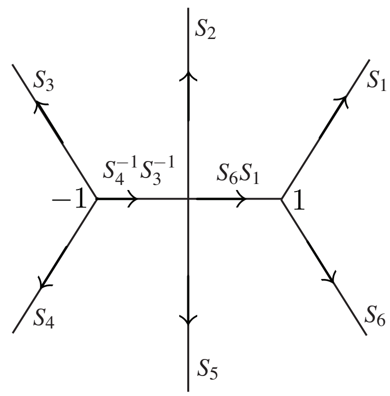


Figure 3: The deformed Riemann-Hilbert problem for the Hastings-McLeod solutions to Painlevé II.

$$\begin{cases} m^{\text{mod}}(\lambda) \text{ is analytic in } \mathbb{C} \setminus [-1,1] \\ m_+^{\text{mod}}(\lambda) = m_-^{\text{mod}}(\lambda) e^{-i\pi\sigma_3/4} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} e^{i\pi\sigma_3/4}, \qquad \lambda \in [-1,1] \\ m^{\text{mod}}(\lambda) = I + O(\lambda^{-1}), \qquad \lambda \to \infty. \end{cases} \tag{48}$$

This problem is solved explicitly by

$$m^{\text{mod}}(\lambda) = \frac{1}{2} e^{-i\pi\sigma_3/4} \begin{pmatrix} f(\lambda) + f(\lambda)^{-1} & f(\lambda) - f(\lambda)^{-1} \\ f(\lambda) - f(\lambda)^{-1} & f(\lambda) + f(\lambda)^{-1} \end{pmatrix} e^{i\pi\sigma_3/4}, \quad f(\lambda) := \left(\frac{\lambda - 1}{\lambda + 1} \right)^{1/4}. \tag{49}$$

Here $f(\lambda)$ has its branch cut on $[-1,1]$ and $f(\lambda) \sim 1$ as $\lambda \to \infty$. Undoing the transformations from $\Psi_2(\lambda; x)$ and using $f_+(0) = e^{i\pi/4}$, $g_+(0) = -i$, and $\lim\limits_{x \to -\infty} m^{(23)}(\lambda; x) = m^{\text{mod}}(\lambda)$ outside of small neighborhoods of ± 1 we have

$$\lim_{x \to -\infty} P(x) e^{t\sigma_3} = \lim_{x \to -\infty} \Psi_2(0; x) e^{itg_+(0)\sigma_3} = e^{i\pi\sigma_3/4} m_+^{\text{mod}}(0) e^{-i\pi\sigma_3/4} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{50}$$

Combining (43) and (50) gives

$$\lim_{x \to -\infty} \begin{pmatrix} e^{-U(+\infty,x)+t} & i \sinh U(+\infty, x) e^{-t} \\ i e^{-U(+\infty,x)+t} & \cosh U(+\infty, x) e^{-t} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{51}$$

The (11) entry is equivalent to

$$\lim_{x \to -\infty} \exp \left(\int_x^{+\infty} u(y) dy + \frac{\sqrt{2}}{3} (-x)^{3/2} \right) = \frac{\sqrt{2}}{2}. \tag{52}$$

Hence, for any fixed c,

$$\lim_{x \to -\infty} \exp \left(\int_c^{+\infty} u(y) dy + \int_x^c u(y) dy + \frac{\sqrt{2}}{3} (-x)^{3/2} \right) = \frac{\sqrt{2}}{2}. \tag{53}$$

From the asymptotics of $u(y)$ we see that $u(y) + \sqrt{|y|/2}$ is integrable at $-\infty$, so

$$\lim_{x \to -\infty} \exp \left(\int_c^{+\infty} u(y) dy + \int_x^c \left(u(y) + \sqrt{\frac{|y|}{2}} \right) dy - \frac{\sqrt{2}}{3} c |c|^{1/2} \right) = \frac{\sqrt{2}}{2}. \tag{54}$$

Taking a logarithm shows

$$\int_c^{+\infty} u(y)dy + \int_{-\infty}^c \left(u(y) - is_1\sqrt{\frac{|y|}{2}} \right) dy = -is_1\frac{\sqrt{2}}{3}c|c|^{1/2} + is_1\frac{1}{2}\log(2) + 2i\pi m \tag{55}$$

for some $m \in \mathbb{Z}$. Since $u(x)$ is purely real, we see $m = 0$, which shows (40). □

3 Purely imaginary solutions

Solutions to (1) are purely imaginary if and only if ([22] page 159) the monodromy data satisfy

$$s_3 = -\overline{s_1}, \quad s_2 = -\overline{s_2}. \tag{56}$$

All purely imaginary solutions are global (page 297 in [22]). There are two distinct asymptotic behaviors:

- The purely imaginary Ablowitz-Segur solutions [40, 41] with monodromy data

  $$s_1 \in \mathbb{R}, \quad s_3 = -s_1, \quad s_2 = 0 \tag{57}$$

  and asymptotics

  $$u(x) = \frac{id}{(-x)^{1/4}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{3}{4}d^2\log(-x) + \phi\right) + O\left(\frac{\log(-x)}{(-x)^{5/4}}\right) \quad \text{as } x \to -\infty, \tag{58}$$

  $$u(x) = is_1\mathrm{Ai}(x) + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right) \quad \text{as } x \to +\infty, \tag{59}$$

  with

  $$\begin{gathered} d^2 := \frac{1}{\pi}\log(1 + |s_1|^2), \quad d > 0, \\ \phi := \frac{3}{2}d^2\log(2) - \frac{\pi}{4} - \arg\Gamma\left(i\frac{d^2}{2}\right) - \arg s_1. \end{gathered} \tag{60}$$

- The generic purely imaginary solutions [27, 14] with monodromy data

  $$\Im(s_1) \neq 0, \quad s_3 = -\overline{s_1}, \quad s_2 = \frac{s_1 - \overline{s_1}}{1 + |s_1|^2} \tag{61}$$

  and asymptotics

  $$u(x) = \frac{id}{(-x)^{1/4}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{3}{4}d^2\log(-x) + \phi\right) + O\left(\frac{\log(-x)}{(-x)^{5/4}}\right), \quad x \to -\infty, \tag{62}$$

  $$u(x) = i\sigma\sqrt{\frac{x}{2}} + \frac{i\sigma\rho}{(2x)^{1/4}} \cos\left(\frac{2\sqrt{2}}{3}x^{3/2} - \frac{3}{2}\rho^2\log x + \theta\right) + O\left(\frac{1}{x}\right), \quad x \to +\infty. \tag{63}$$

  Here d and ϕ are given by (60), and

  $$\begin{gathered} \rho^2 := -\frac{1}{\pi}\log(|s_2|) = \frac{1}{\pi}\log\frac{1 + |s_1|^2}{2|\Im(s_1)|}, \quad \rho > 0, \\ \sigma := -\mathrm{sgn}(\Im(s_1)), \\ \theta := -\frac{3\pi}{4} - \frac{7}{2}\rho^2\log 2 + \arg\Gamma(i\rho^2) + \arg(1 + s_1^2). \end{gathered} \tag{64}$$

Note that the asymptotics as $x \to -\infty$ are exactly the same for both types of purely imaginary solutions. We first find the integral of the purely imaginary Ablowitz-Segur solutions. The result is the same as for the purely real Ablowitz-Segur solutions.

Theorem 3.1. [Purely imaginary Ablowitz-Segur solutions] *Suppose that $u(x)$ is a solution to the Painlevé II equation* (1) *with monodromy data $s_1 \in \mathbb{R}$, $s_3 = -s_1$, $s_2 = 0$ (that is, with asymptotics given by* (58) *and* (59)*). Then*

$$\int_{-\infty}^{+\infty} u(y)dy = \frac{1}{2}\log\left(\frac{1+is_1}{1-is_1}\right) = i\arctan(s_1) = i\arg(1+is_1). \tag{65}$$

Proof. The asymptotic analysis of the purely imaginary Ablowitz-Segur solutions is exactly the same as that for the purely real Ablowitz-Segur solutions. The proof of Theorem 2.1 applies without change through equation (39):

$$\int_{-\infty}^{+\infty} u(y)dy = \frac{1}{2}\log\left(\frac{1+is_1}{1-is_1}\right) + 2i\pi m. \tag{66}$$

Assume $s_3 = -s_1$ and $s_2 = 0$ and parameterize the purely imaginary Ablowitz-Segur solutions $u(x; s_1)$ by s_1. Note that $s_1 = 0$ corresponds to to the solution $u(x; s_1 = 0) \equiv 0$, and in this case clearly $m = 0$. We now show continuity of the total integral $\int_{-\infty}^{\infty} u(x)dx$ with respect to s_1 for $s_1 \in \mathbb{R}$, which shows $m = 0$ in (66). The Fredholm theory for Riemann-Hilbert problems shows that, for fixed x, the solution Ψ to (3) is either meromorphic in s_1 or there is no solution for any s_1 ([22] Corollary 3.1). Furthermore, the associated Riemann-Hilbert problem has a global solution for all $s_1 \in \mathbb{R}$ assuming $s_3 = -s_1$ and $s_2 = 0$ (see [22] Theorem 5.6 and note the condition in (5.5.1) should read $|s_1 + \overline{s_3}| < 2$). Combining these two facts shows that Ψ is analytic in s_1 for the purely imaginary Ablowitz-Segur solutions, and thus $u(x; s_1)$ is continuous in s_1. To show the total integral is continuous in s_1 we fix $L > 0$ large and show

$$\lim_{s_1 \to s_1'} \left(\int_{-\infty}^{-L} \big(u(x;s_1) - u(x;s_1')\big)dx + \int_{-L}^{L} \big(u(x;s_1) - u(x;s_1')\big)dx + \int_{L}^{+\infty} \big(u(x;s_1) - u(x;s_1')\big)dx \right) = 0. \tag{67}$$

The continuity of $u(x; s_1)$ with respect to s_1 shows the limit of the second integral is zero since the region of integration is compact. For the third integral, use (59) to write

$$u(x; s_1) = is_1 \mathrm{Ai}(x) + E^+(x; s_1), \tag{68}$$

where $|E^+(x; s_1)| < B^+(x)$ for some $B^+(x) \in L^1$ uniformly for $s_1 \in [s_1' - \varepsilon, s_1' + \varepsilon]$. So

$$\begin{aligned} &\lim_{s_1 \to s_1'} \int_{L}^{+\infty} \big(u(x;s_1) - u(x;s_1')\big)dx \\ &= \lim_{s_1 \to s_1'} i(s_1 - s_1') \int_{L}^{+\infty} \mathrm{Ai}(x)dx - \lim_{s_1 \to s_1'} \int_{L}^{+\infty} \big(E^+(x;s_1) - E^+(x;s_1')\big)dx \\ &= 0 \end{aligned} \tag{69}$$

by the dominated convergence theorem. For the first integral, use (58) to write

$$u(x; s_1) = \frac{id}{(-x)^{1/4}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{3}{4}d^2\log(-x) + \phi\right) + E^-(x; s_1), \tag{70}$$

where $|E^-(x; s_1)| < B^-(x)$ for some $B^-(x) \in L^1$ uniformly for $s_1 \in [s_1' - \varepsilon, s_1' + \varepsilon]$. Direct computation shows that

$$\begin{aligned} &\int_{-\infty}^{-L} \frac{id}{(-x)^{1/4}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{3}{4}d^2\log(-x) + \phi\right) dx \\ &= \frac{d}{3} e^{-i\phi} L^{3(1-id^2)/4} \left[e^{2i\phi} L^{3id^2/2} E_{(1-id^2)/2}\left(\frac{2}{3}iL^{3/2}\right) - E_{(1+id^2)/2}\left(\frac{2}{3}iL^{3/2}\right)\right], \end{aligned} \tag{71}$$

where $E_n(z) := \int_1^\infty \frac{e^{-zt}}{t^n} dt$ has a branch cut in z on $(-\infty, 0)$. The right-hand side of (71) is continuous in s_1. Therefore, by the dominated convergence theorem,

$$\lim_{s_1 \to s_1'} \int_{-\infty}^{L} \big(u(x; s_1) - u(x; s_1')\big) dx = 0. \tag{72}$$

This verifies equation (67). □

We now compute the integral of the generic purely imaginary solutions. The $O(x^{1/2})$ term in the asymptotic expansion (63) as $x \to +\infty$ is not integrable, so we will subtract it off as in the Hastings-McLeod case. The $O(x^{-1/4})$ term is integrable because of the cosine factor. However, the $O(x^{-1})$ term is not integrable, so it must be computed and subtracted off as well. The explicit form of the $O(x^{-1})$ correction to the asymptotics (63) was formally calculated via the analysis of a certain nonlinear integral equation equivalent to (1) in [32]. The asymptotic expansion for $u(x)$ up to the $O(x^{-1})$ terms turns out to be

$$\begin{aligned} u(x) =& i\sigma\sqrt{\frac{x}{2}} + \frac{i\sigma\rho}{(2x)^{1/4}} \cos\left(\frac{2\sqrt{2}}{3}x^{3/2} - \frac{3}{2}\rho^2 \log x + \theta\right) - \frac{3i\sigma\rho^2}{4x} \\ &+ \frac{i\sigma\rho^2}{4x} \cos\left(2\left[\frac{2\sqrt{2}}{3}x^{3/2} - \frac{3}{2}\rho^2 \log x + \theta\right]\right) + O(x^{-3/2}), \quad x \to +\infty. \end{aligned} \tag{73}$$

With the first two terms already known, the third and the forth terms of this formula (and, in principal, the terms of an arbitrary higher order) can be formally derived via substitution into the Painlevé equation (1) (or to the nonlinear integral equation of [32]). It should be emphasized that even the formal derivation of (73) is quite challenging; indeed, because of the presence of the growing term $\sqrt{x/2}$, it is much more difficult than the similar derivation of the correction terms to the semi-linear asymptotics (21). A serious additional question is the justification of the asymptotics (73) which can be in principal done using a priori information of the structure of the asymptotic series which in turn can be extracted from the Riemann-Hilbert analysis (compare to the approach of [15]). In Section 4 we will present an alternative and rigorous derivation of (73) using the direct asymptotic analysis of the Riemann-Hilbert problem (3). It also should be noticed that, in fact, we do not need to know the $O(x^{-1})$ terms a priori in the proof of Theorem 3.2. The $O(x^{-1})$ term that must be subtracted off to make the integral finite arises naturally during the computation. However, note that the oscillatory term of $O(x^{-1})$ in (73) will not be subtracted off because it is integrable.

Theorem 3.2. [Generic purely imaginary solutions] *Suppose that $u(x)$ is a solution to the Painlevé II equation (1) with monodromy data $\Im(s_1) \neq 0$, $s_3 = -\overline{s_1}$, $s_2 = (s_1 - \overline{s_1})/(1 + |s_1|^2)$ (that is, with asymptotics given by (62) and (63)). Define ρ^2 as in (64). Then, for any $c > 0$, there exists $m \in \mathbb{Z}$ such that*

$$\begin{aligned} &\int_{-\infty}^{c} u(y)dy + \int_{c}^{\infty} \left(u(y) - i\sigma\sqrt{\frac{y}{2}} + i\sigma\frac{3\rho^2}{4y}\right) dy \\ &\quad = i\sigma\left\{\arg(1 + i\sigma s_1) - \frac{5\rho^2}{4}\log 2 + \arg\left(\Gamma\left(\frac{1}{2} + i\frac{\rho^2}{2}\right)\right) + \frac{\sqrt{2}}{3}c^{3/2} - \frac{3\rho^2}{4}\log c + 2\pi m\right\}, \end{aligned} \tag{74}$$

where $\sigma := -\operatorname{sgn}(\Im(s_1))$ and Γ denotes the Gamma function.

This result determines the total integral up to an additive factor of $2\pi i m$ for some $m \in \mathbb{Z}$.

Proof. Since the solutions are integrable for x near $-\infty$, pick $a = -\infty$ and consider $U(-\infty, x)$. For convenience we consider Ψ_1. Set

$$P(x) := \Psi_1(0; x). \tag{75}$$

Now we compute

$$C = \lim_{x \to -\infty} P(x) \tag{76}$$

using the methods in [22]. Start with the scalings

$$z := \frac{\lambda}{(-x)^{1/2}}, \quad t := (-x)^{3/2}, \quad \Psi(z;t) := \Psi(\lambda(z);x), \quad \phi(z) := i\frac{4}{3}z^3 - iz. \tag{77}$$

The solution $\Psi(z;t)$ can be transformed using standard algebraic manipulations to $\Psi^{\text{def}}(z;t)$ which solves the Riemann-Hilbert problem on the deformed contour shown in figure 4, wherein

$$S_L := \begin{pmatrix} 1 & 0 \\ \frac{s_1}{1-s_1s_3} & 1 \end{pmatrix}, \quad S_D := \begin{pmatrix} 1-s_1s_3 & 0 \\ 0 & \frac{1}{1-s_1s_3} \end{pmatrix}, \quad S_U := \begin{pmatrix} 1 & \frac{s_1}{1-s_1s_3} \\ 0 & 1 \end{pmatrix}. \tag{78}$$

The normalization for the deformed Riemann-Hilbert problem is

$$\Psi^{\text{def}}(z;t)e^{\phi(z)t\sigma_3} = I + O(z^{-1}) \text{ as } z \to \infty. \tag{79}$$

In particular,

$$\Psi_1(\lambda;x) = \Psi^{\text{def}}(z;t)\begin{pmatrix} \frac{1}{1-s_1s_3} & \frac{-s_3}{1-s_1s_3} \\ -s_1 & 1 \end{pmatrix} \text{ for } \lambda \in \Omega_1 \cap \{\Im\lambda > 0\} \cap \left\{|\lambda| < \frac{1}{4}\right\} \text{ when } x \leq -1. \tag{80}$$

The jump matrices for $\Psi^{\text{def}}(z;t)e^{t\phi(z)\sigma_3}$ off the real interval $[-\frac{1}{2}, \frac{1}{2}]$ decay to the identity as $t \to \infty$. Indeed,

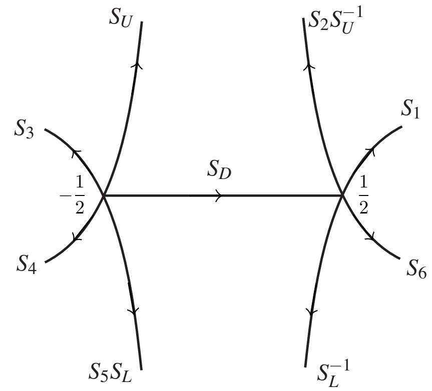


Figure 4: The deformed Riemann-Hilbert problem for the generic purely imaginary solutions and the Ablowitz-Segur solutions to Painlevé II as $x \to -\infty$.

from [22] page 328,

$$\Psi^{\text{def}} = \widehat{\Psi}^D(I + O(t^{-1/2})), \tag{81}$$

where $\widehat{\Psi}^D(z;t)$ solves the model problem

$$\begin{cases} \widehat{\Psi}^D(z;t) \text{ is analytic off } [-\frac{1}{2}, \frac{1}{2}] \\ \widehat{\Psi}^D(z;t) \text{ does not have a non-square integrable singularity at the endpoints } \pm\frac{1}{2} \\ \widehat{\Psi}^D_+(z;t) = \widehat{\Psi}^D_-(z;t)S_D, \qquad z \in [-\frac{1}{2}, \frac{1}{2}] \\ \widehat{\Psi}^D(z;t)e^{t\theta(z)\sigma_3} \to I, \qquad z \to \infty, \quad \theta(z) := i\frac{4}{3}z^3 - iz. \end{cases} \tag{82}$$

This problem is solved by

$$\widehat{\Psi}^D(z;t) = \begin{pmatrix} f(z) & 0 \\ 0 & \frac{1}{f(z)} \end{pmatrix} e^{-t\theta(z)\sigma_3}, \quad f(z) := \left(\frac{z+\frac{1}{2}}{z-\frac{1}{2}}\right)^{\mu}, \quad \mu := \frac{-1}{2\pi i}\log(1-s_1s_3). \tag{83}$$

The function $f(z)$ is defined with its branch cut on $[-\frac{1}{2}, \frac{1}{2}]$ and satisfies $f(z) \to 1$ as $z \to \infty$. It follows that

$$\lim_{x\to-\infty} P(x) = \lim_{x\to-\infty} \Psi_1(0;x) = \lim_{t\to-\infty} \Psi_+^{\text{def}}(0;t) \begin{pmatrix} \frac{1}{1-s_1s_3} & \frac{-s_3}{1-s_1s_3} \\ -s_1 & 1 \end{pmatrix} = \widehat{\Psi}_+^D(0;t) \begin{pmatrix} \frac{1}{1-s_1s_3} & \frac{-s_3}{1-s_1s_3} \\ -s_1 & 1 \end{pmatrix}. \tag{84}$$

Using $f_+(0) = \sqrt{1-s_1s_3}$ gives

$$C = \lim_{x\to-\infty} \Psi_1(0;x) = \frac{1}{\sqrt{1-s_1s_3}} \begin{pmatrix} 1 & -s_3 \\ -s_1 & 1 \end{pmatrix}. \tag{85}$$

We note specifically that this gives

$$(P(x))_{11} + i(P(x))_{21} = \frac{1-is_1}{\sqrt{1+|s_1|^2}} e^{U(-\infty,x)}. \tag{86}$$

Now we analyze $P(x)$ as $x \to +\infty$. This limit does not exist since $u(x)$ is not integrable at $+\infty$. However, the limit of $P(x)$ times an appropriate decaying factor will exist. In [22] (see page 346) it is shown that

$$\Psi(\lambda;x) = (I + O(x^{-3/4}))\widehat{\Psi}(z;x), \tag{87}$$

where $\widehat{\Psi}(z;x)$ is the solution to a model Riemann-Hilbert problem. Let $\widehat{\Psi}_1$ be $\widehat{\Psi}$ in the region Ω_1. Then, by the computations in [22],

$$\widehat{\Psi}_1(0;x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\sigma \\ i\sigma & 1 \end{pmatrix} e^{-i\pi\nu\sigma_3/4} 2^{7\nu\sigma_3/4} (2Q)^{-\sigma_3/2} e^{-it\sqrt{2}\sigma_3/3} t^{\nu\sigma_3/2} \sigma_1 \lim_{\substack{z\to0 \\ z\in\Omega_1}} Z^{\text{RH}}(\zeta(z)), \tag{88}$$

where

$$\begin{gathered} z := \frac{\lambda}{x^{1/2}}, \quad t := x^{3/2}, \quad \nu := \frac{1}{i\pi}\log(i\sigma s_2), \\ Q := i\Gamma(\nu+1)\frac{1+s_1s_2}{\sqrt{2\pi}s_2}, \quad \zeta(z) := 2\sqrt{it\frac{\sqrt{2}}{3} - it\operatorname{sgn}(\Re z)\frac{4}{3}\left(z^2+\frac{1}{2}\right)^{3/2}} \end{gathered} \tag{89}$$

and $Z^{\text{RH}}(\zeta)$ is a function built out of parabolic cylinder functions. Specifically, for $z \in \Omega_1$,

$$Z^{\text{RH}}(\zeta) := 2^{-\sigma_3/2} \begin{pmatrix} D_{-\nu-1}(i\zeta) & D_\nu(\zeta) \\ \frac{d}{d\zeta}D_{-\nu-1}(i\zeta) & \frac{d}{d\zeta}D_\nu(\zeta) \end{pmatrix} \begin{pmatrix} e^{i\pi(\nu+1)/2} & 0 \\ 0 & 1 \end{pmatrix} Q^{\sigma_3/2}, \tag{90}$$

where $D_\nu(\zeta)$ is Whittaker's parabolic cylinder function satisfying

$$\frac{d^2D_\nu}{d\zeta^2} + \left(\nu + \frac{1}{2} - \frac{\zeta^2}{4}\right) D_\nu = 0. \tag{91}$$

Note that $\widehat{\Psi}_1(0;x)$ is uniformly bounded independent of x. From Whittaker and Watson [48], Section 16.5,

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} := \lim_{\substack{z\to0 \\ z\in\Omega_1}} Z^{\text{RH}}(\zeta(z)) = 2^{-\sigma_3/2} \begin{pmatrix} \frac{\Gamma(\frac{1}{2})}{\Gamma(1+\frac{\nu}{2})} 2^{-\nu/2-1/2} & \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{\nu}{2})} 2^{\nu/2} \\ i\frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2})} 2^{-\nu/2-1} & \frac{\Gamma(-\frac{1}{2})}{\Gamma(-\frac{\nu}{2})} 2^{\nu/2-1/2} \end{pmatrix} \begin{pmatrix} e^{i\pi(\nu+1)/2} & 0 \\ 0 & 1 \end{pmatrix} Q^{\sigma_3/2}. \tag{92}$$

Equation (88) implies

$$(\widehat{\Psi}_1(0;x))_{11} + i(\widehat{\Psi}_1(0;x))_{21} = \frac{1}{\sqrt{2}}[i\delta^{-1}(1+\sigma)Z_{11} + \delta(1-\sigma)Z_{21}], \tag{93}$$

where

$$\delta := e^{-i\pi\nu/4}2^{7\nu/4}(2Q)^{-1/2}e^{-it\sqrt{2}/3}t^{\nu/2}. \tag{94}$$

Assume for the moment that $\sigma = +1$. Then

$$(\widehat{\Psi}_1(0;x))_{11} + i(\widehat{\Psi}_1(0;x))_{21} = 2ie^{i\pi\nu/4}2^{-7\nu/4}Q^{1/2}Z_{11}e^{it\sqrt{2}/3}t^{-\nu/2}. \tag{95}$$

Now from (87) and using the fact that $\widehat{\Psi}_1(0;x)$ is uniformly bounded independent of x we find

$$\lim_{x\to+\infty}((P(x)_{11} + iP(x)_{21})e^{-ix^{3/2}\sqrt{2}/3}x^{3\nu/4} = \lim_{x\to+\infty}((\widehat{\Psi}_1(0;x))_{11} + i(\widehat{\Psi}_1(0;x))_{21})e^{-ix^{3/2}\sqrt{2}/3}x^{3\nu/4} \tag{96}$$

$$= 2ie^{i\pi\nu/4}2^{-7\nu/4}Q^{1/2}Z_{11}. \tag{97}$$

Along with (86) this gives

$$\lim_{x\to+\infty}\exp\left(U(-\infty,x) - ix^{3/2}\sqrt{2}/3 + (3\nu/4)\log(x)\right) = \frac{\sqrt{1+|s_1|^2}}{1-is_1}2ie^{i\pi\nu/4}2^{-7\nu/4}Q^{1/2}Z_{11}. \tag{98}$$

Writing $U(-\infty,x) = U(-\infty,c) + U(c,x)$, (98) implies that

$$\lim_{x\to+\infty}\exp\left(\int_{-\infty}^{c}u(y)dy + \int_{c}^{x}\left(u(y) - i\sqrt{\frac{y}{2}+\frac{3\nu}{4y}}\right)dy - i\frac{\sqrt{2}}{3}c^{3/2} + \frac{3\nu}{4}\log c\right) = \frac{\sqrt{1+|s_1|^2}}{1-is_1}2ie^{i\pi\nu/4}2^{-7\nu/4}Q^{1/2}Z_{11} \tag{99}$$

for any $c > 0$ when $\sigma = +1$. It follows that the integral $\int_c^x\left(u(y) - i\sqrt{y/2} + 3\nu/4y\right)dy$ is convergent, and hence

$$\begin{aligned}
&\exp\left(\int_{-\infty}^{c}u(y)dy + \int_{c}^{\infty}\left(u(y) - i\sqrt{\frac{y}{2}+\frac{3\nu}{4y}}\right)dy\right)\\
&= -\frac{\sqrt{1+|s_1|^2}}{1-is_1}\frac{(1+s_1s_2)}{2^{1/2}s_2}\frac{\Gamma(1+\nu)}{\Gamma(1+\frac{\nu}{2})}e^{3i\pi\nu/4}2^{-9\nu/4}e^{i\pi/2}\exp\left(i\frac{\sqrt{2}}{3}c^{3/2} - \frac{3\nu}{4}\log c\right)\\
&= \frac{1+is_1}{2^{1/2}(1+|s_1|^2)^{1/4}|s_1-\overline{s_1}|^{1/4}}\frac{\Gamma(1+\nu)}{\Gamma(1+\frac{\nu}{2})}2^{-9\nu/4}\exp\left(i\frac{\sqrt{2}}{3}c^{3/2} - \frac{3\nu}{4}\log c\right)\\
&= \frac{(1+is_1)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)}{(2\pi)^{1/2}(1+|s_1|^2)^{1/4}|s_1-\overline{s_1}|^{1/4}}2^{-5\nu/4}\exp\left(i\frac{\sqrt{2}}{3}c^{3/2} - \frac{3\nu}{4}\log c\right).
\end{aligned} \tag{100}$$

The first equality follows from the definitions of Q and Z_{11} in (89) and (92), respectively, the second follows from $s_2 = (s_1 - \overline{s_1})/(1+|s_1|^2)$, and the third follows from the identity (see (6.1.18) in [2])

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}}2^{2z}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{101}$$

The fact that the right-hand side of (100) has modulus 1 follows automatically from the fact that $u(x)$ is purely imaginary. However, this can also be checked directly using (6.1.29-31) in [2]. Equation (74) with $\sigma = +1$ follows by taking the logarithm of both sides of (100) and setting $\nu = -i\rho^2$.

Now assume $\sigma = -1$. This result can be obtained from the $\sigma = +1$ case via Lemma 1.1. We also give a direct proof as follows. Using the definition of Z_{21} in (92),

$$(\widehat{\Psi}_1(0;x))_{11} + i(\widehat{\Psi}_1(0;x))_{21} = \frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2})}2^{-1/2}ie^{i\pi(\nu+2)/4}2^{5\nu/4}e^{-it\sqrt{2}/3}t^{\nu/2}. \tag{102}$$

From (87) and the fact that $\widehat{\Psi}_1(0;x)$ is uniformly bounded independent of x, we see

$$\lim_{x\to+\infty}((P(x)_{11}+iP(x)_{21})e^{ix^{3/2}\sqrt{2}/3}x^{-3\nu/4} = \lim_{x\to+\infty}((\widehat{\Psi}_1(0;x))_{11}+i(\widehat{\Psi}_1(0;x))_{21})e^{ix^{3/2}\sqrt{2}/3}x^{-3\nu/4} \tag{103}$$

$$= \frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2})}2^{-1/2}ie^{i\pi(\nu+2)/4}2^{5\nu/4}. \tag{104}$$

From (86),

$$\lim_{x\to+\infty}\exp\Big(U(-\infty,x)+ix^{3/2}\sqrt{2}/3-(3\nu/4)\log(x)\Big) = \frac{\sqrt{1+|s_1|^2}}{1-is_1}\frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2})}2^{-1/2}ie^{i\pi(\nu+2)/4}2^{5\nu/4}. \tag{105}$$

Writing $U(-\infty,x)=U(-\infty,c)+U(c,x)$, (105) shows that

$$\begin{aligned}\lim_{x\to+\infty}\exp\left(\int_{-\infty}^{c}u(y)dy+\int_c^x\left(u(y)+i\sqrt{\frac{y}{2}}-\frac{3\nu}{4y}\right)dy+i\frac{\sqrt{2}}{3}c^{3/2}-\frac{3\nu}{4}\log c\right)&\\ =\frac{\sqrt{1+|s_1|^2}}{1-is_1}\frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{\nu}{2})}2^{-1/2}ie^{i\pi(\nu+2)/4}2^{5\nu/4}&\\ =\frac{(2\pi)^{1/2}(1+|s_1|^2)^{1/4}|s_1-\overline{s_1}|^{1/4}}{(1-is_1)\Gamma(\frac{1}{2}+\frac{\nu}{2})}2^{5\nu/4}&\end{aligned} \tag{106}$$

for any $c>0$ if $\sigma=-1$. Therefore the integral $\int_c^x\Big(u(y)+i\sqrt{y/2}-3\nu/4y\Big)dy$ is convergent, and hence (74) with $\sigma=-1$ follows by taking logarithms and using $\nu=i\rho^2$. □

4 Direct computation of asymptotics of $u(x)$ in the generic purely imaginary solutions

In both [14] and [22] the authors write down asymptotic expansions of the purely imaginary solutions to the Painlevé II equation for large positive x. In this section we calculate the higher order terms for these expansions. Specifically, we will calculate the $\mathcal{O}(x^{-1})$ terms in the asymptotic expansion (63) of the generic purely imaginary solutions as $x\to+\infty$ and show:

Theorem 4.1. *Let $u(x)$ be a generic purely imaginary solution of the Painlevé II equation* (1) *with asymptotic expansion* (62) *as $x\to-\infty$. Then*

$$\begin{aligned}u(x)=&i\sigma\sqrt{\frac{x}{2}}+\frac{i\sigma\rho}{(2x)^{1/4}}\cos\left(\frac{2\sqrt{2}}{3}x^{3/2}-\frac{3}{2}\rho^2\log x+\theta\right)-\frac{3i\sigma\rho^2}{4x}\\&+\frac{i\sigma\rho^2}{4x}\cos\left(2\left[\frac{2\sqrt{2}}{3}x^{3/2}-\frac{3}{2}\rho^2\log x+\theta\right]\right)+O(x^{-3/2}),\quad x\to+\infty,\end{aligned} \tag{107}$$

where σ, ρ, and θ are defined in (64).

Much of the notation is inherited from [22]. We note that

$$\nu=i\rho^2,\quad |\nu|=\rho^2,\quad t=x^{3/2}. \tag{108}$$

Here the solution $u(x)$ to (1) is obtained as

$$u(x)=i\sigma\sqrt{\frac{x}{2}}+2\sqrt{x}\lim_{z\to\infty}\left(z\chi_{12}(z)\right), \tag{109}$$

where $\chi_{12}(z)$ is the 12 entry of the 2×2 matrix valued function that solves the *ratio* Riemann-Hilbert problem

$$\begin{cases} \chi \text{ is analytic in } \mathbb{C}\backslash\gamma \\ \chi(z) \to I \text{ as } z \to \infty \\ \chi_+(z) = \chi_-(z)G(z) \text{ on the contours } \gamma. \end{cases} \tag{110}$$

The jump $G(z)$ is given in [22] (9.5.61) and (9.5.62). As illustrated in figure 5, the contour γ is the union

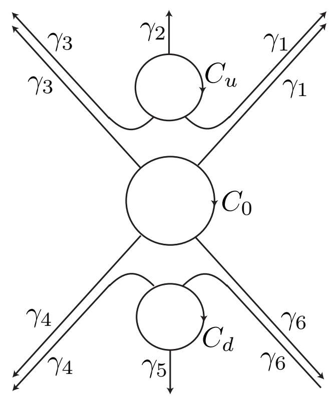


Figure 5: The contour γ for the Riemann-Hilbert problem for $\chi(z)$.

of the several contours γ_i, $i = 1 \cdots 6$ and C_m, $m = 0, u, d$. The contours γ_i are the anti-stokes lines and the contours C_m, $m = 0, u, l$, are small circles oriented clockwise around the origin, $+i/\sqrt{2}$, and $-i/\sqrt{2}$ respectively. On each of these contours the jump $G(z)$ has a different definition and we write $G(z) = G_i(z)$ or $G(z) = G_m(z)$ to denote the corresponding jump on each contour. As in (9.5.73) in [22] one can write

$$\lim_{z\to\infty} (z\chi_{12}(z)) = -\frac{1}{2\pi i}\int_\gamma (G(z) - I)_{12}\, dz - \frac{1}{2\pi i}\int_\gamma ((\chi_-(z) - I)(G(z) - I))_{12}\, dz. \tag{111}$$

To obtain (63) the authors of [22] proved that equation (111) reduces to

$$\lim_{z\to\infty} (z\chi_{12}(z)) = -\frac{1}{2\pi i}\int_{C_0} G_0(z)_{12} dz + \mathcal{O}(x^{-3/2}). \tag{112}$$

See (9.5.74) in [22] and note $\Re\nu = 0$. It is exactly the $\mathcal{O}(x^{-3/2})$ terms that we wish to now compute. To calculate these terms there are several things to check. The following three assertions, once proven, will establish the desired result.

1. In [22], the authors do not compute $-\frac{1}{2\pi i}\int_{C_0} G_0(z)_{12} dz$ explicitly, but they compute the integral $-\frac{1}{2\pi i}\int_{C_0} \breve{G}_0(z)_{12} dz$ where $\breve{G}_0$ is an approximation of G_0. The error from using this approximation is written as $\mathcal{O}(x^{-3/2})$, which could contribute to the $O(x^{-1})$ term in $u(x)$ (note the $\sqrt{x}$ in (109)). However, the error in the off-diagonal entries is actually higher order and does not contribute to the $O(x^{-1})$ term.
2. The contribution to $u(x)$ from the integral $-\frac{1}{2\pi i}\int_\gamma ((\chi_-(z) - I)(G(z) - I))_{12}\, dz$ is $-\frac{\sigma\nu}{4x} + \frac{\sigma\nu}{4x}\cos\left(2|\nu|\log(t) - 4\sqrt{2}t/3 - 2\theta\right)$.

3. The contribution to $u(x)$ from $\int_\gamma G(z)_{12}dz$ is broken down into the sum of the integrals on each component of the contour γ. In [22] it is shown that $||G_i(z)-I|| \leq c\exp(-(2x)^{3/2}|z|^2)$ for some constant c. Consequently, the integrals $\int_{\gamma_i}(G_i(z)-I)_{12}dz$ will not contribute to the term we wish to compute. The contribution from the integral on C_0 is handled in the first assertation. The contribution from $-\frac{1}{2\pi i}\int_{C_u} G_u(z)_{12}dz$ and $-\frac{1}{2\pi i}\int_{C_l} G_l(z)_{12}dz$ is $-\frac{\sigma\nu}{2x}$.

To prove assertion 1 we first proceed to verify the following Lemma:

Lemma 4.2. *For $z \in C_0$, the jump G_0 has the expansion*

$$G_0 = B_0(z)e^{-\frac{it\sqrt{2}}{3}\sigma_3}\left(\breve{M}_0+\breve{M}_0^{(2)}\right)e^{\frac{it\sqrt{2}}{3}\sigma_3}(B_0(z))^{-1}+\mathcal{O}\left(t^{-3/2}\right), \tag{113}$$

where

$$\breve{M}_0 := \begin{pmatrix} 1 & \frac{\nu}{Q\zeta} \\ \frac{Q}{\zeta} & 1 \end{pmatrix}, \quad \breve{M}_0^{(2)} := \begin{pmatrix} \frac{\nu(\nu+1)}{2\zeta^2} & 0 \\ 0 & -\frac{\nu(\nu-1)}{2\zeta^2} \end{pmatrix}. \tag{114}$$

The function $B_0(z)$ is analytic in a neighborhood of the origin and is defined as (see [22])

$$B_0(z) = \begin{cases} \breve{Y}(z)\left(\zeta^\nu(z)\delta(z)\right)^{\sigma_3} & \mathrm{Re}(z)>0 \\ \breve{Y}(z)i\sigma\sigma_1(i\sigma s_2)^{-\sigma_3}\left(\zeta^\nu(z)\delta(z)\right)^{\sigma_3} & \mathrm{Re}(z)<0. \end{cases} \tag{115}$$

The functions $\breve{Y}(z)$ and $\delta(z)$ are given by

$$\begin{aligned} \breve{Y} &= \frac{1}{2}\begin{pmatrix} \beta+\beta^{-1} & \sigma(\beta-\beta^{-1}) \\ \sigma(\beta-\beta^{-1}) & \beta+\beta^{-1} \end{pmatrix}, \quad \beta^2(z) = \left(\frac{z+\frac{i}{\sqrt{2}}}{z-\frac{i}{\sqrt{2}}}\right)^{\frac{1}{2}}, \\ \delta^2(z) &= \left(\frac{\left(z^2+\frac{1}{2}\right)^{\frac{1}{2}}-\frac{1}{\sqrt{2}}}{\left(z^2+\frac{1}{2}\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2}}}\right)^{-\nu}, \quad \nu = \frac{1}{i\pi}\ln(i\sigma s_2), \end{aligned} \tag{116}$$

and $\zeta(z)$ is as in equation (89).

Proof. Consider (9.5.50) and (9.5.43) of [22]. Combining these two facts gives:

$$\Psi_0 = B_0(z)e^{-\frac{it\sqrt{2}}{3}\sigma_3}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}Q^{\frac{-\sigma_3}{2}}\begin{pmatrix} 2+\frac{\nu(\nu+1)}{\zeta^2}+\mathcal{O}\left(\zeta^{-4}\right) & \frac{2\nu}{\zeta}+\mathcal{O}\left(\zeta^{-3}\right) \\ \zeta^{-1}+\frac{\nu^2+3\nu+2}{2\zeta^3}+\mathcal{O}\left(\zeta^{-5}\right) & 1-\frac{\nu(\nu-1)}{2\zeta^2}+\mathcal{O}\left(\zeta^{-4}\right) \end{pmatrix}Q^{\frac{\sigma_3}{2}}e^{\left(\frac{\zeta^2}{4}-\nu\log\zeta\right)\sigma_3}. \tag{117}$$

From (9.5.55) and (9.5.56) in [22] we can then obtain

$$G_0 = \Psi_0\breve{\Psi}^{-1} = B_0(z)e^{-\frac{it\sqrt{2}}{3}\sigma_3}\begin{pmatrix} 1+\frac{\nu(\nu+1)}{2\zeta^2}+\mathcal{O}\left(\zeta^{-4}\right) & \frac{\nu}{Q\zeta}+\mathcal{O}\left(\zeta^{-3}\right) \\ \frac{Q}{\zeta}+Q\frac{\nu^2+3\nu+2}{2\zeta^3}+\mathcal{O}\left(\zeta^{-5}\right) & 1-\frac{\nu(\nu-1)}{2\zeta^2}+\mathcal{O}\left(\zeta^{-4}\right) \end{pmatrix}e^{\frac{it\sqrt{2}}{3}\sigma_3}(B_0(z))^{-1}. \tag{118}$$

From the definition of ζ^3 in (89) one can see that $\zeta^3 = ct^{\frac{3}{2}}z^3\left(1+\mathcal{O}(z)\right)$ for some constant c. Equation (113) then follows. □

Consequently, we have that

$$\int_{C_0} G_0(z)dz = \int_{C_0}\breve{G}_0(z)dz + \int_{C_0}\breve{G}_0^{(2)}(z)dz + \mathcal{O}(t^{-\frac{3}{2}}), \tag{119}$$

where $\breve{G}_0 := B_0(z)e^{-\frac{it\sqrt{2}}{3}\sigma_3}\breve{M}_0 e^{\frac{it\sqrt{2}}{3}\sigma_3}(B_0(z))^{-1}$ and $\breve{G}_0^{(2)} := B_0(z)e^{-\frac{it\sqrt{2}}{3}\sigma_3}\breve{M}_0^{(2)} e^{\frac{it\sqrt{2}}{3}\sigma_3}(B_0(z))^{-1}$. The first integral on the right hand side of equation (119) is computed explicitly in [22] and is what gives rise to the cosine term of the expansion (63). The second integral of (119) does not contribute to the next order term of (112), moreover:

$$\int_{C_0} (\breve{G}_0^{(2)})_{12}(z)dz = 0. \tag{120}$$

To show this, we first write down the (12) entry of $\breve{G}_0^{(2)}$ using the definition of $\breve{M}_0^{(2)}$. Simple algebra yields that

$$\begin{aligned} (\breve{G}_0^{(2)})_{12}(z) &= -\frac{\sigma\nu^2}{2}\frac{\left(\beta^2-\beta^{-2}\right)}{\zeta^2}, \;\; \Re(z)>0, \\ (\breve{G}_0^{(2)})_{12}(z) &= \frac{\sigma\nu^2}{2}\frac{\left(\beta^2-\beta^{-2}\right)}{\zeta^2}, \;\; \Re(z)<0. \end{aligned} \tag{121}$$

The function β has a branch on the imaginary axis. However, since B_0 is analytic in a vicinity of the origin we can deform the contour of integration through the branch so that it does not pass through the interior of C_0. However, as a function of z, ζ^2 is analytic in a vicinity of the origin, but has a zero of multiplicity two at $z=0$. Consequently, when we deform the integral we pick up a residue from the origin. From the definition of ζ in (89) we can write $\zeta^2 = -it4\sqrt{2}z^2(1+\mathcal{O}(z^2))$. With this it is clear that

$$\int_{C_0} G_0{}_{12}^{(2)}(z) = -\frac{\sigma\nu^2}{-it2\sqrt{2}}\int_{C_0}\frac{\left(\beta^2-\beta^{-2}\right)}{z^2}dz = -\frac{\pi\sigma\nu^2}{t\sqrt{2}}\frac{d}{dz}\left.\left(\beta^2-\beta^{-2}\right)\right|_{z=0} = 0.$$

The last inequality is due to the fact that $\beta(0)+\beta^{-3}(0)=0$. This proves the first assertion.

Next we check assertion 2 by showing

$$-\frac{1}{2\pi i}\int_\gamma \left((\chi_-(z)-I)\,(G(z)-I)\right)_{12} dz = -\frac{\sigma\nu}{8t}+\frac{\sigma\nu}{8t}\cos\left(2|\nu|\log(t)-4\sqrt{2}t/3-2\theta\right)+\mathcal{O}\left(t^{-3/2}\right). \tag{122}$$

Let C be the Cauchy operator on $L^2(\gamma)$ defined for $f\in L^2(\gamma)$ by

$$(Cf)(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(s)}{s-z}ds \quad \text{for } z\notin\gamma. \tag{123}$$

Let C_- denote the boundary limit of C defined for $f\in L^2(\gamma)$ by

$$(C_-f)(z) := \lim_{z'\to z}(Cf)(z'), \tag{124}$$

where z' is on the right-hand side of γ and $z\in\gamma$. Define, for $f\in L^2(\gamma)$,

$$(C_{G-I}f)(z) := C_-[(G-I)f](z) \quad \text{for } z\in\gamma. \tag{125}$$

Note that $||C_{G-I}||_{L^2(\gamma)\to L^2(\gamma)} \le ||C_-||_{L^2(\gamma)\to L^2(\gamma)}||G-I||_{L^\infty(\gamma)}$. Using the fact ((9.5.69) in [22]) that the jump matrix G satisfies

$$||G-I||_{L^2(\gamma)\cap L^\infty(\gamma)} \le \frac{c}{\sqrt{t}} \tag{126}$$

for some constant c we have

$$||C_{G-I}||_{L^2(\gamma)\to L^2(\gamma)} \le \frac{c}{\sqrt{t}} \tag{127}$$

as C_- is a bounded operator on $L^2(\gamma)$.

Suppose $\mu(s)$ satisfies $\mu - I \in L^2(\gamma)$ and $(1-C_{G-I})(\mu-I) = C_{G-I}I$. Note that from (127), $1-C_{G-I}$ is invertible for all x sufficiently large. So, for x sufficiently large,

$$\begin{aligned} ||\mu - I||_{L^2(\gamma)} &\le ||(1-C_{G-I})^{-1}C_{G-I}I||_{L^2(\gamma)} \le 2||C_{G-I}I||_{L^2(\gamma)} \\ &\le 2||C_-||_{L^2(\gamma)\to L^2(\gamma)}||G-I||_{L^\infty(\gamma)} \le \frac{c}{\sqrt{t}}. \end{aligned} \tag{128}$$

From standard Riemann-Hilbert theory (see, for instance, [8]), for $z \notin \gamma$,

$$\chi(z) = I + \frac{1}{2\pi i}\int_\gamma \frac{\mu(s)(G(s)-I)}{s-z}ds = I + \frac{1}{2\pi i}\int_\gamma \frac{G(s)-I}{s-z}ds + E(z), \tag{129}$$

where

$$E(z) := \frac{1}{2\pi i}\int_\gamma \frac{(\mu(s)-I)(G(s)-I)}{s-z}ds = C[(\mu-I)(G-I)](z). \tag{130}$$

Defining $E_-(z) := C_-[(\mu-I)(G-I)](z)$, we have

$$\begin{aligned} \left| -\frac{1}{2\pi i}\int_\gamma E_-(z)(G(z)-I)dz \right| &\le c||E_-||_{L^2(\gamma)}||G-I||_{L^2(\gamma)} \\ &\le ||C_-||_{L^2(\gamma)\to L^2(\gamma)}||\mu-I||_{L^2(\gamma)}||G-I||_{L^\infty(\gamma)}||G-I||_{L^2(\gamma)} \\ &\le \frac{c}{t^{3/2}}. \end{aligned} \tag{131}$$

Therefore

$$-\frac{1}{2\pi i}\int_\gamma (\chi_-(z)-I)(G(z)-I)dz = -\frac{1}{(2\pi i)^2}\int_\gamma\int_\gamma \frac{(G(s)-I)(G(z)-I)}{s-z_-}dsdz + \mathcal{O}\left(t^{-3/2}\right). \tag{132}$$

As (see (9.5.64) and (9.5.65) in [22])

$$||G-I||_{L^2(\gamma\setminus C_0)\cap L^\infty(\gamma\setminus C_0)} \le \frac{c}{t} \tag{133}$$

for some constant c, in (132) we can restrict the integrals to C_0 without adding a larger error term:

$$-\frac{1}{2\pi i}\int_\gamma (\chi_-(z)-I)(G(z)-I)dz = -\frac{1}{(2\pi i)^2}\int_{C_0}\int_{C_0} \frac{(G_0(s)-I)(G_0(z)-I)}{s-z_-}dsdz + \mathcal{O}\left(t^{-3/2}\right), \tag{134}$$

where G_0 denotes G on C_0. Furthermore, as (see (9.5.66) in [22])

$$||G_0 - \breve{G}_0||_{L^2(C_0)\cap L^\infty(C_0)} \le \frac{c}{t}, \tag{135}$$

where $\breve{G}_0$ is defined in (9.5.58) of [22], we can also replace G_0 by $\breve{G}_0$:

$$-\frac{1}{2\pi i}\int_\gamma (\chi_-(z)-I)(G(z)-I)dz = -\frac{1}{(2\pi i)^2}\int_{C_0}\int_{C_0} \frac{(\breve{G}_0(s)-I)(\breve{G}_0(z)-I)}{s-z_-}dsdz + \mathcal{O}\left(t^{-3/2}\right). \tag{136}$$

Now we evaluate this double integral explicitly. The function $\breve{G}_0(z)$ is defined ((9.5.58) of [22]) by

$$\breve{G}_0(z) = B_0(z)\begin{pmatrix} 1 & \frac{\nu}{Q\zeta(z)}e^{-i2\sqrt{2}t/3} \\ \frac{Q}{\zeta(z)}e^{i2\sqrt{2}t/3} & 1 \end{pmatrix} B_0^{-1}(z), \tag{137}$$

where Q and ν are constants given in (89) above and $\zeta(z)$ and $B_0(z)$ (see (115)) are holomorphic in $\overline{C_0}$. The error term $\mathcal{O}(z)$ in (9.5.47) of [22] is actually $\mathcal{O}(z^2)$, and the function $\zeta(z)$ satisfies

$$\zeta(z) = e^{-i\pi/4}\sqrt{t}2^{5/4}z(1+\mathcal{O}(z^2)), \quad z \sim 0. \tag{138}$$

The function $B_0(z)$ satisfies ((9.5.53) in [22])

$$B_0(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\sigma \\ i\sigma & 1 \end{pmatrix} e^{-i\pi\nu\sigma_3/4} t^{\nu\sigma_3/2} 2^{7\nu\sigma_3/4} \tag{139}$$

where $\sigma = -\mathrm{sgn}(\Im s_1)$. Note that $\breve{G}_0(s)$ has a pole at $s = 0$, with the residue (see (9.5.76) in [22])

$$\begin{aligned} A :=& \operatorname*{Res}_{z=0} \breve{G}(z) \\ =& B_0(0) \begin{pmatrix} 0 & \frac{\nu}{Q\zeta'(0)} e^{-2i\sqrt{2}t/3} \\ \frac{Q}{\zeta'(0)} e^{2i\sqrt{2}t/3} & 0 \end{pmatrix} B_0(0)^{-1} \\ =& \frac{1}{2t^{1/2}} \begin{pmatrix} i\sigma(qt^{-\nu}e^{2\sqrt{2}it/3} - pt^{\nu}e^{-2\sqrt{2}it/3}) & qt^{-\nu}e^{2\sqrt{2}it/3} + pt^{\nu}e^{-2\sqrt{2}it/3} \\ qt^{-\nu}e^{2\sqrt{2}it/3} + pt^{\nu}e^{-2\sqrt{2}it/3} & -i\sigma(qt^{-\nu}e^{2\sqrt{2}it/3} - pt^{\nu}e^{-2\sqrt{2}it/3}) \end{pmatrix}, \end{aligned} \tag{140}$$

where

$$p = i\sigma \frac{2^{-3/4}\sqrt{\pi}e^{i\pi\nu/2}}{(1+s_1 s_2)\Gamma(\nu)} e^{3i\pi/4} 2^{7\nu/2}, \quad q = \overline{p}. \tag{141}$$

There are a few typographical errors in (9.5.76) and (9.5.78) of [22]. In (9.5.76), the diagonal entries of the middle matrix which is conjugated by $B_0(0)$ should both be 0. The diagonal entries of the middle matrix in the last equality of (9.5.76) should also both be 0. In (9.5.78), $\Gamma(\nu)$ should be replaced by $\Gamma(-\nu)$.

From the residues at $s = z$ and $s = 0$,

$$-\frac{1}{2\pi i} \int_{C_0} \int_{C_0} \frac{(\breve{G}_0(s) - I)(\breve{G}_0(z) - I)}{s - z_-} ds dz = -\frac{1}{2\pi i} \int_{C_0} (\breve{G}_0(z) - I)^2 dz + \frac{1}{2\pi i} \int_{C_0} \frac{A(\breve{G}_0(z) - I)}{z} dz + \mathcal{O}(t^{-\frac{3}{2}}). \tag{142}$$

Noting that the (12) entry of

$$(\breve{G}_0(z) - I)^2 = \frac{\nu}{\zeta^2(z)} I \tag{143}$$

is zero, we see the second integral on the right-hand side of (142) does not contribute to the evaluation of $u(x)$. Next, another residue calculation shows

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_0} \frac{A(\breve{G}_0(z) - I)}{z} dz &= \frac{1}{2\pi i} \int_{C_0} \frac{1}{z^2} AB_0(z) \begin{pmatrix} 0 & \frac{\nu z}{Q\zeta} e^{-2i\sqrt{2}t/3} \\ \frac{Qz}{\zeta} e^{2i\sqrt{2}t/3} & 0 \end{pmatrix} B_0(z)^{-1} dz \\ &= A[B_0'(0)B_0(0)^{-1}, A], \end{aligned} \tag{144}$$

where $[M, N] := MN - NM$. From (115), a direct calculation shows that

$$\begin{aligned} B_0'(0)B_0(0)^{-1} &= \breve{Y}'(0)\breve{Y}(0)^{-1} + \frac{(\zeta^\nu \delta)'(0)}{(\zeta^\nu \delta)(0)} \breve{Y}(0)\sigma_3 \breve{Y}(0)^{-1} \\ &= -\frac{i}{\sqrt{2}} \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{145}$$

From (136), (140), (142), and (145),

$$\begin{aligned} \left(-\frac{1}{2\pi i} \int_\gamma (\chi_-(z) - I)(G(z) - I) dz \right)_{12} &= \left(A \left[-\frac{i\sigma}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A \right] \right)_{12} + \mathcal{O}\left(t^{-\frac{3}{2}}\right) \\ &= -\frac{i\sigma}{2^{3/2}t} \left(-i\nu 2^{-3/2} + q^2 e^{4\sqrt{2}it/3 - 2\nu\log(t)} + p^2 e^{-4\sqrt{2}it/3 + 2\nu\log(t)} \right) + \mathcal{O}\left(t^{-\frac{3}{2}}\right) \\ &= -\frac{\sigma\nu}{8t} + \frac{\sigma\nu}{8t} \cos\left(2|\nu|\log(t) - 4\sqrt{2}t/3 - 2\theta\right) + \mathcal{O}\left(t^{-\frac{3}{2}}\right), \end{aligned} \tag{146}$$

where θ is defined in (64). The last equality in (146) uses

$$\Gamma(1+iy) = \frac{\pi y}{\sinh y} \text{ for } y > 0 \quad \text{and} \quad s_2 = \frac{s_1 - \overline{s_1}}{1+|s_1|^2}. \tag{147}$$

We proceed to check assertion 3. The jump matrices G_u and G_l are written as $\Psi^u \breve{\Psi}^{-1}$ and $\Psi^l \breve{\Psi}^{-1}$. G_u and G_l satisfy the symmetry

$$\int_{C_l} G_l(z) dz = -\sigma_2 \left(\int_{C_u} G_u(z) dz \right) \sigma_2. \tag{148}$$

Using this symmetry we note that

$$\int_{C_u} G_u(z)_{12} dz + \int_{C_l} G_l(z)_{12} dz = \int_{C_u} G_u(z)_{12} + G_u(z)_{21} dz. \tag{149}$$

This allows us to work only with G_u. $\breve{\Psi}$ and Ψ^u are explicit functions (see (9.5.24) and (9.5.34) of [22]) the later being constructed using Airy functions (see (9.5.30) in [22]). Using the leading and second order asymptotics of the Airy function and its derivative (see Abramowitz and Stegun [2]) one can write:

$$G_u = \left[I + \tfrac{3}{4\zeta^{\frac{3}{2}}} \breve{Y} \delta^{\sigma_3} (is_2)^{\sigma_3/2} \begin{pmatrix} c_1 + d_1 & -c_1 + d_1 \\ c_1 - d_1 & -(c_1 + d_1) \end{pmatrix} (is_2)^{-\sigma_3/2} \delta^{-\sigma_3} \breve{Y}^{-1} \right] \left(I + \mathcal{O}(|\zeta|^{-3}) \right). \tag{150}$$

where $\breve{Y}(z)$ and $\delta(z)$ are defined in (116) and

$$\zeta = 2^{\frac{2}{3}} t^{\frac{2}{3}} e^{-\frac{\pi i}{3}} \left(z^2 + \frac{1}{2} \right), \quad c_1 = \frac{5}{72}, \quad d_1 = -\frac{7}{72}. \tag{151}$$

The leading-order contribution from the integral $\int_{C_u} (G_u(z) - I) dz$ will come from integrating

$$\frac{3}{4\zeta^{\frac{3}{2}}} \breve{Y} \delta^{\sigma_3} (is_2)^{\sigma_3/2} \begin{pmatrix} c_1 + d_1 & -c_1 + d_1 \\ c_1 - d_1 & -(c_1 + d_1) \end{pmatrix} (is_2)^{-\sigma_3/2} \delta^{-\sigma_3} \breve{Y}^{-1}. \tag{152}$$

In particular we are interested in the sum of the (12) and the (21) entries of this matrix, which can be written out as

$$(12) + (21) = -\frac{i}{16 x^{\frac{3}{2}} (z^2 + 1/2)^{\frac{3}{2}}} \left(\delta^2 (is_2) - \delta^{-2} (is_2)^{-1} \right). \tag{153}$$

We expand $\delta(z)$ for z near $i/\sqrt{2}$, giving

$$\delta^2(z) \quad = \quad \frac{1}{i\sigma s_2} \left(1 + 2^{\frac{7}{4}} e^{\pi i/4} \nu (z - i/\sqrt{2})^{\frac{1}{2}} + 8\nu^2 e^{\pi i/2} (z - i/\sqrt{2}) + \mathcal{O}\left((z - i/\sqrt{2})^{\frac{3}{2}} \right) \right), \tag{154}$$

Consequently,

$$(is_2)\delta^2 - (is_2)^{-1} \delta^{-2} = \sigma \left(2^{\frac{11}{4}} e^{\pi i/4} \nu (z - i/\sqrt{2})^{\frac{1}{2}} + \mathcal{O}\left((z - i\sqrt{2})^{\frac{3}{2}} \right) \right). \tag{155}$$

Additionally,

$$\left(1 + \frac{\left(z - \frac{i}{\sqrt{2}} \right)}{i\sqrt{2}} \right)^{-3/2} = 1 - \frac{3}{i2\sqrt{2}} \left(z - \frac{i}{\sqrt{2}} \right) + \mathcal{O}\left(\left(z - \frac{i}{\sqrt{2}} \right)^2 \right). \tag{156}$$

Inserting the expansions (155) and (156) we obtain

$$
\begin{aligned}
&\int_{C_u} \frac{(is_2)\delta(z)^2 - (is_2)^{-1}\delta^{-2}(z)}{\left(z^2 + \frac{1}{2}\right)^{3/2}} dz \\
&= \int_{C_u} \frac{\sigma \left(2^{\frac{11}{4}} e^{\frac{\pi i}{4}} \nu (z - \frac{i}{\sqrt{2}})^{\frac{1}{2}} + \mathcal{O}\left((z - \frac{i}{\sqrt{2}})^{\frac{3}{2}}\right)\right)}{(z - \frac{i}{\sqrt{2}})^{\frac{3}{2}} (i\sqrt{2})^{\frac{3}{2}}} \left(1 - \frac{3}{i2\sqrt{2}}\left(z - \frac{i}{\sqrt{2}}\right) + \mathcal{O}\left(\left(z - \frac{i}{\sqrt{2}}\right)^2\right)\right) dz \qquad (157) \\
&= \frac{\sigma 2^{\frac{11}{4}} e^{\frac{\pi i}{4}} \nu}{(i\sqrt{2})^{\frac{3}{2}}} \int_{C_u} \frac{1}{z - \frac{i}{\sqrt{2}}} dz \\
&= -8\sigma\nu\pi.
\end{aligned}
$$

Using the definition of G_u, (157), and (153) we have that:

$$
-\frac{1}{2\pi i} \int_{C_u} G_u(z)_{12} dz - \frac{1}{2\pi i} \int_{C_l} G_l(z)_{12} dz = -\frac{\sigma\nu}{4x^{\frac{3}{2}}} + \mathcal{O}(x^{-3}). \qquad (158)
$$

Together, assertions 1, 2, and 3 and (63) establish Theorem 4.1.

5 The GOE and GSE sine-kernel constants

Define J to be the interval $(-1, 1)$. Let $\mathbf{K^{(x)}}$ be the integral operator on $L^2(J, dz)$ with kernel

$$
K^{(x)}(z, z'; x) := \frac{\sin x(z - z')}{\pi(z - z')}. \qquad (159)
$$

Also let $\mathbf{K_\pm^{(x)}}$ be the integral operators on $L^2((0,1), dz)$ with kernels

$$
K_\pm^{(x)} := \frac{1}{\pi}\left(\frac{\sin x(z - z')}{z - z'} \pm \frac{\sin x(z + z')}{z + z'}\right). \qquad (160)
$$

Then

$$
P^{(x)} := \det(1 - \mathbf{K^{(x)}}) \qquad (161)
$$

is the limit (as $N \to \infty$) of the probability that an $N \times N$ matrix drawn from the Gaussian Unitary Ensemble has no eigenvalues in $(-\frac{x}{\pi}, \frac{x}{\pi})$ after proper scaling so that the mean spacing of eigenvalues in the bulk is normalized to 1. Also define the determinants

$$
D_\pm(x) := \det(1 - \mathbf{K_\pm^{(x)}}). \qquad (162)
$$

Then $D_+(x)$ and $\frac{1}{2}(D_+(2x) + D_-(2x))$ are, respectively, the limits of the probabilities that a matrix drawn from the Gaussian Orthogonal or Gaussian Simplectic Ensembles has no eigenvalues in $(0, \frac{x}{\pi})$ after scaling so the bulk spacing of eigenvalues is normalized to 1. Dyson [16] conjectured and Ehrhardt [18] recently proved that

Theorem 5.1.

$$
\log D_\pm = -\frac{x^2}{4} \mp \frac{x}{2} - \frac{\log x}{8} + \frac{\log 2}{24} \pm \frac{\log 2}{4} + \frac{3}{2}\zeta'(-1) + o(1) \text{ as } x \to +\infty, \qquad (163)
$$

where ζ is the Riemann zeta function.

The $o(1)$ terms are given by an explicit, asymptotic series. We give a short alternative proof of this theorem that will follow from Lemmas 5.2 and 5.3. To begin, we express $\log D_{\pm}(x)$ in terms of the definite integral of a solution to the Painlevé V equation. This function arises in the solution of a Riemann-Hilbert problem (studied in [11]) that is associated with the sine kernel. Set $J := (-1,1)$ and let $m(z;x)$ satisfy the Riemann-Hilbert problem (see (1.11) in [11])

$$\begin{cases} m(z;x) \text{ is analytic for } z \notin J \\ m_+(z;x) = m_-(z;x)\begin{pmatrix} 0 & e^{2ixz} \\ -e^{-2ixz} & 2 \end{pmatrix} \text{ on } \overline{J} \\ m(z;x) = I + O(\frac{1}{z}) \text{ as } z \to \infty. \end{cases} \tag{164}$$

Here J is oriented left to right. Define $m_1(x)$ by

$$m(z;x) = I + \frac{m_1(x)}{z} + O\left(\frac{1}{z^2}\right) \text{ as } z \to \infty. \tag{165}$$

Then set (see (4.31) in [11])

$$\xi(x) := 2i(m_1(x))_{21} = -2i(m_1(x))_{12}. \tag{166}$$

It is shown in [11] that $\xi(x)$ is related to a solution of the Painlevé V equation. Indeed, let $u(x)$ be the regular at $x=0$ solution of the Painlevé V equation

$$\frac{d^2u}{dx^2} = \left(\frac{du}{dx}\right)^2 \frac{3u-1}{2u(u-1)} + \frac{2u(u+1)}{u-1} + \frac{2iu}{x} - \frac{1}{x}\frac{du}{dx}, \tag{167}$$

characterized by the following behavior at $x=0$:

$$u(x) = 1 + 2ix - \frac{2\pi + 2i}{\pi}x^2 + O(x^3), \quad x \to 0, \tag{168}$$

and put

$$v(x) = \sqrt{u(2x)} = 1 + 2ix - \frac{2\pi + 4i}{\pi}x^2 + O(x^3). \tag{169}$$

Then

$$\xi(x) = \frac{2iv(x) - v'(x)}{v^2(x) - 1}. \tag{170}$$

Alternatively, one can use the Hirota-Jimbo-Miwa-Okomoto σ-form of Painlevé V (see [30]),

$$\left(x\frac{d^2\sigma}{dx^2}\right) = -16\left(\sigma - x\frac{d\sigma}{dx} - \frac{1}{4}\left(\frac{d\sigma}{dx}\right)^2\right)\left(\sigma - x\frac{d\sigma}{dx}\right), \tag{171}$$

and choose the (regular for all positive x) solution $\sigma(x)$ satisfying the initial conditions

$$\sigma(x) = -\frac{2}{\pi}x - \frac{4}{\pi^2}x^2 + O(x^3), \quad x \to 0. \tag{172}$$

The relation of $\sigma(x)$ to the function $\xi(x)$ is given by the formula (see [11][4]),

$$\frac{d}{dx}\left(\frac{\sigma(x)}{x}\right) = -\xi^2(x). \tag{173}$$

[4] In [11], the symbol $\theta(x)$ is used instead of $\sigma(x)$, the symbol $y(x)$ is used instead of $v(x)$, and the symbol $\omega(x)$ instead of $u(x)$.

It also worth noticing that the function $\sigma(x)$, similar to the function $\xi(z)$, can be determined via the solution $m(z;x)$ of the Riemann-Hilbert problem (164) via the equation (see (4.55) of [11])

$$\sigma(x) = -2ix(m_1(x))_{11}. \tag{174}$$

The central role of the function $\xi(x)$ in the analysis of the determinants $D_\pm(x)$ is based on the following important fact.

Define (see [11] (4.38))

$$Q^\pm(x) := \xi^2(x) \pm \xi'(x). \tag{175}$$

Then,

$$Q^\pm(x) = -2\frac{d^2}{dx^2}\big(\log D_\pm(x)\big). \tag{176}$$

This equation is proved in [11] (see equation (4.125) of that work) using Dyson's results [16] concerning the spectral analysis of the 1-D Schrödinger operators with the potentials determined by the second logarithmic derivative of the determinants $D_\pm(x)$. In the Appendix, we give an alternative derivation of (176) based solely on the Riemann-Hilbert problem (164).

Equations (176) are companion equations to the equation

$$\xi^2(x) = -\frac{d^2}{dx^2}\log P^{(x)}, \tag{177}$$

which in turn follows from the relation

$$\sigma(x) = x\frac{d}{dx}\log P^{(x)}. \tag{178}$$

This is one of the key formulas concerning the sine-kenel determinant $P^{(x)}$. It was first discovered by Jimbo, Miwa, Mori, and Sato in [31]. In [11] it was re-derived using the Riemann-Hilbert problem (164) (see also Appendix). One more derivation of (178) was obtained earlier by Tracy and Widom in [42].

We shall also need the important formula

$$P^{(x)} = D_+D_-, \tag{179}$$

whose proof via the general operator technique is given in [38]. An alternative proof of (179) via Riemann-Hilbert techniques is presented in [11], page 206.

We now proceed to establish the above mentioned evaluation of $\log D_\pm(x)$ and the asymptotics of these functions in terms of the integrals of the Painlevé V transcendents, i.e. in terms of the function $\xi(z)$.

Lemma 5.2.

$$\log D_\pm(x) = -\frac{x^2}{4} - \frac{\log x}{8} + \frac{\log 2}{24} + \frac{3}{2}\zeta'(-1) \mp \frac{1}{2}\int_0^x \xi(y)dy + o(1). \tag{180}$$

Proof. From [11], page 206 we have

$$\frac{d}{dx}P^{(x)}\bigg|_{x=0} := \frac{d}{dx}\left(\log D_+D_-\right)\bigg|_{x=0} = -\frac{2}{\pi} \quad \text{and} \quad \frac{d}{dx}\left(\log\frac{D_+}{D_-}\right)\bigg|_{x=0} = -\frac{2}{\pi}, \tag{181}$$

and so[5]

$$\frac{d}{dx}\big(\log D_+\big)\bigg|_{x=0} = -\frac{2}{\pi} \quad \text{and} \quad \frac{d}{dx}\big(\log D_-\big)\bigg|_{x=0} = 0. \tag{182}$$

[5]Equations (181) follow also from the direct small-x expansions of the Fredholm determinants $D_\pm(x)$, which can be easily obtained with the help of the identity

$$\log D_\pm = \text{trace}\ \log\left(1 - \mathbf{K}_\pm^{(x)}\right).$$

Integrating (176) twice and using (182) and $D_\pm(0)=1$ gives

$$\log D_+(x) = -\frac{1}{2}\int_0^x\int_0^y \xi^2(s)dsdy - \frac{1}{2}\int_0^x\int_0^y \xi'(s)dsdy - \frac{2}{\pi}x, \tag{183}$$

$$\log D_-(x) = -\frac{1}{2}\int_0^x\int_0^y \xi^2(s)dsdy + \frac{1}{2}\int_0^x\int_0^y \xi'(s)dsdy. \tag{184}$$

In view of (177), the integral involving $\xi^2(s)$ can be expressed in terms of the determinant $P^{(x)}$. Indeed, taking into account the first equation in (181) and the equation $P_0=1$, we derive from (177) that

$$\int_0^x\int_0^y \xi^2(s)dsdy + \frac{2}{\pi}x = -\log P^{(x)}. \tag{185}$$

Simultaneously, from (169) and (170) it follows that $\xi(0)=2/\pi$, and hence

$$\int_0^x\int_0^y \xi'(s)dsdy = \int_0^x \xi(y)dy - \frac{2}{\pi}x. \tag{186}$$

Combining (183), (184), (185), and (186) we arrive at the following representations for the logarithms of the determinants $D_\pm(x)$:

$$\log D_\pm(x) = \frac{1}{2}\log P^{(x)} \mp \frac{1}{2}\int_0^x \xi(y)dy. \tag{187}$$

The asymptotic expansion

$$\log P^{(x)} = -\frac{x^2}{2} - \frac{1}{4}\log x + \frac{\log 2}{12} + 3\zeta'(-1) + o(1) \text{ as } x\to+\infty \tag{188}$$

was conjectured by Dyson [16] and proven by three different methods by Krasovsky [35], Ehrhardt [17], and Deift, Its, Krasovsky, and Zhou [12]. Substituting (188) into (187) we obtain (180) and complete the proof of the Lemma. □

Now we compute $\int_0^x \xi(y)dy$.

Lemma 5.3. *We have*

$$U(0,x) := \int_0^x \xi(y)dy = x - \frac{\log 2}{2} + o(1) \text{ as } x\to+\infty. \tag{189}$$

Proof. If $m(z)$ satisfies the Riemann-Hilbert problem (164) then $\psi(z) := m(z)e^{ixz\sigma_3}$ satisfies (see (4.11) and (4.30) in [11])

$$\frac{d\psi}{dx} = (iz\sigma_3 + \xi\sigma_1)\psi. \tag{190}$$

The function $\psi(z)$ solves the Riemann-Hilbert problem ((4.2) in [11])

$$\begin{cases} \psi \text{ is analytic for } z\notin J \\ \psi_+ = \psi_- \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \text{ on } \overline{J} \\ \psi(z)e^{-ixz\sigma_3} = I + O(\frac{1}{z}) \text{ as } z\to\infty. \end{cases} \tag{191}$$

Set

$$R(x) := \psi_+(0;x). \tag{192}$$

Since $R(x)$ satisfies the differential equation

$$\frac{dR}{dx} = \xi\sigma_1 R \tag{193}$$

we have

$$R(x) = e^{U(0,x)\sigma_1} C = \begin{pmatrix} \cosh U & \sinh U \\ \sinh U & \cosh U \end{pmatrix} C \tag{194}$$

for some constant matrix C. Note $C = R(0)$, which can be computed exactly. From Lemma 4.5 in [11],

$$\psi(z; x = 0) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{i}{2\pi} \log\left(\frac{z+1}{z-1}\right) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{195}$$

where the principle branch of $\log\left(\frac{z+1}{z-1}\right)$ is chosen. Since $\lim_{z\to 0^+} \log\left(\frac{z+1}{z-1}\right) = -i\pi$, where the limit is taken from the upper half-plane, we have

$$C = R(0) = \lim_{z\to 0^+} \psi(z; x = 0) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}. \tag{196}$$

Hence

$$R(x) = \frac{1}{2} \begin{pmatrix} e^{-U(0,x)} & 2e^{U(0,x)} - e^{-U(0,x)} \\ -e^{-U(0,x)} & 2e^{U(0,x)} + e^{-U(0,x)} \end{pmatrix}. \tag{197}$$

Now $\lim_{x\to+\infty} R(x)$ is analyzed via the nonlinear steepest-descent method for Riemann-Hilbert problems as in [11]. Define $g(z) := \sqrt{z^2 - 1}$ with branch cut J and $g(z) \sim z$ as $z \to \infty$. Then

$$f(z; x) := \psi(z; x) e^{-ixg(z)\sigma_3} \tag{198}$$

satisfies the Riemann-Hilbert problem

$$\begin{cases} f \text{ is analytic for } z \notin J \\ f_+ = f_- \begin{pmatrix} 0 & 1 \\ -1 & 2e^{2ixg_+(z)} \end{pmatrix} \text{ on } \overline{J} \\ f(z) = I + O(\frac{1}{z}) \text{ as } z \to \infty. \end{cases} \tag{199}$$

Now since $\Im(g_+(z)) > 0$ on J, the jump matrix in (199) decays to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as $x \to +\infty$. Care must be taken because the decay is not uniform in x near ± 1. Nevertheless, it is shown in [11] that $\lim_{x\to+\infty} f_+(0; x) = f_+^\infty(0)$, where $f^\infty(z)$ is independent of x and satisfies the Riemann-Hilbert problem

$$\begin{cases} f^\infty \text{ is analytic for } z \notin J \\ f_+^\infty = f_-^\infty \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ on } \overline{J} \\ f^\infty(z) = I + O(\frac{1}{z}) \text{ as } z \to \infty. \end{cases} \tag{200}$$

Define

$$\beta(z) := \left(\frac{z-1}{z+1}\right)^{1/4} \tag{201}$$

with branch cut on J and so that $\beta(z) = 1 + O(\frac{1}{z})$ as $z \to \infty$. Then the Riemann-Hilbert problem (200) is solved by

$$f^\infty(z) = \begin{pmatrix} \frac{1}{2}(\beta + \beta^{-1}) & \frac{1}{2i}(\beta - \beta^{-1}) \\ -\frac{1}{2i}(\beta - \beta^{-1}) & \frac{1}{2}(\beta + \beta^{-1}) \end{pmatrix}. \tag{202}$$

Using $g_+(0) = i$ and $\beta_+(0) = e^{i\pi/4}$ gives

$$\lim_{x\to+\infty} R(x) e^{x\sigma_3} = \lim_{x\to+\infty} \lim_{z\to 0^+} \psi(z; x) e^{-ixg(z)\sigma_3} = \lim_{x\to+\infty} f_+(0) = f_+^\infty(0) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{203}$$

The (11) entry of this equation shows

$$e^{-U(0,x)+x} = \sqrt{2} + o(1) \text{ as } x \to +\infty, \tag{204}$$

and so

$$\int_0^x \xi(y)dy = x - \frac{1}{2}\log 2 + 2\pi i m + o(1) \text{ as } x \to +\infty \tag{205}$$

for some $m \in \mathbb{Z}$. Since the left-hand side of (180) is real, $m = 0$, which establishes (189). $\square$

Taken together Lemmas 5.2 and 5.3 immediately prove Theorem 5.1. Simultaneously, we have obtained the following Painlevé V analog of the Painlevé II total-integrals theorems of Sections 2 and 3.

Theorem 5.4. [A special fifth Painlevé transcendent] *Suppose that $u(x)$ is a solution to the Painlevé V equation* (167) *characterized by the Cauchy condition (168), and let $\xi(x)$ be the function defined by $u(x)$ according to equation (170). Then*

$$\int_0^\infty (1 - \xi(y))dy = \frac{1}{2}\log 2. \tag{206}$$

The function $\xi(x)$ can be alternatively defined by equation (173) in terms of the solution $\sigma(x)$ of the σ-version (171) of the fifth Painlevé equation characterized by the initial condition (172).

6 The first integrals of the mKdV equation

The evaluation of the total integrals of the global solutions of Painlevé equations performed in the previous sections was based on the analysis of the solution $\Psi(\lambda; x)$ of the relevant Riemann-Hilbert problems at the point $\lambda = 0$. One can wonder then what would come (if anything) from the investigation of the higher terms of the expansion of the Ψ-function at $\lambda = \infty$. It turns out that if we look at these terms then, instead of the total integrals of the Painlevé functions themselves, we will be able to evaluate explicitly the (properly regularized) total integrals of certain polynomials of u and its derivatives that play a central role in the theory of the modified Korteweg-de Vries (mKdV) equation

$$u_t - 6u^2 u_x + u_{xxx} = 0. \tag{207}$$

We remind the reader (see [3]) that the second Painlevé transcendents provide this equation with the important class of self-similar solutions. Indeed, if $u(x)$ is a solution of the Painlevé equation (1) then the formula

$$u(x,t) = \frac{1}{(3t)^{1/3}} u\left(\frac{x}{(3t)^{1/3}}\right), \tag{208}$$

gives a solution of the mKdV equation (207).

A fundamental fact about equation (207) is that it defines (for more detail see e.g. [1], [19]) on the proper functional spaces, e.g. on the Schwartz space, an infinite-dimensional completely integrable Hamiltonian system that possesses an infinite number of independent and commuting first integrals of the form

$$I_n \equiv \int_{-\infty}^{\infty} \alpha_{2n} dx, \quad n = 1, 2, 3... \tag{209}$$

Here, each conserved density α_k is a polynomial of u and its derivatives up to the order k that can be found explicitly via the following recurrence relations:

$$\alpha_0 = -\frac{i}{2}u^2, \tag{210}$$

$$\alpha_1 = -\frac{1}{4}u u_x, \tag{211}$$

$$\alpha_2 = \frac{i}{8}\left(uu_{xx} - u^4\right), \tag{212}$$

$$\alpha_{k+1} = \frac{i}{2}\left(\frac{u_x}{u}\alpha_k - \frac{d\alpha_k}{dx} + \sum_{l,m\geq 0;\ l+m=k-1} \alpha_l\alpha_m\right), \quad k \geq 1. \tag{213}$$

It can be observed that all the α's with odd subscripts are total derivatives (cf. (211)), hence the appearance of only α_{2n} in the description of the nontrivial first integrals (209).

Suppose now that $u(x)$ is a global (for real x) solution of the Painlevé equation (1). Then each α_k can be transformed to a polynomial of u, u_x, and x,

$$\alpha_k \equiv \alpha_k(u, u_x, x).$$

Our aim in this section is to evaluate the properly regularized total integrals of $\alpha_k(u, u_x, x)$. Obviously, we need to concentrate on the α's with even subscripts only. The remarkable fact is that, when calculated for the Painlevé functions, the α_{2n} become total derivatives (of certain polynomials of u, u_x, and x) as well. This is well known in modern Painlevé theory (see, e.g., [7] and [37]). We shall now outline the procedure of finding the relevant antiderivatives. To this end let us recall the origin of the recurrence system (210)-(213).

Even before their emergence in soliton theory, the densities $\alpha_k(x)$ were very well known in the scattering theory of the Dirac equation (9) with a rapidly decaying potential $u(x)$ (see e.g. [36] and earlier references therein). The densities $\alpha_k(x)$ appear in the so-called *trace formulae* which equate integrals (209) with the moments of the logarithm of the absolute value of the transmission coefficient associated with the potential $u(x)$. We will discuss the trace formulae in more detail in Section 7. What is important for us in this section is the main ingredient of the trace formulae derivation, i.e. the *Riccati equation* associated with the Dirac equation (9). The Riccati equation appears after one transforms the first order matrix differential equation (9) to the second order scalar differential equation for the entry $\Psi_{11}(\lambda; x)$,

$$\Psi_{11,xx} - \frac{u_x}{u}\Psi_{11,x} + \left(\lambda^2 - i\lambda\frac{u_x}{u} - u^2\right)\Psi_{11} = 0. \tag{214}$$

If we now write the function $\Psi_{11}(\lambda; x)$ in the form

$$\Psi_{11}(\lambda; x) = \exp\left(-\frac{4i}{3}\lambda^3 - i\lambda x - L(\lambda; x)\right) \tag{215}$$

and put

$$\alpha(\lambda; x) = L_x(\lambda; x), \tag{216}$$

then this substitution would indeed bring equation (214) to the following Riccati type differential equation for $\alpha(\lambda; x)$:

$$\frac{d\alpha}{dx} - 2i\lambda\alpha - \alpha^2 - \frac{u_x}{u}\alpha + u^2 = 0. \tag{217}$$

Assume now that the function $\alpha(\lambda; x)$ admits the differentiable asymptotic expansion

$$\alpha(\lambda; x) \sim \sum_{k=0}^{\infty} \frac{\alpha_k(x)}{\lambda^{k+1}} \quad \text{as} \quad \lambda \to \infty. \tag{218}$$

This is certainly true in both cases of our interest, i.e. in the case of the Schwartz function $u(x)$ and in the case of the Painlevé function $u(x)$. The recurrence system (210)-(213) appears now as a result of the substitution of the series (218) into Riccati equation (217).

Let us now expand the function $L(\lambda; x)$ in the neighborhood of $\lambda = \infty$,

$$L(\lambda; x) \sim \sum_{k=0}^{\infty} \frac{L_k(x)}{\lambda^{k+1}}, \quad \text{as} \quad \lambda \to \infty. \tag{219}$$

We have that

$$\alpha_k(x) = \frac{d}{dx} L_k(x). \tag{220}$$

A principal point now is that, in the case of the Painlevé function $u(x)$, all the coefficients $L_k(x)$ are polynomials of u, u_x, and x. This important fact follows from the possibility, in the Painlevé case, of using the first equation of the Lax pair (8)-(9), in addition to the second one, in order to study the asymptotic series (219).

Technically, it is more convenient to start with the asymptotic series for the whole matrix function $\Psi(\lambda; x)$,

$$\Psi(\lambda; x) e^{\theta(\lambda;x)\sigma_3} \sim I + \sum_{k=1}^{\infty} \frac{m_k(x)}{\lambda^k} \quad \text{as} \quad \lambda \to \infty. \tag{221}$$

The existence and differentiability of the series follows from the general properties of the Riemann-Hilbert problem (3) (see e.g. [6], see also [22]). Moreover, the entries of the matrix coefficients m_k are polynomials of $u \equiv 2(m_1)_{12}$, u_x, and x. The recurrence procedure that allows one to evaluate these polynomials is the following.

As in [29] (see also Chapter 1 of [22]), we rewrite the formal series from (221) as

$$I + \sum_{k=1}^{\infty} \frac{m_k}{\lambda^k} \equiv \left(I + \sum_{k=1}^{\infty} \frac{F_k}{\lambda^k} \right) \exp\left(\sum_{k=1}^{\infty} \frac{\Lambda_k}{\lambda^k} \right), \tag{222}$$

where all the matrices Λ_k and F_k are assumed to be diagonal and diagonal-free, respectively. Then, from the differential equation (8) we easily get[6]

$$\begin{gathered} F_1 = \frac{u}{2}\sigma_1, \quad F_2 = -\frac{u_x}{4}\sigma_2, \quad F_3 = -\frac{1}{4}\left(xu + u^3\right)\sigma_1, \\ F_4 = \frac{1}{16}\left(u + xu_x + u^2 u_x\right)\sigma_2, \quad \Lambda_1 = \frac{i}{2}(u^4 + xu^2 - u_x^2)\sigma_3, \end{gathered} \tag{223}$$

and the recurrence relation for the rest of the coefficients,

$$\begin{aligned} 4i[\sigma_3, F_{k+3}] - k\Lambda_k =& kF_k - (2u_x\sigma_1 - 2iu^2\sigma_3)F_{k+1} \\ & - 4u\sigma_2 F_{k+2} + \sum_{l,m\geq 1;\ l+m=k} m\Lambda_m F_l, \quad k > 1. \end{aligned} \tag{224}$$

Note that taking the diagonal part of the last equation we determine Λ_k for $k > 1$ while the off-diagonal part yields F_k for $k > 4$. The coefficients m_k of the original series are determined, once again by recurrence, via the identity (222). Indeed,

$$m_1 = F_1 + \Lambda_1 = \frac{u}{2}\sigma_1 + \frac{i}{2}(u^4 + xu^2 - u_x^2)\sigma_3, \tag{225}$$

$$\begin{aligned} m_2 =& F_2 + \Lambda_2 + \frac{1}{2}\Lambda_1^2 + F_1\Lambda_1 \\ =& \frac{1}{8}\left(u^2 - (u^4 + xu^2 - u_x^2)^2\right) I - \frac{1}{4}\left(u_x - u(u^4 + xu^2 - u_x^2)\right)\sigma_2, \end{aligned} \tag{226}$$

and so on.

[6]It is worth noticing that the polynomial

$$H = i(\Lambda_1)_{11} \equiv \frac{1}{2}\left(u_x^2 - xu^2 - u^4\right)$$

is the Hamiltonian for Painlevé II equation (1) with respect to the usual choice of the canonical variables: $q = u$, $p = u_x$. For more on the Hamiltonian aspects of the theory of Painlevé equations, which we feel should have a strong relation to the topic of this paper, we refer the reader to the papers [39], [29], and [24].

Let us now come back to the series (219). The coefficients $L_k(x)$, which we have been after, are recursively determined from the already known $m_k(x)$ via the formal identity

$$\exp\left(-\sum_{k=0}^{\infty}\frac{L_k(x)}{\lambda^{k+1}}\right) \equiv 1+\sum_{k=1}^{\infty}\frac{(m_k(x))_{11}}{\lambda^k}. \tag{227}$$

It follows then that all coefficients $L_k(x)$ are indeed polynomials of u, u_x, and x. In particular, we have that

$$L_0 = -(m_1)_{11} = -\frac{i}{2}(u^4+xu^2-u_x^2), \tag{228}$$

$$L_1 = \frac{1}{2}L_0^2-(m_2)_{11} = -\frac{1}{8}u^2, \tag{229}$$

$$L_2 = -\frac{1}{6}L_0^2+L_0L_1-(m_3)_{11}. \tag{230}$$

Equation (220) tells us that the polynomials $L_k(u,u_x,x)$ defined by (227) are the antiderivatives of the polynomials $\alpha_k(u,u_x,x)$ defined by (210)-(213). In fact, only half of these relations - the ones corresponding to even k's - are of interest; whereas the ones that correspond to odd k's are just identities (cf. (229) and (211)). Hence, the polynomials $L_{2n}(u,u_x,x)$ are exactly the antiderivatives we have been looking for. We are ready now to proceed with the evaluation of the regularized total integrals of $\alpha_{2n}(u,u_x,x)$.

Let u be either a purely real Ablowitz-Segur or Hastings-McLeod solution. Then we can integrate (220) from x to $+\infty$ and obtain the relations

$$\int_x^{+\infty}\alpha_{2n}(u,u_y,y)dy = -L_{2n}(u,u_x,x), \quad n=0,1,2,... \tag{231}$$

In particular, the first relation reads[7]

$$\int_x^{+\infty}u^2(y)dy = u_x^2-xu^2-u^4. \tag{232}$$

Suppose now that $u(x)$ is the Ablowitz-Segur solution (20)-(23). Then to regularize the above integral at $x=-\infty$ we need to subtract the term $-\beta/\sqrt{|y|}$ from $u^2(y)$. Simultaneously, the right hand side of (232) satisfies the estimates

$$u_x^2-xu^2-u^4 = -2\beta|x|^{1/2}+o(1) \quad \text{as} \quad x\to-\infty.$$

Therefore, for any $c\in\mathbb{R}$ we have that

$$\begin{aligned}\int_c^{+\infty}u^2(y)dy+\int_x^c\left(u^2(y)+\frac{\beta}{\sqrt{|y|}}\right)dy-2\beta\,\mathrm{sgn}(c)|c|^{1/2}-2\beta|x|^{1/2} =& u_x^2-xu^2-u^4\\ =& -2\beta|x|^{1/2}+o(1) \quad \text{as} \quad x\to-\infty.\end{aligned} \tag{233}$$

Hence, we obtain the following total integral formula:

$$\int_c^{+\infty}u^2(y)dy+\int_{-\infty}^c\left(u^2(y)+\frac{\beta}{\sqrt{|y|}}\right)dy = 2\beta\,\mathrm{sgn}(c)|c|^{1/2}. \tag{234}$$

In the case of the Hastings-McLeod solution, we need to subtract off the term $-\frac{y}{2}$ in order to make the integral convergent at $x=-\infty$. The resulting total integral relation reads

$$\int_c^{+\infty}u^2(y)dy+\int_{-\infty}^c\left(u^2(y)+\frac{y}{2}\right)dy = \frac{c^2}{4} \tag{235}$$

[7] Of course, equation (232) can be checked by direct differentiation.

for any $c \in \mathbb{R}$. Similar analysis can be performed with equation (231) for any n, and it yields the total integral relation of the form

$$\int_c^{+\infty} \alpha_{2n}(u, u_y, y)dy + \int_{-\infty}^{c} \left(\alpha_{2n}(u, u_y, y) - \frac{dF_n(y)}{dy}\right) dy = -F_n(c) \tag{236}$$

for any $c \leq 0$. Here, the function $F_n(x)$ is uniquely defined by the asymptotic relation[8],

$$L_{2n}(u(x), u_x(x), x) = F_n(x) + o(1) \quad \text{as} \quad x \to -\infty. \tag{237}$$

In particular,

$$F_0(x) = \begin{cases} 2\beta(-x)^{1/2} & \text{for the Ablowitz-Segur solution} \\ -x^2/4 & \text{for the Hastings-McLeod solution.} \end{cases} \tag{238}$$

In order to explicitly write the regularizing function $F_n(x)$ for large values of the number n one needs to know more terms in the asymptotics of the solution $u(x)$. In the case of the Ablowitz-Segur and Hastings-McLeod solutions these terms can be relatively easily obtained from the substitution of a-priori ansatzs (whose existence is vouched for by the Riemann-Hilbert analysis) into the Painlevé equation (1) (see [14])). We also note that the total integral formulae, which are similar to (234)-(236), can be obtained for the case of the purely imaginary solutions $u(x)$ as well. In the generic purely imaginary case one needs the regularization at $x = +\infty$ as well, and to determine the higher terms of the relevant asymptotic expansions is now a serious technical problem for large values of the number n.

There is an interesting feature in which equations (234)-(236) differ from equations (27), (40), (65), and (74) describing the total integrals of the function u itself. Let us combine in all these equations the integral terms with the terms generated by the regularization procedure and use the symbol

$$v.p \int_{-\infty}^{+\infty}$$

to denote this combination. That is, we put

$$v.p \int_{-\infty}^{+\infty} \alpha_{2n}(u, u_y, y)dy := \int_c^{+\infty} \alpha_{2n}(u, u_y, y)dy + \int_{-\infty}^{c} \left(\alpha_{2n}(u, u_y, y) - \frac{dF_n(y)}{dy}\right) dy + F_n(c) \tag{239}$$

for the purely real Ablowitz-Segur and Hastings-McLeod solutions,

$$v.p \int_{-\infty}^{+\infty} u(y)dy := \int_c^{+\infty} u(y)dy + \int_{-\infty}^{c} \left(u(y) - is_1\sqrt{\frac{|y|}{2}}\right) dy + is_1\frac{\sqrt{2}}{3}c|c|^{1/2} \tag{240}$$

for the Hastings-McLeod solution, and

$$v.p \int_{-\infty}^{+\infty} u(y)dy := \int_{-\infty}^{c} u(y)dy + \int_c^{+\infty} \left(u(y) - i\sigma\sqrt{\frac{y}{2}} + i\sigma\frac{3\rho^2}{4y}\right) dy - i\sigma\frac{\sqrt{2}}{3}c^{3/2} + i\sigma\frac{3\rho^2}{4}\log c \tag{241}$$

for the generic purely imaginary solution. The total integrals of the Ablowitz-Segur solutions do not need any regularization, so we have

$$v.p \int_{-\infty}^{+\infty} u(y)dy := \int_{-\infty}^{+\infty} u(y)dy \tag{242}$$

for the Ablowitz-Segur solutions. The point we want to make now is that while the regularized total integrals of the solutions themselves are nontrivial quantities depending on the solution integrated, the regularized

[8] In the derivation of (236) we need the differentiability of the estimate (237) which can be shown to be a consequence of the differentiability of the basic asymptotics for the solution $u(x)$.

total integrals of the densities $\alpha_{2n}(u, u_x, x)$ are all identically zero[9]. The explanation of this phenomenon lies in the fact that all the polynomials $\alpha_{2n}(u, u_x, x)$ that have been calculated for the Painlevé function $u(x)$ become the total derivatives of other polynomials - the polynomials $L_k(u, u_x, x)$. Therefore, the evaluation of the total integrals of $\alpha_n(u, u_x, x)$ becomes rather trivial due to (very nontrivial!) fact that we know the global asymptotics of the Painlevé functions. In the case of the total integral of the function u itself the situation is much different. The antiderivative of u is actually given by equation (10) and it is *not* a polynomial in u and u_x. The antiderivative is explicit, but it is given in terms of the solution of the associated Lax pair. Therefore, in order to evaluate the integral of u the knowledge of its asymptotics is not enough. We have to know the asymptotics of the associated Ψ-function. The latter we extract from the asymptotic analysis of the Riemann-Hilbert problem associated with the Painlevé equation[10](1).

Remark 1. *The vanishing of all regularized total integrals of $\alpha_{2n}(u, u_x, x)$, which has been established in this section, perhaps can be also derived using the meromorphicity of the second Painlevé functions and the fact that the densities $\alpha_{2n}(u(x), u_x(x), x)$ all have zero residues at the poles of $u(x)$ (Treves' type theorem [46]).*

Remark 2. *As we have already seen, the evaluation of the total integral of the function $u(x)$ is a more difficult task than the evaluation of the total integrals of $\alpha_{2n}(u(x), u_x(x), x)$, e.g. the evaluation of the $u^2(x)$. Even more difficult, though still possible ([35], [17], [12], [4]), is the evaluation of the total integrals of the combination $xu^2(x)$ which, in case of $u(x)$ being the Hastings-McLeod solution, appears in connection with the analysis of the Tracy-Widom distribution functions. The integral mentioned is neither one of the polynomials $\alpha_{2n}(u, u_x, x)$ nor it can be extracted from the behavior of the Ψ-function at $\lambda = 0$. Evaluation of this integral involves an extra discretization of the original Painlevé equation and an extra Riemann-Hilbert analysis of certain Toeplitz ([4]) or Hankel ([12]) determinants. The corresponding answer reads*

$$\int_c^{+\infty} yu^2(y)dy + \int_{-\infty}^c \left(yu^2(y) + \frac{y^2}{2} + \frac{1}{8y} \right) dy = \frac{c^3}{6} + \frac{1}{8}\log|c| - \frac{1}{8} + \frac{\log 2}{24} + \zeta'(-1). \tag{243}$$

A similar "most difficult" integral in the case of the solution $\sigma(x)$ of the Painlevé V equation we dealt with in Section 5 follows from the formulae (178) and (188), and it reads

$$\int_0^c \frac{\sigma(y)}{y} dy + \int_c^{\infty} \left(\frac{\sigma(y)}{y} + y + \frac{1}{4y} \right) dy = -\frac{c^2}{2} - \frac{1}{4}\log|c| + \frac{\log 2}{12} + 3\zeta'(-1). \tag{244}$$

We brings the reader's attention to the appearance of the Riemann zeta-function in both equations.

7 Trace formulae

Let $u(x)$ be a purely real Ablowitz-Segur solution of the Painlevé equation (1), and let $L(\lambda; x)$ be as in (215) from Section 6. From the asymptotic analysis performed in Section 3 the following estimate for $L(\lambda; x)$ follows (cf. (81), (83)):

$$L\left(z(-x)^{1/2}; x\right) = -\mu \log \frac{z + 1/2}{z - 1/2} + O\left(\frac{1}{(-x)^{3/4}}\right), \tag{245}$$

as $x \to -\infty$, uniformly for $|z| > 1$. Here, we remind the reader that

$$\mu = -\frac{1}{2\pi i} \log\left(1 - |s_1|^2\right), \quad -1 < is_1 < 1. \tag{246}$$

[9]The particular statement that $v.p \int_{-\infty}^{+\infty} u^2(x)dx = 0$ was first pointed out to the fourth author by J. B. McLeod; it plays an important role in the analysis of the double scaling limit in the Hermitian matrix models with quartic potential - see [5].

[10]The evaluation of the integrals of $\alpha_k(u, u_x, x)$ also needs the asymptotic solution of the Riemann-Hilbert problem. Indeed, the global asymptotics of the function $u(x)$ necessary for this evaluation are obtained from the solution of the Riemann-Hilbert problem.

Expanding both sides of (245) over the negative powers of z we arrive at the relations

$$\frac{1}{(-x)^{\frac{2n+1}{2}}}L_{2n}(x) = \frac{1}{2\pi i}\frac{1}{2^{2n}(2n+1)}\log\left(1-|s_1|^2\right) + O\left(\frac{1}{(-x)^{3/4}}\right), \tag{247}$$

or, recalling (231),

$$\frac{1}{(-x)^{\frac{2n+1}{2}}}\int_x^{+\infty}\alpha_{2n}(u,u_y,y)dy = -\frac{1}{2\pi i}\frac{1}{2^{2n}(2n+1)}\log\left(1-|s_1|^2\right) + O\left(\frac{1}{(-x)^{3/4}}\right) \tag{248}$$

$$\text{as} \quad x\to-\infty, \quad n=0,1,2,...$$

The left-hand side of (248) suggests yet another way (comparing to the one used in Section 6) to regularize the total integrals of α_{2n}. Namely, we can put

$$reg\int_{-\infty}^{+\infty}\alpha_{2n}(u,u_y,y)dy := \lim_{x\to-\infty}\left[\frac{1}{(-x)^{\frac{2n+1}{2}}}\int_x^{+\infty}\alpha_{2n}(u,u_y,y)dy\right]. \tag{249}$$

From (248) it follows then that

$$reg\int_{-\infty}^{+\infty}\alpha_{2n}(u,u_y,y)dy = -\frac{1}{2\pi i}\frac{1}{2^{2n}(2n+1)}\log\left(1-|s_1|^2\right), \quad n=0,1,2,... \tag{250}$$

There is a striking similarity of relations (250) with the classical trace formulae of the theory of the Dirac operator (9) with the potential $u(x)$ belonging to the Schwartz class. Indeed, the Dirac operator trace formulae are

$$\int_{-\infty}^{+\infty}\alpha_{2n}(u,u_y,u_{yy},...)dy = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\log\left(1-|r(\lambda)|^2\right)\lambda^{2n}d\lambda, \quad n=0,1,2,..., \tag{251}$$

where the *reflection coefficient* $r(\lambda)$ is defined via the following relation (see e.g. [19]; see also [10]),

$$\Phi_-^{-1}(\lambda;x)\Phi_+(\lambda;x) = \begin{pmatrix} 1-|r(\lambda)|^2 & -\overline{r(\lambda)} \\ r(\lambda) & 1 \end{pmatrix}, \quad \lambda\in\mathbb{R}, \tag{252}$$

where, in turn, $\Phi(\lambda;x)$ is a unique solution of the Dirac equation (9) satisfying the conditions

$$\Phi(\lambda;x) \text{ is analytic in } \mathbb{C}\setminus\mathbb{R} \tag{253}$$

and

$$\Phi(\lambda;x)e^{i\lambda x\sigma_3}\to I \quad \text{as} \quad \lambda\to\infty. \tag{254}$$

In other words, $r(\lambda)$ is defined through the jump matrix of the Riemann-Hilbert problem corresponding to the Dirac operator in the formalism of the inverse scattering problem (for more detail see again [19]). Equations (250) can be formally obtained from the trace formulae (251) by putting in the latter

$$\log\left(1-|r(\lambda)|^2\right) \equiv \log\left(1-|s_1|^2\right)\theta\left(\frac{1}{4}-\lambda^2\right),$$

where $\theta(s)$ is the Heaviside step function, and replacing simultaneously in the left hand side the symbol $\int$ by the symbol $reg\int$. We also note that the interval $[-1/2,1/2]$ plays a central role in the asymptotic analysis of the Painlevé Riemann-Hilbert problem (3) as $x\to-\infty$ - see Section 3.

The relation between the formulae (250) and (251) can be made less formal if one observes that under the restrictions (20) on the monodromy data corresponding to the purely real Ablowitz-Segur case one can define the solution $\Phi(\lambda;x)$ of the associated Dirac operator (9) that would *almost* have the properties (253) and (254). Indeed, if we put

$$\Phi(\lambda;x) = \begin{cases} \Psi_2(\lambda;x)e^{\frac{4i}{3}\lambda^3\sigma_3} & \text{for } \Im\lambda > 0, \\ \Psi_6(\lambda;x)e^{\frac{4i}{3}\lambda^3\sigma_3} & \text{for } \Im\lambda < 0, \end{cases} \tag{255}$$

then we would have that

$$\Phi(\lambda;x) \text{ is analytic in } \mathbb{C}\setminus\mathbb{R} \tag{256}$$

and

$$\Phi(\lambda;x)e^{i\lambda x\sigma_3} \to I \quad \text{as} \quad \lambda\to\infty, \quad \Im\lambda \neq 0 \tag{257}$$

(note the last inequality!). Simultaneously, in place of (252) we get the equation

$$\Phi_-^{-1}(\lambda;x)\Phi_+(\lambda;x) = \begin{pmatrix} 1-|s_1|^2 & -\overline{s_1} \\ s_1 & 1 \end{pmatrix}, \quad \lambda\in\mathbb{R}. \tag{258}$$

Hence we arrive at the *almost* non-formal identification,

$$r(\lambda) \equiv s_1. \tag{259}$$

In conclusion, we note that the "Painlevé trace formulae" (250) can be also used to evaluate the total integrals $v.p.\int \alpha_{2n}dx$ that we worked out in Section 6. However, as with the method of that Section, in order to evaluate the integrals for large values of the number n we need to find higher corrections to the estimate (248).

8 Appendix

In this appendix we present an alternative to the derivation of equation (176) in [11]. Instead of using the spectral results of [16] we shall rely solely on the relation between the sine-kernel determinant $\mathbf{K^{(x)}}$ and the Riemann-Hilbert problem (164). This relation is based on the representation of the sine-kernel $K^{(x)}(z,z')$ in the "integrable form"[11] (see e.g. Section 2 of [11])

$$K^{(x)}(z,z') = \frac{f^T(z)g(z')}{z-z'}, \quad f^T(z)g(z) = 0, \tag{260}$$

with the column vector-functions $f(z)$ and $g(z)$ defined by the equations

$$f(z) \equiv \Big(f_1(z),\, f_2(z)\Big)^T := \Big(e^{izx},\, e^{-izx}\Big)^T, \quad g(z) \equiv \Big(g_1(z),\, g_2(z)\Big)^T := \frac{1}{2\pi i}\Big(e^{-izx},\, -e^{izx}\Big)^T \tag{261}$$

A first key point is that the kernel $R^{(x)}(z,z')$ of the resolvent, $\mathbf{R^{(x)}} = (1-\mathbf{K^{(x)}})^{-1} - 1 \equiv -(1-\mathbf{K^{(x)}})^{-1}\mathbf{K^{(x)}}$ has the same "integrable" structure (see [28], [45], and Section 2 in [11]) as the one indicated in (260), i.e.,

$$R^{(x)}(z,z') = \frac{F^T(z)G(z')}{z-z'}, \tag{262}$$

where the components $F_j(z)$ and $G_j(z)$, $j=1,2$, of the column vector-functions $F(z)$ and $G(z)$ are defined by the relations[12]

$$F_j = \Big(1-\mathbf{K^{(x)}}\Big)^{-1} f_j, \quad G_j = \Big(1-(\mathbf{K^{(x)}})^{\mathbf{T}}\Big)^{-1} g_j, \quad j=1,2. \tag{263}$$

[11] The theory of "integrable integral operators" was pushed forward in [28] and built upon the ideas of [31]. It was further developed in [45], [23], [9]. Some of the important elements of the modern theory of integrable operators were already implicitly present in the earlier work [47].

[12] It is shown in [11], Section 2, that the operator $1-\mathbf{K^{(x)}}$ is indeed invertible for all positive x.

In other words, the functions $F_j(z)$ and $G_j(z)$ are the solutions of the integral Fredholm equations

$$F_j(z) - \int_{-1}^{1} K^{(x)}(z,z')F_j(z')dz' = f_j(z), \quad j = 1,2, \tag{264}$$

and

$$G_j(z) - \int_{-1}^{1} K^{(x)}(z',z)G_j(z')dz' = g_j(z), \quad j = 1,2, \tag{265}$$

respectively. It is worth noticing that in our concrete example of the integrable kernel the following symmetry identities hold:

$$F_1(-z) = F_2(z), \quad F_2(-z) = F_1(z). \tag{266}$$

The second principal observation is that the vector functions $F(z)$ and $G(z)$ can be alternatively evaluated via the algebraic equations (see [28] and Section 2 of [11])

$$F(z) = m_\pm(z)f(z), \quad G(z) = \Big(m^T_\pm(z)\Big)^{-1}g(z), \tag{267}$$

where $m(z)$ is the solution of the Riemann-Hilbert problem posed on the interval $[-1,1]$ with the jump matrix $V(z)$ defined by the formula

$$V(z) = I - 2\pi i f(z)g^T(z). \tag{268}$$

(We note that in virtue of the second equation in (260), $m_+(z)f(z) = m_-(z)f(z)$ and $\Big(m^T_+(z)\Big)^{-1}g(z) = \Big(m^T_-(z)\Big)^{-1}g(z)$.) By a direct calculation, using (261), we see at once that in our case

$$V(z) = \begin{pmatrix} 0 & e^{2ixz} \\ -e^{-2ixz} & 2 \end{pmatrix},$$

hence $m(z)$ is exactly the solution of the Riemann-Hilbert problem (164).

The last piece of the general theory that we will need is the inversion of equations (267), i.e., the formula expressing $m(z)$ in terms of $F(z)$ (see again, e.g., Section 2 of [11]):

$$m(z) = I - \int_{-1}^{1} F(z')g^T(z')\frac{dz'}{z'-z}, \quad z \notin [-1,1]. \tag{269}$$

From this equation it follows, in particular, that the matrix coefficient $m_1(x)$ in the expansion (165) admits a representation in the form

$$m_1(x) = \int_{-1}^{1} F(z)g^T(z)dz. \tag{270}$$

Consider now the determinants $D_\pm(x)$ and let us try to evaluate their logarithmic derivatives with respect to x following the same line of arguments as presented on pages 167-168 of [11]. We have

$$\frac{d}{dx}\log D_\pm(x) = -\mathrm{trace}\left(\Big(1 - \mathbf{K}^{(\mathbf{x})}_\pm\Big)^{-1}\frac{d}{dx}\mathbf{K}^{(\mathbf{x})}_\pm\right). \tag{271}$$

A simple calculation shows that

$$\frac{d}{dx}\mathbf{K}^{(\mathbf{x})}_\pm = \frac{1}{\pi}\Big(\cos x(z-z') \pm \cos x(z+z')\Big)$$

$$= i\Big(f_1(z) \pm f_2(z)\Big)g_1(z) \mp i\Big(f_1(z) \pm f_2(z)\Big)g_1(z). \tag{272}$$

On the other hand, taking into account the symmetries (266), the integral equations (264) can be rewritten as

$$F_1(z) - \int_0^1 K^{(x)}(z,z')F_1(z')dz' - \int_0^1 K^{(x)}(z,-z')F_2(z')dz' = f_1(z) \tag{273}$$

and

$$F_2(z) - \int_0^1 K^{(x)}(z,z')F_2(z')dz' - \int_0^1 K^{(x)}(z,-z')F_1(z')dz' = f_2(z). \tag{274}$$

By summing and subtracting (273) and (274), we obtain the integral equations for the combinations $\Big(F_1(z) \pm F_2(z)\Big)$:

$$\Big(F_1(z) \pm F_2(z)\Big) - \int_0^1 K_{\pm}^{(x)}(z,z')\Big(F_1(z') \pm F_2(z')\Big)dz' = \Big(f_1(z) \pm f_2(z)\Big). \tag{275}$$

From these equations we read that

$$\Big(1 - \mathbf{K}_{\pm}^{(\mathbf{x})}\Big)^{-1}(f_1 \pm f_2) = F_1 \pm F_2. \tag{276}$$

Equations (276) and (272) imply that the operator $\Big(1 - \mathbf{K}_{\pm}^{(\mathbf{x})}\Big)^{-1} \frac{d}{dx}\mathbf{K}_{\pm}^{(\mathbf{x})}$ has kernel

$$i\Big(F_1(z) \pm F_2(z)\Big)g_1(z') \mp i\Big(F_1(z) \pm F_2(z)\Big)g_2(z'),$$

which yields the formula

$$\frac{d}{dx}\log D_{\pm}(x) = -i\int_0^1 \Big(F_1(z) \pm F_2(z)\Big)g_1(z)dz \pm i\int_0^1 \Big(F_1(z) \pm F_2(z)\Big)g_2(z)dz. \tag{277}$$

Taking into account the symmetry relations (266) one more time (and similar symmetries for $g_1(z)$ and $g_2(z)$) we rewrite (277) as

$$\frac{d}{dx}\log D_{\pm}(x) = -i\int_{-1}^1 F_1(z)g_1(z)dz \pm i\int_{-1}^1 F_1(z)g_2(z)dz. \tag{278}$$

With the help of the identity (270), we transform (278) into the relation

$$\frac{d}{dx}\log D_{\pm}(x) = -i\Big((m_1(x))_{11} \mp (m_1(x))_{12}\Big). \tag{279}$$

To complete the proof of (176) we only need to recall definition (166) of the function $\xi(x)$ and notice that (173) and (174) lead to the equation (cf. (4.33) of [11])

$$2i\frac{d}{dx}(m_1(x))_{11} = \xi^2(x). \tag{280}$$

References

[1] Ablowitz, M. and Clarkson, P. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Math. Soc. Lecture Notes Series, **149**, Cambridge University Press, Cambridge, 1991.

[2] Abramowitz, M. and Stegun, I. *Handbook of Mathematical Functions*. Dover Publications, New York, 1965.

[3] Ablowitz, M. and Segur, H. Exact linearization of a Painlevé transcendent. *Phys. Rev. Lett.* **38** (1977), 1103–1106.

[4] Baik, J., Buckingham, R., and DiFranco, J. Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function. *Comm. Math. Phys.* **280** (2008), 463–497.

[5] Bleher, P. and Its, A. Double scaling limit in the random matrix model: the Riemann-Hilbert approach. *Comm. Pure Appl. Math.* **56** (2003), 0433–0516.

[6] Bolibruch, A., Its, A., and Kapaev, A. On the Riemann- Hilbert-Birkhoff inverse monodromy problem and the Painlevé equations. *Algebra and Analysis*, **16** (2004), 121–162.

[7] Clarkson, P., Joshi, N., and Pickering, A. Bäcklund transformation for the second Painlevé hierarchy: a modified truncation approach. *Inverse Problems* **15** (1999), 175–187.

[8] Deift, P. *Orthogonal Polynomials and Random Matrices: a Riemann-Hilbert Approach.* American Mathematial Society, Providence, 1998.

[9] Deift, P. Integrable operators, in *Differential Operators and Spectral Theory,* Amer. Math. Soc. Transl., Ser. 2, **189** pp. 69–84, 1999.

[10] Deift, P., Its, A. and Zhou, X. Long-time asymptotics for integrable nonlinear wave equations. in *Important Developments in Soliton Theory,* ed. A.S. Fokas and V.E. Zakharov, Springer-Verlag, pp. 181-204 (1993).

[11] Deift, P., Its, A., and Zhou, X. A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics. *Ann. of Math.* **146** (1997), 149–235.

[12] Deift, P., Its, A., Krasovsky, I., and Zhou, X. The Widom-Dyson constant for the gap probability in random matrix theory. *J. Comput. Appl. Math.* **202** (2007), 26–47.

[13] Deift, P. and Zhou, X. Long-time behavior of the non-focusing nonlinear Schrödinger equation - a case study. New series, Lectures in Mathematical Sciences **5**, Tokyo, 1994.

[14] Deift, P. and Zhou, X. Asymptotics for the Painlevé II equation. *Comm. Pure Appl. Math.* **48** (1995), 277–337.

[15] Deift, P. and Zhou, X. Long-time asymptotics for integrable systems. Higher order theory. *Comm. Math. Phys.* **165** (1995), 175–191.

[16] Dyson, F. Fredholm determinants and inverse scattering problems. *Comm. Math. Phys.* **47** (1976), 171–183.

[17] Ehrhardt, T. Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel. *Comm. Math. Phys.* **262** (2006), 317–341.

[18] Ehrhardt, T. Dyson's constants in the asymptotics of the determinants of Wiener-Hopf-Hankel operators with the sine kernel. *Comm. Math. Phys.* **272** (2007), 683–698.

[19] Faddeev, L. and Takhtadjan, L. *Hamiltonian methods in the theory of solitons*, Springer Verlag, Berlin, Heidelberg, 1987.

[20] Flaschka, H. and Newell, A. Monodromy- and spectrum-preserving deformations I. *Comm. Math. Phys.* **76** (1980), 65–116.

[21] Fokas, A. and Ablowitz, M. On the initial value problem of the second Painlevé transcendent. *Comm. Math. Phys.* **91** (1983), 381–403.

[22] Fokas, A., Its, A., Kapaev, A., and Novokshenov, V. *Painlevé Transcendents: The Riemann-Hilbert Approach.* American Mathematical Society Mathematical Surveys and Monographs **128**, Providence, 2006.

[23] Harnad, J. and Its, A. Integrable Fredholm operators and dual isomonodromic deformations, *Comm. Math. Phys.* **226** (2002), 497–530

[24] Harnad, J. and Wisse, M. Loop algebra moment maps and Hamiltonian models for the Painlevé transcendants. *Fields Inst. Commun.* **7** (1993), 155–169

[25] Hastings, S. and McLeod, J. A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation. *Arch. Rational Mech. Anal.* **73** (1980), 31–51.

[26] Its, A. and Novokshenov, V. *The Isomonodromic Deformation Method in the Theory of Painlevé Equations.* Lecture Notes in Mathematics **1191**, Springer-Verlag, Berlin, 1986.

[27] Its, A. and Kapaev, A. The method of isomonodromy deformations and connection formulas for the second Painlevé transcendent. *Izv. Akad. Nauk SSR Ser. Mat.* **51** (1987), 78 (Russian); *Math. USSR-Izv.* **31** (1988), 193–207 (English).

[28] Its, A., Izergin, A., Korepin, V., and Slavnov, N. Differential equations for quantum correlation functions, *J. Mod. Phys. B* **1003**, (1990); Proc. of a Conf. "Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory", CMA, ANU, Canberra, Australia, July 10-14, 1989, eds. M.N. Barber, P.A. Pearce, World Scientific, 1990, pp. 303-338.

[29] Jimbo, M., Tetsuji, M., and Ueno, K. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. *Physica D* **2** (1981), 306–362.

[30] Jimbo, M. and Miwa, T. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II. *Physica D* **2** (1981), 407–448.

[31] Jimbo, M., Miwa, T., Môri, Y., and Sato, M. Density matrix of impenetrable bose gas and the fifth Painlevé transcendent, *Physica D* **1** (1980), 80–158.

[32] Kapaev, A. Asymptotics of the second Painlevé transcendents, PhD thesis, Leningrad University, 1987

[33] Kapaev, A. Irregular singular point of the second Painlevé function and the nonlinear Stokes phenomenon, *Zap. Nauch. Semin. LOMI*, **187** (1991), 139–170.

[34] Kapaev, A. Global asymptotics of the second Painlevé transcendent, *Phys. Lett. A*, **167** (1992), 356–362.

[35] Krasovsky, I. Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle. *Int. Math. Res. Not.* **25** (2004), 1249–1272.

[36] Levitan B. and Sargsyan I. Operatory ShturmaLiuvillya i Diraka. M.: Nauka, 1988.

[37] Mazzocco, M. and Mo, M. The Hamiltonian structure of the second Painlevé hierarchy, arxiv:nlin/0610066.

[38] Metha, M. *Random Matrices*, 6th edition, Academic Press, San Diego (1994).

[39] Okamoto, K. Polynomial Hamiltonians associated with Painlevé equations II. Differential equations satisfied by polynomial Hamiltonians, *Proc. Japan Acad. Ser. A Math. Sci.*, **56** (1980), 367–371.

[40] Segur, H. and Ablowitz, M. Asymptotic solutions of the Korteweg de Vries equation. *Stud. Appl. Math.* **571** (1977), 13–44.

[41] Segur, H. and Ablowitz, M. Asymptotic solutions of nonlinear evolution equations and a Painlevé transcendent. *Physica D* **3** (1981), 165–184.

[42] Tracy, C. and Widom, H. Introduction to random matrices, In: *Geometric and Quantum Aspects of Integrable Systems* (G. F. Helminck, ed.), Lect. Notes in Phys., **424** (1993), 103–130.

[43] Tracy, C. and Widom, H. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** (1994), 151–174.

[44] Tracy, C. and Widom, H. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.* **177** (1996), 727–754.

[45] Tracy, C. and Widom, H. Fredholm determinants, differential equations and matrix models, *Comm. Math. Phys.* **163**, (1994), 33–72.

[46] Trves, F. An algebraic characterization of the Korteweg-de Vries hierarchy, *Duke Math. Journal* **108** (2001), 251-294.

[47] Sakhnovich, L. Operators similar to unitary operators, *Functional Anal.and Appl.* **2** (1968), 48–60.

[48] Whittaker, E. and Watson, G. *A Course of Modern Analysis.* Cambridge University Press, Cambridge, 1927.