Explicit constructions of centrally symmetric k-neighborly polytopes and large strictly antipodal sets

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Abstract

We present explicit constructions of centrally symmetric 2-neighborly d-dimensional polytopes with about $3^{d/2} \approx (1.73)^d$ vertices and of centrally symmetric k-neighborly d-polytopes with about $2^{3d/20k^2 2^k}$ vertices. Using this result, we construct for a fixed $k \geq 2$ and arbitrarily large d and N, a centrally symmetric d-polytope with N vertices that has at least $\left(1 - k^2 \cdot (\gamma_k)^d\right) \binom{N}{k}$ faces of dimension $k-1$, where $\gamma_2 = 1/\sqrt{3} \approx 0.58$ and $\gamma_k = 2^{-3/20k^2 2^k}$ for $k \geq 3$. Another application is a construction of a set of $3^{\lfloor d/2-1 \rfloor} - 1$ points in $\mathbb{R}^d$ every two of which are strictly antipodal as well as a construction of an n-point set (for an arbitrarily large n) in $\mathbb{R}^d$ with many pairs of strictly antipodal points. The two latter results significantly improve the previous bounds by Talata, and Makai and Martini, respectively.

1 Introduction

1.1 Cs neighborliness

What is the maximum number of k-dimensional faces that a centrally symmetric d-dimensional polytope with N vertices can have? While the answer in the class of all polytopes is classic by now [15], very little is known in the centrally symmetric case. Here we present several constructions that significantly improve existing lower bounds on this number.

Recall that a *polytope* is the convex hull of a set of finitely many points in $\mathbb{R}^d$. The dimension of a polytope P is the dimension of its affine hull. A polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* (cs, for short) if $P = -P$. A cs polytope P is *k-neighborly* if every set of k vertices of P no two of which are antipodes forms the vertex set of a $(k-1)$-face of P.

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It was proved in [12] that a cs 2-neighborly d-dimensional polytope cannot have more than 2^d vertices. On the other hand, a construction from [4] showed that there exist such polytopes with about $3^{d/4} \approx (1.316)^d$ vertices. In Theorem 3.2(1) we present a construction of a cs 2-neighborly d-polytope with about $3^{d/2} \approx (1.73)^d$ vertices.

More generally, it was verified in [3] that a cs d-dimensional polytope with N vertices cannot have more than $\left(1 - 0.5^d\right) \frac{N^2}{2}$ edges. However, a construction from [4] produced cs d-dimensional polytopes with N vertices and about $\left(1 - 3^{-d/4}\right) \frac{N^2}{2} \approx \left(1 - 0.77^d\right) \frac{N^2}{2}$ edges. In Theorem 3.2(2), we improve this bound by constructing a cs d-dimensional polytope with N vertices (for an arbitrarily large N) and at least $\left(1 - 3^{-\lfloor d/2-1 \rfloor}\right) \binom{N}{2} \approx \left(1 - 0.58^d\right) \frac{N^2}{2}$ edges.

For higher-dimensional faces even less is known. It follows from the results of [12] that no cs k-neighborly d-polytope can have more than $\lfloor d \cdot 2^{Cd/k} \rfloor$ vertices, where $C > 0$ is some absolute constant. At the same time, papers [12, 16] used a randomized construction to prove existence of k-neighborly cs d-dimensional polytopes with $\lfloor d \cdot 2^{cd/k} \rfloor$ vertices for some absolute constant $c > 0$. However, for $k > 2$ no deterministic construction of a d-dimensional k-neighborly cs polytope with $2^{\Omega(d)}$ vertices is known. In Theorem 5.2 and Remark 5.3 we present a deterministic construction of a cs k-neighborly d-polytope with at least $2^{c_k d}$ vertices where $c_k = 3/20k^2 2^k$. We then use this result in Corollary 5.4 to construct for a fixed k and arbitrarily large N and d, a cs d-polytope with N vertices that has a record number of $(k-1)$-dimensional faces. Our construction relies on the notion of k-independent families [1, 9] (see also [2]).

Through Gale duality m-dimensional subspaces of $\mathbb{R}^N$ correspond to $(N-m)$-dimensional cs polytopes with $2N$ vertices. If the subspace is "almost Euclidean" (meaning that the ratio of the ℓ^1 and ℓ^2 norms of nonzero vectors of the subspace remains within certain bounds, see [12] for technical details), then the corresponding polytope turns out to be k-neighborly. Despite considerable efforts, see for example [11], no explicit constructions of "almost Euclidean" subspaces is known for m anywhere close to N. Our polytopes give rise to subspaces of $\mathbb{R}^N$ of codimension $O(\log N)$ and it would be interesting to find out if the resulting subspaces are indeed "almost Euclidean".

1.2 Antipodal points

Our results on cs polytopes provide new bounds on several problems related to strict antipodality. Let $X \subset \mathbb{R}^d$ be a set that affinely spans $\mathbb{R}^d$. A pair of points $u, v \in X$ is called *strictly antipodal* if there exist two distinct parallel hyperplanes H and H' such that $X \cap H = \{u\}$, $X \cap H' = \{v\}$, and X lies in the slab between H and H'. Denote by $A'(d)$ the maximum size of a set $X \subset \mathbb{R}^d$ having the property that every pair of points of X is strictly antipodal, by $A'_d(Y)$ the number of strictly antipodal pairs of a given set Y, and by $A'_d(n)$ the maximum size of $A'_d(Y)$ taken over all n-element subsets Y of $\mathbb{R}^d$. (Our notation follows the recent survey paper [14].)

The notion of strict antipodality was introduced in 1962 by Danzer and Grünbaum [7] who verified that $2d-1 \leq A'(d) \leq 2^d$ and conjectured that $A'(d) = 2d-1$. However, twenty years later, Erdős and Füredi [8] used a probabilistic argument to prove that $A'(d)$ is exponential in d. Their result was improved by Talata (see [6, Lemma 9.11.2]) who found an explicit construction showing that for $d \geq 3$,

$$A'(d) \geq \lfloor (\sqrt[3]{3})^d / 3 \rfloor.$$

Talata also announced that $(\sqrt[3]{3})^d/3$ in the above formula can be replaced with $(\sqrt[4]{5})^d/4$. (It is worth remarking that Erdős and Füredi established existence of an *acute set* in $\mathbb{R}^d$ that has an exponential size in d. As every acute set has the property that all of its pairs of vertices are strictly antipodal, their result implied an exponential lower bound on $A'(d)$. A significant improvement of the Erdős–Füredi bound on the maximum size of an acute set in $\mathbb{R}^d$ was recently found by Harangi [10].)

Regarding the value of $A'_d(n)$, Makai and Martini [13] showed that for $d \geq 4$,

$$\left(1 - \frac{\text{const}}{(1.0044)^d}\right)\frac{n^2}{2} - O(1) \leq A'_d(n) \leq \left(1 - \frac{1}{2^d - 1}\right)\frac{n^2}{2}.$$

Here we observe that an appropriately chosen half of the vertex set of a cs d-polytope with many edges has a large number of strictly antipodal pairs of points. Consequently, our construction of cs d-polytopes with many edges implies — see Theorem 4.1 — that

$$A'(d) \geq 3^{\lfloor d/2-1 \rfloor} - 1 \quad \text{and} \quad A'_d(n) \geq \left(1 - \frac{1}{3^{\lfloor d/2-1 \rfloor} - 1}\right)\frac{n^2}{2} - O(n) \quad \text{for all } d \geq 4.$$

The rest of the paper is structured as follows. In Section 2 we review several facts and definitions related to the symmetric moment curve. In Section 3, we present our construction of a cs 2-neighborly d-polytope with many vertices as well as that of a cs d-polytope with arbitrarily many vertices and a record number of edges. Section 4 is devoted to applications of these results to problems on strict antipodality. Finally, in Section 5 we provide a deterministic construction of a cs k-neighborly d-polytope and of a cs d-polytope with arbitrarily many vertices and a record number of $(k-1)$-faces.

2 The symmetric moment curve

In this section we collect several definitions and results needed for the proofs. We start with the notion of the symmetric moment curve on which all our constructions are based. The *symmetric moment curve* $U_k : \mathbb{R} \longrightarrow \mathbb{R}^{2k}$ is defined by

$$U_k(t) = (\cos t,\ \sin t,\ \cos 3t,\ \sin 3t,\ \ldots,\ \cos(2k-1)t,\ \sin(2k-1)t)\,. \tag{2.1}$$

Since

$$U_k(t) = U_k(t + 2\pi) \quad \text{for all} \quad t,$$

from this point on, we consider $U_k(t)$ to be defined on the unit circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. We note that t and $t + \pi$ form a pair of antipodes for all $t \in \mathbb{S}$ and that

$$U_k(t + \pi) = -U_k(t) \quad \text{for all} \quad t \in \mathbb{S}.$$

The value of an affine function $A : \mathbb{R}^{2k} \longrightarrow \mathbb{R}$ on the symmetric moment curve U_k is represented by a trigonometric polynomial of degree at most $2k - 1$ that has the following form

$$f(t) = c + \sum_{j=1}^{k} a_j \cos(2j-1)t + \sum_{j=1}^{k} b_j \sin(2j-1)t, \quad \text{where } a_j, b_j, c \in \mathbb{R}.$$

Starting with any trigonometric polynomial $f : \mathbb{S} \longrightarrow \mathbb{R}$, $f(t) = c + \sum_{j=1}^{d} a_j \cos(jt) + \sum_{j=1}^{d} b_j \sin(jt)$ of degree at most d and substituting $z = e^{it}$ gives rise to a complex polynomial

$$\mathcal{P}(f)(z) := z^d \left(c + \sum_{j=1}^{d} a_j \frac{z^j + z^{-j}}{2} + \sum_{j=1}^{d} b_j \frac{z^j - z^{-j}}{2i} \right). \tag{2.2}$$

This polynomial has degree at most $2d$, it is self-inversive (that is, the coefficient of z^j is conjugate to that of z^{2d-j}), and $t^* \in \mathbb{S}$ is a root of $f(t)$ if and only if e^{it^*} is a root of $\mathcal{P}(f)(z)$ (see [3] and [5] for more details). In particular, $f(t)$ cannot have more than $2d$ roots (counted with multiplicities).

The following result concerning the convex hull of the symmetric moment curve was proved in [5]. In what follows we talk about exposed faces, that is, intersections of convex bodies with supporting affine hyperplanes.

Theorem 2.1. *Let $\mathcal{B}_k \subset \mathbb{R}^{2k}$,*

$$\mathcal{B}_k = \operatorname{conv}\left(U_k(t) : \quad t \in \mathbb{S}\right),$$

be the convex hull of the symmetric moment curve. Then for every positive integer k there exists a number

$$\frac{\pi}{2} \ < \ \alpha_k \ < \ \pi$$

such that for an arbitrary open arc $\Gamma \subset \mathbb{S}$ of length α_k and arbitrary distinct $n \leq k$ points $t_1, \ldots, t_n \in \Gamma$, the set

$$\operatorname{conv}\left(U_k\left(t_1\right), \ldots, U_k\left(t_n\right)\right)$$

is a face of $\mathcal{B}_k$.

For $k = 2$ with $\alpha_2 = 2\pi/3$ this result is due to Smilansky [17].

We also frequently use the following well-known fact about polytopes: if $T : \mathbb{R}^{d'} \longrightarrow \mathbb{R}^{d''}$ is a linear transformation and $P \subset \mathbb{R}^{d'}$ is a polytope, then $Q = T(P)$ is also a polytope and for every face F of Q the inverse image of F,

$$T^{-1}(F) = \left\{x \in P : \quad T(x) \in F\right\},$$

is a face of P; this face is the convex hull of the vertices of P mapped by T into vertices of F.

3 Centrally symmetric polytopes with many edges

In this section we provide a construction of a cs 2-neighborly polytope of dimension d and with about $3^{d/2} \approx (1.73)^d$ vertices as well a construction of a cs d-polytope with N vertices (for an arbitrarily large N) that has about $\left(1 - 3^{-d/2}\right)\binom{N}{2} \approx \left(1 - 0.58^d\right)\binom{N}{2}$ edges. Our construction is a slight modification of the one from [4]; however our new trick allows us to halve the dimension of the polytope from [4] while keeping the number of vertices almost the same as before.

For an integer $m \geq 1$, consider the curve

$$\Phi_m : \mathbb{S} \longrightarrow \mathbb{R}^{2(m+1)}, \quad \text{where} \quad \Phi_m(t) := \big(\cos t, \sin t, \cos 3t, \sin 3t, \ldots, \cos(3^m t), \sin(3^m t)\big). \tag{3.1}$$

Note that $\Phi_1 = U_2$, see eq. (2.1). The key to our construction is the following observation.

Lemma 3.1. *For an integer $m \geq 1$ and a finite set $C \subset \mathbb{S}$, define*

$$P(C,m) = \text{conv}\left(\Phi_m(t) : t \in C\right).$$

Then $P(C,m)$ is a polytope of dimension at most $2(m+1)$ that has $|C|$ vertices. Moreover, if the elements of C satisfy

$$3^i t_1 \not\equiv 3^i t_2 \mod 2\pi \quad \textit{for all } t_1, t_2 \in C \textit{ such that } t_1 \neq t_2, \textit{ and all } i = 1, 2, \ldots, m-1, \tag{3.2}$$

then for every pair of distinct points $t_1, t_2 \in C$ that lie on an open arc of length $\pi(1 - \frac{1}{3^m})$, the interval $[\Phi_m(t_1), \Phi_m(t_2)]$ is an edge of $P(C,m)$.

Proof: To show that $P(C,m)$ has $|C|$ vertices, we consider the projection $\mathbb{R}^{2(m+1)} \longrightarrow \mathbb{R}^4$ that forgets all but the first four coordinates. Since $\Phi_1 = U_2$, the image of $P(C,m)$ is the polytope

$$P(C,1) = \text{conv}\left(U_2(t) : t \in C\right).$$

By Theorem 2.1, the polytope $P(C,1)$ has $|C|$ distinct vertices: $U_2(t)$ for $t \in C$. Furthermore, the inverse image of each vertex $U_2(t)$ of $C(m,1)$ in $P(C,m)$ consists of a single vertex $\Phi_m(t)$ of $P(C,m)$. Therefore, $\Phi_m(t)$ for $t \in C$ are all the vertices of P_m without duplicates.

To prove the statement about edges, we proceed by induction on m. As $\Phi_1 = U_2$, the $m = 1$ case follows from [17] (see Theorem 2.1 above and the sentence following it).

Suppose now that $m \geq 2$. Let t_1, t_2 be two distinct elements of C that lie on an open arc of length $\pi(1 - \frac{1}{3^m})$. There are two cases to consider.

Case I: t_1, t_2 lie on an open arc of length $2\pi/3$. In this case, the above projection of $\mathbb{R}^{2(m+1)}$ onto $\mathbb{R}^4$ maps $P(C,m)$ onto $P(C,1)$, and according to the base of induction, $[\Phi_1(t_1), \Phi_2(t_2)]$ is an edge of $P(C,1)$. Since the inverse image of a vertex $\Phi_1(t)$ of $P(C,1)$ in $P(C,m)$ consists of a single vertex $\Phi_m(t)$ of $P(C,m)$, we conclude that $[\Phi_m(t_1), \Phi_m(t_2)]$ is an edge of $P(C,m)$.

Case II: t_1, t_2 lie on an open arc of length $\pi(1 - \frac{1}{3^m})$, but not on an arc of length $2\pi/3$. (Observe that since $3t_1 \not\equiv 3t_2 \mod 2\pi$, the points t_1 and t_2 may not form an arc of length exactly $2\pi/3$.) Then $3t_1$ and $3t_2$ do not coincide and lie on an open arc of length $\pi(1 - \frac{1}{3^{m-1}})$. Consider the projection of $\mathbb{R}^{2(m+1)}$ onto $\mathbb{R}^{2m}$ that forgets the first two coordinates. The image of $P(C,m)$ under this projection is

$$P(3C, m-1), \quad \text{where} \quad 3C := \{3t \mod 2\pi : t \in C\} \subset \mathbb{S},$$

and since the pair $(3C, m-1)$ satisfies eq. (3.2), by the induction hypothesis, the interval

$$[\Phi_{m-1}(3t_1), \Phi_{m-1}(3t_2)]$$

is an edge of $P(3C, m-1)$. By eq. (3.2), the inverse image of a vertex $\Phi_{m-1}(3t)$ of $P(3C, m-1)$ in $P(C,m)$ consists of a single vertex $\Phi_m(t)$ of $P(C,m)$, and hence we infer that $[\Phi_m(t_1), \Phi_m(t_2)]$ is an edge of $P(C,m)$. $\square$

We are now in a position to state and prove the main result of this section. We follow the notation of Lemma 3.1.

Theorem 3.2. *Fix integers $m \geq 2$ and $s \geq 2$. Let $A_m \subset \mathbb{S}$ be the set of $2(3^m - 1)$ equally spaced points:*

$$A_m = \left\{ \frac{\pi(j-1)}{3^m - 1} : \quad j = 1, \ldots, 2(3^m - 1) \right\},$$

and let $A_{m,s} \subset \mathbb{S}$ be the set of $2(3^m - 1)$ clusters of s points each, chosen in such a way that for all $j = 1, \ldots, 2(3^m - 1)$, the j-th cluster lies on an arc of length 10^{-m} that contains the point $\frac{\pi(j-1)}{3^m-1}$, and the entire set $A_{m,s}$ is centrally symmetric. Then

1. *The polytope $P(A_m, m)$ is a centrally symmetric 2-neighborly polytope of dimension $2(m+1)$ that has $2(3^m - 1)$ vertices.*
2. *The polytope $P(A_{m,s}, m)$ is a centrally symmetric $2(m+1)$-dimensional polytope that has $N := 2s(3^m - 1)$ vertices and at least $N(N - s - 1)/2 > (1 - 3^{-m})\binom{N}{2}$ edges.*

Proof: To see that $P(A_m, m)$ is centrally symmetric, note that the transformation

$$t \mapsto t + \pi \mod 2\pi$$

maps A_m onto itself and also that $\Phi_m(t+\pi) = -\Phi_m(t)$. The same argument applies to $P(A_{m,s}, m)$.

We now show that the dimension of $P(A_m, m)$ is $2(m+1)$. If not, then the points $\Phi_m(t) : t \in A_m$ are all in an affine hyperplane in $\mathbb{R}^{2(m+1)}$, and hence the $2(3^m - 1)$ elements of A_m are roots of a trigonometric polynomial of the form

$$f(t) = c + \sum_{j=0}^{m} a_j \cos(3^j t) + \sum_{j=0}^{m} b_j \sin(3^j t).$$

Moreover, a_m and b_m cannot both be zero as by our assumption $f(t)$ has at least $2(3^m - 1)$ roots, and so the degree of $f(t)$ is at least $3^m - 1 > 3^{m-1}$. Thus the complex polynomial $\mathcal{P}(f)$ defined by eq. (2.2) is of the form

$$\mathcal{P}(f)(z) = d_m z^{2 \cdot 3^m} + d_{m-1} z^{3^m + 3^{m-1}} + d_{m-2} z^{3^m + 3^{m-2}} + \cdots + c z^{3^m} + \cdots + \overline{d_m}, \quad \text{where } d_m \neq 0.$$

Note that since $m > 1$, $3^m + 3^{m-1} < 2 \cdot 3^m - 2$. In particular, the coefficients of $z^{2 \cdot 3^m - 1}$ and $z^{2 \cdot 3^m - 2}$ are both equal to 0. Therefore, the sum of all the roots (counted with multiplicities) of $\mathcal{P}(f)$ as well as the sum of their squares is 0. As $\deg \mathcal{P}(f) = 2 \cdot 3^m$, the (multi)set of roots of $\mathcal{P}(f)$ consists of $\{e^{it} : t \in A_m\}$ together with two additional roots, denote them by ζ_1 and ζ_2. The complex numbers $e^{it} : t \in A_m$ form a geometric progression, and it is straightforward to check that

$$\sum_{t \in A_m} e^{it} = 0 \quad \text{and} \quad \sum_{t \in A_m} e^{2it} = 0.$$

Hence for the sum of all the roots of $\mathcal{P}(f)$ and for the sum of their squares to be zero, we must have

$$\zeta_1 + \zeta_2 = 0 \quad \text{and} \quad \zeta_1^2 + \zeta_2^2 = 0.$$

Thus $\zeta_1 = \zeta_2 = 0$, and so the constant term of $\mathcal{P}(f)$ is zero. This however contradicts the fact that the constant term of $\mathcal{P}(f)$ equals $\overline{d_m}$, where $d_m \neq 0$. Therefore, the polytope $P(A_m, m)$ is full-dimensional.

Finally, to see that $P(A_m, m)$ is 2-neighborly, observe that it follows from the definition of A_m that if $t_1, t_2 \in A_m$ are not antipodes, then they lie on a closed arc of length $\pi(1 - \frac{1}{3^m-1})$, and hence also on an an open arc of length $\pi(1 - \frac{1}{3^m})$. In addition, since $3^m - 1$ is relatively prime to 3, we obtain that for every two distinct elements t_1, t_2 of A_m, $3^i t_1 \not\equiv 3^i t_2 \mod 2\pi$ (for $i = 1, \ldots, m-1$). Part (1) of the theorem is then immediate from Lemma 3.1.

To compute the dimension of $P(A_{m,s}, m)$, note that if it is smaller than $2(m+1)$, then $P(A_{m,s}, m)$ is a subset of an affine hyperplane in $\mathbb{R}^{2(m+1)}$. As all vertices of this polytope lie on the curve Φ_m, such a hyperplane corresponds to a trigonometric polynomial of degree 3^m that has at least $N = 2s(3^m - 1) \geq 4(3^m - 1) > 2 \cdot 3^m$ roots. This is however impossible, as no nonzero trigonometric polynomial of degree D has more than $2D$ roots.

To finish the proof of Part (2), note that since each cluster of $A_{m,s}$ lies on an open arc of length

$$10^{-m} < \frac{\pi}{2} \left(\frac{1}{3^m - 1} - \frac{1}{3^m} \right)$$

that contains the corresponding element of A_m, and since multiplication by 3^i modulo 2π maps A_m bijectively onto itself, it follows that

- $3^i t_1 \not\equiv 3^i t_2 \mod 2\pi$ (for $i = 1, \ldots, m-1$) holds for all distinct $t_1, t_2 \in A_{m,s}$. (Indeed, for t_1, t_2 from the same cluster, the points $3^i t_1$ and $3^i t_2$ of $\mathbb{S}$ do not coincide as $3^m/10^m < 2\pi$, and for t_1, t_2 from different clusters, $3^i t_1$ and $3^i t_2$ do not coincide as the distance between them along $\mathbb{S}$ is at least $\frac{\pi}{3^m-1} - \frac{2 \cdot 3^m}{10^m} > 0$.)
- Every two points $t_1, t_2 \in A_{m,s}$ lie on an open arc of length $\pi(1 - \frac{1}{3^m})$ as long as they do not belong to a pair of opposite clusters.

Thus Lemma 3.1 applies and shows that the interval $[\Phi_m(t_1), \Phi_m(t_2)]$ is an edge of $P(A_{m,s}, m)$ for all $t_1, t_2 \in A_{m,s}$ that are not from opposite clusters. In other words, each vertex of $P(A_{m,s}, m)$ is incident with at least $N - s - 1$ edges. This yields the promised bound on the number of edges of $P(A_{m,s}, m)$ and completes the proof of Part (2). □

4 Applications to strict antipodality problems

In this section we observe that an appropriately chosen half of the vertex set of any cs $2k$-neighborly d-dimensional polytope has a large number of pairwise strictly antipodal $(k-1)$-simplices. The results of the previous section then imply new lower bounds on questions related to strict antipodality. Specifically, in the following theorem we improve both Talata's and Makai–Martini's bounds.

Theorem 4.1.

1. *For every $m \geq 1$, there exists a set $X_m \subset \mathbb{R}^{2(m+1)}$ of size $3^m - 1$ that affinely spans $\mathbb{R}^{2(m+1)}$ and such that each pair of points of X_m is strictly antipodal. Thus, $A'(d) \geq 3^{\lfloor d/2-1 \rfloor} - 1$ for all $d \geq 4$.*

2. *For all positive integers m and s, there exists a set $Y_{m,s} \subset \mathbb{R}^{2(m+1)}$ of size $n := s(3^m - 1)$ that has at least*
$$\left(1 - \frac{1}{3^m - 1}\right) \cdot \frac{n^2}{2}$$
pairs of antipodal points. Thus, $A'_d(n) \geq \left(1 - \frac{1}{3^{\lfloor d/2-1 \rfloor}-1}\right) \cdot \frac{n^2}{2} - O(n)$ for all $d \geq 4$ and n.

One can generalize the notion of strictly antipodal points in the following way: for a set $X \subset \mathbb{R}^d$ that affinely spans $\mathbb{R}^d$, we say that two simplices, σ and σ', spanned by the points of X are strictly antipodal if there exist two distinct parallel hyperplanes H and H' such that X lies in the slab defined by H and H', $H \cap \operatorname{conv}(X) = \sigma$, and $H' \cap \operatorname{conv}(X) = \sigma'$. Makai and Martini [13] asked about the maximum number of pairwise strictly antipodal $(k-1)$-simplices in $\mathbb{R}^d$. The following result gives a lower bound to their question.

Theorem 4.2. *There exists a set of $\lfloor (d/2) \cdot 2^{cd/k} \rfloor$ points in $\mathbb{R}^d$ with the property that every two disjoint k-subsets of X form the vertex sets of strictly antipodal $(k-1)$-simplices. In particular, there exists a set of $\lfloor \frac{d}{2k} \cdot 2^{cd/k} \rfloor$ pairwise strictly antipodal $(k-1)$-simplices in $\mathbb{R}^d$. Here $c > 0$ is an absolute constant.*

The key to our proofs are results of Section 3 and paper [12] along with the following observation.

Lemma 4.3. *Let $P \subset \mathbb{R}^d$ be a full-dimensional cs polytope on the vertex set $V = X \sqcup (-X)$. If U_1, U_2 are subsets of X such that $U_1 \cup (-U_2)$ is the vertex set of a $(|U_1| + |U_2| - 1)$-face of P, then $\sigma_1 := \operatorname{conv}(U_1)$ and $\sigma_2 := \operatorname{conv}(U_2)$ are strictly antipodal simplices spanned by points of X. In particular, if P is 2-neighborly, then every pair of vertices of X is strictly antipodal, and, more generally, if P is $2k$-neighborly, then every two disjoint k-subsets of X form a pair of strictly antipodal $(k-1)$-simplices.*

Proof: Since $\tau_1 := \operatorname{conv}(U_1 \cup (-U_2))$ is a face of P, there exists a supporting hyperplane H_1 of P that defines τ_1: specifically, P is contained in one of the closed half-spaces bounded by H_1 and $P \cap H_1 = \tau_1$. As P is centrally symmetric, the hyperplane $H_2 := -H_1 = \{x \in \mathbb{R}^d \ : -x \in H_1\}$ is a supporting hyperplane of P that defines the opposite face, $\tau_2 := \operatorname{conv}((-U_1) \cup U_2)$. Thus P, and hence also X, is contained in the slab between H_1 and H_2. Moreover, since U_1, U_2 are subsets of X, it follows that $-U_1$ and $-U_2$ are contained in $-X$, and hence disjoint from X. Therefore,

$$H_i \cap \operatorname{conv}(X) = H_i \cap P \cap \operatorname{conv}(X) = \tau_i \cap \operatorname{conv}(X) = \operatorname{conv}(U_i) = \sigma_i \quad \text{for } i = 1, 2.$$

The result follows. $\square$

Proof of Theorem 4.1: Consider the sets A_m and $A_{m,s}$ of Theorem 3.2. Define

$$A_m^+ = \{t \in A_m : 0 \leq t < \pi\},$$

and define $A_{m,s}^+$ by taking the union of those clusters of $A_{m,s}$ that lie on small arcs around the points of A_m^+. In particular, $|A_m^+| = 3^m - 1$ and $|A_{m,s}^+| = s(3^m - 1)$. Let

$$X_m := \left\{\Phi_m(t) : t \in A_m^+\right\} \subset \mathbb{R}^{2(m+1)} \quad \text{and} \quad Y_{m,s} := \left\{\Phi_m(t) : t \in A_{m,s}^+\right\} \subset \mathbb{R}^{2(m+1)}.$$

Theorem 3.2 and Lemma 4.3 imply that each pair of points of X_m is strictly antipodal, and each pair of points of $Y_{m,s}$ that are not from the same cluster is strictly antipodal. The claim follows. $\square$

Proof of Theorem 4.2: It was proved in [12, 16] (by using a probabilistic construction) that if k, d, and N satisfy

$$k \leq \frac{cd}{1 + \log \frac{N}{d}},$$

where $c > 0$ is some absolute constant, then there exists a d-dimensional cs polytope on $2N$ vertices that is $2k$-neighborly. Solving this inequality for N, implies existence of a d-dimensional cs polytope on $\lfloor d \cdot 2^{cd/k} \rfloor$ vertices that is $2k$-neighborly. This together with Lemma 4.3 yields the result. $\square$

5 Constructing k-neighborly cs polytopes

The goal of this section is to present a deterministic construction of a cs k-neighborly d-polytope with at least $2^{c_k d}$ for $c_k = 3/20k^2 2^k$ vertices. This requires the following facts and definitions.

A family $\mathcal{F}$ of subsets of $[m] := \{1, 2, \ldots, m\}$ is called *k-independent* if for every k distinct subsets $I_1, \ldots, I_k$ of $\mathcal{F}$ all 2^k intersections

$$\bigcap_{j=1}^{k} J_j, \quad \text{where } J_j = I_j \text{ or } J_j = I_j^c := [m] \setminus I_j, \text{ are non-empty.}$$

The crucial component of our construction is a deterministic construction of k-independent families of size larger than $2^{m/5(k-1)2^k}$ given in [9].

For a subset I of $[m]$ and a given number $a \in \{0, 1\}$, we (recursively) define a sequence $x(I, a) = (x_0, x_1, \ldots, x_m)$ of **zeros and ones** according to the following rule:

$$x_0 = x_0(I, a) := a \quad \text{and} \quad x_n = x_n(I, a) \equiv \begin{cases} \sum_{j=0}^{n-1} x_j & \text{if } n \notin I \\ 1 + \sum_{j=0}^{n-1} x_j & \text{if } n \in I \end{cases} \mod 2 \quad \text{for } n \geq 1. \quad (5.1)$$

We also set

$$t(I, a) := \pi \sum_{j=0}^{m} \frac{x_j}{3^j} \in \mathbb{S}. \quad (5.2)$$

A few observations are in order. First, it follows from (5.1) that $x(I, a) \neq x(J, a)$ if $I \neq J$, and that $x(I, a)$ and $x(I^c, 1-a)$ agree in all but the 0-th component, where they disagree. Hence

$$t(I, a) = t(I^c, 1-a) + \pi \mod 2\pi.$$

Second, since $\sum_{j=1}^{\infty} \frac{1}{3^j} = \frac{1}{2}$ and since all components of $x(I, a)$ are zeros and ones, we infer from eq. (5.2) that for all $1 \leq n \leq m$ and all $0 \leq \epsilon \leq 1/3^{m+1}$, the point $3^n \cdot (t(I, a) + \pi\epsilon)$ of $\mathbb{S}$ either lies on the arc $[0, \pi/2)$ or on the arc $[\pi, 3\pi/2)$ depending on the parity of

$$\sum_{j=0}^{n} 3^{n-j} x_j(I, a) \equiv \sum_{j=0}^{n} x_j(I, a) \mod 2.$$

As, by (5.1), $\sum_{j=0}^{n} x_j(I, a)$ is even if $n \notin I$ and is odd if $n \in I$, we obtain that

$$3^n \cdot (t(I, a) + \pi\epsilon) \in [\pi, 3\pi/2) \mod 2\pi \quad \text{for all } n \in I \text{ and } a \in \{0, 1\}. \quad (5.3)$$

The relevance of k-independent sets to cs k-neighborly polytopes is explained by the following lemma along with Theorem 2.1.

Lemma 5.1. *Let $\mathcal{F}$ be a k-independent family of subsets of $[m]$, let $\epsilon_I \in [0, 1/3^{m+1}]$ for $I \in \mathcal{F}$, and let*

$$V^\epsilon(\mathcal{F}) = \bigcup_{I \in \mathcal{F}} \{t(I, 0) + \pi\epsilon_I, \, t(I^c, 1) + \pi\epsilon_I\} \subset \mathbb{S}.$$

Then for every k distinct points $t_1, \ldots, t_k$ of $V^\epsilon(\mathcal{F})$ no two of which are antipodes, there exists an integer $n \in [m]$ such that the subset $\{3^n t_1, \ldots, 3^n t_k\}$ of $\mathbb{S}$ is entirely contained in $[\pi, 3\pi/2)$.

Proof: As $t_1, \dots, t_k$ are elements of $V^\epsilon(\mathcal{F})$, by relabeling them if necessary, we can assume that

$$t_j = \begin{cases} t(I_j, 0) + \pi\epsilon_{I_j} & \text{if } 1 \le j \le q \\ t(I_j^c, 1) + \pi\epsilon_{I_j} & \text{if } q < j \le k \end{cases}$$

for some $0 \le q \le k$ and $I_1, \dots, I_k \in \mathcal{F}$. Moreover, the sets $I_1, \dots, I_k$ are distinct, since $t_1, \dots, t_k$ are distinct and no two of them are antipodes. As $\mathcal{F}$ is a k-independent family, the intersection $(\cap_{j=1}^q I_j) \cap (\cap_{j=q+1}^k I_j^c)$ is non-empty. The result follows, since by eq. (5.3), for any element n of this intersection, $\{3^n t_1, \dots, 3^n t_k\} \subset [\pi, 3\pi/2)$. $\square$

For $I \in \mathcal{F}$, define $\epsilon_I = \epsilon_{I^c} := \sum_{i \in I} 10^{-i-m}$. Then

$$3^n t_1 \not\equiv 3^n t_2 \mod 2\pi \quad \text{for all } t_1, t_2 \in V^\epsilon(\mathcal{F}) \text{ such that } t_1 \ne t_2, \text{ and all } 1 \le n \le m. \tag{5.4}$$

Indeed, if t_1 and t_2 are antipodes, then so are $3^n t_1$ and $3^n t_2$, and (5.4) follows. If t_1 and t_2 are not antipodes, then there exist two distinct and not complementary subsets I, J of $[m]$ such that $t_1 = t(I,a) + \pi\epsilon_I$ and $t_2 = t(J,b) + \pi\epsilon_J$ for some $a, b \in \{0,1\}$. Hence, by definition of ϵ_I and ϵ_J,

$$\pi/10^{2m} < 3^n \cdot \pi|\epsilon_I - \epsilon_J| < \pi(3/10)^m,$$

while by definition of $t(I,a)$ and $t(J,b)$, the distance between the points $3^n \cdot t(I,a)$ and $3^n \cdot t(J,b)$ of $\mathbb{S}$ along $\mathbb{S}$ is either 0 or at least $\pi/3^m$. In either case, it follows that the distance between $3^n\left(t(I,a) + \pi\epsilon_I\right)$ and $3^n\left(t(J,b) + \pi\epsilon_J\right)$ is positive, yielding eq. (5.4).

We are now in a position to present our construction of k-neighborly cs polytopes. The construction is similar to that in Theorem 3.2, except that it is based on the set $V^\epsilon(\mathcal{F}) \subset \mathbb{S}$, where $\mathcal{F}$ is a k-independent family of subsets of $[m]$, instead of $A_m \subset \mathbb{S}$, and on a modification of Φ_m to a curve that involves U_k instead of U_2.

Let $U_k : \mathbb{S} \longrightarrow \mathbb{R}^{2k}$ be the curve defined by eq. (2.1). In analogy with the curve Φ_m (see eq. (3.1)), for integers $m \ge 0$ and $k \ge 3$, define the curve

$$\Psi_{k,m} : \mathbb{S} \longrightarrow \mathbb{R}^{2k(m+1)} \quad \text{by} \quad \Psi_{k,m}(t) := \left(U_k(t), U_k(3t), U_k(3^2 t), \dots, U_k(3^m t)\right). \tag{5.5}$$

Thus, $\Psi_{k,0} = U_k$ and $\Psi_{k,m}(t+\pi) = -\Psi_{k,m}(t)$.

The following theorem is the main result of this section. We use the same notation as in Lemma 5.1. Also, mimicking the notation of Lemma 3.1, for a subset C of $\mathbb{S}$, we denote by $P_k(C,m)$ the polytope $\mathrm{conv}(\Psi_{k,m}(t) : \ t \in C)$.

Theorem 5.2. *Let $m \ge 1$ and $k \ge 3$ be fixed integers, let $\mathcal{F}$ be a k-independent family of subsets of $[m]$, and let $\epsilon_I = \sum_{i \in I} 10^{-i-m}$ for $I \in \mathcal{F}$. Then the polytope*

$$P_k(V^\epsilon(\mathcal{F}), m) := \mathrm{conv}\left(\Psi_{k,m}(t) : \quad t \in V^\epsilon(\mathcal{F})\right)$$

is a cs k-neighborly polytope of dimension at most $2k(m+1) - 2m\lfloor (k+1)/3 \rfloor$ that has $2|\mathcal{F}|$ vertices.

Remark 5.3. For a fixed k and an arbitrarily large m, a deterministic algorithm from [9] produces a k-independent family $\mathcal{F}$ of subsets of $[m]$ such that $|\mathcal{F}| > 2^{m/5(k-1)2^k}$. Combining this with Theorem 5.2 results in a cs neighborly polytope of dimension $d \approx \frac{4}{3}km$ and more than $2^{3d/20k^2 2^k}$ vertices. Of a special interest is the case of $k = 3$: the algorithm from [9] provides a 3-independent family of size $\approx 2^{0.092m}$, which together with Theorem 5.2 yields a deterministic construction of a cs 3-neighborly polytope of dimension $\le d$ and with about $2^{0.023d}$ vertices.

Proof of Theorem 5.2: As in the proof of Theorem 3.2, the polytope $P_k(V^\epsilon(\mathcal{F}), m)$ is centrally symmetric since $V^\epsilon(\mathcal{F})$ is a cs subset of $\mathbb{S}$ and since $\Psi_{k,m}(t+\pi) = -\Psi_{k,m}(t)$.

Also as in the proof of Theorem 3.2, the fact that $P_k(V^\epsilon(\mathcal{F}), m)$ has $2|\mathcal{F}|$ vertices follows by considering the projection $\mathbb{R}^{2k(m+1)} \longrightarrow \mathbb{R}^{2k}$ that forgets all but the first $2k$ coordinates. Indeed, the image of $P_k(V^\epsilon(\mathcal{F}), m)$ under this projection is the polytope

$$P_k(V^\epsilon(\mathcal{F}), 0) = \operatorname{conv}\left(U_k(t) : \, t \in V^\epsilon(\mathcal{F})\right),$$

and this latter polytope has $2|\mathcal{F}|$ vertices (by Theorem 2.1).

To prove k-neighborliness of $P_k(V^\epsilon(\mathcal{F}), m)$, let $t_1, \dots, t_k \in V^\epsilon(\mathcal{F})$ be k distinct points no two of which are antipodes. By Lemma 5.1, there exists an integer $1 \leq n \leq m$ such that the points $3^n t_1, \dots, 3^n t_k$ of $\mathbb{S}$ are all contained in the arc $[\pi, 3\pi/2)$. Consider the projection $\mathbb{R}^{2k(m+1)} \longrightarrow \mathbb{R}^{2k(m+1-n)}$ that forgets the first $2kn$ coordinates followed by the projection $\mathbb{R}^{2k(m+1-n)} \longrightarrow \mathbb{R}^{2k}$ that forgets all but the first $2k$ coordinates. The image of $P_k(V^\epsilon(\mathcal{F}), m)$ under this composite projection is

$$P_k(3^n V^\epsilon(\mathcal{F}), 0) = \operatorname{conv}(U_k(3^n t) : \quad t \in V^\epsilon(\mathcal{F})),$$

and, since $\{3^n t_1, \dots, 3^n t_k\} \subset [\pi, 3\pi/2)$, Theorem 2.1 implies that the set $\{U_k(3^n t_i) : i = 1, \dots, k\}$ is the vertex set of a $(k-1)$-face of this latter polytope. As, by eq. (5.4), the inverse image of a vertex $U_k(3^n t)$ of $P_k(3^n V^\epsilon(\mathcal{F}), 0)$ in $P_k(V^\epsilon(\mathcal{F}), m)$ consists of a single vertex $\Psi_{k,m}(t)$ of $P_k(V^\epsilon(\mathcal{F}), m)$, we obtain that $\{\Psi_{k,m}(t_i) : i = 1, \dots, k\}$ is the vertex set of a $(k-1)$-face of $P_k(V^\epsilon(\mathcal{F}), m)$. This completes the proof of k-neighborliness of $P_k(V^\epsilon(\mathcal{F}), m)$.

To bound the dimension of $P_k(V^\epsilon(\mathcal{F}), m)$, observe that the third coordinate of $U_k(t)$ coincides with the first coordinate of $U_k(3t)$ while the fourth coordinate of $U_k(t)$ coincides with the second coordinate of $U_k(3t)$, etc. Thus $P_k(V^\epsilon(\mathcal{F}), m)$ is in a subspace of $\mathbb{R}^{2k(m+1)}$, and to bound the dimension of this subspace we must account for all repeated coordinates. This can be done exactly as in [4, Lemma 2.3]. We leave details to our readers. $\square$

Fix $s \geq 2$, and let $V^\epsilon(\mathcal{F}, s)$ be a centrally symmetric subset of $\mathbb{S}$ obtained by replacing each point $t \in V^\epsilon(\mathcal{F})$ (in Theorem 5.2) with a cluster of s points that all lie on a sufficiently small open arc containing t. Then the proof of Theorem 5.2 implies that the polytope $P_k(V^\epsilon(\mathcal{F}, s), m)$ is a cs polytope with $N := 2s|\mathcal{F}|$ vertices, of dimension at most $2k(m+1) - 2m\lfloor (k+1)/3 \rfloor$, and such that every k vertices of this polytope no two of which are from opposite clusters form the vertex set of a $(k-1)$-face. Choose a k-element set from the union of these $2|\mathcal{F}|$ clusters (of s points each) at random from the uniform distribution. Then the probability that this set has no two points from opposite clusters is at least

$$\prod_{i=0}^{k-1} \frac{(2|\mathcal{F}| - i)s - i}{2|\mathcal{F}|s - i} \geq \prod_{i=0}^{k-1} \left(1 - \frac{i}{|\mathcal{F}|}\right) \geq 1 - \frac{k^2}{|\mathcal{F}|}.$$

Thus, the resulting polytope has at least

$$\left(1 - \frac{k^2}{|\mathcal{F}|}\right) \binom{N}{k}$$

$(k-1)$-faces. Combining this estimate with Remark 5.3, we obtain

Corollary 5.4. *For a fixed k and arbitrarily large N and d, there exists a cs d-dimensional polytope with N vertices and at least*

$$\left(1 - k^2 \left(2^{-3/20k^2 2^k}\right)^d\right) \binom{N}{k}$$

$(k-1)$-faces.

This corollary improves [4, Cor. 1,4] asserting existence of cs d-polytopes with N vertices and at least $\left(1 - (\delta_k)^d\right) \binom{N}{k}$ faces of dimension $k-1$, where $\delta_k \approx (1 - 5^{-k+1})^{5/(24k+4)}$.

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