CONVEX GEOMETRY OF ORBITS

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Abstract. We study metric properties of convex bodies B and their polars B°, where B is the convex hull of an orbit under the action of a compact group G. Examples include the Traveling Salesman Polytope in polyhedral combinatorics ($G = S_n$, the symmetric group), the set of non-negative polynomials in real algebraic geometry ($G = SO(n)$, the special orthogonal group), and the convex hull of the Grassmannian and the unit comass ball in the theory of calibrated geometries ($G = SO(n)$, but with a different action). We compute the radius of the largest ball contained in the symmetric Traveling Salesman Polytope, give a reasonably tight estimate for the radius of the Euclidean ball containing the unit comass ball and review (sometimes with simpler and unified proofs) recent results on the structure of the set of non-negative polynomials (the radius of the inscribed ball, volume estimates, and relations to the sums of squares). Our main tool is a new simple description of the ellipsoid of the largest volume contained in B°.

1. Introduction and Examples

Let G be a compact group acting in a finite-dimensional real vector space V and let $v \in V$ be a point. The main object of this paper is the convex hull

$$B = B(v) = \operatorname{conv}\bigl(gv : \ g \in G\bigr)$$

of the orbit as well as its polar

$$B^\circ = B^\circ(v) = \bigl\{\ell \in V^* : \quad \ell(gv) \leq 1 \quad \text{for all} \quad g \in G\bigr\}.$$

Objects such as B and B° appear in many different contexts. We give three examples below.

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(1.1) Example: Combinatorial optimization polytopes. Let $G = S_n$ be the symmetric group, that is, the group of permutations of $\{1, \ldots, n\}$. Then $B(v)$ is a polytope and varying V and v, one can obtain various polytopes of interest in combinatorial optimization. This idea is due to A.M. Vershik (see [BV88]) and some polytopes of this kind were studied in [Ba92].

Here we describe perhaps the most famous polytope in this family, the Traveling Salesman Polytope (see, for example, Chapter 58 of [Sc03]), which exists in two major versions, symmetric and asymmetric. Let V be the space of $n \times n$ real matrices $A = (a_{ij})$ and let S_n act in V by simultaneous permutations of rows and columns: $(ga)_{ij} = a_{g^{-1}(i)g^{-1}(j)}$ (we assume that $n \geq 4$). Let us choose v such that $v_{ij} = 1$ provided $|i - j| = 1 \mod n$ and $v_{ij} = 0$ otherwise. Then, as g ranges over the symmetric group S_n, matrix gv ranges over the adjacency matrices of Hamiltonian cycles in a complete undirected graph with n vertices. The convex hull $B(v)$ is called the *symmetric Traveling Salesman Polytope* (we denote it by ST_n). It has $(n-1)!/2$ vertices and its dimension is $(n^2 - 3n)/2$.

Let us choose $v \in V$ such that $v_{ij} = 1$ provided $i - j = 1 \mod n$ and $v_{ij} = 0$ otherwise. Then, as g ranges over the symmetric group S_n, matrix gv ranges over the adjacency matrices of Hamiltonian circuits in a complete directed graph with n vertices. The convex hull $B(v)$ is called the *asymmetric Traveling Salesman Polytope* (we denote it by AT_n). It has $(n-1)!$ vertices and its dimension is $n^2 - 3n + 1$.

A lot of effort has been put into understanding of the facial structure of the symmetric and asymmetric Traveling Salesman Polytopes, in particular, what are the linear inequalities that define the facets of AT_n and ST_n, see Chapter 58 of [Sc03]. It follows from the computational complexity theory that in some sense one *cannot* describe efficiently the facets of the Traveling Salesman Polytope. More precisely, if *NP* $\neq$ *co-NP* (as is widely believed), then there is no polynomial time algorithm, which, given an inequality, decides if it determines a facet of the Traveling Salesman Polytope, symmetric or asymmetric, see, for example, Section 5.12 of [Sc03]. In a similar spirit, Billera and Sarangarajan proved that any 0-1 polytope (that is, a polytope whose vertices are 0-1 vectors), appears as a face of some AT_n (up to an affine equivalence) [BS96].

(1.2) Example: Non-negative polynomials. Let us fix positive integers n and k. We are interested in homogeneous polynomials $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ of degree $2k$ that are non-negative for all $x = (x_1, \ldots, x_n)$. Such polynomials form a convex cone and we consider its compact base:

$$\text{(1.2.1)} \qquad \begin{aligned} Pos_{2k,n} = \Big\{ p : \quad & p(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \\ & \int_{\mathbb{S}^{n-1}} p(x) \ dx = 1 \Big\}, \end{aligned}$$

where dx is the rotation invariant probability measure on the unit sphere $\mathbb{S}^{n-1}$.

It is not hard to see that $\dim Pos_{2k,n} = \binom{n+2k-1}{2k} - 1$.

It is convenient to consider a translation $Pos'_{2k,n}$, $p \longmapsto p - (x_1^2 + \ldots + x_n^2)^k$ of $Pos_{2k,n}$:

$$\begin{aligned} Pos'_{2k,n} = \Big\{p: \quad p(x) \geq -1 \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \\ \int_{\mathbb{S}^{n-1}} p(x) \ dx = 0\Big\}. \end{aligned} \tag{1.2.2}$$

Let $U_{m,n}$ be the real vector space of all homogeneous polynomials $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ of degree m such that the average value of p on $\mathbb{S}^{n-1}$ is 0. Then, for $m = 2k$, the set $Pos'_{2k,n}$ is a full-dimensional convex body in $U_{2k,n}$.

One can view $Pos'_{2k,n}$ as the *negative polar* $-B^\circ(v)$ of some orbit.

We consider the m-th tensor power $(\mathbb{R}^n)^{\otimes m}$ of $\mathbb{R}^n$, which we view as the vector space of all m-dimensional arrays $\big(x_{i_1,\ldots,i_m} : 1 \leq i_1, \ldots, i_m \leq n\big)$. For $x \in \mathbb{R}^n$, let $y = x^{\otimes m}$ be the tensor with the coordinates $y_{i_1,\ldots,i_m} = x_{i_1} \cdots x_{i_m}$. The group $G = SO(n)$ of orientation preserving orthogonal transformations of $\mathbb{R}^n$ acts in $(\mathbb{R}^n)^{\otimes m}$ by the m-th tensor power of its natural action in $\mathbb{R}^n$. In particular, $gy = (gx)^{\otimes m}$ for $y = x^{\otimes m}$.

Let us choose $e \in \mathbb{S}^{n-1}$ and let $w = e^{\otimes m}$. Then the orbit $\{gw : \ g \in G\}$ consists of the tensors $x^{\otimes m}$, where x ranges over the unit sphere in $\mathbb{R}^n$. The orbit $\{gw : \ g \in G\}$ lies in the symmetric part of $(\mathbb{R}^n)^{\otimes m}$. Let $q = \displaystyle\int_{\mathbb{S}^{n-1}} gw \ dg$ be the center of the orbit. If m is odd then $q = 0$ and if $m = 2k$ is even then q is a positive multiple of $(x_1^2 + \ldots + x_n^2)^k$. We translate the orbit by shifting q to the origin, so in the end we consider the convex hull B of the orbit of $v = w - q$:

$$B = \operatorname{conv}\big(gv : \quad g \in G\big).$$

A homogeneous polynomial

$$p(x_1, \ldots, x_n) = \sum_{1 \leq i_1, \ldots, i_m \leq n} c_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m}$$

of degree m, viewed as a function on the unit sphere in $\mathbb{R}^n$, is identified with the restriction onto the orbit $\big\{gw : \ g \in G\big\}$ of the linear functional $\ell : (\mathbb{R}^n)^{\otimes m} \longrightarrow \mathbb{R}$ defined by the coefficients $c_{i_1,\ldots,i_m}$. Consequently, the linear functionals ℓ on B are in one-to-one correspondence with the polynomials $p \in U_{m,n}$. Moreover, for $m = 2k$, the negative polar $-B^\circ$ is identified with $Pos'_{2k,n}$. If m is odd, then $B^\circ = -B^\circ$ is the set of polynomials p such that $|p(x)| \leq 1$ for all $x \in \mathbb{S}^{n-1}$.

The facial structure of $Pos_{2k,n}$ is well-understood if $k = 1$ or if $n = 2$, see, for example, Section II.11 (for $n = 2$) and Section II.12 (for $k = 1$) of [Ba02a]. In particular, for $k = 1$, the set $Pos_{2,n}$ is the convex body of positive semidefinite

n-variate quadratic forms of trace n. The faces of $Pos_{2,n}$ are parameterized by the subspaces of $\mathbb{R}^n$: if $L \subset \mathbb{R}^n$ is a subspace then the corresponding face is

$$F_L = \Big\{ p \in Pos_{2,n} : \quad p(x) = 0 \quad \text{for all} \quad x \in L \Big\}$$

and $\dim F_L = r(r+1)/2 - 1$, where $r = \operatorname{codim} L$. Interestingly, for large n, the set $Pos_{2,n}$ is a counterexample to famous Borsuk's conjecture [K95].

For any $k \geq 2$, the situation is much more complicated: the *membership problem* for $Pos_{2k,n}$:

given a polynomial, decide whether it belongs to $Pos_{2k,n}$,

is NP-hard, which indicates that the facial structure of $Pos_{2k,n}$ is probably hard to describe.

(1.3) Example: Convex hulls of Grassmannians and calibrations. Let $G_m(\mathbb{R}^n)$ be the Grassmannian of all oriented m-dimensional subspaces of $\mathbb{R}^n$. Let us consider $G_m(\mathbb{R}^n)$ as a subset of $V_{m,n} = \bigwedge^m \mathbb{R}^n$ via the Plücker embedding. Namely, let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. We make $V_{m,n}$ a Euclidean space by choosing an orthonormal basis $e_{i_1} \wedge \ldots \wedge e_{i_m}$ for $1 \leq i_1 < \ldots < i_m \leq n$. Thus the coordinates of a subspace $x \in G_m(\mathbb{R}^n)$ are indexed by m-subsets $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ of $\{1, \ldots, n\}$ and the coordinate $x_{i_1, \ldots, i_m}$ is equal to the oriented volume of the parallelepiped spanned by the orthogonal projection of $e_{1_1}, \ldots, e_{i_m}$ onto x. This identifies $G_m(\mathbb{R}^n)$ with a subset of the unit sphere in $V_{m,n}$. The convex hull $B = \operatorname{conv}(G_m(\mathbb{R}^n))$, called the *unit mass ball*, turns out to be of interest in the theory of calibrations and area-minimizing surfaces: a face of B gives rise to a family of m-dimensional area-minimizing surfaces whose tangent planes belong to the face, see [HL82] and [F88]. The *comass* of a linear functional $\ell : V_{m,n} \longrightarrow \mathbb{R}$ is the maximum value of ℓ on $G_m(\mathbb{R}^n)$. A *calibration* is a linear functional $\ell : V_{m,n} \longrightarrow \mathbb{R}$ of comass 1. The polar B° is called the *unit comass ball*.

One can easily view $G_m(\mathbb{R}^n)$ as an orbit. We let $G = SO(n)$, the group of orientation-preserving orthogonal transformations of $\mathbb{R}^n$, and consider the action of $SO(n)$ in $V_{m,n}$ by the m-th exterior power of its defining action in $\mathbb{R}^n$. Choosing $v = e_1 \wedge \ldots \wedge e_m$, we observe that $G_m(\mathbb{R}^n)$ is the orbit $\{gv : \ g \in G\}$. It is easy to see that $\dim \operatorname{conv}\big(G_m(\mathbb{R}^n)\big) = \binom{n}{m}$.

This example was suggested to the authors by B. Sturmfels and J. Sullivan.

The facial structure of the convex hull of $G_m(\mathbb{R}^n)$ is understood for $m \leq 2$, for $m \geq n-2$ and for some special values of m and n, see [HL82], [HM86] and [F88]. If $m = 2$, then the faces of the unit mass ball are as follows: let us choose an even-dimensional subspace $U \subset \mathbb{R}^m$ and an orthogonal complex structure on U, thus identifying $U = \mathbb{C}^{2k}$ for some k. Then the corresponding face of $\operatorname{conv}(G_m(\mathbb{R}^n))$ is the convex hull of all oriented planes in U identified with complex lines in $\mathbb{C}^{2k}$.

In general, it appears to be difficult to describe the facial structure of the unit mass ball. The authors do not know the complexity status of the *membership problem* for the unit mass ball:

given a point $x \in \bigwedge^m \mathbb{R}^n$, decide if it lies in $\operatorname{conv}(G_m(\mathbb{R}^n))$,

but suspect that the problem is NP-hard if $m \geq 3$ is fixed and n is allowed to grow.

The above examples suggest that the boundary of B and B° can get very complicated, so there is little hope in understanding the combinatorics (the facial structure) of general convex hulls of orbits and their polars. Instead, we study metric properties of convex hulls. Our approach is through approximation of a complicated convex body by a simpler one.

As is known, every convex body contains a unique ellipsoid $E_{\max}$ of the maximum volume and is contained in a unique ellipsoid $E_{\min}$ of the minimum volume, see [B97]. Thus ellipsoids $E_{\max}$ and $E_{\min}$ provide reasonable "first approximations" to a convex body.

The main result of Section 2 is Theorem 2.4 which states that the maximum volume ellipsoid of B° consists of the linear functionals $\ell : V \longrightarrow \mathbb{R}$ such that the average value of ℓ^2 on the orbit does not exceed $(\dim V)^{-1}$. We compute the minimum- and maximum- volume ellipsoids of the symmetric Traveling Salesman Polytope, which both turn out to be balls under the "natural" Euclidean metric and ellipsoid $E_{\min}$ of the asymmetric Traveling Salesman Polytope, which turns out to be slightly stretched in the direction of the skew-symmetric matrices. As an immediate corollary of Theorem 2.4, we obtain the description of the maximum volume ellipsoid of the set of non-negative polynomials (Example 1.2), as a ball of radius $\left(\binom{n+2k-1}{2k} - 1\right)^{-1/2}$ in the L^2-metric. We also compute the minimum volume ellipsoid of the convex hull of the Grassmannian and hence the maximum volume ellipsoid of the unit comass ball (Example 1.3).

In Section 3, we obtain some inequalities which allow us to approximate the maximum value of a linear functional ℓ on the orbit by an L^p-norm of ℓ. We apply those inequalities in Section 4. We obtain a reasonably tight estimate of the radius of the Euclidean ball containing the unit comass ball and show that the classical Kähler and special Lagrangian faces of the Grassmannian, are, in fact, rather "shallow" (Example 1.3). Also, we review (with some proofs and some sketches) the recent results of [Bl03], which show that for most values of n and k the set of non-negative n-variate polynomials of degree $2k$ is much larger than its subset consisting of the sums of squares of polynomials of degree k.

2. Approximation by Ellipsoids

Let $B \subset V$ be a convex body in a finite-dimensional real vector space. We assume that $\dim B = \dim V$. Among all ellipsoids contained in B there is a unique ellipsoid $E_{\max}$ of the maximum volume, which we call the maximum volume ellipsoid of B and which is also called the John ellipsoid of B or the Löwner-John ellipsoid of B. Similarly, among all ellipsoids containing B there is a unique ellipsoid $E_{\min}$ of the minimum volume, which we call the minimum volume ellipsoid of B and which is also called the Löwner or the Löwner-John ellipsoid. The maximum and minimum volume ellipsoids of B do not depend on the volume form chosen in V, they are intrinsic to B.

Assuming that the center of E_{max} is the origin, we have

$$E_{\text{max}} \subset B \subset (\dim B)\, E_{\text{max}}.$$

If B is symmetric about the origin, that is, if $B = -B$ then the bound can be strengthened:

$$E_{\text{max}} \subset B \subset \left(\sqrt{\dim B}\right) E_{\text{max}}.$$

More generally, let us suppose that E_{max} is centered at the origin. The *symmetry coefficient* of B with respect to the origin is the largest $\alpha > 0$ such that $-\alpha B \subset B$. Then we have

$$E_{\text{max}} \subset B \subset \left(\sqrt{\frac{\dim B}{\alpha}}\right) E_{\text{max}},$$

where α is the symmetry coefficient of B with respect to the origin.

Similarly, assuming that E_{min} is centered at the origin, we have

$$(\dim B)^{-1}\, E_{\text{min}} \subset B \subset E_{\text{min}}.$$

If, additionally, α is the symmetry coefficient of B with respect to the origin, then

$$\left(\sqrt{\frac{\alpha}{\dim B}}\right) E_{\text{min}} \subset B \subset E_{\text{min}}.$$

In particular, if B is symmetric about the origin, then

$$(\dim B)^{-1/2}\, E_{\text{min}} \subset B \subset E_{\text{min}}.$$

These, and other interesting properties of the minimum- and maximum- volume ellipsoids can be found in [B97], see also the original paper [J48], [Bl03], and Chapter V of [Ba02b].

Suppose that a compact group G acts in V by linear transformations and that B is invariant under the action: $gB = B$ for all $g \in G$. Let $\langle \cdot, \cdot \rangle$ be a G-invariant scalar product in V, so G acts in V by isometries. Since the ellipsoids E_{max} and E_{min} associated with B are unique, they also have to be invariant under the action of G. If the group of symmetries of B is sufficiently rich, we may be able to describe E_{max} or E_{min} precisely.

The following simple observation will be used throughout this section. Let us suppose that the action of G in V is irreducible: if $W \subset V$ is a G-invariant subspace, then either $W = \{0\}$ or $W = V$. Then, the ellipsoids E_{max} and E_{min} of a G-invariant convex body B are necessarily balls centered at the origin:

$$E_{\text{max}} = \left\{x \in V:\ \langle x, x\rangle \leq r^2\right\} \quad \text{and} \quad E_{\text{min}} = \left\{x \in V:\ \langle x, x\rangle \leq R^2\right\}$$

for some $r, R > 0$.

Indeed, since the action of G is irreducible, the origin is the only G-invariant point and hence both $E_{\max}$ and $E_{\min}$ must be centered at the origin. Furthermore, an ellipsoid $E \subset V$ centered at the origin is defined by the inequality $E = \{x : q(x) \leq 1\}$, where $q : V \longrightarrow \mathbb{R}$ is a positive definite quadratic form. If E is G-invariant, then $q(gx) = q(x)$ for all $g \in G$ and hence the eigenspaces of q must be G-invariant. Since the action of G is irreducible, there is only one eigenspace which coincides with V, from which $q(x) = \lambda\langle x, x\rangle$ for some $\lambda > 0$ and all $x \in V$ and E is a ball.

This simple observation allows us to compute ellipsoids $E_{\max}$ and $E_{\min}$ of the Symmetric Traveling Salesman Polytope (Example 1.1).

(2.1) Example: The minimum and maximum volume ellipsoids of the symmetric Traveling Salesman Polytope. In this case, V is the space of $n \times n$ real matrices, on which the symmetric group S_n acts by simultaneous permutations of rows and columns, see Example 1.1. Let us introduce an S_n-invariant scalar product by

$$\langle a, b\rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij} \quad \text{for} \quad a = (a_{ij}) \quad \text{and} \quad b = (b_{ij})$$

and the corresponding Euclidean norm $\|a\| = \sqrt{\langle a, a\rangle}$. It is not hard to see that the affine hull of the symmetric Traveling Salesman Polytope ST_n consists of the symmetric matrices with 0 diagonal and row and column sums equal to 2, from which one can deduce the formula $\dim ST_n = (n^2 - 3n)/2$. Let us make the affine hull of ST_n a vector space by choosing the origin at $c = (c_{ij})$ with $c_{ij} = 2/(n-1)$ for $i \neq j$ and $c_{ii} = 0$, the only fixed point of the action. One can see that the action of S_n on the affine hull of ST_n is irreducible and corresponds to the Young diagram $(n-2, 2)$, see, for example, Chapter 4 of [FH91].

Hence the maximum- and minimum- volume ellipsoids of ST_n must be balls in the affine hull of ST_n centered at c. Moreover, since the boundary of the minimum volume ellipsoid $E_{\min}$ must contain the vertices of ST_n, we conclude that the radius of the ball representing $E_{\min}$ is equal to $\sqrt{2n(n-3)/(n-1)}$.

One can compute the symmetry coefficient of ST_n with respect to the center c. Suppose that $n \geq 5$. Let us choose a vertex v of ST_n and let us consider the functional $\ell(x) = \langle v - c, x - c\rangle$ on ST_n. The maximum value of $2n(n-3)/(n-1)$ is attained at $x = v$ while the minimum value of $-4n/(n-1)$ is attained at the face F_v of ST_n with the vertices h such that $\langle v, h\rangle = 0$ (combinatorially, h correspond to Hamiltonian cycles in the graph obtained from the complete graph on n vertices by deleting the edges of the Hamiltonian cycle encoded by v). Moreover, one can show that for $\lambda = 2/(n-3)$, we have $-\lambda(v-c) + c \in F_v$. This implies that the coefficient of symmetry of ST_n with respect to c is equal to $2/(n-3)$. Therefore ST_n contains the ball centered at c and of the radius $\sqrt{8/\big((n-1)(n-3)\big)}$ (for $n \geq 5$).

One can observe that the ball centered at c and of the radius $\sqrt{8/\big((n-1)(n-3)\big)}$ touches the boundary of ST_n. Indeed, let $b=(b_{ij})$ be the centroid of the set of vertices x of ST_n with $x_{12}=x_{21}=0$. Then

$$b_{ij}=\begin{cases} 0 & \text{if } 1 \leq i, j \leq 2 \\ \frac{2}{n-2} & \text{if } i=1,2 \text{ and } j>2 \quad \text{or} \quad j=1,2 \text{ and } i>2 \\ \frac{2(n-4)}{(n-2)(n-3)} & \text{if } i, j \geq 3, \end{cases}$$

and the distance from c to b is precisely $\sqrt{8/\big((n-1)(n-3)\big)}$.

Hence for $n \geq 5$ the maximum volume ellipsoid $E_{\max}$ is the ball centered at c of the radius $\sqrt{8/\big((n-1)(n-3)\big)}$.

Some bounds on the radius of the largest inscribed ball for a polytope from a particular family of combinatorially defined polytopes are computed in [V95]. The family of polytopes includes the symmetric Traveling Salesman Polytope, although in its case the bound from [V95] is not optimal.

If the action of G in the ambient space V is not irreducible, the situation is more complicated. For one thing, there is more than one (up to a scaling factor) G-invariant scalar product, hence the notion of a "ball" is not really defined. However, we are still able to describe the minimum volume ellipsoid of the convex hull of an orbit.

Without loss of generality, we assume that the orbit $\{gv: \ g \in G\}$ spans V affinely. Let $\langle \cdot, \cdot \rangle$ be a G-invariant scalar product in V. As is known, V can be decomposed into the direct sum of pairwise orthogonal invariant subspaces V_i, such that the action of G in each V_i is irreducible. It is important to note that the decomposition is *not* unique: non-uniqueness appears when some of V_i are isomorphic, that, is, when there exists an isomorphism $V_i \longrightarrow V_j$ which commutes with G. If the decomposition is unique, we say that the action of G is *multiplicity-free.*

Since the orbit spans V affinely, the orthogonal projection v_i of v onto each V_i must be non-zero (if $v_i=0$ then the orbit lies in $V_i^{\perp}$). Also, the origin in V must be the only invariant point of the action of G (otherwise, the orbit is contained in the hyperplane $\langle x, u\rangle = \langle v, u \rangle$, where $u \in V$ is a non-zero vector fixed by the action of G).

(2.2) Theorem. *Let B be the convex hull of the orbit of a vector $v \in V$:*

$$B=\operatorname{conv}\Big(gv: \ g \in G\Big).$$

Suppose that the affine hull of B is V.

Then there exists a decomposition

$$V=\bigoplus_i V_i$$

of V into the direct sum of pairwise orthogonal irreducible components with the following properties.

The minimum volume ellipsoid $E_{\min}$ of B is defined by the inequality

$$E_{\min} = \Big\{x : \quad \sum_i \frac{\dim V_i}{\dim V} \cdot \frac{\langle x_i, x_i\rangle}{\langle v_i, v_i\rangle} \leq 1\Big\}, \tag{2.2.1}$$

where x_i (resp. v_i) is the orthogonal projection of x (resp. v) onto V_i.

We have

$$\int_G \langle x, gv\rangle^2 \ dg = \sum_i \frac{\langle x_i, x_i\rangle\langle v_i, v_i\rangle}{\dim V_i} \quad \textit{for all} \quad x \in V, \tag{2.2.2}$$

where dg is the Haar probability measure on G.

Proof. Let us consider the quadratic form $q : V \longrightarrow \mathbb{R}$ defined by

$$q(x) = \int_G \langle x, gv\rangle^2 \ dg.$$

We observe that q is G-invariant, that is, $q(gx) = q(x)$ for all $x \in V$ and all $g \in G$. Therefore, the eigenspaces of q are G-invariant. Writing the eigenspaces as direct sums of pairwise orthogonal invariant subspaces where the action of G is irreducible, we obtain a decomposition $V = \bigoplus_i V_i$ such that

$$q(x) = \sum_i \lambda_i \langle x_i, x_i\rangle \quad \text{for all} \quad x \in V$$

and some $\lambda_i \geq 0$. Recall that $v_i \neq 0$ for all i since the orbit $\{gv : g \in G\}$ spans V affinely.

To compute λ_i, we substitute $x \in V_i$ and observe that the trace of

$$q_i(x) = \int_G \langle x, gv_i\rangle^2 \ dg$$

as a quadratic form $q_i : V_i \longrightarrow \mathbb{R}$ is equal to $\langle v_i, v_i\rangle$. Hence we must have $\lambda_i = \langle v_i, v_i\rangle / \dim V_i$, which proves (2.2.2), cf. [Ba02a].

We will also use the polarized form of (2.2.2):

$$\int_G \langle x, gv\rangle\langle y, gv\rangle \ dg = \sum_i \frac{\langle x_i, y_i\rangle\langle v_i, v_i\rangle}{\dim V_i}, \tag{2.2.3}$$

obtained by applying (2.2.2) to $q(x + y) - q(x) - q(y)$.

Next, we observe that the ellipsoid E defined by the inequality (2.2.1) contains the orbit $\big\{gv : \ g \in G\big\}$ on its boundary and hence contains B.

Our goal is to show that E is the minimum volume ellipsoid. It is convenient to introduce a new scalar product:

$$(a, b) = \sum_i \frac{\dim V_i}{\dim V} \cdot \frac{\langle a_i, b_i \rangle}{\langle v_i, v_i \rangle} \quad \text{for all} \quad a, b \in V.$$

Obviously $(\cdot, \cdot)$ is a G-invariant scalar product. Furthermore, the ellipsoid E defined by (2.2.1) is the unit ball in the scalar product $(\cdot, \cdot)$.

Now,

$$(c, gv) = \sum_i \frac{\dim V_i}{\dim V} \cdot \frac{\langle c_i, gv \rangle}{\langle v_i, v_i \rangle}$$

and hence

$$(c, gv)^2 = \sum_{i,j} \frac{(\dim V_i)(\dim V_j)}{(\dim V)^2} \cdot \frac{\langle c_i, gv \rangle \langle c_j, gv \rangle}{\langle v_i, v_i \rangle^2}.$$

Integrating and using (2.2.3), we get

$$\int_G (c, gv)^2 \, dg = \frac{1}{\dim V} \sum_i \frac{\dim V_i}{\dim V} \cdot \frac{\langle c_i, c_i \rangle}{\langle v_i, v_i \rangle} = \frac{(c, c)}{\dim V}. \tag{2.2.4}$$

Since the origin is the only fixed point of the action of G, the minimum volume ellipsoid should be centered at the origin.

Let $e_1, \ldots, e_k$ for $k = \dim V$ be an orthonormal basis with respect to the scalar product $(\cdot, \cdot)$. Suppose that $E' \subset V$ is an ellipsoid defined by

$$E' = \Big\{ x \in V : \quad \sum_{j=1}^k \frac{(x, e_j)^2}{\alpha_j^2} \leq 1 \Big\}$$

for some $\alpha_1, \ldots, \alpha_k > 0$. To show that E is the minimum volume ellipsoid, it suffices to show that as long as E' contains the orbit $\big\{ gv : \ g \in G \big\}$, we must have $\operatorname{vol} E' \geq \operatorname{vol} E$, which is equivalent to $\alpha_1 \cdots \alpha_k \geq 1$.

Indeed, since $gv \in E'$, we must have

$$\sum_{j=1}^k \frac{(e_j, gv)^2}{\alpha_j^2} \leq 1 \quad \text{for all} \quad g \in G.$$

Integrating, we obtain

$$\sum_{j=1}^k \frac{1}{\alpha_j^2} \int_G (e_j, gv)^2 \, dg \leq 1.$$

Applying (2.2.4), we get

$$\frac{1}{\dim V} \sum_{j=1}^k \frac{1}{\alpha_j^2} \leq 1.$$

Since $k = \dim V$, from the inequality between the arithmetic and geometric means, we get that $\alpha_1 \ldots \alpha_k \geq 1$, which completes the proof. $\square$

Remark. We note that in the part of the proof where we compare the volumes of E' and E, we reproduce the "sufficiency" (that is, "the easy") part of John's criterion for optimality of an ellipsoid, cf., for example, [B97].

Theorem 2.2 allows us to compute the minimum volume ellipsoid of the asymmetric Traveling Salesman Polytope, see Example 1.1.

(2.3) Example: the minimum volume ellipsoid of the asymmetric Traveling Salesman Polytope. In this case (see Examples 1.1 and 2.1), V is the space of $n \times n$ matrices with the scalar product and the action of the symmetric group S_n defined as in Example 2.1. On can observe that the affine hull of AT_n consists of the matrices with zero diagonal and row and column sums equal to 1, from which one can deduce the formula $\dim AT_n = n^2 - 3n + 1$.

The affine hull of AT_n is S_n-invariant. We make the affine hull of AT_n a vector space by choosing the origin at $c = (c_{ij})$ with $c_{ij} = 1/(n-1)$ for $i \neq j$ and $c_{ii} = 0$, the only fixed point of the action. The action of S_n on the affine hull of AT_n is reducible and multiplicity-free, so there is no ambiguity in choosing the irreducible components. The affine hull is the sum of two irreducible invariant subspaces V_s and V_a.

Subspace V_s consists of the matrices $x + c$, where x is a symmetric matrix with zero diagonal and zero row and column sums. One can see that the action of S_n in V_s is irreducible and corresponds to the Young diagram $(n-2, 2)$, see, for example, Chapter 4 of [FH91]. We have $\dim V_s = (n^2 - 3n)/2$

Subspace V_s consists of the matrices matrices $x+c$, where x is a skew-symmetric matrix with zero row and column sums. One can see that the action of S_n in V_s is irreducible and corresponds to the Young diagram $(n-2, 1, 1)$, see, for example, Chapter 4 of [FH91]. We have $\dim V_s = (n-1)(n-2)/2$.

The orthogonal projection onto V_s is defined by $x \longmapsto (x + x^t)/2$, while the orthogonal projection onto V_a is defined by $x \longmapsto (x - x^t)/2 + c$.

Applying Theorem 2.2, we conclude that the minimum volume ellipsoid of AT_n is defined in the affine hull of AT_n by the inequality:

$$\begin{gathered}(n-1) \sum_{1 \leq i \neq j \leq n} \left(\frac{x_{ij} + x_{ji}}{2} - \frac{1}{n-1} \right)^2 \\ + \frac{(n-1)(n-2)}{n} \sum_{1 \leq i \neq j \leq n} \left(\frac{x_{ij} - x_{ji}}{2} \right)^2 \leq n^2 - 3n + 1.\end{gathered}$$

Thus one can say that the minimum volume ellipsoid of the asymmetric Traveling Salesman Polytope is slightly stretched in the direction of skew-symmetric matrices.

The dual version of Theorem 2.2 is especially simple.

(2.4) Theorem. *Let G be a compact group acting in a finite-dimensional real vector space V. Let B be the convex hull of the orbit of a vector $v \in V$:*

$$B = \operatorname{conv}\Big(gv : \ g \in G\Big).$$

Suppose that the affine hull of B is V.

Let V^* be the dual to V and let

$$B^\circ = \Big\{\ell \in V^* : \quad \ell(x) \leq 1 \quad \textit{for all} \quad x \in B\Big\}$$

be the polar of B. Then the maximum volume ellipsoid of B° is defined by the inequality

$$E_{\max} = \Big\{\ell \in V^* : \quad \int_G \ell^2(gv)\ dg \leq \frac{1}{\dim V}\Big\}.$$

Proof. Let us introduce a G-invariant scalar product $\langle \cdot, \cdot \rangle$ in V, thus identifying V and V^*. Then

$$B^\circ = \Big\{c \in V : \quad \langle c, gv \rangle \leq 1 \quad \text{for all} \quad g \in G\Big\}.$$

Since the origin is the only point fixed by the action of G, the maximum volume ellipsoid $E_{\max}$ of B° is centered at the origin. Therefore, $E_{\max}$ must be the polar of the minimum volume ellipsoid of B.

Let $V = \bigoplus_i V_i$ be the decomposition of Theorem 2.2. Since $E_{\max}$ is the polar of the ellipsoid $E_{\min}$ associated with B, from (2.2.1), we get

$$E_{\max} = \Big\{c : \quad \dim V \sum_i \frac{\langle c_i, c_i \rangle \langle v_i, v_i \rangle}{\dim V_i} \leq 1\Big\}.$$

Applying (2.2.2), we get

$$E_{\max} = \Big\{c : \quad \int_G \langle c, gv \rangle^2\ dg \leq \frac{1}{\dim V}\Big\},$$

which completes the proof. $\square$

Remark. Let G be a compact group acting in a finite-dimensional real vector space V and let $v \in V$ be a point such that the orbit $\big\{gv : \ g \in V\big\}$ spans V affinely. Then the dual space V^* acquires a natural scalar product

$$\langle \ell_1, \ell_2 \rangle = \int_G \ell_1(gv)\ell_2(gv)\ dg$$

induced by the scalar product in $L^2(G)$. Theorem 2.4 states that the maximum volume ellipsoid of the polar of the orbit is the ball of radius $(\dim V)^{-1/2}$ in this scalar product.

By duality, V acquires the dual scalar product (which we denote below by $\langle \cdot, \cdot \rangle$ as well). It is a constant multiple of the product $(\cdot, \cdot)$ introduced in the proof of Theorem 2.2: $\langle u_1, u_2 \rangle = (\dim V)(u_1, u_2)$. We have $\langle v, v \rangle = \dim V$ and the minimum volume ellipsoid of the convex hull of the orbit of v is the ball of radius $\sqrt{\dim V}$.

As an immediate application of Theorem 2.4, we compute the maximum volume ellipsoid of the set of non-negative polynomials, see Example 1.2.

(2.5) Example: the maximum volume ellipsoid of the set of non-negative polynomials. In this case, $U^*_{2k,n}$ is the space of all homogeneous polynomials $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ of degree $2k$ with the zero average on the unit sphere $\mathbb{S}^{n-1}$, so $\dim U^*_{2k,n} = \binom{n+2k-1}{2k} - 1$. We view such a polynomial p as a linear functional ℓ on an orbit $\{gv : g \in G\}$ in the action of the orthogonal group $G = SO(n)$ in $(\mathbb{R}^n)^{\otimes 2k}$ and the shifted set $Pos'_{2k,n}$ of non-negative polynomials as the negative polar $-B^\circ$ of the orbit, see Example 1.2. In particular, under this identification $p \longleftrightarrow \ell$, we have

$$\int_{\mathbb{S}^{n-1}} p^2(x)\ dx = \int_G \ell^2(gv)\ dg,$$

where dx and dg are the Haar probability measures on $\mathbb{S}^{n-1}$ and $SO(n)$ respectively.

Applying Theorem 2.4 to $-B^\circ$, we conclude that the maximum volume ellipsoid of $-B^\circ = Pos'_{2k,n}$ consists of the polynomials p such that

$$\int_{\mathbb{S}^{n-1}} p(x)\ dx = 0 \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} p^2(x)\ dx \leq \left(\binom{n+2k-1}{2k} - 1 \right)^{-1}.$$

Consequently, the maximum volume ellipsoid of $Pos_{2k,n}$ consists of the polynomials p such that

$$\int_{\mathbb{S}^{n-1}} p(x)\ dx = 1 \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} \left(p(x) - 1\right)^2\ dx \leq \left(\binom{n+2k-1}{2k} - 1 \right)^{-1}.$$

Geometrically, the maximum volume ellipsoid of $Pos_{2k,n}$ can be described as follows. Let us introduce a scalar product in the space of polynomials by

$$\langle f,\ g \rangle = \int_{\mathbb{S}^{n-1}} f(x) g(x)\ dx,$$

where dx is the rotation invariant probability measure, as above. Then the maximum volume ellipsoid of $Pos_{2k,n}$ is the ball centered at $r(x) = (x_1^2 + \ldots + x_n^2)^k$ and of the radius $\left(\binom{n+2k-1}{2k} - 1 \right)^{-1/2}$. This result was first obtained by more direct and complicated computations in [Bl02]. In the same paper, G. Blekherman also determined the coefficient of symmetry of $Pos_{2k,n}$ (with respect to the center r), it turns out to be equal to $\left(\binom{n+k-1}{k}-1\right)^{-1}$.

It follows then that $Pos_{2k,n}$ is contained in the ball centered at r and of the radius $\left(\binom{n+k-1}{k}-1\right)^{1/2}$. This estimate is poor if k is fixed and n is allowed to grow: as follows from results of Duoandikoetxea [D87], for any fixed k, the set $Pos_{2k,n}$ is contained in a ball of a fixed radius, as n grows. However, the estimate gives the right logarithmic order if $k \gg n$, which one can observe by inspecting a polynomial $p \in Pos_{2k,n}$ that is the $2k$-th power of a linear function.

We conclude this section by computing the the minimum volume ellipsoid of the convex hull of the Grassmannian and, consequently, the maximum volume ellipsoid of the unit comass ball, see Example 1.3.

(2.6) Example: the minimum volume ellipsoid of the convex hull of the Grassmannian. In this case, $V_{m,n} = \bigwedge^m \mathbb{R}^n$ with the orthonormal basis $e_I = e_{i_1} \wedge \ldots \wedge e_{i_m}$, where I is an m-subset $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ of the set $\{1, \ldots, n\}$ and $e_1, \ldots, e_n$ is the standard orthonormal basis of $\mathbb{R}^n$.

Let $\langle \cdot, \cdot \rangle$ be the corresponding scalar product in $V_{m,n}$, so that

$$\langle a, b \rangle = \sum_I a_I b_I,$$

where I ranges over all m-subsets of $\{1, \ldots, n\}$. The scalar product allows us to identify $V_{m,n}^*$ with $V_{m,n}$. First, we find the maximum volume ellipsoid of the unit comass ball B°, that is the polar of the convex hull $B = \operatorname{conv}(G_m(\mathbb{R}^n))$ of the Grassmannian.

A linear functional $a \in V_{m,n}^* = V_{m,n}$ is defined by its coefficients a_I. To apply Theorem 2.4, we have to compute

$$\int_{SO(n)} \langle a, gv \rangle^2 \, dg = \int_{G_m(\mathbb{R}^n)} \langle a, x \rangle^2 \, dx,$$

where dx is the Haar probability measure on the Grassmannian $G_m(\mathbb{R}^n)$. We note that

$$\int_{G_m(\mathbb{R}^n)} \langle e_I, x \rangle \langle e_J, x \rangle \, dx = 0$$

for $I \neq J$, since for $i \in I \setminus J$, the reflection $e_i \longmapsto -e_i$ of $\mathbb{R}^n$ induces an isometry of $V_{m,n}$, which maps $G_m(\mathbb{R}^n)$ onto itself, reverses the sign of $\langle e_I, x \rangle$ and does not change $\langle e_J, x \rangle$. Also,

$$\int_{G_m(\mathbb{R}^n)} \langle e_I, x \rangle^2 \, dx = \binom{n}{m}^{-1},$$

since the integral does not depend on I and $\sum_I \langle e_I, x \rangle^2 = 1$ for all $x \in G_m(\mathbb{R}^n)$.

By Theorem 2.4, we conclude that the maximum volume ellipsoid of the unit comass ball B° is defined by the inequality

$$E_{\max} = \Big\{a \in V_{m,n} : \quad \sum_I a_I^2 \leq 1\Big\},$$

that is, the unit ball in the Euclidean metric of $V_{m,n}$. Since B° is centrally symmetric, we conclude that B° is contained in the ball of radius $\binom{n}{m}^{1/2}$. As follows from Theorem 4.1, this estimate is optimal up to a factor of $\big(mn \ln(m+1)\big)^{1/2}$.

Consequently, the convex hull B of the Grassmannian is contained in the unit ball of $V_{m,n}$, which is the minimum volume ellipsoid of B, and contains a ball of radius $\binom{n}{m}^{-1/2}$. Again, the estimate of the radius of the inner ball is optimal up to a factor of $\big(mn \ln(m+1)\big)^{1/2}$.

3. Higher Order Estimates

The following construction can be used to get a better understanding of metric properties of an orbit $\big\{gv : \ g \in G\big\}$. Let us choose a positive integer k and let us consider the k-th tensor power

$$V^{\otimes k} = \underbrace{V \otimes \ldots \otimes V}_{k \text{ times}}.$$

The group G acts in $V^{\otimes k}$ by the k-th tensor power of its action in V: on decomposable tensors we have

$$g(v_1 \otimes \ldots \otimes v_k) = g(v_1) \otimes \ldots \otimes g(v_k).$$

Let us consider the orbit $\big\{gv^{\otimes k} : \ g \in G\big\}$ for

$$v^{\otimes k} = \underbrace{v \otimes \ldots \otimes v}_{k \text{ times}}.$$

Then, a linear functional on the orbit of $v^{\otimes k}$ is a polynomial of degree k on the orbit of v and hence we can extract some new "higher order" information about the orbit of v by applying already developed methods to the orbit of $v^{\otimes k}$. An important observation is that the orbit $\big\{gv^{\otimes k} : \ g \in G\big\}$ lies in the symmetric part of $V^{\otimes k}$, so the dimension of the affine hull of the orbit of $v^{\otimes k}$ does not exceed $\binom{\dim V + k - 1}{k}$.

(3.1) Theorem. *Let G be a compact group acting in a finite-dimensional real vector space V, let $v \in V$ be a point, and let $\ell : V \longrightarrow \mathbb{R}$ be a linear functional. Let us define*

$$f : G \longrightarrow \mathbb{R} \qquad by \qquad f(g) = \ell(gv).$$

For an integer $k > 0$, let d_k be the dimension of the subspace spanned by the orbit $\{gv^{\otimes k} : \ g \in G\}$ in $V^{\otimes k}$. In particular, $d_k \leq \binom{\dim V + k - 1}{k}$. Let

$$\|f\|_{2k} = \left(\int_G f^{2k}(g) \ dg \right)^{\frac{1}{2k}}.$$

(1) *Suppose that k is odd and that*

$$\int_G f^k(g) \ dg = 0.$$

Then

$$d_k^{-\frac{1}{2k}} \|f\|_{2k} \leq \max_{g \in G} f(g) \leq d_k^{\frac{1}{2k}} \|f\|_{2k}.$$

(2) *We have*

$$\|f\|_{2k} \leq \max_{g \in G} |f(g)| \leq d_k^{\frac{1}{2k}} \|f\|_{2k}.$$

Proof. Without loss of generality, we assume that $f \not\equiv 0$.

Let

$$B_k(v) = \operatorname{conv}\left(gv^{\otimes k} : \ g \in G\right)$$

be the convex hull of the orbit of $v^{\otimes k}$. We have $\dim B_k(v) \leq d_k$.

Let $\ell^{\otimes k} \in (V^*)^{\otimes k}$ be the k-th tensor power of the linear functional $\ell \in V^*$. Thus $f^k(g) = \ell^{\otimes k}\left(gv^{\otimes k}\right)$.

To prove Part (1), we note that since k is odd,

$$\max_{g \in G} f^k(g) = \left(\max_{g \in G} f(g) \right)^k.$$

Let

$$u = \int_G g\left(v^{\otimes k}\right) \ dg$$

be the center of $B_k(v)$. Since the average value of $f^k(g)$ is equal to 0, we have $\ell^{\otimes k}(u) = 0$ and hence $\ell^{\otimes k}(x) = \ell^{\otimes k}(x - u)$ for all $x \in V^{\otimes k}$. Let us translate $B_k(v)' = B_k(v) - u$ to the origin and let us consider the maximum volume ellipsoid E of the polar of $B_k(v)'$ in its affine hull. By Theorem 2.4, we have

$$E = \left\{ \mathcal{L} \in \left(V^{\otimes k}\right)^* : \quad \int_G \mathcal{L}^2\left(gv^{\otimes k} - u\right) \ dg \leq \frac{1}{\dim B_k(v)} \right\}.$$

Since the ellipsoid E is contained in the polar of $B_k(v)'$, for any linear functional $\mathcal{L} : V^{\otimes k} \longrightarrow \mathbb{R}$, the inequality

$$\int_G \mathcal{L}^2\left(gv^{\otimes k} - u\right) \ dg \leq \frac{1}{d_k} \leq \frac{1}{\dim B_k(v)}$$

implies the inequality

$$\max_{g \in G} \mathcal{L}\left(gv^{\otimes k} - u\right) \leq 1.$$

Choosing $\mathcal{L} = \lambda \ell^{\otimes k}$ with $\lambda = d_k^{-1/2} \|f\|_{2k}^{-k}$, we get the upper bound for $\max_{g \in G} f(g)$.

Since the ellipsoid $(\dim E)E$ contains the polar of $B_k(v)'$, for any linear functional $\mathcal{L} : V^{\otimes k} \longrightarrow \mathbb{R}$, the inequality

$$\max_{g \in G} \mathcal{L}\left(gv^{\otimes k} - u\right) \leq 1$$

implies the inequality

$$\int_G \mathcal{L}^2\left(gv^{\otimes k} - u\right) \, dg \leq \dim B_k(v) \leq d_k.$$

Choosing $\mathcal{L} = \lambda \ell^{\otimes k}$ with any $\lambda > \|f\|_{2k}^{-k} d_k^{1/2}$, we obtain the lower bound for $\max_{g \in G} f(g)$.

The proof of Part (2) is similar. We modify the definition of $B_k(v)$ by letting

$$B_k(v) = \operatorname{conv}\bigl(gv^{\otimes k}, -gv^{\otimes k} : \quad g \in G\bigr).$$

The set $B_k(v)$ so defined can be considered as the convex hull of an orbit of $G \times \mathbb{Z}_2$ and is centrally symmetric, so the ellipsoid $(\sqrt{\dim E})E$ contains the polar of $B_k(v)$.

Part (2) is also proven by a different method in [Ba02a]. □

Remark. Since $d_k \leq \binom{\dim V + k - 1}{k}$, the upper and lower bounds in Theorem 3.1 are asymptotically equivalent as long as $k^{-1} \dim V \longrightarrow 0$. In many interesting cases we have $d_k \ll \binom{\dim V + k - 1}{k}$, which results in stronger inequalities.

Polynomials on the unit sphere

As is discussed in Examples 1.2 and 2.5, the restriction of a homogeneous polynomial $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of degree m onto the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ can be viewed as the restriction of a linear functional $\ell : (\mathbb{R}^n)^{\otimes m} \longrightarrow \mathbb{R}$ onto the orbit of a vector $v = e^{\otimes m}$ for some $e \in \mathbb{S}^{n-1}$ in the action of the special orthogonal group $SO(n)$. In this case, $v^{\otimes k} = e^{\otimes mk}$ spans the symmetric part of $(\mathbb{R}^n)^{mk}$, so we have $d_k = \binom{n+mk-1}{mk}$ in Theorem 3.1.

Hence Part (1) of Theorem 3.1 implies that if f is an n-variate homogeneous polynomial of degree m such that

$$\int_{\mathbb{S}^{n-1}} f^k(x) \, dx = 0,$$

where dx is the rotation invariant probability measure on $\mathbb{S}^{n-1}$, then

$$\binom{n + mk - 1}{mk}^{-\frac{1}{2k}} \|f\|_{2k} \leq \max_{x \in \mathbb{S}^{n-1}} f(x) \leq \binom{n + mk - 1}{mk}^{\frac{1}{2k}} \|f\|_{2k},$$

where

$$\|f\|_{2k} = \left(\int_{\mathbb{S}^{n-1}} f^{2k}(x) \, dx\right)^{\frac{1}{2k}}.$$

We obtain the following corollary.

(3.2) Corollary. *Let us choose $k \geq n \ln(m+1)$. Then*

$$\|f\|_{2k} \leq \max_{x \in \mathbb{S}^{n-1}} |f(x)| \leq \alpha \|f\|_{2k},$$

for some absolute constant $\alpha > 0$ and all homogeneous polynomials $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of degree m.

Proof. Applying Part(2) of Theorem 3.1 as above, we conclude that for any homogeneous polynomial $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of degree m,

$$\|f\|_{2k} \leq \max_{x \in \mathbb{S}^{n-1}} |f(x)| \leq \binom{n+mk-1}{mk}^{\frac{1}{2k}} \|f\|_{2k}$$

(this inequality is also proven in [Ba02a]).

Let

$$H(x) = x \ln \frac{1}{x} + (1-x) \ln \frac{1}{1-x} \quad \text{for} \quad 0 \leq x \leq 1$$

be the entropy function. The result now follows from the estimate

$$\binom{a}{b} \leq \exp\big\{aH(b/a)\big\},$$

see, for example, Theorem 1.4.5 of [L99]. □

Our next application concerns calibrations, see Examples 1.3 and 2.6.

(3.3) Theorem. *Let $G_m(\mathbb{R}^n) \subset \bigwedge^m \mathbb{R}^n$ be the Plücker embedding of the Grassmannian of oriented m-subspaces of $\mathbb{R}^n$. Let $\ell : \bigwedge^m \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linear functional. Let*

$$\|\ell\|_{2k} = \left(\int_{G_m(\mathbb{R}^n)} \ell^{2k}(x) \ dx \right)^{\frac{1}{2k}},$$

where dx is the Haar probability measure on $G_m(\mathbb{R}^n)$. Then, for any positive integer k,

$$\|\ell\|_{2k} \leq \max_{x \in G_m(\mathbb{R}^n)} |\ell(x)| \leq (d_k)^{\frac{1}{2k}} \|\ell\|_{2k},$$

$$\textit{where} \quad d_k = \prod_{i=1}^{m} \prod_{j=1}^{k} \frac{n+j-i}{m+k-i-j+1}.$$

Proof. As we discussed in Example 1.3, the Grassmannian $G_m(\mathbb{R}^n)$ can be viewed as the orbit of $v = e_1 \wedge \ldots \wedge e_m$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, under the action of the special orthogonal group $SO(n)$ by the m-th exterior power of its defining representation in $\mathbb{R}^n$. We are going to apply Part (2) of Theorem 3.1 and for that we need to estimate the dimension of the subspace spanned by the orbit of $v^{\otimes k}$. First, we identify $\bigwedge^m \mathbb{R}^n$ with the subspace of skew-symmetric tensors in $(\mathbb{R}^n)^{\otimes m}$ and v with the point

$$\sum_{\sigma \in S_m} (\text{sgn } \sigma) e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(m)},$$

where S_m is the symmetric group of all permutations of $\{1, \ldots, m\}$.

Let us consider $W = (\mathbb{R}^n)^{\otimes mk}$. We introduce the right action of the symmetric group S_{mk} on W by permutations of the factors in the tensor product:

$$\big(u_1 \otimes \ldots \otimes u_{mk}\big)\sigma = u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(mk)}.$$

For $i = 1, \ldots, m$, let $R_i \subset S_{mk}$ be the subgroup permuting the numbers $1 \leq a \leq mk$ such that $a \equiv i \mod m$ and leaving all other numbers intact and for $j = 1, \ldots, k$, let $C_i \subset S_{mk}$ be the subgroup permuting the numbers $m(i-1)+1 \leq a \leq mi$ and leaving all other numbers intact.

Let $w = e_1 \otimes \ldots \otimes e_m$. Then

$$v^{\otimes k} = (k!)^{-m} w^{\otimes k} \left(\sum_{\sigma \in R_1 \times \ldots \times R_m} \sigma \right) \left(\sum_{\sigma \in C_1 \times \ldots \times C_k} (\text{sgn } \sigma)\sigma \right).$$

It follows then that $v^{\otimes k}$ generates the GL_n-module indexed by the rectangular $m \times k$ Young diagram, so its dimension d_k is given by the formula of the Theorem, see Chapter 6 of [FH91]. $\square$

(3.4) Corollary. *Under the conditions of Theorem 3.3, let $k \geq mn \ln(m+1)$. Then*

$$\|\ell\|_{2k} \leq \textit{comass of } \ell \leq \alpha \|\ell\|_{2k}$$

for some absolute constant $\alpha > 0$.

Proof. We have

$$\begin{aligned} d_k &\leq \prod_{i=1}^{m} \prod_{j=1}^{k} \frac{n+j-i}{k-j+1} \leq \left(\prod_{j=1}^{k} \frac{n+j-1}{k-j+1} \right)^m = \binom{n+k-1}{n-1}^m \\ &\leq \exp\left\{ m(n+k-1) H\left(\frac{n-1}{n+k-1} \right) \right\}, \end{aligned}$$

cf. Corollary 3.2. The proof now follows. $\square$

To understand the convex geometry of an orbit, we would like to compute the maximum value of a "typical" linear functional on the orbit. Theorem 3.1 allows us to replace the maximum value by an L^p norm. To estimate the average value of an L^p norm, we use the following simple computation.

(3.5) Lemma. *Let G be a compact group acting in a d-dimensional real vector space V endowed with a G-invariant scalar product $\langle \cdot, \cdot \rangle$ and let $v \in V$ be a point. Let $\mathbb{S}^{d-1} \subset V$ be the unit sphere endowed with the Haar probability measure dc. Then, for every positive integer k, we have*

$$\int_{\mathbb{S}^{d-1}} \left(\int_G \langle c, gv \rangle^{2k} \, dg \right)^{\frac{1}{2k}} \, dc \leq \sqrt{\frac{2k \langle v, v \rangle}{d}}.$$

Proof. Applying Hölder's inequality, we get

$$\int_{\mathbb{S}^{d-1}} \left(\int_G \langle c, gv \rangle^{2k} \, dg \right)^{\frac{1}{2k}} \, dc \leq \left(\int_{\mathbb{S}^{d-1}} \int_G \langle c, gv \rangle^{2k} \, dg \, dc \right)^{\frac{1}{2k}}.$$

Interchanging the integrals, we get

$$\int_{\mathbb{S}^{d-1}} \int_G \langle c, gv \rangle^{2k} \, dg \, dc = \int_G \left(\int_{\mathbb{S}^{d-1}} \langle c, gv \rangle^{2k} \, dc \right) \, dg. \tag{3.5.1}$$

Now we observe that the integral inside has the same value for all $g \in G$. Therefore, (3.5.1) is equal to

$$\int_{\mathbb{S}^{d-1}} \langle c, v \rangle^{2k} \, dc = \langle v, v \rangle^k \frac{\Gamma(d/2)\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k+d/2)},$$

see, for example, [Ba02a].

Now we use that $\Gamma(k+1/2) \leq \Gamma(k+1) \leq k^k$ and

$$\frac{\Gamma(d/2)}{\Gamma(k+d/2)} = \frac{1}{(d/2)(d/2+1)\cdots(d/2+k-1)} \leq (d/2)^{-k}.$$

□

4. Some Geometric Corollaries

The metric structure of the unit comass ball

Let $V_{m,n} = \bigwedge^m \mathbb{R}^n$ with the orthonormal basis $e_I = e_{i_1} \wedge \ldots \wedge e_{i_m}$, where I is an m-subset $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ of the set $\{1, \ldots, n\}$, and the corresponding scalar product $\langle \cdot, \cdot \rangle$. Let $G_m(\mathbb{R}^n) \subset V_{m,n}$ be the Plücker embedding of the Grassmannian of oriented m-subspaces of $\mathbb{R}^n$, let $B = \operatorname{conv}(G_m(\mathbb{R}^n))$ be the unit mass ball, and let $B^\circ \subset V^*_{m,n} = V_{m,n}$ be the unit comass ball, consisting of the linear functionals with the maximum value on $G_m(\mathbb{R}^n)$ not exceeding 1, see Examples 1.3 and 2.6.

The most well-known example of a linear functional $\ell : V_{m,n} \longrightarrow \mathbb{R}$ of comass 1 is given by an exterior power of the Kähler form. Let us suppose that m and n are even, so $m = 2p$ and $n = 2q$. Let

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4 + \ldots + e_{q-1} \wedge e_q \quad \text{and}$$
$$f = \frac{1}{p!} \underbrace{\omega \wedge \ldots \wedge \omega}_{p \text{ times}} \in V_{m,n}.$$

Then

$$\max_{x \in G_m(\mathbb{R}^n)} \langle f, x \rangle = 1,$$

and, moreover, the subspaces $x \in G_m(\mathbb{R}^n)$ where the maximum value 1 is attained look as follows. We identify $\mathbb{R}^n$ with $\mathbb{C}^q$ by identifying

$$\mathbb{R}e_1 \oplus \mathbb{R}e_2 = \mathbb{R}e_3 \oplus \mathbb{R}e_4 = \ldots = \mathbb{R}e_{q-1} \oplus \mathbb{R}e_q = \mathbb{C}.$$

Then the subspaces $x \in G_m(\mathbb{R}^n)$ with $\langle f, x \rangle = 1$ are exactly those identified with the complex p-dimensional subspaces of $\mathbb{C}^q$, see [HL82].

We note that the Euclidean length $\langle f, f \rangle^{1/2}$ of f is equal to $\binom{q}{p}^{1/2}$. In particular, if $m = 2p$ is fixed and $n = 2q$ grows, the length of f grows as $n^{p/2} = n^{m/4}$.

Another example is provided by the special Lagrangian calibration a. In this case, $n = 2m$ and

$$a = \operatorname{Re}\, (e_1 + ie_2) \wedge \ldots \wedge (e_{2m-1} + ie_{2m}).$$

The length $\langle a, a \rangle^{1/2}$ of a is $\left(\sum_{j \leq m/2} \binom{m}{2j} \right)^{1/2}$. The maximum value of $\langle a, x \rangle$ for $x \in G_m(\mathbb{R}^n)$ is 1 and it is attained on the "special Lagrangian subspaces", see [HL82].

The following result shows that there exist calibrations with a much larger Euclidean length than that of the power f of the Kähler form or the special Lagrangian calibration a.

(4.1) Theorem.

(1) *Let $c \in V_{m,n}$ be a vector such that*

$$\max_{x \in G_m(\mathbb{R}^n)} \langle c, x \rangle = 1.$$

Then

$$\langle c, c \rangle \leq \binom{n}{m}.$$

(2) *There exists* $c \in V_{m,n}$ *such that*

$$\max_{x \in G_m(\mathbb{R}^n)} \langle c, x \rangle = 1$$

and

$$\langle c, c \rangle \geq \alpha \big(nm \ln(m+1)\big)^{-1} \binom{n}{m},$$

where $\alpha > 0$ *is an absolute constant.*

Proof. Part (1) follows since the convex hull of the Grassmannian contains a ball of radius $\binom{n}{m}^{-1/2}$, see Example 2.6.

To prove Part (2), let us choose $k = \lceil mn \ln(m+1) \rceil$ in Lemma 3.5. Then, by Corollary 3.4, the maximum value of $\langle c, x \rangle$ for $x \in G_m(\mathbb{R}^n)$ is approximated by

$$\left(\int_{G_m(\mathbb{R}^n)} \langle c, x \rangle^{2k} \, dx \right)^{\frac{1}{2k}}$$

within a constant factor. We apply Lemma 3.5 with $V = V_{m,n}$, $d = \binom{n}{m}$, $G = SO(n)$, and $v = e_1 \wedge \ldots \wedge e_m$. Hence $\langle v, v \rangle = 1$ and there exists $c \in V_{m,n}$ with $\langle c, c \rangle = 1$ and such that

$$\left(\int_{G_m(\mathbb{R}^n)} \langle c, x \rangle^{2k} \, dx \right)^{\frac{1}{2k}} \leq \sqrt{2k} \binom{n}{m}^{-1/2}.$$

Rescaling c to a comass 1 functional, we complete the proof of Part (2). $\square$

For $m = 2$ the estimate of Part (2) is exact up to an absolute constant, as witnessed by the Kähler calibration. However, for $m \geq 3$, the calibration c of Part (2) has a larger length than the Kähler or special Lagrangian calibrations. The gap only increases when m and n grow. The distance to the origin of the supporting hyperplane $\langle c, x \rangle = 1$ of the face of the convex hull of the Grassmannian is equal to $\langle c, c \rangle^{-1/2}$ so the faces defined by longer calibrations c are closer to the origin. Thus, the faces spanned by complex subspaces or the faces spanned by special Lagrangian subspaces are much more "shallow" than the faces defined by calibrations c in Part (2) of the Theorem. We do not know if those "deep" faces are related to any interesting geometry. Intuitively, the closer the face to the origin, the larger piece of the Grassmannian it contains, so it is quite possible that some interesting classes of manifolds are associated with the "long" calibrations c.

The volume of the set of non-negative polynomials

Let $U_{m,n}$ be the space of real homogeneous polynomials p of degree m in n variables such that the average value of p on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is 0, so $\dim U_{m,n} = \binom{n+m-1}{m} - 1$ for m even and $\dim U_{m,n} = \binom{n+m-1}{m}$ for m odd. As before, we make $U_{m,n}$ a Euclidean space with the L^2 inner product

$$\langle f, g \rangle = \int_{\mathbb{S}^{n-1}} f(x)g(x)\ dx.$$

We obtain the following corollary.

(4.2) Corollary. *Let $\Sigma_{m,n} \subset U_{m,n}$ be the unit sphere, consisting of the polynomials with L^2-norm equal to 1. For a polynomial $p \in U_{m,n}$, let*

$$\|p\|_\infty = \max_{x \in \mathbb{S}^{n-1}} |p(x)|.$$

Then

$$\int_{\Sigma_{m,n}} \|p\|_\infty\ dp \leq \alpha\sqrt{n \ln(m+1)}$$

for some absolute constant $\alpha > 0$.

Proof. Let us choose $k = \lceil n \ln(m+1) \rceil$. Then, by Corollary 3.2, $\|p\|_{2k}$ approximates $\|p\|_\infty$ within an absolute constant.

Now we use Lemma 3.5. As in Examples 1.2 and 2.5, we identify space $U_{m,n}$ with the space of linear functionals $\langle c, gv \rangle$ on the orbit $\{gv :\ g \in SO(n)\}$ of v. By the remark after the proof of Theorem 2.4, we have $\langle v, v \rangle = \dim U_{m,n}$. The proof now follows. $\square$

Thus the L^∞-norm of a typical n-variate polynomial of degree m of the unit L^2-norm in $U_{m,n}$ is $O\big(\sqrt{n \ln(m+1)}\big)$. In contrast, the L^∞ norm of a *particular* polynomial can be of the order of $n^{m/2}$, that is, substantially bigger.

Corollary 4.2 was used by the second author to obtain a bound on the volume of the set of non-negative polynomials.

Let us consider the shifted set $Pos'_{2k,n} \subset U_{2k,n}$ of non-negative polynomials defined by (1.2.2). We measure the size of a set $X \subset U_{2k,n}$ by the quantity $\left(\dfrac{\operatorname{vol} X}{\operatorname{vol} K}\right)^{1/d}$, where $d = \dim U_{2k,n}$ and K is the unit ball in $U_{2k,n}$, which is more "robust" than just the volume $\operatorname{vol} X$, as it takes into account the effect of a high dimension, cf. Chapter 6 of [P89].

The following result is from [Bl03], we made some trivial improvement in the dependence on the degree $2k$.

(4.3) Theorem. *Let $Pos'_{2k,n} \subset U_{2k,n}$ be the shifted set of non-negative polynomials, let $K \subset U_{2k,n}$ be the unit ball and let $d = \dim U_{2k,n} = \binom{n+2k-1}{2k} - 1$. Then*

$$\left(\frac{\operatorname{vol} Pos_{2k,n}}{\operatorname{vol} K}\right)^{1/d} \geq \alpha\big(n \ln(2k+1)\big)^{-1/2}$$

for some absolute constant $\alpha > 0$.

Proof. Let $\Sigma_{2k,n} \subset U_{2k,n}$ be the unit sphere. Let $p \in \Sigma_{2k,n}$ be a point. The ray $\lambda p : \lambda \geq 0$ intersects the boundary of $Pos'_{2k,n}$ at a point p_1 such that $\min_{x \in \mathbb{S}^{n-1}} p_1(x) = -1$, so the length of the interval $[0, p_1]$ is $|\min_{x \in \mathbb{S}^{n-1}} p(x)| \leq \|p\|_\infty$.

Hence

$$\left(\frac{\operatorname{vol} Pos'_{2k,n}}{\operatorname{vol} K}\right)^{1/d} = \left(\int_{\Sigma_{2k,n}} |\min_{x \in \mathbb{S}^{n-1}} p(x)|^{-d} \ dp\right)^{1/d} \geq \left(\int_{\Sigma_{2k,n}} \|p\|_\infty^{-d} \ dp\right)^{1/d}$$

$$\geq \int_{\Sigma_{2k,n}} \|p\|_\infty^{-1} \ dp \geq \left(\int_{\Sigma_{2k,n}} \|p\|_\infty \ dp\right)^{-1},$$

by the consecutive application of Hölder's and Jensen's inequalities, so the proof follows by Corollary 4.2. $\square$

We defined $Pos_{2k,n}$ as the set of non-negative polynomials with the average value 1 on the unit sphere, see (1.2.1). There is an important subset $Sq_{2k,n} \subset Pos_{2k,n}$, consisting of the polynomials that are sums of squares of homogeneous polynomials of degree k. It is known that $Pos_{2k,n} = Sq_{2k,n}$ if $k = 1$, $n = 2$, or $k = 2$ and $n = 3$, see Chapter 6 of [BCR98]. The following result from [Bl03] shows that, in general, $Sq_{2k,n}$ is a rather small subset of $Pos_{2k,n}$.

Translating $p \longmapsto p - (x_1^2 + \ldots + x_n^2)^k$, we identify $Sq_{2k,n}$ with a subset $Sq'_{2k,n}$ of $U_{2k,n}$.

(4.4) Theorem. *Let $Sq'_{2k,n} \subset U_{2k,n}$ be the shifted set of sums of squares, let $K \subset U_{2k,n}$ be the unit ball and let $d = \dim U_{2k,n} = \binom{n+2k-1}{2k} - 1$. Then*

$$\left(\frac{\operatorname{vol} Sq_{2k,n}}{\operatorname{vol} K}\right)^{1/d} \leq \alpha 2^{4k} \binom{n+k-1}{k}^{1/2} \binom{n+2k-1}{2k}^{-1/2}$$

for some absolute constant $\alpha > 0$.

In particular, if k is fixed and n grows, the upper bound has the form $c(k) n^{-k/2}$ for some $c(k) > 0$.

The proof is based on bounding the right hand side of the inequality of Theorem 4.4 by the average width of $Sq_{2k,n}$, cf. Section 6.2 of [S93]. The average width is represented by the integral

$$\int_{\Sigma_{2k,n}} \left(\max_{f \in \Sigma_{k,n}} \langle g, f^2 \rangle - \min_{f \in \Sigma_{k,n}} \langle g, f^2 \rangle\right) \ dg.$$

By Corollary 3.2, we can bound the integrand by

$$\alpha \left(\int_{\Sigma_{k,n}} \langle g, f^2 \rangle^{2q} \, df \right)^{\frac{1}{2q}}$$

for some absolute constant α and $q = \binom{n+k-1}{k}$ and proceed as in the proof of Lemma 3.5.

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