MAXIMUM ENTROPY GAUSSIAN APPROXIMATIONS FOR THE NUMBER OF INTEGER POINTS AND VOLUMES OF POLYTOPES

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We describe a maximum entropy approach for computing volumes and counting integer points in polyhedra. To estimate the number of points from a particular set $X \subset \mathbb{R}^n$ in a polyhedron $P \subset \mathbb{R}^n$, by solving a certain entropy maximization problem, we construct a probability distribution on the set X such that a) the probability mass function is constant on the set $P \cap X$ and b) the expectation of the distribution lies in P. This allows us to apply Central Limit Theorem type arguments to deduce computationally efficient approximations for the number of integer points, volumes, and the number of 0-1 vectors in the polytope. As an application, we obtain asymptotic formulas for volumes of multi-index transportation polytopes and for the number of multi-way contingency tables.

1. Introduction

In this paper, we address the problems of computing the volume and counting the number of integer points in a given polytope. These problems have a long history (see for example, surveys [GK94], [DL05] and [Ve05]) and, generally speaking, are computationally hard. We describe a maximum entropy approach which, in a number of non-trivial cases, allows one to obtain good quality approximations by solving certain specially constructed convex optimization problems on polytopes. Those optimization problems can be solved quite efficiently, in theory and in practice, by interior point methods, see [NN94].

The essence of our approach is as follows: given a discrete set $S \subset \mathbb{R}^n$ of interest, such as the set $\mathbb{Z}_+^n$ of all non-negative integer points or the set $\{0, 1\}^n$ of all 0-1 points, and an affine subspace $A \subset \mathbb{R}^n$ we want to compute or estimate the number $|S \cap A|$ of points in A. For that, we construct a probability measure μ on S with

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the property that the probability mass function is constant on the set $A \cap S$ and the expectation of μ lies in A. These two properties allow us to apply Local Central Limit Theorem type arguments to estimate $|S \cap A|$. The measure μ turns out to be the measure of the largest entropy on S with the expectation in A, so that constructing μ reduces to solving a convex optimization problem. We also consider a continuous version of the problem, where S is the non-negative orthant $\mathbb{R}^n_+$ and our goal is to estimate the volume of the set $S \cap A$.

Our approach is similar in spirit to that of E.T. Jaynes [Ja57] (see also [Go63]), who, motivated by problems of statistical mechanics, formulated a general principle of estimating the average value of a functional g with respect to an unknown probability distribution on a discrete set S of states provided the average values of some other functionals $f_1, \ldots, f_r$ on S are given. He suggested estimating g by its expectation with respect to the maximum entropy probability distribution on S such that the expectations of f_i have prescribed values. Our situation fits this general framework when, for example, S is the set $\mathbb{Z}^n_+$ of non-negative integer vectors, f_i are the equations defining an affine subspace A, functional g is some quantity of interest, while the unknown probability distribution on S is the counting measure on $S \cap A$ (in interesting cases, the set $S \cap A$ is complicated enough so that we may justifiably think of the counting measure on $S \cap A$ as of an unknown measure).

(1.1) Definitions and notation. In what follows, $\mathbb{R}^n$ is Euclidean space with the standard integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. A polyhedron $P \subset \mathbb{R}^n$ is defined as the set of solutions $x = (\xi_1, \ldots, \xi_n)$ to a vector equation

$$\xi_1 a_1 + \ldots + \xi_n a_n = b, \tag{1.1.1}$$

where $a_1, \ldots, a_n; b \in \mathbb{R}^d$ are d-dimensional vectors for $d < n$, and inequalities

$$\xi_1, \ldots, \xi_n \ \geq \ 0. \tag{1.1.2}$$

We assume that vectors $a_1, \ldots, a_n$ span $\mathbb{R}^d$, in which case the affine subspace defined by (1.1.1) has dimension $n - d$. We also assume that P has a non-empty interior, that is, contains a point $x = (\xi_1, \ldots, \xi_n)$, where inequalities (1.1.2) are strict. One of our goals is to compute the $(n-d)$-dimensional volume $\operatorname{vol} P$ of P with respect to the Lebesgue measure in the affine subspace (1.1.1) induced from $\mathbb{R}^n$. Often, we use a shorthand $Ax = b, x \geq 0$ for (1.1.1)–(1.1.2), where $A = [a_1, \ldots, a_n]$ is the matrix with the columns $a_1, \ldots, a_n$ and x is thought of as a column vector $x = [\xi_1, \ldots, \xi_n]^T$.

We are also interested in the number $|P \cap \mathbb{Z}^n|$ of integer points in P. In this case, we assume that vectors $a_1, \ldots, a_n$ and b are integer, that is, $a_1, \ldots, a_n; b \in \mathbb{Z}^d$. The number $|P \cap \mathbb{Z}^n|$ as a function of vector b in (1.1.1) is known as the *vector partition function* associated with vectors $a_1, \ldots, a_n$, see for example, [BV97].

Finally, we consider a version of the integer point counting problem where we are interested in 0-1 vectors only. Namely, let $\{0, 1\}^n$ be the set (Boolean cube) of all vectors in $\mathbb{R}^n$ with the coordinates 0 and 1. We estimate $|P \cap \{0, 1\}^n|$.

(1.2) The maximum entropy approach. Let us consider the integer counting problem first. One of the most straightforward approaches to computing $|P \cap \mathbb{Z}^n|$ approximately is via the Monte Carlo method. As in Section 1.1, we think of P as defined by a system $Ax = b, x \geq 0$. One can place P in a sufficiently large axis-parallel integer box B in the non-negative orthant $\mathbb{R}^n_+$ of $\mathbb{R}^n$, sample integer points from B independently at random and count what proportion of points lands in P. It is well understood that the method is very inefficient if P occupies a small fraction of B, in which case the sampled points will not land in P unless we use great many samples, see for example, Chapter 11 of [MR95]. Let X be a random vector distributed uniformly on the set of integer points in box B. One can try to circumvent sampling entirely by considering the random vector $Y = AX$ and interpreting the number of integer points in P in terms of the probability mass function of Y at b. One can hope then, in the spirit of the Central Limit Theorem, that since the coordinates of Y are linear combinations of independent coordinates $x_1, \ldots, x_n$ of X, the distribution of Y is somewhat close to the Gaussian and hence the probability mass function of Y at b can be approximated by the Gaussian density. The problem with this approach is that, generally speaking, the expectation $\mathbf{E}\, Y$ will be very far from the target vector b, so one tries to apply the Central Limit Theorem on the tail of the distribution, which is precisely where it is not applicable.

We propose an "exponential tilting" remedy, see, for example, Section 13.7 of [Te99], to this naive Monte Carlo approach. Namely, by solving a convex optimization problem on P, we construct a multivariate geometric random variable X such that

(1.2.1) The probability mass function of X is constant on the set $P \cap \mathbb{Z}^n$ of integer points in P;

(1.2.2) We have $\mathbf{E}\, X \in P$, or, equivalently, $\mathbf{E}\, Y = b$ for $Y = AX$.

Condition (1.2.1) allows us to express the number $|P \cap \mathbb{Z}^n|$ of integer points in P in terms of the probability mass function of Y, while condition (1.2.2) allows us to prove a Local Central Limit Theorem for Y in a variety of situations. We have $X = (x_1, \ldots, x_n)$ where x_j are independent geometric random variables with expectations ζ_j such that $z = (\zeta_1, \ldots, \zeta_n)$ is the unique point maximizing the value of the strictly concave function, the entropy of X,

$$g(x) = \sum_{j=1}^{n} \Big((\xi_j + 1) \ln (\xi_j + 1) - \xi_j \ln \xi_j \Big)$$

on P. In this case, the probability mass function of X at every point of $P \cap \mathbb{Z}^n$ is equal to $e^{-g(z)}$; see Theorem 3.1 for the precise statement.

Similarly, to estimate the number of 0-1 vectors in P, we construct a multivariate Bernoulli random variable X, such that (1.2.2) holds while (1.2.1) is replaced by

(1.2.3) The probability mass function of X is constant on the set $P \cap \{0,1\}^n$ of 0-1 vectors in P.

In this case, $X = (x_1, \ldots, x_n)$, where x_j are independent Bernoulli random variables with expectations ζ_j such that $z = (\zeta_1, \ldots, \zeta_n)$ is the unique point maximizing the value of the strictly concave function, the entropy of X,

$$h(x) = \sum_{j=1}^{n} \left(\xi_j \ln \frac{1}{\xi_j} + (1 - \xi_j) \ln \frac{1}{1 - \xi_j} \right)$$

on the truncated polytope

$$P \cap \Big\{ 0 \leq \xi_j \leq 1 : \quad \text{for} \quad j = 1, \ldots, n \Big\}.$$

In this case, the probability mass function of X at every point of $P \cap \{0,1\}^n$ is equal to $e^{-h(z)}$; see Theorem 3.3 for the precise statement.

Finally, to approximate the volume of P, we construct a multivariate exponential random variable X such that (1.2.2) holds and (1.2.1) is naturally replaced by

(1.2.4) The density of X is constant on P.

Condition (1.2.4) allows us to express the volume of P in terms of the density of $Y = AX$ at $Y = b$, while (1.2.2) allows us to establish a Local Central Limit Theorem for Y in a number of cases. In this case, each coordinate x_j is sampled independently from the exponential distribution with expectation ζ_j such that $z = (\zeta_1, \ldots, \zeta_n)$ is the unique point maximizing the value of the strictly concave function, the entropy of X,

$$f(x) = n + \sum_{j=1}^{n} \ln \xi_j$$

on P. In this case, the density of X at every point of P is equal to $e^{-f(z)}$; see Theorem 3.6 for the precise statement. In optimization, the point z is known as the *analytic center* of P and it played a central role in the development of interior point methods, see [Re88].

Summarizing, in each of the three cases (counting integer points, counting 0-1 points and computing volumes), we construct a random d-dimensional vector Y, which is a linear combination of n independent (discrete or continuous) random vectors. We express the quantity of interest as the probability mass function (in the discrete case) or density (in the continuous case) of Y at its expectation $b = \mathbf{E}\, Y$, multiplied by constants $e^{g(z)}$, $e^{h(z)}$ or $e^{f(z)}$ respectively. Using a Local Central Limit argument, we consider a Gaussian d-dimensional vector Y^* with the same expectation and covariance matrix as Y and approximate the density of Y by the density of Y^* in the continuous case (see Section 3.7) and the probability mass function of a lattice random vector Y by the density of Y^* multiplied by the volume of the fundamental domain in the discrete case (see Section 3.2).

These three examples (counting integer points, counting 0-1 vectors, and computing volumes) are important particular cases of a general approach to counting through the solution to an entropy maximization problem (cf. Theorem 3.5) with the subsequent asymptotic analysis of multivariate integrals needed to establish Local Central Limit Theorem type results. Although the intuition for our formulas is supplied by probability, the formulas we obtain are entirely deterministic. This makes our approach very different from Monte Carlo type algorithms (see, for example, Chapter 11 of [MR95] and [C+05]).

2. Main results

(2.1) Gaussian approximation for volume. Let $P \subset \mathbb{R}^n$ be a polytope, defined by a system $Ax = b, x \geq 0$, where A is an $d \times n$ matrix with the columns $a_1, \ldots, a_n$. We assume that $\operatorname{rank} A = d < n$. We find the point $z = (\zeta_1, \ldots, \zeta_n)$ maximizing

$$f(x) = n + \sum_{j=1}^{n} \ln \xi_j, \qquad x = (\xi_1, \ldots, \xi_n),$$

on P. Let B be the $d \times n$ matrix with the columns $\zeta_1 a_1, \ldots, \zeta_n a_n$. We approximate the volume of P by the Gaussian formula

$$\text{(2.1.1)} \qquad \operatorname{vol} P \approx \frac{1}{(2\pi)^{d/2}} \left(\frac{\det AA^T}{\det BB^T} \right)^{1/2} e^{f(z)}.$$

We consider the standard scalar product $\langle \cdot, \cdot \rangle$ and the corresponding Euclidean norm $\| \cdot \|$ in $\mathbb{R}^d$.

We prove the following main result.

(2.2) Theorem. *Let us consider a quadratic form $q : \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by*

$$q(t) = \frac{1}{2} \sum_{j=1}^{n} \zeta_j^2 \langle a_j, t \rangle^2.$$

Suppose that for some $\lambda > 0$ we have

$$q(t) \geq \lambda \|t\|^2 \quad \textit{for all} \quad t \in \mathbb{R}^d$$

and that for some $\theta > 0$ we have

$$\zeta_j \|a_j\| \leq \theta \quad \textit{for} \quad j = 1, \ldots, n.$$

Then there exists an absolute constant γ such that the following holds: let $0 < \epsilon \leq 1/2$ be a number and suppose that

$$\lambda \ \geq \ \gamma \theta^2 \epsilon^{-2} \left(d + \ln \frac{1}{\epsilon} \right)^2 \ln \left(\frac{n}{\epsilon} \right).$$

Then the number

$$\frac{1}{(2\pi)^{d/2}} \left(\frac{\det AA^T}{\det BB^T} \right)^{1/2} e^{f(z)}$$

approximates $\operatorname{vol} P$ *within relative error* ϵ*.*

Let us consider the columns $a_1, \ldots, a_n$ of A as vectors from Euclidean space $\mathbb{R}^d$ endowed with the standard scalar product $\langle \cdot, \cdot \rangle$. The quadratic form q defines the moment of inertia of the set of vectors $\{\zeta_1 a_1, \ldots, \zeta_n a_n\}$, see, for example, [Ba97]. By requiring that the smallest eigenvalue of q is sufficiently large compared to the lengths of the vectors $\zeta_j a_j$, we require that the set is sufficiently "round". For a sufficiently generic (random) set of n vectors, we will have $q(t)$ roughly proportional to $\|t\|^2$ and hence λ will be of the order of $nd^{-1} \max_{j=1,\ldots,n} \zeta_j^2 \|a_j\|^2$.

We prove Theorem 2.2 in Section 6.

In Section 4, we apply Theorem 2.2 to approximate the volume of a multi-index transportation polytope, see, for example, [Y+84], that is, the polytope P of ν-dimensional $k_1 \times \ldots \times k_\nu$ arrays of non-negative numbers $(\xi_{j_1 \ldots j_\nu})$ with $1 \leq j_i \leq k_i$ for $i = 1, \ldots, \nu$ with prescribed sums along the coordinate hyperplanes $j_i = j$. We show that Theorem 2.2 implies that asymptotically the volume of P is given by a Gaussian formula (2.1.1) as long as $\nu \geq 5$. We suspect that the Gaussian approximation holds as long as $\nu \geq 3$, but the proof would require some additional considerations beyond those of Theorem 2.2. In particular, for $\nu \geq 5$ we obtain the asymptotic formula for the volume of the polytope of polystochastic tensors, see [Gr92].

For $\nu = 2$ polytope P is the usual transportation polytope. Interestingly, its volume is *not* given by the Gaussian formula, cf. [CM07b].

In [Ba09], a much cruder asymptotic formula $\ln(\operatorname{vol} P) \approx f(z)$ was proved under much weaker assumptions.

(2.3) Gaussian approximation for the number of integer points. For a polytope P, defined by a system $Ax = b, x \geq 0$, we find the point $z = (\zeta_1, \ldots, \zeta_n)$ maximizing

$$g(x) = \sum_{j=1}^{n} \Big((\xi_j + 1) \ln (\xi_j + 1) - \xi_j \ln \xi_j \Big), \qquad x = (\xi_1, \ldots, \xi_n),$$

on P. Assuming that $a_1, \ldots, a_n \in \mathbb{Z}^d$ are the columns of A, we define B as the $d \times n$ matrix whose j-th column is $\left(\zeta_j^2 + \zeta_j\right)^{1/2} a_j$ for $j = 1, \ldots, n$.

We assume that A is an integer $d \times n$ matrix of rank $d < n$. Let $\Lambda = A\left(\mathbb{Z}^n\right)$ be image of the standard lattice, $\Lambda \subset \mathbb{Z}^d$. We approximate the number of integer points in P by the Gaussian formula

$$|P \cap \mathbb{Z}^n| \approx \frac{e^{g(z)} \det \Lambda}{(2\pi)^{d/2} (\det BB^T)^{1/2}}. \tag{2.3.1}$$

In this paper, we consider the simplest case of $\Lambda = \mathbb{Z}^d$, which is equivalent to the greatest common divisor of the $d \times d$ minors of A being equal to 1.

Together with the Euclidean norm $\| \cdot \|$ in $\mathbb{R}^d$, we consider the ℓ^1 and ℓ^∞ norms:

$$\|t\|_1 = \sum_{i=1}^d |\tau_i| \quad \text{and} \quad \|t\|_\infty = \max_{i=1,\dots,d} |\tau_i| \quad \text{where} \quad t = (\tau_1, \dots, \tau_d).$$

Clearly, we have

$$\|t\|_1 \ \geq \ \|t\| \ \geq \ \|t\|_\infty \quad \text{for all} \quad t \in \mathbb{R}^d.$$

Compared to the case of volume estimates (Sections 2.1–2.2), we acquire an additive error which is governed by the arithmetic of the problem.

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{Z}^d$. We prove the following main result.

(2.4) Theorem. *Let us consider a quadratic form $q: \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by*

$$q(t) = \frac{1}{2} \sum_{j=1}^n \left(\zeta_j + \zeta_j^2\right) \langle a_j, t \rangle^2.$$

For $i = 1, \dots, d$ let us choose a non-empty finite set $Y_i \subset \mathbb{Z}^n$ such that $Ay = e_i$ for all $y \in Y_i$ and let us define a quadratic form $\psi_i: \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\psi_i(x) = \frac{1}{|Y_i|} \sum_{y \in Y_i} \langle y, x \rangle^2.$$

Suppose that for some $\lambda \geq 0$ we have

$$q(t) \geq \lambda \|t\|^2 \quad \textit{for all} \quad t \in \mathbb{R}^d,$$

that for some $\rho > 0$ we have

$$\psi_i(x) \leq \rho \|x\|^2 \quad \textit{for all} \quad x \in \mathbb{R}^n \quad \textit{and} \quad i = 1, \dots, d,$$

that for some $\theta \geq 1$ we have

$$\|a_j\|_1 \leq \theta \sqrt{\frac{\zeta_j}{\left(1 + \zeta_j\right)^3}} \quad \textit{for} \quad j = 1, \dots, n$$

and that

$$\zeta_j(1+\zeta_j) \geq \alpha \quad \textit{for} \quad j=1, \ldots, n$$

and some $\alpha \geq 0$.

Then, for some absolute constant $\gamma > 0$ *and for any* $0 \leq \epsilon \leq 1/2$, *as long as*

$$\lambda \geq \gamma \epsilon^{-2} \theta^2 \left(d + \ln \frac{1}{\epsilon}\right)^2 \ln\left(\frac{n}{\epsilon}\right),$$

we have

$$|P \cap \mathbb{Z}^n| = e^{g(z)} \left(\frac{\kappa}{(2\pi)^{d/2} \left(\det BB^T\right)^{1/2}} + \Delta \right),$$

where

$$1 - \epsilon \leq \kappa \leq 1 + \epsilon$$

and

$$|\Delta| \leq \left(1 + \frac{2}{5}\alpha\pi^2\right)^{-m} \quad \textit{for} \quad m = \left\lfloor \frac{1}{16\pi^2 \rho \theta^2} \right\rfloor.$$

While the condition on the smallest eigenvalue of quadratic form q is very similar to that of Theorem 2.2 and is linked to the metric properties of P, the appearance of quadratic forms ψ_i is explained by the arithmetic features of P. Let us choose $1 \leq i \leq d$ and let us consider the affine subspace $\mathcal{A}_i$ of the points $x \in \mathbb{R}^n$ such that $Ax = e_i$. Let $\Lambda_i = \mathcal{A}_i \cap \mathbb{Z}^n$ be the point lattice in $\mathcal{A}_i$. We would like to choose a set $Y_i \subset \Lambda_i$ in such a way that the maximum eigenvalue ρ_i of the form ψ_i, which defines the moment of inertia of Y_i, see [Ba97], becomes as small as possible, $\rho_i \ll 1$, so that the additive error term Δ becomes negligibly small compared to the Gaussian term $(2\pi)^{-d/2} \left(\det BB^T\right)^{-1/2}$. For that, we would like the set Y_i to consist of short vectors and to look reasonably round. Let us consider the ball $B_r = \{x \in \mathbb{R}^n : \ \|x\| \leq r\}$ of radius r and choose $Y_i = B_r \cap \Lambda_i$. If the lattice points Y_i are sufficiently regular in $B_r \cap \mathcal{A}_i$ then the moment of inertia of Y_i is roughly the moment of inertia of the section $B_r \cap \mathcal{A}_i$, from which it follows that the maximum eigenvalue of ψ_i is about $r^2/\dim \mathcal{A}_i = r^2/(n-d)$. Roughly, we get

$$\rho \approx \frac{r^2}{(n-d)},$$

where r is the smallest radius of the ball B_r such that the lattice points $B_r \cap \Lambda_i$ are distributed regularly in every section $B_r \cap \mathcal{A}_i$ for $i = 1, \ldots, d$.

We prove Theorem 2.4 in Section 8.

In Section 5, we apply Theorem 2.4 to approximate the number of 1-margin multi-way contingency tables, see for example, [Go63] and [DO04], that is, ν-dimensional $k_1 \times \ldots \times k_\nu$ arrays of non-negative integers $(\xi_{j_1 \ldots j_\nu})$ with $1 \leq j_i \leq k_i$ for $i = 1, \ldots, \nu$ with prescribed sums along coordinate hyperplanes $j_i = j$. We show that Theorem 2.4 implies that asymptotically the number of such arrays is given by a Gaussian formula (2.3.1) as long as $\nu \geq 5$. We suspect that the Gaussian approximation holds as long as $\nu \geq 3$, but the proof would require some additional considerations beyond those of Theorem 2.4.

In [Ba09], a much cruder asymptotic formula $\ln |P \cap \mathbb{Z}^n| \approx g(z)$ is shown to hold for *flow polytopes* P (a class of polytopes extending transportation polytopes for $\nu = 2$).

A. Yong [Yo08] at our request computed a number of examples, and then J. A. De Loera [DL09a] and [DL09b] conducted extensive numerical experiments. Here is one of the examples, originating in [DE85] and then often used as a benchmark for various computational approaches:

we want to estimate the number of 4×4 non-negative integer matrices with the row sums $220, 215, 93$ and 64 and the column sums $108, 286, 71$ and 127. The exact number of such matrices is $1225914276768514 \approx 1.23 \times 10^{15}$. Framing the problem as the problem of counting integer points in a polytope in the most straightforward way, we obtain an over-determined system $Ax = b$ (note that the row and column sums of a matrix are not independent). Throwing away one constraint and applying formula (2.3.1), we obtain 1.30×10^{15}, which overestimates the true number by about 6%. The precision is not bad, given that we are applying the Gaussian approximation to the probability mass-function of the sum of 16 independent random 7-dimensional integer vectors, see also Section 3.2.

Here is another example from [DL09b]:

we want to estimate the number of $3 \times 3 \times 3$ arrays of non-negative integers with the prescribed sums $[31, 22, 87], [50, 13, 77], [42, 87, 11]$ along the affine coordinate hyperplanes, cf. Sections 4 and 5. The exact number of such arrays is $8846838772161591 \approx 8.85 \times 10^{15}$. Again, the constraints are not independent and this time we throw away two constraints. The relative error of the approximation given by formula (2.3.1) is about 0.185%. This time, we are applying the Gaussian approximation to the probability mass function of the sum of 27 independent random 7-dimensional integer vectors. It is therefore not surprising that the precision improves; see Section 3.2.

Regarding the CPU time used, De Loera writes in [DL09b]: "Overall the evaluation step takes a negligible amount of time in all instances, so we do not record any time of computation."

(2.5) Gaussian approximation for the number of 0-1 points. For a polytope P defined by a system $Ax = b$, $0 \leq x \leq 1$ (shorthand for $0 \leq \xi_j \leq 1$ for $x = (\xi_1, \ldots, \xi_n)$), we find the point $z = (\zeta_1, \ldots, \zeta_n)$ maximizing

$$h(x) = \sum_{j=1}^{n} \left(\xi_j \ln \frac{1}{\xi_j} + (1 - \xi_j) \ln \frac{1}{1 - \xi_j} \right), \qquad x = (\xi_1, \ldots, \xi_n),$$

on P. Assuming that A is an integer matrix of rank $d < n$ with the columns $a_1, \ldots, a_n \in \mathbb{Z}^d$, we compute the $d \times n$ matrix B whose j-th column is

$\left(\zeta_j - \zeta_j^2\right)^{1/2} a_j$. We approximate the number of 0-1 vectors in P by the Gaussian formula

$$(2.5.1) \qquad |P \cap \{0,1\}^n| \approx \frac{e^{h(z)} \det \Lambda}{(2\pi)^{d/2} (\det BB^T)^{1/2}},$$

where $\Lambda = A\left(\mathbb{Z}^n\right)$. Again, we consider the simplest case of $\Lambda = \mathbb{Z}^d$. We prove the following main result.

(2.6) Theorem. *Let us consider a quadratic form $q : \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by*

$$q(t) = \frac{1}{2} \sum_{j=1}^n \left(\zeta_j - \zeta_j^2\right) \langle a_j, t \rangle^2.$$

For $i = 1, \ldots, d$ let us choose a non-empty finite set $Y_i \subset \mathbb{Z}^n$ such that $Ay = e_i$ for all $y \in Y_i$ and let us define a quadratic form $\psi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\psi_i(x) = \frac{1}{|Y_i|} \sum_{y \in Y_i} \langle y, x \rangle^2.$$

Suppose that for some $\lambda > 0$ we have

$$q(t) \geq \lambda \|t\|^2 \quad \text{for all} \quad t \in \mathbb{R}^d,$$

that for some $\rho > 0$ we have

$$\psi_i(x) \leq \rho \|x\|^2 \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{and} \quad i = 1, \ldots, d,$$

that for some $\theta \geq 1$ we have

$$\|a_j\|_1 \ \leq \ \theta \sqrt{\zeta_j \left(1 - \zeta_j\right)} \quad \text{for} \quad j = 1, \ldots, n$$

and that for some $0 < \alpha \leq 1/4$ we have

$$\zeta_j (1 - \zeta_j) \geq \alpha \quad \text{for} \quad j = 1, \ldots, n.$$

Then, for some absolute constant $\gamma > 0$ and for any $0 < \epsilon \leq 1/2$, as long as

$$\lambda \geq \gamma \epsilon^{-2} \theta^2 \left(d + \ln \frac{1}{\epsilon}\right)^2 \ln\left(\frac{n}{\epsilon}\right),$$

we have

$$|P \cap \{0,1\}^n| = e^{h(z)} \left(\frac{\kappa}{(2\pi)^{d/2} \left(\det BB^T\right)^{1/2}} + \Delta\right),$$

where

$$1-\epsilon \ \leq \kappa \ \leq 1+\epsilon$$

and

$$|\Delta| \leq \exp\left\{-\frac{\alpha}{80\theta^2\rho}\right\}.$$

We note that in [Ba08] a much cruder asymptotic formula $\ln|P \cap \{0,1\}^n| \approx h(z)$ is shown to hold for flow polytopes P.

We prove Theorem 2.6 in Section 7.

In Section 5, we apply Theorem 2.6 to approximate the number of binary 1-margin multi-way contingency tables, see for example, [Go63] and [DO04], that is, ν-dimensional $k_1 \times \ldots \times k_\nu$ arrays $(\xi_{j_1 \ldots j_\nu})$ of 0's and 1's with $1 \leq j_i \leq k_i$ for $i=1,\ldots,\nu$ with prescribed sums along coordinate hyperplanes $j_i = j$. Alternatively, the number of such arrays is the number of ν-partite uniform hypergraphs with prescribed degrees of all vertices. We show that Theorem 2.6 implies that asymptotically the number of such arrays is given by the Gaussian formula (2.5.1) as long as $\nu \geq 5$. We suspect that the Gaussian approximation holds as long as $\nu \geq 3$, but the proof would require some additional considerations beyond those of Theorem 2.6.

3. Maximum entropy

We start with the problem of integer point counting.

Let us fix positive numbers p and q such that $p+q=1$. We recall that a discrete random variable x has geometric distribution if

$$\mathbf{Pr}\left\{x=k\right\} = pq^k \quad \text{for} \quad k=0,1,\ldots.$$

For the expectation and variance of x we have

$$\mathbf{E}\, x = \frac{q}{p} \quad \text{and} \quad \mathbf{var}\, x = \frac{q}{p^2}$$

respectively. Conversely, if $\mathbf{E}\, x = \zeta$ for some $\zeta > 0$ then

$$p = \frac{1}{1+\zeta}, \quad q = \frac{\zeta}{1+\zeta} \quad \text{and} \quad \mathbf{var}\, x = \zeta + \zeta^2.$$

Our first main result is as follows.

(3.1) Theorem. *Let $P \subset \mathbb{R}^n$ be the intersection of an affine subspace in $\mathbb{R}^n$ and the non-negative orthant $\mathbb{R}^n_+$. Suppose that P is bounded and has a non-empty interior, that is contains a point $y=(\eta_1,\ldots,\eta_n)$ where $\eta_j > 0$ for $j=1,\ldots,n$.*

Then the strictly concave function

$$g(x) = \sum_{j=1}^n \Big((\xi_j+1)\ln(\xi_j+1) - \xi_j \ln \xi_j\Big) \quad \textit{for} \quad x=(\xi_1,\ldots,\xi_n)$$

attains its maximum value on P at a unique point $z = (\zeta_1, \dots, \zeta_n)$ such that $\zeta_j > 0$ for $j = 1, \dots, n$.

Suppose now that x_j are independent geometric random variables with expectations ζ_j for $j = 1, \dots, n$. Let $X = (x_1, \dots, x_n)$. Then the probability mass function of X is constant on $P \cap \mathbb{Z}^n$ and equal to $e^{-g(z)}$ at every $x \in P \cap \mathbb{Z}^n$. In particular,

$$|P \cap \mathbb{Z}^n| = e^{g(z)} \mathbf{Pr}\big\{X \in P\big\}.$$

Proof. It is straightforward to check that g is strictly concave on the non-negative orthant $\mathbb{R}^n_+$, so it attains its maximum on P at a unique point $z = (\zeta_1, \dots, \zeta_n)$. Let us show that $\zeta_j > 0$. Since P has a non-empty interior, there is a point $y = (\eta_1, \dots, \eta_n)$ with $\eta_j > 0$ for $j = 1, \dots, n$. We note that

$$\frac{\partial}{\partial \xi_j} g = \ln\left(\frac{\xi_j + 1}{\xi_j}\right),$$

which is finite for $\xi_j > 0$ and equals $+\infty$ for $\xi_j = 0$ (we consider the right derivative in this case). Therefore, if $\zeta_j = 0$ for some j then $g\big((1-\epsilon)z + \epsilon y\big) > g(z)$ for all sufficiently small $\epsilon > 0$, which is a contradiction.

Suppose that the affine hull of P is defined by a system of linear equations

$$\sum_{j=1}^n \alpha_{ij} \xi_j = \beta_i \quad \text{for} \quad i = 1, \dots, d.$$

Since z is an interior maximum point, the gradient of g at z is orthogonal to the affine hull of P, so we have

$$\ln\left(\frac{1+\zeta_j}{\zeta_j}\right) = \sum_{i=1}^d \lambda_i \alpha_{ij} \quad \text{for} \quad j = 1, \dots, n$$

and some $\lambda_1, \dots, \lambda_d$. Therefore, for any $x \in P$, $x = (\xi_1, \dots, \xi_n)$, we have

$$\sum_{j=1}^n \xi_j \ln\left(\frac{1+\zeta_j}{\zeta_j}\right) = \sum_{j=1}^n \sum_{i=1}^d \lambda_i \xi_j \alpha_{ij} = \sum_{i=1}^d \lambda_i \beta_i,$$

or, equivalently,

$$\prod_{j=1}^n \left(\frac{1+\zeta_j}{\zeta_j}\right)^{\xi_j} = \exp\left\{\sum_{i=1}^d \lambda_i \beta_i\right\}. \tag{3.1.1}$$

Substituting $\xi_j = \zeta_j$ for $j = 1, \dots, n$, we obtain

$$\prod_{j=1}^n \left(\frac{1+\zeta_j}{\zeta_j}\right)^{\zeta_j} = \exp\left\{\sum_{i=1}^d \lambda_i \beta_i\right\}. \tag{3.1.2}$$

From (3.1.1) and (3.1.2), we deduce

$$\left(\prod_{j=1}^{n}\left(\frac{\zeta_j}{1+\zeta_j}\right)^{\xi_j}\right)\left(\prod_{j=1}^{n}\frac{1}{1+\zeta_j}\right)=\exp\left\{-\sum_{i=1}^{d}\lambda_i\beta_i\right\}\left(\prod_{j=1}^{n}\frac{1}{1+\zeta_j}\right)$$
$$=\prod_{j=1}^{n}\frac{\zeta_j^{\zeta_j}}{\left(1+\zeta_j\right)^{1+\zeta_j}}=e^{-g(z)}.$$

The last identity states that the probability mass function of X is equal to $e^{-g(z)}$ for every integer point $x \in P$. □

One can observe that the random variable X of Theorem 3.1 has the maximum entropy distribution among all distributions on $\mathbb{Z}_+^n$ subject to the constraint $\mathbf{E}\,X \in P$.

(3.2) The Gaussian heuristic for the number of integer points. Below we provide an informal justification for the Gaussian approximation formula (2.3.1).

Let P be a polytope and let X be a random vector as in Theorem 3.1. Suppose that P is defined by a system $Ax = b, x \geq 0$, where $A = (\alpha_{ij})$ is a $d \times n$ matrix of rank $d < n$. Let $Y = AX$, so $Y = (y_1, \ldots, y_d)$, where

$$y_i = \sum_{j=1}^{n} \alpha_{ij} x_j \quad \text{for} \quad i = 1, \ldots, d.$$

By Theorem 3.1,

$$|P \cap \mathbb{Z}^n| = e^{g(z)}\mathbf{Pr}\left\{Y = b\right\}$$

and

$$\mathbf{E}\,Y = Az = b.$$

Moreover, the covariance matrix $Q = (q_{ij})$ of Y is computed as follows:

$$q_{ij} = \mathbf{cov}\,(y_i, y_j) = \sum_{k=1}^{n} \alpha_{ik}\alpha_{jk}\mathbf{var}\,x_k = \sum_{k=1}^{n} \alpha_{ik}\alpha_{jk}\left(\zeta_k + \zeta_k^2\right).$$

We would like to approximate the discrete random variable Y by the Gaussian random variable Y^* with the same expectation b and covariance matrix Q. We assume now that A is an integer matrix and let $\Lambda = \left\{Ax : \ x \in \mathbb{Z}^n\right\}$. Hence $\Lambda \subset \mathbb{Z}^d$ is a d-dimensional lattice. Let $\Pi \subset \mathbb{R}^d$ be a fundamental domain of Λ, so $\operatorname{vol}\Pi = \det\Lambda$. For example, we can choose Π to be the set of points in $\mathbb{R}^d$ that are closer to the origin than to any other point in Λ. Then we can write

$$|P \cap \mathbb{Z}^n| = e^{g(z)}\mathbf{Pr}\left\{Y \in b + \Pi\right\}.$$

Assuming that the probability density of Y^* does not vary much on $b+\Pi$ and that the probability mass function of Y at $Y=b$ is well approximated by the integral of the density of Y^* over $b+\Pi$, we obtain (2.3.1).

Next, we consider the problem of counting 0-1 vectors.

Let p and q be positive numbers such that $p+q=1$. We recall that a discrete random variable x has Bernoulli distribution if

$$\mathbf{Pr}\,\{x=0\}=p \quad \text{and} \quad \mathbf{Pr}\,\{x=1\}=q.$$

We have

$$\mathbf{E}\,x=q \quad \text{and} \quad \mathbf{var}\,x=qp.$$

Conversely, if $\mathbf{E}\,x=\zeta$ for some $0<\zeta<1$ then

$$p=1-\zeta, \quad q=\zeta \quad \text{and} \quad \mathbf{var}\,x=\zeta-\zeta^2.$$

Our second main result is as follows.

(3.3) Theorem. *Let $P \subset \mathbb{R}^n$ be the intersection of an affine subspace in $\mathbb{R}^n$ and the unit cube $\big\{0 \le \xi_j \le 1 : \quad j=1,\ldots,n\big\}$. Suppose that P has a non-empty interior, that is, contains a point $y=(\eta_1,\ldots,\eta_n)$ where $0<\eta_j<1$ for $j=1,\ldots,n$. Then the strictly concave function*

$$h(x)=\sum_{j=1}^n \left(\xi_j \ln \frac{1}{\xi_j} + (1-\xi_j)\ln \frac{1}{1-\xi_j}\right) \quad \text{for} \quad x=(\xi_1,\ldots,\xi_n)$$

attains its maximum value on P at a unique point $z=(\zeta_1,\ldots,\zeta_n)$ such that $0<\zeta_j<1$ for $j=1,\ldots,n$.

Suppose now that x_j are independent Bernoulli random variables with expectations ζ_j for $j=1,\ldots,n$. Let $X=(x_1,\ldots,x_n)$. Then the probability mass function of X is constant on $P\cap\{0,1\}^n$ and equal to $e^{-h(z)}$ for every $x\in P\cap\{0,1\}^n$. In particular,

$$|P\cap\{0,1\}^n| \;=\; e^{h(z)}\mathbf{Pr}\,\big\{X\in P\big\}.$$

□

One can observe that X has the maximum entropy distribution among all distributions on $\{0,1\}^n$ subject to the constraint $\mathbf{E}\,X\in P$. The proof is very similar to that of Theorem 3.1. Besides, Theorem 3.3 follows from a more general Theorem 3.5 below.

(3.4) Comparison with the Monte Carlo method. Suppose we want to sample a random 0-1 point from the uniform distribution on $P\cap\{0,1\}^n$. The standard Monte Carlo rejection method consists in sampling a random 0-1 point x, accepting x if $x\in P$ and sampling a new point if $x\notin P$. The probability of hitting P

is, therefore, $2^{-n}|P \cap \mathbb{Z}^n|$. It is easy to see that the largest possible value of h in Theorem 3.3 is $n \ln 2$ and is attained at $\zeta_1 = \ldots = \zeta_n = 1/2$. Therefore, the rejection sampling using the maximum entropy Bernoulli distribution of Theorem 3.3 is at least as efficient as the standard Monte Carlo approach and is essentially more efficient if the value of $h(z)$ is small.

Applying a similar logic as in Section 3.2, we obtain the Gaussian heuristic approximation of (2.5.1).

We notice that

$$h(\xi) = \xi \ln \frac{1}{\xi} + (1-\xi) \ln \frac{1}{1-\xi}$$

is the entropy of the Bernoulli distribution with expectation ξ while

$$g(\xi) = (\xi+1)\ln(\xi+1) - \xi \ln \xi$$

is the entropy of the geometric distribution with expectation ξ. One can suggest the following general maximum entropy approach, cf. also a similar computation in [Ja57].

(3.5) Theorem. *Let $S \subset \mathbb{R}^n$ be a finite set and let $\operatorname{conv}(S)$ be the convex hull of S. Let us assume that $\operatorname{conv}(S)$ has a non-empty interior. For $x \in \operatorname{conv}(S)$, let us define $\phi(x)$ to be the maximum entropy of a probability distribution on S with expectation x, that is,*

$$\begin{aligned} \phi(x) = \max & \sum_{s \in S} p_s \ln \frac{1}{p_s} \\ \textit{Subject to: } & \sum_{s \in S} p_s = 1 \\ & \sum_{s \in S} s p_s = x \\ & p_s \geq 0 \quad \textit{for all} \quad s \in S. \end{aligned}$$

Then $\phi(x)$ is a strictly concave continuous function on $\operatorname{conv}(S)$.

Let $A \subset \mathbb{R}^n$ be an affine subspace intersecting the interior of $\operatorname{conv}(S)$. Then ϕ attains its maximum value on $A \cap \operatorname{conv}(S)$ at a unique point z in the interior of $\operatorname{conv}(S)$. There is a unique probability distribution μ on S with entropy $\phi(z)$ and expectation in A. Furthermore, the probability mass function of μ is constant on the points of $S \cap A$ and equal to $e^{-\phi(z)}$:

$$\mu\{s\} = e^{-\phi(z)} \quad \textit{for all} \quad s \in S \cap A.$$

In particular,

$$|S \cap A| = e^{\phi(z)} \mu\{S \cap A\}.$$

Proof. Let

$$H\Big(p_s : \quad s \in S\Big) = \sum_{s \in S} p_s \ln \frac{1}{p_s}$$

be the entropy of the probability distribution $\{p_s\}$ on S.

Continuity and strict concavity of ϕ follows from continuity and strict concavity of H. Similarly, uniqueness of μ follows from the strict concavity of H.

Since

$$\frac{\partial}{\partial p_s} H = \ln \frac{1}{p_s} - 1,$$

which is finite for $p_s > 0$ and is equal to $+\infty$ for $p_s = 0$ (we consider the right derivative), we conclude that for the optimal distribution μ we have $p_s > 0$ for all s.

Suppose that A is defined by linear equations

$$\langle a_i, x\rangle = \beta_i \quad \text{for} \quad i = 1, \ldots, d,$$

where $a_i \in \mathbb{R}^n$ are vectors, $\beta_i \in \mathbb{R}$ are numbers and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. Thus the measure μ is the solution to the following optimization problem:

$$\begin{aligned}
&\sum_{s \in S} p_s \ln \frac{1}{p_s} \longrightarrow \max \\
\text{Subject to:} \quad &\sum_{s \in S} p_s = 1 \\
&\sum_{s \in S} \langle a_i, s\rangle p_s = \beta_i \quad \text{for} \quad i = 1, \ldots, d \\
&p_s \geq 0 \quad \text{for all} \quad s \in S.
\end{aligned}$$

Writing the optimality conditions, we conclude that for some $\lambda_0, \lambda_1, \ldots, \lambda_d$ we have

$$\ln p_s = \lambda_0 + \sum_{i=1}^{d} \lambda_i \langle a_i, s\rangle.$$

Therefore,

$$p_s = \exp\left\{\lambda_0 + \sum_{i=1}^{d} \lambda_i \langle a_i, s\rangle\right\}.$$

In particular, for $s \in A$ we have

$$p_s = \exp\left\{\lambda_0 + \sum_{i=1}^{d} \lambda_i \beta_i\right\}.$$

On the other hand,

$$
\begin{aligned}
\phi(z) =& H\Big(p_s : \quad s \in S\Big) \\
=& -\sum_{s \in S} p_s \left(\lambda_0 + \sum_{i=1}^d \lambda_i \langle a_i, s\rangle\right) \\
=& -\lambda_0 - \sum_{i=1}^d \lambda_i \beta_i,
\end{aligned}
$$

which completes the proof. □

Finally, we discuss a continuous version of the maximum entropy approach.

We recall that x is an exponential random variable with expectation $\zeta > 0$ if the density function ψ of x is defined by

$$
\psi(\tau) = \begin{cases} (1/\zeta)e^{-\tau/\zeta} & \text{for } \ \tau \geq 0 \\ 0 & \text{for } \tau < 0. \end{cases}
$$

We have

$$
\mathbf{E}\, x = \zeta \quad \text{and} \quad \mathbf{var}\, x = \zeta^2.
$$

The characteristic function of x is defined by

$$
\mathbf{E}\, e^{i\tau x} = \frac{1}{1 - i\zeta\tau} \quad \text{for} \quad \tau \in \mathbb{R}.
$$

(3.6) Theorem. *Let $P \subset \mathbb{R}^n$ be the intersection of an affine subspace in $\mathbb{R}^n$ and a non-negative orthant $\mathbb{R}^n_+$. Suppose that P is bounded and has a non-empty interior. Then the strictly concave function*

$$
f(x) = n + \sum_{j=1}^n \ln \xi_j \quad \textit{for} \quad x = (\xi_1, \ldots, \xi_n)
$$

attains its unique maximum on P at a point $z = (\zeta_1, \ldots, \zeta_n)$, where $\zeta_j > 0$ for $j = 1, \ldots, n$.

Suppose now that x_j are independent exponential random variables with expectations ζ_j for $j = 1, \ldots, n$. Let $X = (x_1, \ldots, x_n)$. Then the density of X is constant on P and for every $x \in P$ is equal to $e^{-f(z)}$.

Proof. As in the proof of Theorem 3.1, we establish that $\zeta_j > 0$ for $j = 1, \ldots, n$. Consequently, the gradient of f at z must be orthogonal to the affine span of P. Assume that P is defined by a system of linear equations

$$
\sum_{j=1}^n \alpha_{ij} \xi_j = \beta_i \quad \text{for} \quad i = 1, \ldots, d.
$$

Then

$$\frac{1}{\zeta_j} = \sum_{i=1}^{d} \lambda_i \alpha_{ij} \quad \text{for} \quad j = 1, \ldots, n.$$

Therefore, for any $x \in P$, $x = (\xi_1, \ldots, \xi_n)$, we have

$$\sum_{j=1}^{n} \frac{\xi_j}{\zeta_j} = \sum_{i=1}^{d} \left(\sum_{j=1}^{n} \alpha_{ij} \xi_j \right) = \sum_{i=1}^{d} \lambda_i \beta_i.$$

In particular, substituting $\xi_j = \zeta_j$, we obtain

$$\sum_{j=1}^{n} \frac{\xi_j}{\zeta_j} = n.$$

Therefore, the density of X at $x \in P$ is equal to

$$\left(\prod_{j=1}^{n} \frac{1}{\zeta_j} \right) \exp \left\{ -\sum_{j=1}^{n} \frac{\xi_j}{\zeta_j} \right\} = e^{-f(z)}.$$

□

Again, X has the maximum entropy distribution among all distributions on $\mathbb{R}^n_+$ subject to the constraint $\mathbf{E}\, X \in P$.

(3.7) The Gaussian heuristic for volumes. Below we provide an informal justification of the Gaussian approximation formula (2.1.1)

Let P be a polytope and let $x_1, \ldots, x_n$ be the random variables as in Theorem 3.6. Suppose that P is defined by a system $Ax = b$, $x \geq 0$, where $A = (\alpha_{ij})$ is a $d \times n$ matrix of rank $d < n$. Let $Y = AX$, so $Y = (y_1, \ldots, y_d)$, where

$$y_i = \sum_{j=1}^{n} \alpha_{ij} x_j \quad \text{for} \quad i = 1, \ldots, d.$$

In view of Theorem 3.6, the density of Y at b is equal to

$$(\operatorname{vol} P) e^{-f(z)} \left(\det AA^T \right)^{-1/2}$$

(we measure $\operatorname{vol} P$ as the $(n-d)$-dimensional volume with respect to the Euclidean structure induced from $\mathbb{R}^n$).

We have $\mathbf{E}\, Y = b$. The covariance matrix $Q = (q_{ij})$ of Y is computed as follows:

$$q_{ij} = \mathbf{cov}\ (y_i, y_j) = \sum_{k=1}^{n} \alpha_{ik} \alpha_{jk} \mathbf{var}\, x_k = \sum_{k=1}^{n} \alpha_{ik} \alpha_{jk} \zeta_k^2.$$

Assuming that the distribution of Y at $Y = b$ is well approximated by the Gaussian distribution, we obtain formula (2.1.1).

(3.8) Extensions to exponential sums and exponential integrals. Let $\ell : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\ell(x) = \gamma_1 \xi_1 + \ldots + \gamma_n \xi_n, \quad \text{where} \quad x = (\xi_1, \ldots, \xi_n),$$

be a linear function. For a (not necessarily bounded) polyhedron $P \subset \mathbb{R}^n$, defined as the intersection of an affine subspace and the non-negative orthant $\mathbb{R}^n_+$, we consider the sums

$$\sum_{x \in P \cap \mathbb{Z}^n} \exp\{\ell(x)\}, \qquad \sum_{x \in \in P \cap \{0,1\}^n} \exp\{\ell(x)\}$$
$$\text{and the integral} \quad \int_P \exp\{\ell(x)\} \ dx. \tag{3.8.1}$$

It is not hard to show that the infinite sum and the integral converge as long as ℓ is bounded from above on P and attains its maximum on P at a bounded face of P. Let us modify the functions

$$g \longmapsto g_\ell := g + \ell, \quad h := h_\ell + \ell, \quad \text{and} \quad f \longmapsto f_\ell := f + \ell$$

of Theorems 3.1, 3.3 and 3.6 respectively. Since the functions g_ℓ, h_ℓ and f_ℓ are strictly concave, the optimum in Theorems 3.1, 3.3 or 3.6 is attained at a unique point $z \in P$ and we define random vectors X with $\mathbf{E}\, X = z$ in the same way. Then the sums and the integral of (3.8.1) are equal to $\mathbf{Pr}\{X \in P\}$ multiplied by $\exp\{g_\ell(z)\}$, $\exp\{h_\ell(z)\}$ and $\exp\{f_\ell(z)\}$ respectively.

4. Volumes of multi-index transportation polytopes

We apply Theorem 2.2 to compute volumes of multi-index transportation polytopes.

Let us fix an integer $\nu \geq 2$ and let us choose integers $k_1, \ldots, k_\nu > 1$. We consider the polytope of P of $k_1 \times \ldots \times k_\nu$ arrays of non-negative numbers $\xi_{j_1 \ldots j_\nu}$, where $1 \leq j_i \leq k_i$ for $i = 1, \ldots, \nu$, with prescribed sums along the affine coordinate hyperplanes. Thus P lies in the non-negative orthant $\mathbb{R}_+^{k_1 \cdots k_\nu}$ and is defined by $k_1 + \ldots + k_\nu$ linear equations. The equations are not independent since if the add the sums over each family of parallel affine coordinate hyperplanes, we obtain the total sum N of the entries of the array.

We define P by the following non-redundant system of equations and inequalities. Given positive numbers β_{ij} (sums along the affine coordinate hyperplanes), where $1 \leq j \leq k_i$ for $i = 1, \ldots, \nu$ and such that

$$\sum_j \beta_{ij} = N$$

for some N and all $i = 1, \ldots, \nu$, we define P by the inequalities

$$\xi_{j_1 \ldots j_\nu} \geq 0 \quad \text{for all} \quad j_1, \ldots, j_\nu$$

and equations

$$
(4.1) \qquad
\begin{gathered}
\sum_{j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_\nu} \xi_{j_1 \dots j_{i-1}, j, j_{i+1} \dots j_\nu} = \beta_{ij} \\
\text{for} \quad i = 1, \dots, \nu \quad \text{and} \quad 1 \leq j \leq k_i - 1 \quad \text{and} \\
\sum_{j_1, \dots, j_\nu} \xi_{j_1 \dots j_\nu} = N.
\end{gathered}
$$

Let us choose a pair of indices $1 \leq i \leq \nu$ and $1 \leq j \leq k_i - 1$. We call the first sum in (4.1) the *j-th sectional sum in direction i*. Hence for each direction $i = 1, \dots, \nu$ we prescribe all but the last one sectional sum and also prescribe the total sum of the entries of the array.

We observe that every column a of the matrix A of the system (4.1) contains at most $\nu + 1$ non-zero entries (necessarily equal to 1), so $\|a\| \leq \sqrt{\nu + 1}$.

Let $z = (\zeta_{j_1 \dots j_\nu})$ be the point maximizing

$$f(z) = k_1 \cdots k_\nu + \sum_{j_1, \dots, j_\nu} \ln \xi_{j_1 \dots j_\nu}$$

on P. We describe the quadratic form $q : \mathbb{R}^d \longrightarrow \mathbb{R}$ which Theorem 2.2 associates with system (4.1). We have $d = k_1 + \ldots + k_\nu - \nu + 1$ and it is convenient to think of $\mathbb{R}^d$ as of a particular coordinate subspace of a bigger space $V = \mathbb{R}^{k_1} \oplus \ldots \oplus \mathbb{R}^{k_\nu} \oplus \mathbb{R}$. Namely, we think of V as of the set of vectors (t, ω), where

$$t = (\tau_{ij}) \quad \text{for} \quad 1 \leq j \leq k_i \quad \text{and} \quad i = 1, \dots, \nu,$$

and τ_{ij} and ω are real numbers. We identify $\mathbb{R}^d$ with the coordinate subspace defined by the equations

$$\tau_{1k_1} = \tau_{2k_2} = \ldots = \tau_{\nu k_\nu} = 0.$$

Next, we define a quadratic form $p : V \longrightarrow \mathbb{R}$ by

$$p(t, \omega) = \frac{1}{2} \sum_{j_1, \dots, j_\nu} \zeta^2_{j_1 \dots j_\nu} \left(\tau_{1j_1} + \ldots + \tau_{\nu j_\nu} + \omega\right)^2.$$

Then the quadratic form q of Theorem 2.2 is the restriction of p onto $\mathbb{R}^d$.

To bound the eigenvalues of q from below, we consider a simpler quadratic form $\hat{q}$ which is the restriction of

$$\hat{p}(t, \omega) = \sum_{j_1, \dots, j_\nu} \left(\tau_{1j_1} + \ldots + \tau_{\nu j_\nu} + \omega\right)^2$$

onto $\mathbb{R}^d$.

For $i=1, \ldots, \nu$, let us consider the (k_i-2)-dimensional subspace $H_i \subset \mathbb{R}^d$ defined by the equations

$$\sum_{j=1}^{k_i-1} \tau_{ij}=0, \quad \tau_{i'j}=0 \quad \text{for} \quad i' \neq i \quad \text{and all} \quad j, \quad \text{and} \quad \omega=0.$$

Then H_i is an eigenspace of $\hat{q}$ with the eigenvalue

$$\lambda_i = k_1 \cdots k_{i-1} k_{i+1} \cdots k_\nu,$$

since the gradient of $\hat{q}$ at $x \in H_i$ is equal to $2\lambda_i x$. Let $L \subset \mathbb{R}^d$ be the orthogonal complement to $H_1 \oplus \ldots \oplus H_\nu$ in $\mathbb{R}^d$. Then $\dim L = \nu+1$ and L consists of the vectors

$$\Bigl(\underbrace{\alpha_1, \ldots, \alpha_1}_{k_1-1 \text{ times}}, 0; \underbrace{\alpha_2, \ldots, \alpha_2}_{k_2-1 \text{ times}}, 0; \ldots, \underbrace{\alpha_\nu, \ldots, \alpha_\nu}_{k_\nu-1 \text{ times}}, 0; \omega\Bigr)$$

for some real $\alpha_1, \ldots, \alpha_\nu; \omega$. Denoting

$$\begin{aligned} &\mu_0 = (k_1-1) \cdots (k_\nu-1) \quad \text{and} \\ &\mu_i = (k_1-1) \cdots (k_{i-1}-1)\,(k_{i+1}-1) \cdots (k_\nu-1)\,, \end{aligned}$$

We observe that the restriction of $\hat{q}$ onto L satisfies

$$\begin{aligned} &\hat{q}\Bigl(\underbrace{\alpha_1, \ldots, \alpha_1}_{k_1-1 \text{ times}}, 0; \underbrace{\alpha_2, \ldots, \alpha_2}_{k_2-1 \text{ times}}, 0; \ldots, \underbrace{\alpha_\nu, \ldots, \alpha_\nu}_{k_\nu-1 \text{ times}}, 0; \omega\Bigr) \\ &\quad \geq \ \mu_0 \left(\alpha_1+\ldots+\alpha_\nu+\omega\right)^2 + \sum_{i=1}^{\nu} \mu_i \left(\alpha_1+\ldots+\alpha_{i-1}+\alpha_{i+1}+\ldots+\alpha_\nu+\omega\right)^2. \end{aligned}$$

Since

$$\begin{aligned} &\left(\alpha_1+\ldots+\alpha_\nu+\omega\right)^2 + \sum_{i=1}^{\nu} \left(\alpha_1+\ldots+\alpha_{i-1}+\alpha_{i+1}+\ldots+\alpha_\nu+\omega\right)^2 \\ &\qquad \geq \ \delta \left(\omega^2 + \sum_{i=1}^{\nu} \alpha_i^2\right) \end{aligned}$$

for some $\delta = \delta(\nu) > 0$ and all $\alpha_1, \ldots, \alpha_\nu$ and ω, we conclude that the eigenvalues of $\hat{q}$ exceed

$$\delta(\nu) \min_{i=1,\ldots,\nu} \left(k_i-1\right)^{-2} \prod_{j=1}^{\nu} \left(k_j-1\right),$$

where $\delta(\nu) > 0$ is a constant depending on ν alone.

Suppose now that ν is fixed and let us consider a sequence of polytopes P_n where $k_1, \ldots, k_\nu$ grow roughly proportionately with n and where the coordinates $\zeta_{j_1 \ldots j_\nu}$ remain in the interval between two positive constants. Then the minimum eigenvalue of the quadratic form q in Theorem 2.2 grows as $\Omega\left(n^{\nu-2}\right)$. In particular, if $\nu \geq 5$ then Theorem 2.2 implies that the Gaussian formula (2.1.1) approximates the volume of P_n with a relative error which approaches 0 as n grows.

As an example, let us consider the (dilated) polytope P_k of *polystochastic tensors*, that is $k \times \ldots \times k$ arrays of non-negative numbers with all sums along affine coordinate hyperplanes equal to $k^{\nu-1}$, cf. [Gr92]. By symmetry, we must have

$$\zeta_{j_1 \ldots j_\nu} = 1.$$

Theorem 2.2 implies that for $\nu \geq 5$

$$\operatorname{vol} P_k = \big(1 + o(1)\big) \frac{e^{k^\nu}}{(2\pi)^{(\nu k - \nu + 1)/2}} \quad \text{as} \quad k \longrightarrow +\infty.$$

Interestingly, for $\nu = 2$, where our analysis is not applicable, the formula is smaller by a factor of $e^{1/3}$ than the true asymptotic value computed in [CM07b]. For $\nu = 2$ there is an Edgeworth correction factor to the Gaussian density, cf. [BH09a] and [BH09b].

5. The number of multi-way contingency tables

We apply Theorems 2.4 and 2.6 to compute the number of multi-way contingency tables. The smallest eigenvalue of the quadratic form q is bounded as in Section 4 and hence our main goal is to bound the additive error Δ.

Let us consider the ν-index transportation polytope P of Section 4. We assume that the affine span of P is defined by system (4.1), where numbers β_{ij} are all integer. The integer points in P are called sometimes *multi-way* contingency tables while 0-1 points are called binary multi-way contingency tables, see [Go63] and [DO04].

To bound the additive error term Δ in Theorems 2.4 and 2.6, we construct a set Y_{ij} of $k_1 \times \ldots \times k_\nu$ arrays y of integers such that the total sum of entries of y is 0, the j-th sectional sum in the i-th direction is 1 and all other sectional sums are 0, where by "all other" we mean all but the k_i-th sectional sums in every direction $i = 1, \ldots, \nu$. For that, let us choose $\nu - 1$ integers $m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_\nu$, where

$$1 \leq m_1 \leq k_1, \quad \ldots, \quad 1 \leq m_{i-1} \leq k_{i-1}, \quad 1 \leq m_{i+1} \leq k_{i+1}, \quad \ldots, \quad 1 \leq m_\nu \leq k_\nu$$

and define $y = (\eta_{j_1 \ldots j_\nu})$ by letting

$$\eta_{m_1 \ldots m_{i-1}, j, m_{i+1} \ldots m_\nu} = 1, \quad \eta_{m_1 \ldots m_{i-1}, k_i, m_{i+1} \ldots m_\nu} = -1$$

and letting all other coordinates of y equal to 0.

Thus the set Y_{ij} contains $k_1 \cdots k_{i-1} k_{i+1} \cdots k_\nu$ elements y, and the corresponding quadratic form ψ_{ij} can be written as

$$\begin{aligned}\psi_{ij}(x) =& \frac{1}{|Y_{ij}|} \sum_{m_1,\dots,m_{i-1},m_{i+1},\dots,m_\nu} \left(\xi_{m_1 \cdots m_{i-1},j,m_{i+1} \cdots m_\nu} - \xi_{m_1 \cdots m_{i-1},k_i,m_{i+1} \cdots m_\nu}\right)^2 \\ &\text{for} \quad x = (\xi_{j_1 \dots j_\nu}),\end{aligned}$$

from which the maximum eigenvalue ρ_{ij} of ψ_{ij} is $2/k_1 \cdots k_{i-1} k_{i+1} \cdots k_\nu$.

Next, we construct a set Y_0 of arrays y of $k_1 \cdots k_\nu$ integers $(\eta_{j_1 \dots j_\nu})$ such that the total sum of entries of y is 1 while all sectional sums, with a possible exception of the k_i-th sectional sum in every direction i, are equal 0. For that, let us choose ν integers $m_1, \dots, m_\nu$, where

$$1 \leq m_1 \leq k_1 - 1, \quad \dots, \quad 1 \leq m_\nu \leq k_\nu - 1$$

and define $y = (\eta_{j_1,\dots,j_\nu})$ by letting

$$\begin{aligned} y_{m_1 \dots m_\nu} &= 1 - \nu \\ y_{k_1, m_2 \dots m_\nu} &= 1 \\ y_{m_1, k_2, m_3 \dots m_\nu} &= 1 \\ &\cdots\cdots\cdots \\ y_{m_1 \dots m_{\nu-1}, k_\nu} &= 1 \end{aligned}$$

and by letting all other coordinates equal to 0.

The set Y_0 contains $(k_1 - 1) \cdots (k_\nu - 1)$ elements and the corresponding quadratic form ψ_0 of Theorems 2.4 and 2.6 can be written as

$$\begin{aligned}\psi_0(x) =& \frac{1}{|Y_0|} \sum_{y \in Y_0} \langle y, x \rangle^2 \\ =& \frac{1}{|Y_0|} \sum_{\substack{1 \leq m_1 \leq k_1 - 1 \\ \cdots\cdots \\ 1 \leq m_\nu \leq k_\nu - 1}} \left((1-\nu)\xi_{m_1 \dots m_\nu} + \xi_{k_1, m_2 \dots m_\nu} + \dots + \xi_{m_1 \dots m_{\nu-1}, k_\nu}\right)^2 \\ \leq& \frac{(\nu+1)}{|Y_0|} \sum_{\substack{1 \leq m_1 \leq k_1 - 1 \\ \cdots\cdots \\ 1 \leq m_\nu \leq k_\nu - 1}} \left((1-\nu)^2 \xi^2_{m_1 \dots m_\nu} + \xi^2_{k_1, m_2 \dots m_\nu} + \dots + \xi^2_{m_1 \dots m_{\nu-1}, k_\nu}\right).\end{aligned}$$

Therefore, the maximum eigenvalue ρ_0 of ψ_0 does not exceed

$$(\nu+1)(\nu-1)^2 \max_{i=1,\dots,\nu} \left\{ \frac{1}{(k_1 - 1) \cdots (k_{i-1} - 1)\,(k_{i+1} - 1) \cdots (k_\nu - 1)} \right\},$$

and the same bound can be used for the value of ρ in Theorems 2.4 and 2.6.

Suppose now that ν is fixed and let us consider a sequence of polytopes P_n where $k_1, \dots, k_\nu$ grow roughly proportionately with n. Then in Theorems 2.4 and 2.6 we have

$$\rho = O\left(\frac{1}{n^{\nu-1}}\right).$$

Let us apply Theorem 2.6 for counting multi-way binary contingency tables. We assume, additionally, that for the point $z = (\zeta_{j_1 \dots j_\nu})$ maximizing

$$f(x) = \sum_{j_1, \dots, j_\nu} \xi_{j_1 \dots j_\nu} \ln \frac{1}{\xi_{j_1 \dots j_\nu}} + (1 - \xi_{j_1 \dots j_\nu}) \ln \frac{1}{1 - \xi_{j_1 \dots j_\nu}}$$

on the transportation polytope P_n we have

$$1 - \delta \ \geq \ \zeta_{j_1, \dots, j_\nu} \ \geq \ \delta$$

for some constant $1/2 > \delta > 0$ and all $j_1, \dots, j_\nu$. Then we can bound the additive term by

$$|\Delta| \leq \exp\left\{-\gamma \delta n^{\nu-1}\right\}$$

for some constant $\gamma > 0$. On the other hand, by Hadamard's inequality,

$$\det BB^T = n^{O(n)}.$$

Therefore, for $\nu \geq 3$, the additive term Δ is negligible compared to the Gaussian term. From Section 4, we conclude that for $\nu \geq 5$ the relative error for the number of multi-way binary contingency tables in P_n for the Gaussian approximation formula (2.5.1) approaches 0 as n grows.

Similarly, we apply Theorem 2.4 for counting multi-way contingency tables. Here we assume, additionally, that for the point $z = (\zeta_{j_1 \dots j_\nu})$ maximizing

$$f(x) = \sum_{j_1, \dots, j_\nu} (\xi_{j_1 \dots j_\nu} + 1) \ln (\xi_{j_1 \dots j_\nu} + 1) - \xi_{j_1 \dots j_\nu} \ln \xi_{j_1 \dots j_\nu}$$

on the transportation polytope P_n the numbers $\zeta_{j_1 \dots j_\nu}$ lie between two positive constants. As in the case of binary tables, we conclude that for $\nu \geq 3$, the additive error term Δ is negligible compared to the Gaussian approximation term as $n \longrightarrow +\infty$. Therefore, for $\nu \geq 5$ the relative error for the number of multi-way contingency tables in P_n for the Gaussian approximation formula (2.3.1) approaches 0 as n grows.

Computations show that in the case of $k_1 = \dots = k_\nu = k$ for the matrix A of constraints in Theorems 2.4 and 2.6 we have

$$\det AA^T = k^{(\nu^2 - \nu)(k-1)}.$$

Hence we obtain, for example, that the number of non-negative integer ν-way $k \times \ldots \times k$ contingency tables with all sectional sums equal to $r = \alpha k^{\nu-1}$ is

$$\Big(1+o(1)\Big)\left((\alpha+1)^{\alpha+1}\alpha^{-\alpha}\right)^{k^{\nu}}\left(2\pi\alpha^2+2\pi\alpha\right)^{-(k\nu-\nu+1)/2}k^{(\nu-\nu^2)(k-1)/2}$$

provided $\nu \geq 5$, $k \longrightarrow +\infty$ and α stays between two positive constants.

For $\nu = 2$, the integer points in polytope P are the two-way contingency tables. There are several articles in the statistical literature estimating the number of such tables. For example, Good [Go76] uses an approximation based on the negative binomial distribution, when all the row sums are equal and all the column sums are equal, Diaconis and Efron [DE85] examine the distribution of the χ^2 statistic for uniformly distributed tables with given margins, Chen, Diaconis, Holmes and Liu [C+05] consider importance sampling methods for sampling from the tables that lead to good estimates of the number of tables when the sample size is large enough. Zipunnikov, Booth and Yoshida [Z+09] use independent geometric variables with the same parameters that we use, although they do not give the maximum entropy justification for the use. In addition, their estimate of the total number of tables differs from ours by being based on a conditional Gaussian estimate of the number of tables with the specified column sums, given the specified row sums, multiplied by an estimate similar to that of [Go76] of the number of tables with the specified row sums. In the case of $\nu = 2$, both the estimate of [Z+09] and the maximum entropy Gaussian estimate differ asymptotically from the number of tables by a constant order factor, which we show in [BH09a] and [BH09b] may be corrected by an Edgeworth term involving third and fourth moments; see also Canfield and McKay [CM07a] for the correction in the case where all the row sums are equal and all the column sums are equal.

Similarly, the number of binary ν-way $k \times \ldots \times k$ binary contingency tables with all sectional sums equal to $r = \alpha k^{\nu-1}$ is

$$\Big(1+o(1)\Big)\left(\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{-k^{\nu}}\left(2\pi\alpha-2\pi\alpha^2\right)^{-(k\nu-\nu+1)/2}k^{(\nu-\nu^2)(k-1)/2}$$

as long as $\nu \geq 5$, $k \longrightarrow +\infty$ and α remains separated from 0 and 1. Again, for $\nu = 2$ the formula is off by a constant factor from the asymptotic obtained in [C+08].

6. Proof of Theorem 2.2

We treat Theorem 2.2 as a Local Central Limit Theorem type result and prove it using the method of characteristic functions, see, for example, Chapter VII of [Pe75]. In contrast to the setting of [Pe75], we have to deal with sums of independent random vectors where both the number of vectors *and* their dimension may vary.

We start with some standard technical results.

(6.1) Lemma. *Let $x_1, \ldots, x_n$ be independent exponential random variables such that $\mathbf{E}\, x_j = \zeta_j$ for $j = 1, \ldots, n$, let $a_1, \ldots, a_n \in \mathbb{R}^d$ be vectors which span $\mathbb{R}^d$ and let $Y = x_1 a_1 + \ldots + x_n a_n$. Then the density of Y at $b \in \mathbb{R}^d_+$ is equal to*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle b, t\rangle} \left(\prod_{j=1}^n \frac{1}{1 - i\zeta_j \langle a_j, t\rangle} \right) \ dt.$$

Proof. The characteristic function of Y is

$$\mathbf{E}\, e^{i\langle Y, t\rangle} = \prod_{j=1}^n \frac{1}{1 - i\zeta_j \langle a_j, t\rangle}.$$

The proof now follows by the inverse Fourier transform formula. □

We need some standard estimates.

(6.2) Lemma. *Let $q : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a positive definite quadratic form and let $\omega > 0$ be a number.*

(1) *Suppose that $\omega \geq 3$. Then*

$$\int_{t:\ q(t) \geq \omega d} e^{-q(t)} \ dt \ \leq \ e^{-\omega d/2} \int_{\mathbb{R}^d} e^{-q(t)} \ dt.$$

(2) *Suppose that for some $\lambda > 0$ we have*

$$q(t) \ \geq \ \lambda \|t\|^2 \quad \text{for all} \quad t \in \mathbb{R}^d.$$

Let $a \in \mathbb{R}^d$ be a vector. Then

$$\int_{t:\ |\langle a, t\rangle| > \omega \|a\|} e^{-q(t)} \ dt \ \leq \ e^{-\lambda\omega^2} \int_{\mathbb{R}^d} e^{-q(t)} \ dt.$$

Proof. We use the Laplace transform method. For every $1 > \alpha > 0$ we have

$$\begin{aligned} \int_{t:\ q(t) \geq \omega d} e^{-q(t)} \ dt \ &\leq \int_{t:\ q(t) \geq \omega d} \exp\left\{\alpha\big(q(t) - \omega d\big) - q(t)\right\} \ dt \\ &\leq e^{-\alpha\omega d} \int_{\mathbb{R}^d} \exp\big\{-(1-\alpha) q(t)\big\} \ dt \\ &= \frac{e^{-\alpha\omega d}}{(1-\alpha)^{d/2}} \int_{\mathbb{R}^d} e^{-q(t)} \ dt. \end{aligned}$$

Optimizing on α, we choose $\alpha = 1 - 1/2\omega$ to conclude that

$$\int_{t:\ q(t) \geq \omega d} e^{-q(t)} \ dt \ \leq \ \exp\left\{-\omega d + \frac{d}{2} + \frac{d}{2} \ln(2\omega)\right\} \int_{\mathbb{R}^d} e^{-q(t)} \ dt.$$

Since

$$\ln(2\omega) \leq \omega - 1 \quad \text{for} \quad \omega \geq 3,$$

Part (1) follows.

Without loss of generality we assume that $a \neq 0$ in Part (2). Let us consider the Gaussian probability distribution on $\mathbb{R}^d$ with the density proportional to e^{-q}. Then $z = \langle a, t \rangle$ is a Gaussian random variable such that $\mathbf{E}\, z = 0$ and $\mathbf{var}\, z \leq \|a\|^2/2\lambda$. Part (2) now follows from the inequality

$$\mathbf{Pr}\left\{ |y| \geq \tau \right\} \ \leq \ e^{-\tau^2/2}$$

for the standard Gaussian random variable y. $\square$

(6.3) Lemma. *For $\rho \geq 0$ and $k > d$ we have*

$$\int_{t \in \mathbb{R}^d: \ \|t\| \geq \rho} \left(1 + \|t\|^2\right)^{-k/2} \ dt \ \leq \ \frac{2\pi^{d/2}}{\Gamma(d/2)(k-d)} \left(1 + \rho^2\right)^{(d-k)/2}.$$

Proof. Let $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ be the unit sphere in $\mathbb{R}^d$. We recall the formula for the surface area of $\mathbb{S}^{d-1}$:

$$\left|\mathbb{S}^{d-1}\right| = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

We have

$$\begin{aligned}\int_{t \in \mathbb{R}^d: \ \|t\| \geq \rho} \left(1 + \|t\|^2\right)^{-k/2} \ dt =& \left|\mathbb{S}^{d-1}\right| \int_{\rho}^{+\infty} \left(1 + \tau^2\right)^{-k/2} \tau^{d-1} \ d\tau \\ \leq& \left|\mathbb{S}^{d-1}\right| \int_{\rho}^{+\infty} \left(1 + \tau^2\right)^{(d-k-2)/2} \tau \ d\tau,\end{aligned}$$

where we used that

$$\tau^{d-1} = \tau\tau^{d-2} \leq \tau \left(1 + \tau^2\right)^{(d-2)/2}.$$

The proof now follows. $\square$

Now we are ready to prove Theorem 2.2.

(6.4) Proof of Theorem 2.2. Scaling vectors a_j if necessary, without loss of generality we may assume that $\theta = 1$.

From Section 3.7 and Lemma 6.1, we have

$$\operatorname{vol} P = e^{f(z)} \left(\det AA^T\right)^{1/2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle b, t\rangle} \left(\prod_{j=1}^{n} \frac{1}{1 - i\zeta_j \langle a_j, t \rangle} \right) \ dt.$$

Hence our goal is to estimate the integral and, in particular, to compare it with

$$\int_{\mathbb{R}^d} e^{-q(t)} \ dt = (2\pi)^{d/2} \left(\det BB^T\right)^{-1/2}.$$

Let us denote

$$F(t) = e^{-i\langle b,t\rangle} \left(\prod_{j=1}^{n} \frac{1}{1 - i\zeta_j \langle a_j, t\rangle}\right) \quad \text{for} \quad t \in \mathbb{R}^d.$$

Let

$$\sigma = 4d + 10 \ln \frac{1}{\epsilon}.$$

We estimate the integral separately over the three regions:

the outer region $\|t\| \geq 1/2$

the inner region $q(t) \leq \sigma$

the middle region $\|t\| < 1/2$ and $q(t) > \sigma$.

We note that for a sufficiently large constant γ we have $q(t) > \sigma$ in the outer region, we have $\|t\| < 1/2$ in the inner region and the three regions form a partition of $\mathbb{R}^d$.

We start with the outer region $\|t\| \geq 1/2$. Our goal is to show that the integral is negligible there.

We have

$$|F(t)| = \left(\prod_{j=1}^{n} \frac{1}{1 + \zeta_j^2 \langle a_j, t\rangle^2}\right)^{1/2}.$$

Let us denote

$$\xi_j = \zeta_j^2 \langle a_j, t\rangle^2 \quad \text{for} \quad j = 1, \ldots, n.$$

The minimum value of the log-concave function

$$\prod_{j=1}^{n} (1 + \xi_j)$$

on the polytope

$$\sum_{j=1}^{n} \xi_j \ \geq \ 2\lambda \|t\|^2 \quad \text{and} \quad 0 \ \leq \ \xi_j \ \leq \ \|t\|^2$$

is attained at an extreme point of the polytope, that is, at a point where all but possibly one coordinate ξ_j is either 0 or $\|t\|^2$. Therefore,

$$\left(\prod_{j=1}^n \frac{1}{1+\zeta_j^2 \langle a_j, t\rangle^2} \right)^{1/2} \leq \left(1+\|t\|^2\right)^{-\lambda+1/2}.$$

Applying Lemma 6.3, we conclude that

$$\int_{t \in \mathbb{R}^d:\ \|t\| \geq 1/2} |F(t)|\ dt \ \leq \ \frac{2\pi^{d/2}}{\Gamma(d/2)(2\lambda-d-1)} \left(\frac{5}{4}\right)^{(d-2\lambda+1)/2}.$$

By the Binet-Cauchy formula and the Hadamard bound,

$$\det BB^T \leq \binom{n}{d} \leq n^d.$$

It follows then that for a sufficiently large absolute constant γ and the value of the integral over the outer region does not exceed $(\epsilon/10)(2\pi)^{d/2} \det(BB^T)^{-1/2}$.

Next, we estimate the integral over the middle region with $\|t\| < 1/2$ and $q(t) > \sigma$. Again, our goal is to show that the integral is negligible.

From the estimate

$$\left| \ln(1+\xi) - \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3} \right| \ \leq \ \frac{|\xi|^4}{2} \quad \text{for all complex} \quad |\xi| \leq \frac{1}{2},$$

we can write

$$\ln\left(1 - i\zeta_j \langle a_j, t\rangle\right) = -i\zeta_j \langle a_j, t\rangle + \frac{1}{2}\zeta_j^2 \langle a_j, t\rangle^2 + \frac{i}{3}\zeta_j^3 \langle a_j, t\rangle^3 + g_j(t)\zeta_j^4 \langle a_j, t\rangle^4,$$

where

$$|g_j(t)| \ \leq \ \frac{1}{2} \quad \text{for} \quad j=1, \ldots, n.$$

Since

$$\sum_{j=1}^n \zeta_j a_j = b,$$

we have

$$\begin{aligned} F(t) &= \exp\left\{-q(t) - if(t) + g(t)\right\} \\ &\text{where} \quad f(t) = \frac{1}{3}\sum_{j=1}^n \zeta_j^3 \langle a_j, t\rangle^3 \quad \text{and} \\ &|g(t)| \leq \frac{1}{2}\sum_{j=1}^n \zeta_j^4 \langle a_j, t\rangle^4. \end{aligned} \tag{6.4.1}$$

In particular,

$$|F(t)| \leq e^{-3q(t)/4} \quad \text{provided} \quad \|t\| \leq 1/2.$$

Therefore, by Part (1) of Lemma 6.2 we have

$$\begin{aligned}\left| \int\limits_{\substack{\|t\| \leq 1/2 \\ q(t) > \sigma}} F(t) \ dt \right| \leq & \int_{t:\ q(t) > \sigma} e^{-3q(t)/4} \ dt \\ \leq & e^{-3d/2} \epsilon^3 \int_{\mathbb{R}^d} e^{-3q(t)/4} \ dt \\ \leq & \epsilon^3 \int_{\mathbb{R}^d} e^{-q(t)} \ dt.\end{aligned}$$

Finally, we estimate the integral over the inner region where $q(t) < \sigma$ and, necessarily, $\|t\| < 1/2$. Here our goal is to show that the integral is very close to $\displaystyle\int_{\mathbb{R}^d} e^{-q(t)} \ dt$.

From (6.4.1), we obtain

$$\begin{aligned}\left| \int_{t:\ q(t) < \sigma} F(t) \ dt - \int_{t:\ q(t) < \sigma} e^{-q(t)} \ dt \right| \\ \leq \int_{t:\ q(t) < \sigma} e^{-q(t)} \left| e^{-if(t) + g(t)} - 1 \right| \ dt.\end{aligned} \tag{6.4.2}$$

If $q(t) < \sigma$ then $\|t\|^2 \leq \sigma/\lambda$ and hence

$$|g(t)| \ \leq \ \frac{1}{2} \sum_{j=1}^{n} \zeta_j^4 \langle a_j, t \rangle^4 \ \leq \ \frac{\sigma}{2\lambda} \sum_{j=1}^{n} \zeta_j^2 \langle a_j, t \rangle^2 = \frac{\sigma^2}{\lambda}.$$

Thus for all sufficiently large γ, we have $|g(t)| \leq \epsilon/10$.

Let

$$X = \left\{ t: \quad q(t) < \sigma \quad \text{and} \quad \zeta_j |\langle a_j, t \rangle| \ \leq \frac{\epsilon}{10\sigma} \quad \text{for} \quad j = 1, \ldots, n \right\}.$$

By Part (2) of Lemma 6.2, for all sufficiently large γ, we have

$$\int_{\mathbb{R}^d \setminus X} e^{-q(t)} \ dt \ \leq \ \frac{\epsilon}{10} \int_{\mathbb{R}^d} e^{-q(t)} \ dt$$

whereas for $t \in X$ we have

$$|f(t)| \ \leq \ \frac{1}{3} \sum_{j=1}^{n} \zeta_j^3 \left| \langle a_j, t \rangle \right|^3 \ \leq \ \frac{\epsilon}{30\sigma} \sum_{j=1}^{n} \zeta_j^2 \langle a_j, t \rangle^2 \ \leq \frac{\epsilon}{15}.$$

Estimating

$$\left|e^{-if(t)+g(t)}-1\right| \leq \frac{\epsilon}{3} \quad \text{for} \quad t \in X \quad \text{and} \quad \left|e^{-if(t)+g(t)}-1\right| \leq 3 \quad \text{for} \quad t \notin X$$

we deduce from (6.4.2) that

$$\begin{aligned}\left|\int_{t:\ q(t)<\sigma} F(t)\ dt - \int_{t:\ q(t)<\sigma} e^{-q(t)}\ dt\right| \leq & 3\int_{\mathbb{R}^d\setminus X} e^{-q(t)}\ dt + \frac{\epsilon}{3}\int_X e^{-q(t)}\ dt \\ \leq & \frac{2\epsilon}{3}\int_{\mathbb{R}^d} e^{-q(t)}\ dt.\end{aligned}$$

Since by Part (1) of Lemma 6.2, we have

$$\int_{t:\ q(t)>\sigma} e^{-q(t)}\ dt \ \leq \ e^{-2d}\epsilon^5 \int_{\mathbb{R}^d} e^{-q(t)}\ dt,$$

the proof follows. □

7. Proof of Theorem 2.6

First, we represent the number of 0-1 points as an integral.

(7.1) Lemma. *Let p_j, q_j be positive numbers such that $p_j+q_j=1$ for $j=1,\ldots,n$ and let μ be the Bernoulli measure on the set $\{0,1\}^n$ of 0-1 vectors:*

$$\mu\{x\} = \prod_{j=1}^n p_j^{1-\xi_j} q_j^{\xi_j} \quad \text{for} \quad x=(\xi_1,\ldots,\xi_n).$$

Let $P \subset \mathbb{R}^n$ be a polyhedron defined by a vector equation

$$\xi_1 a_1 + \ldots + \xi_n a_n = b$$

for some integer vectors $a_1,\ldots,a_n; b \in \mathbb{Z}^d$ and inequalities

$$0 \ \leq \ \xi_1,\ldots,\xi_n \ \leq \ 1.$$

Let $\Pi \subset \mathbb{R}^d$ be the parallelepiped consisting of the points $t=(\tau_1,\ldots,\tau_d)$ such that

$$-\pi \ \leq \ \tau_k \ \leq \ \pi \quad \text{for} \quad k=1,\ldots,d.$$

Then, for

$$\mu(P) = \sum_{x \in P\cap\{0,1\}^n} \mu\{x\}$$

we have

$$\mu(P) = \frac{1}{(2\pi)^d} \int_\Pi e^{-i\langle t,b\rangle} \prod_{j=1}^n \left(p_j + q_j e^{i\langle a_j,t\rangle}\right) \ dt.$$

Here $\langle\cdot,\cdot\rangle$ is the standard scalar product in $\mathbb{R}^d$ and dt is the Lebesgue measure on $\mathbb{R}^d$.

Proof. The result follows from the expansion

$$\prod_{j=1}^n \left(p_j + q_j e^{i\langle a_j,t\rangle}\right) = \sum_{\substack{x\in\{0,1\}^n \\ x=(\xi_1,\ldots,\xi_n)}} \exp\left\{i\langle \xi_1 a_1 + \ldots + \xi_n a_n, \ t\rangle\right\} \prod_{j=1}^n p_j^{1-\xi_j} q_j^{\xi_j}$$

and the identity

$$\frac{1}{(2\pi)^d} \int_\Pi e^{i\langle u,t\rangle} \ dt = \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{if } \ u \in \mathbb{Z}^d \setminus \{0\}. \end{cases}$$

□

The integrand

$$\prod_{j=1}^n \left(p_j + q_j e^{i\langle a_j,t\rangle}\right)$$

is the characteristic function of $Y = AX$ where X is the multivariate Bernoulli random variable and A is the matrix with the columns $a_1, \ldots, a_n$.

The following result is crucial for bounding the additive error Δ.

(7.2) Lemma. *Let A be a $d\times n$ integer matrix with the columns $a_1, \ldots, a_n \in \mathbb{Z}^d$. For $k = 1, \ldots, d$, let $Y_k \subset \mathbb{Z}^n$ be a non-empty finite set such that $Ay = e_k$ for all $y \in Y_k$, where e_k is the k-th standard basis vector. Let $\psi_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form,*

$$\psi_k(x) = \frac{1}{|Y_k|} \sum_{y\in Y_k} \langle y, x\rangle^2 \quad \text{for} \quad x \in \mathbb{R}^n,$$

and let ρ_k be the maximum eigenvalue of of ψ_k.

Suppose further that $0 < \zeta_1, \ldots, \zeta_n < 1$ are numbers such that

$$\zeta_j(1-\zeta_j) \geq \alpha \quad \text{for some} \quad 0 < \alpha \leq 1/4 \quad \text{and} \quad j = 1, \ldots, n.$$

Then for $t = (\tau_1, \ldots, \tau_d)$ where $-\pi \leq \tau_k \leq \pi$ for $k = 1, \ldots, d$ we have

$$\left| \prod_{j=1}^n \left(1 - \zeta_j + \zeta_j e^{i\langle a_j,t\rangle}\right) \right| \ \leq \ \exp\left\{-\frac{\alpha\tau_k^2}{5\rho_k}\right\}.$$

Proof. Let us denote

$$F(t)=\prod_{j=1}^{n}\left(1-\zeta_j+\zeta_j e^{i\langle a_j,t\rangle}\right).$$

Then

$$|F(t)|^2=\prod_{j=1}^{n}\left((1-\zeta_j)^2+2\zeta_j(1-\zeta_j)\cos\langle a_j,t\rangle+\zeta_j^2\right).$$

For real numbers ξ,η, we write

$$\xi\equiv\eta \mod 2\pi$$

if $\xi-\eta$ is an integer multiple of 2π. Let

$$-\pi\le\gamma_j\le\pi \quad \text{for} \quad j=1,\ldots,n$$

be numbers such that

$$\langle a_j,t\rangle\equiv\gamma_j \mod 2\pi \quad \text{for} \quad j=1,\ldots,n.$$

Hence we can write

$$|F(t)|^2=\prod_{j=1}^{n}\left((1-\zeta_j)^2+2\zeta_j(1-\zeta_j)\cos\gamma_j+\zeta_j^2\right).$$

Since

$$\cos\gamma \ \le\ 1-\frac{\gamma^2}{5} \quad \text{for} \quad -\pi\le\gamma\le\pi,$$

we have

$$|F(t)|^2\le\prod_{j=1}^{n}\left(1-\frac{2\zeta_j(1-\zeta_j)}{5}\gamma_j^2\right)\ \le\ \exp\left\{-\frac{2\alpha}{5}\sum_{j=1}^{n}\gamma_j^2\right\}. \tag{7.2.1}$$

Let

$$c=(\gamma_1,\ldots,\gamma_n), \quad c\in\mathbb{R}^n.$$

Then for all $y\in Y_k$ we have

$$\tau_k=\langle e_k,t\rangle=\langle Ay,t\rangle=\langle y,A^*t\rangle\equiv\langle y,c\rangle \mod 2\pi,$$

where A^* is the transpose matrix of A. Since $|\tau_k|\le\pi$, we have

$$|\langle y,c\rangle|\ \ge\ |\tau_k| \quad \text{for all} \quad y\in Y_k.$$

Therefore,

$$\|c\|^2\ \ge\ \frac{1}{\rho_k}\psi_k(c)\ =\ \frac{1}{\rho_k|Y_k|}\sum_{y\in Y_k}\langle y,c\rangle^2\ \ge\ \frac{\tau_k^2}{\rho_k}.$$

The proof follows by (7.2.1). □

(7.3) Proof of Theorem 2.6. By Theorem 3.3 and Lemma 7.1, we write

$$|P \cap \{0,1\}^n| = \frac{e^{h(z)}}{(2\pi)^d} \int_{\Pi} e^{-i\langle b,t\rangle} \prod_{j=1}^n \left(1 - \zeta_j + \zeta_j e^{i\langle a_j,t\rangle}\right) \ dt, \tag{7.3.1}$$

where Π is the parallelepiped consisting of the points $t = (\tau_1, \ldots, \tau_d)$ with $-\pi \leq \tau_k \leq \pi$ for $k = 1, \ldots, d$.

Let us denote

$$F(t) = e^{-i\langle b,t\rangle} \prod_{j=1}^n \left(1 - \zeta_j + \zeta_j e^{i\langle a_j,t\rangle}\right).$$

If

$$\|t\|_\infty \leq \frac{1}{4\theta},$$

we have

$$|\langle a_j, t\rangle| \leq \frac{1}{4} \quad \text{for} \quad j = 1, \ldots, n.$$

Using the estimate

$$\left|e^{i\xi} - 1 - i\xi + \frac{\xi^2}{2} + i\frac{\xi^3}{6}\right| \leq \frac{\xi^4}{24} \quad \text{for all real} \quad \xi,$$

we can write

$$e^{i\langle a_j,t\rangle} = 1 + i\langle a_j,t\rangle - \frac{\langle a_j,t\rangle^2}{2} - i\frac{\langle a_j,t\rangle^3}{6} + g_j(t)\langle a_j,t\rangle^4,$$
$$\text{where} \quad |g_j(t)| \leq \frac{1}{24} \quad \text{for} \quad j = 1, \ldots, n.$$

Therefore,

$$F(t) = e^{-i\langle b,t\rangle} \prod_{j=1}^n \left(1 + i\zeta_j\langle a_j,t\rangle - \zeta_j\frac{\langle a_j,t\rangle^2}{2} - i\zeta_j\frac{\langle a_j,t\rangle^3}{6} + \zeta_j g_j(t)\langle a_j,t\rangle^4\right).$$

Furthermore, using the estimates

$$\left|\ln(1+\xi) - \xi + \frac{\xi^2}{2} - \frac{\xi^3}{3}\right| \leq \frac{|\xi|^4}{2} \quad \text{for all complex} \quad |\xi| \leq 1/2$$

and that

$$\sum_{j=1}^n \zeta_j a_j = b_j,$$

we can write

$$
\begin{aligned}
& F(t) = e^{-q(t)+if(t)+g(t)}, \\
\text{(7.3.2)} \qquad \text{where} \quad & f(t) = \frac{1}{6} \sum_{j=1}^{n} (2\zeta_j - 1) \left(\zeta_j - \zeta_j^2\right) \langle a_j, t \rangle^3 \quad \text{and} \\
& |g(t)| \leq 2 \sum_{j=1}^{n} \langle a_j, t \rangle^4.
\end{aligned}
$$

In particular,

$$
|g(t)| \ \leq \ \frac{1}{4} q(t) \quad \text{provided} \quad \|t\|_\infty \ \leq \ \frac{1}{4\theta}.
$$

Let

$$
\sigma = 4d + 10 \ln \frac{1}{\epsilon}.
$$

We split the integral (7.3.1) over three regions.

The outer region:

$$
\|t\|_\infty \geq \frac{1}{4\theta}.
$$

We let

$$
\Delta = \frac{1}{(2\pi)^d} \int_{\substack{t \in \Pi \\ \|t\|_\infty \geq 1/4\theta}} F(t) \ dt,
$$

and use Lemma 7.2 to bound $|\Delta|$.

The middle region:

$$
q(t) \geq \sigma \quad \text{and} \quad \|t\|_\infty \leq \frac{1}{4\theta}.
$$

From (7.3.2) we obtain

$$
|F(t)| \leq e^{-3q(t)/4}
$$

and as in the proof of Theorem 2.2 (see Section 6.4), we show that the integral over the region is asymptotically negligible for all sufficiently large γ.

The inner region:

$$
q(t) < \sigma.
$$

Here we have

$$
\|t\|_\infty \ \leq \ \|t\| \ \leq \ \frac{\sigma}{\sqrt{\lambda}} \leq \frac{1}{4\theta}
$$

provided γ is sufficiently large.

If $q(t) < \sigma$ then $\|t\|_\infty \leq \|t\| \leq \sqrt{\sigma/\lambda}$ and

$$
|g(t)| \ \leq \ 2 \sum_{j=1}^{n} \langle a_j, t \rangle^4 \ \leq \ 2\theta^2 \frac{\sigma}{\lambda} \sum_{j=1}^{n} \left(\zeta_j - \zeta_j^2\right) \langle a_j, t \rangle^2 \ \leq 4 \ \frac{\theta^2 \sigma^2}{\lambda}.
$$

In particular, if constant γ is large enough, we have $|g(t)| \leq \epsilon/10$.

As in Section 6.4, we define

$$X = \left\{ t: \quad q(t) < \sigma \quad \text{and} \quad |\langle a_j, t \rangle| \leq \frac{\epsilon}{10\sigma} \quad \text{for} \quad j = 1, \ldots, n \right\}.$$

Hence for $t \in X$ we have

$$|f(t)| \ \leq \ \frac{1}{6} \sum_{j=1}^{n} \left(\zeta_j - \zeta_j^2\right) |\langle a_j, t \rangle|^3 \ \leq \ \frac{\epsilon}{60\sigma} \sum_{j=1}^{n} \left(\zeta_j - \zeta_j^2\right) \langle a_j, t \rangle^2 \ \leq \ \frac{\epsilon}{30}.$$

By Part (2) of Lemma 6.2, for all sufficiently large γ, we have

$$\int_{\mathbb{R}^d \setminus X} e^{-q(t)} \ dt \ \leq \ \frac{\epsilon}{10} \int_{\mathbb{R}^d} e^{-q(t)} \ dt$$

and the proof is finished as in Section 6.4. □

8. Proof of Theorem 2.4

We begin with an integral representation for the number of integer points.

(8.1) Lemma. *Let p_j, q_j be positive numbers such that $p_j + q_j = 1$ for $j = 1, \ldots, n$ and let μ be the geometric measure on the set $\mathbb{Z}_+^n$ of non-negative integer vectors:*

$$\mu\{x\} = \prod_{j=1}^{n} p_j q_j^{\xi_j} \quad \textit{for} \quad x = (\xi_1, \ldots, \xi_n).$$

Let $P \subset \mathbb{R}^n$ be a polyhedron defined by a vector equation

$$\xi_1 a_1 + \ldots + \xi_n a_n = b$$

for some integer vectors $a_1, \ldots, a_n; b \in \mathbb{Z}^d$ and inequalities

$$\xi_1, \ldots, \xi_n \geq 0.$$

Let $\Pi \subset \mathbb{R}^d$ be the parallelepiped consisting of the points $t = (\tau_1, \ldots, \tau_d)$ such that

$$-\pi \ \leq \ \tau_k \ \leq \ \pi \quad \textit{for} \quad k = 1, \ldots, d.$$

Then, for

$$\mu(P) = \sum_{x \in P \cap \mathbb{Z}^n} \mu\{x\}$$

we have

$$\mu(P) = \frac{1}{(2\pi)^d} \int_{\Pi} e^{-i\langle t, b \rangle} \prod_{j=1}^{n} \frac{p_j}{1 - q_j e^{i\langle a_j, t \rangle}} \ dt.$$

Here $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^d$ and dt is the Lebesgue measure in $\mathbb{R}^d$.

Proof. As in the proof of Lemma 7.1, the result follows from the multiple geometric expansion

$$\prod_{j=1}^{n} \frac{p_j}{1 - q_j e^{i\langle a_j, t\rangle}} = \sum_{\substack{x \in \mathbb{Z}_+^n \\ x=(\xi_1, \ldots, \xi_n)}} \exp\bigl\{i\langle \xi_1 a_1 + \ldots + \xi_n a_n,\ t\rangle\bigr\} \prod_{j=1}^{n} p_j q_j^{\xi_j}.$$

$\square$

The integrand

$$\prod_{j=1}^{n} \frac{p_j}{1 - q_j e^{i\langle a_j, t\rangle}}$$

is, of course, the characteristic function of $Y = AX$, where X is the multivariate geometric random variable and A is the matrix with the columns $a_1, \ldots, a_n$.

The following result is an analogue of Lemma 7.2.

(8.2) Lemma. *Let A be a $d \times n$ integer matrix with the columns $a_1, \ldots, a_n \in \mathbb{Z}^d$. For $k = 1, \ldots, d$ let $Y_k \subset \mathbb{Z}^d$ be a non-empty finite set such that $Ay = e_k$ for all $y \in Y_k$, where e_k is the k-th standard basis vector in $\mathbb{Z}^d$. Let $\psi_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a quadratic form,*

$$\psi_k(x) = \frac{1}{|Y_k|} \sum_{y \in Y_k} \langle y, x\rangle^2 \quad \text{for} \quad x \in \mathbb{R}^n,$$

and let ρ_k be the maximum eigenvalue of ψ_k. Suppose further that $\zeta_1, \ldots, \zeta_n > 0$ are numbers such that

$$\zeta_j(1 + \zeta_j) \geq \alpha \quad \text{for some} \quad \alpha > 0 \quad \text{and} \quad j = 1, \ldots, n.$$

Then for $t = (\tau_1, \ldots, \tau_d)$ where $-\pi \leq \tau_k \leq \pi$ for $k = 1, \ldots, d$, we have

$$\left| \prod_{j=1}^{n} \frac{1}{1 + \zeta_j - \zeta_j e^{i\langle a_j, t\rangle}} \right| \leq \left(1 + \frac{2}{5}\alpha\pi^2\right)^{-m_k} \quad \text{where} \quad m_k = \left\lfloor \frac{\tau_k^2}{\rho_k \pi^2} \right\rfloor.$$

Proof. Let us denote

$$F(t) = \prod_{j=1}^{n} \frac{1}{1 + \zeta_j - \zeta_j e^{i\langle a_j, t\rangle}}.$$

Then

$$|F(t)|^2 = \prod_{j=1}^{n} \frac{1}{1 + 2\zeta_j\,(1 + \zeta_j)\,(1 - \cos\langle a_j, t\rangle)}.$$

Let

$$-\pi \ \leq \ \gamma_j \ \leq \ \pi \quad \text{for} \quad j=1, \dots, n$$

be numbers such that

$$\gamma_j \equiv \langle a_j, t \rangle \mod 2\pi \quad \text{for} \quad j=1, \dots, n.$$

Hence we can write

$$\begin{aligned} |F(t)|^2 = & \prod_{j=1}^n \frac{1}{1+2\zeta_j\left(1+\zeta_j\right)\left(1-\cos\gamma_j\right)} \\ \leq & \prod_{j=1}^n \frac{1}{1+2\alpha(1-\cos\gamma_j)}. \end{aligned}$$

Since

$$\cos\gamma \ \leq \ 1-\frac{\gamma^2}{5} \quad \text{for} \quad -\pi \leq \gamma \leq \pi,$$

we estimate

$$|F(t)|^2 \ \leq \ \prod_{j=1}^n \left(1+\frac{2}{5}\alpha\gamma_j^2\right)^{-1}. \tag{8.2.1}$$

Let

$$c = (\gamma_1, \dots, \gamma_n).$$

As in the proof of Lemma 7.2, we obtain

$$\|c\|^2 \ \geq \ \frac{\tau_k^2}{\rho_k}.$$

Let us denote $\xi_j = \gamma_j^2$ for $j=1, \dots, n$. The minimum of the log-concave function

$$\sum_{j=1}^n \ln\left(1+\frac{2}{5}\alpha\xi_j\right)$$

on the polytope defined by the inequalities $0 \leq \xi_j \leq \pi^2$ for $j=1, \dots, n$ and

$$\sum_{j=1}^n \xi_j \ \geq \ \frac{\tau_k^2}{\rho_k}$$

is attained at an extreme point of the polytope, where all but possibly one coordinate ξ_j is either 0 or π^2. The number of non-zero coordinates ξ_j is at least $\tau_k^2/\rho_k\pi^2$ and the proof follows by (8.2.1). □

(8.3) Proof of Theorem 2.4. By Theorem 3.1 and Lemma 8.1, we have

$$|P \cap \mathbb{Z}^n| = \frac{e^{g(z)}}{(2\pi)^d} \int_{\Pi} e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1+\zeta_j - \zeta_j e^{i\langle a_j, t\rangle}}\, dt, \tag{8.3.1}$$

where Π is the parallelepiped consisting of the points $t = (\tau_1, \ldots, \tau_d)$ with $-\pi \leq \tau_k \leq \pi$ for $k = 1, \ldots, d$.

Let us denote

$$F(t) = e^{-i\langle t,b\rangle} \prod_{j=1}^{n} \frac{1}{1+\zeta_j - \zeta_j e^{i\langle a_j, t\rangle}}.$$

Similarly to the proof of Theorem 2.6 (see Section 7.3), assuming that $\|t\|_\infty \leq 1/4\theta$, we write

$$F(t) = e^{-q(t) - if(t) + g(t)},$$

$$\text{where} \quad f(t) = \frac{1}{6}\sum_{j=1}^{n} \left(\zeta_j + \zeta_j^2\right)(2\zeta_j + 1)\langle a_j, t\rangle^3 \quad \text{and}$$

$$|g(t)| \leq 2\sum_{j=1}^{n} (1+\zeta_j)^4 \langle a_j, t\rangle^4.$$

We let

$$\sigma = 4d + 10 \ln \frac{1}{\epsilon}$$

and as in the proof of Theorem 2.6 (see Section 7.3), we split the integral (8.3.1) over the three regions:

the outer region: $\|t\|_\infty \geq 1/4\theta$,
the middle region: $q(t) \geq \sigma$ and $\|t\|_\infty \leq 1/4\theta$ and
the inner region: $q(t) < \sigma$.

For the outer region, we let

$$\Delta = \frac{1}{(2\pi)^d} \int\limits_{\substack{t \in \Pi \\ \|t\|_\infty \geq 1/4\theta}} F(t)\, dt$$

and use Lemma 8.2 to bound Δ.

We have

$$|F(t)| \;\leq\; e^{-3q(t)/4}$$

in the middle region and we bound the integral there as in Section 7.3.

In the inner region, we have $\|t\|_\infty \leq \|t\| \leq \sqrt{\sigma/\lambda}$ and

$$|g(t)| \ \leq \ 2\sum_{j=1}^n (1+\zeta_j)^4 \langle a_j, t\rangle^4 \ \leq \ 2\frac{\theta^2\sigma}{\lambda}\sum_{j=1}^n \left(\zeta_j+\zeta_j^2\right)\langle a_j, t\rangle^2 \ \leq \ 4\frac{\theta^2\sigma^2}{\lambda}.$$

We define

$$X = \left\{t: \quad q(t) < \sigma \quad \text{and} \quad (2\zeta_j+1)\left|\langle a_j, t\rangle\right| \leq \frac{\epsilon}{10\sigma} \quad \text{for} \quad j=1, \ldots, n\right\}$$

and note that for $t \in X$ we have

$$|f(t)| \ \leq \ \frac{1}{6}\sum_{j=1}^n (2\zeta_j+1)\left(\zeta_j+\zeta_j^2\right)\left|\langle a_j, t\rangle\right|^3 \ \leq \ \frac{\epsilon}{60\sigma}\sum_{j=1}^n \left(\zeta_j+\zeta_j^2\right)\langle a_j, t\rangle \ \leq \ \frac{\epsilon}{30}.$$

The proof is finished as in Section 7.3. □

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