

CENTRALLY SYMMETRIC POLYTOPES WITH MANY FACES

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ABSTRACT. We present explicit constructions of centrally symmetric polytopes with many faces: (1) we construct a d -dimensional centrally symmetric polytope P with about $3^{d/4} \approx (1.316)^d$ vertices such that every pair of non-antipodal vertices of P spans an edge of P , (2) for an integer $k \geq 2$, we construct a d -dimensional centrally symmetric polytope P of an arbitrarily high dimension d and with an arbitrarily large number N of vertices such that for some $0 < \delta_k < 1$ at least $(1 - (\delta_k)^d) \binom{N}{k}$ k -subsets of the set of vertices span faces of P , and (3) for an integer $k \geq 2$ and $\alpha > 0$, we construct a centrally symmetric polytope Q with an arbitrarily large number of vertices N and of dimension $d = k^{1+o(1)}$ such that at least $(1 - k^{-\alpha}) \binom{N}{k}$ k -subsets of the set of vertices span faces of Q .

1. INTRODUCTION AND MAIN RESULTS

A *polytope* is the convex hull of a set of finitely many points in \mathbb{R}^d . A polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* if $P = -P$. We present explicit constructions of centrally symmetric polytopes with many faces. Recall that a *face* of a convex body is the intersection of the body with a supporting affine hyperplane, see, for example, Chapter II of [Ba02].

A construction of *cyclic polytopes*, which goes back to Carathéodory [Ca11] and was studied by Motzkin [Mo57] and Gale [Ga63], presents a family of polytopes in \mathbb{R}^d with an arbitrarily large number N of vertices, such that the convex hull of every set of $k \leq d/2$ vertices is a face of P . Such a polytope is obtained as the convex hull of a set of N distinct points on the moment curve (t, t^2, \dots, t^d) in \mathbb{R}^d .

The situation with centrally symmetric polytopes is far less understood. A centrally symmetric polytope P is called *k -neighborly* if the convex hull of every set $\{v_1, \dots, v_k\}$ of k vertices of P , not containing a pair of antipodal vertices $v_i = -v_j$, is a face of P . In contrast with polytopes without symmetry, even 2-neighborly

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centrally symmetric polytopes cannot have too many vertices: it was shown in [LN06] that no d -dimensional 2-neighborly centrally symmetric polytope has more than 2^d vertices. Moreover, as was verified in [BN08], the number $f_1(P)$ of edges (1-dimensional faces) of an arbitrary centrally symmetric polytope $P \subset \mathbb{R}^d$ with N vertices satisfies

$$f_1(P) \leq \frac{N^2}{2} (1 - 2^{-d}).$$

Let $f_k(P)$ denote the number of k -dimensional faces of a polytope P . Even more generally, [BN08] proved that for a d -dimensional centrally symmetric polytope P with N vertices,

$$f_{k-1}(P) \leq \frac{N}{N-1} (1 - 2^{-d}) \binom{N}{k}, \quad \text{provided } k \leq d/2.$$

In particular, as the number N of vertices grows while the dimension d of the polytope stays fixed, the fraction of k -tuples v_1, \dots, v_k of vertices of P that do not form the vertex set of a $(k-1)$ -dimensional face of P remains bounded from below by roughly 2^{-d} .

Besides being of intrinsic interest, centrally symmetric polytopes with many faces appear in problems of sparse signal reconstruction, see [Do04], [RV05], and also Section 5. Typically, such polytopes are obtained through a randomized construction, for example, as the orthogonal projection of a high-dimensional cross-polytope (octahedron) onto a random subspace, see [LN06] and [DT09].

In this paper, we present explicit deterministic constructions. First, we construct a d -dimensional 2-neighborly centrally symmetric polytope with roughly $3^{d/4} \approx (1.316)^d$ vertices. Then, for any fixed $k \geq 2$, we verify (again by presenting an explicit construction) that there exists $0 < \delta_k < 1$ such that for an arbitrarily large d and for an arbitrarily large even N , there is a d -dimensional centrally symmetric polytope P with N vertices satisfying

$$f_{k-1}(P) \geq (1 - (\delta_k)^d) \binom{N}{k}.$$

Our construction guarantees that one can take

$$\text{any } \delta_2 > 3^{-1/4} \approx 0.77 \quad \text{and any } \delta_k > (1 - 5^{-k+1})^{5/(24k+4)} \quad \text{for } k > 2$$

provided N and d are sufficiently large. Finally, for an integer $k \geq 2$ and $\alpha > 0$ we construct a centrally symmetric polytope Q of dimension $k^{1+o(1)}$ with an arbitrarily large number of vertices N such that

$$f_{k-1}(Q) \geq (1 - k^{-\alpha}) \binom{N}{k}.$$

We note that the random projection construction cannot produce polytopes with the last two properties since if N is very large compared to d , the projection of a cross-polytope in \mathbb{R}^N onto a random d -dimensional subspace is very close to a Euclidean ball, and hence has few faces relative to the number of vertices, cf. [DT09]. Our constructions are based on the symmetric moment curve introduced in [BN08] and further studied in [B+11].

(1.1) The symmetric moment curve. We define the *symmetric moment curve* $U_k(t) \in \mathbb{R}^{2k}$ by

$$(1.1.1) \quad U_k(t) = \left(\cos t, \sin t, \cos 3t, \sin 3t, \dots, \cos(2k-1)t, \sin(2k-1)t \right)$$

for $t \in \mathbb{R}$. Since

$$U_k(t) = U_k(t + 2\pi) \quad \text{for all } t,$$

from this point on, we consider $U_k(t)$ to be defined on the unit circle

$$\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}.$$

We note that t and $t + \pi$ form a pair of antipodal points for all $t \in \mathbb{S}$ and that

$$U_k(t + \pi) = -U_k(t) \quad \text{for all } t \in \mathbb{S}.$$

First, we construct a 2-neighborly centrally symmetric polytope using the curve

$$U_3(t) = \left(\cos t, \sin t, \cos 3t, \sin 3t, \cos 5t, \sin 5t \right).$$

(1.2) Theorem. *For a non-negative integer m , consider the map*

$$\Psi_m : \mathbb{S} \longrightarrow \mathbb{R}^{6(m+1)} \quad \text{defined by} \quad \Psi_m(t) = \left(U_3(t), U_3(3t), \dots, U_3(3^m t) \right).$$

Let $A_m \subset \mathbb{S}$ be the set of $4 \cdot 3^{m+1}$ equally spaced points,

$$A_m = \left\{ \frac{2\pi j}{4 \cdot 3^{m+1}}, \quad j = 0, \dots, 4 \cdot 3^{m+1} - 1 \right\},$$

and let

$$P_m = \text{conv} \left(\Psi_m(t) : t \in A_m \right).$$

Then P_m is a centrally symmetric polytope of dimension $d = 4m + 6$ that has $4 \cdot 3^{m+1}$ vertices: $\Psi_m(t)$ for $t \in A_m$. Moreover, for $t_1, t_2 \in A_m$ such that $t_1 \neq t_2$ and $t_1 \neq t_2 + \pi \pmod{2\pi}$, the interval

$$[\Psi_m(t_1), \Psi_m(t_2)]$$

is an edge of P_m .

Our construction of a centrally symmetric polytope with N vertices and about $(1 - 3^{-d/4}) \binom{N}{2}$ edges for an arbitrarily large N is a slight modification of the construction presented in Theorem 1.2 — see Remark 3.2. On the other hand, to construct a centrally symmetric polytope with many $(k-1)$ -dimensional faces for $k > 2$, we need to use the curve (1.1.1) to the full extent.

(1.3) Theorem. Fix an integer $k \geq 1$. For a non-negative integer m , consider the map $\Psi_{k,m} : \mathbb{S} \longrightarrow \mathbb{R}^{6k(m+1)}$ defined by

$$\Psi_{k,m}(t) = \left(U_{3k}(t), U_{3k}(5t), \dots, U_{3k}(5^m t) \right).$$

For a positive even integer n , let $A_{m,n} \subset \mathbb{S}$ be the set of $n5^m$ equally spaced points,

$$A_{m,n} = \left\{ \frac{2\pi j}{n5^m} : j = 0, \dots, n5^m - 1 \right\},$$

and let

$$P = P_{k,m,n} = \text{conv} \left(\Psi_{k,m}(t) : t \in A_{m,n} \right).$$

Then

- (1) The polytope $P \subset \mathbb{R}^{6k(m+1)}$ is a centrally symmetric polytope with $n5^m$ distinct vertices:

$$\Psi_{k,m}(t) \quad \text{for } t \in A_{m,n}$$

and of dimension $d \leq 6k(m+1) - 2m \lfloor (3k+2)/5 \rfloor$; moreover, if $n > 2(6k-1)$, then the dimension of P is equal to $6k(m+1) - 2m \lfloor (3k+2)/5 \rfloor$.

- (2) Let t_1, \dots, t_k be points chosen independently at random from the uniform distribution in $A_{m,n}$ (in particular, some of t_i may coincide). Then the probability that

$$\text{conv} \left(\Psi_{k,m}(t_1), \dots, \Psi_{k,m}(t_k) \right)$$

is not a face of P does not exceed

$$(1 - 5^{-k+1})^m.$$

We obtain the following corollary.

(1.4) Corollary. Let $P_{k,m,n}$ be the polytope of Theorem 1.3 with $N = n5^m$ vertices and dimension $d \leq 6k(m+1) - 2m \lfloor (3k+2)/5 \rfloor$. Then

$$f_{k-1}(P_{k,m,n}) \geq \binom{N}{k} - (1 - 5^{-k+1})^m \frac{N^k}{k!}.$$

The construction of Theorem 1.3 produces a family of centrally symmetric polytopes of an increasing dimension d and with an arbitrarily large number of vertices such that for any fixed $k \geq 1$, the probability $p_{d,k}$ that k randomly chosen vertices of the polytope do not span a face decreases exponentially in d . However, it does not start doing so very quickly: for instance, to make $p_{d,k} < 1/2$ we need to choose d as high as $2^{\Omega(k)}$.

Using a trick which the authors learned from Imre Bárány (cf. Section 7.3 of [BN08]), we construct new families of polytopes with many faces of a reasonably high dimension. Namely, we can make $p_{d,k} < d^{-\alpha}$ for any fixed $\alpha > 0$ by using d as low as $k^{1+o(1)}$.

(1.5) Theorem. Fix positive integers k, m, n and r , where n is even. Let $P = P_{k,m,n}$ be the polytope of Theorem 1.3, so that $P \subset \mathbb{R}^{6k(m+1)}$ is a centrally symmetric polytope with $n5^m$ vertices. For $d = 6kr(m+1)$, identify \mathbb{R}^d with a direct sum of r copies of $\mathbb{R}^{6k(m+1)}$, each containing a copy of P . Let Q be the convex hull of the r copies of P ; in particular, $Q \subset \mathbb{R}^d$ is a centrally symmetric polytope with $rn5^m$ vertices.

If

$$r < \min \left\{ (k+1)!, \left(\frac{5^{k-1}}{5^{k-1}-1} \right)^m \right\},$$

then the probability that r vertices of Q , chosen independently at random from the uniform distribution on the set of vertices of Q , span a face of Q is at least

$$\left(1 - \frac{r}{(k+1)!} \right) \left(1 - r(1 - 5^{-k+1})^m \right).$$

If we now fix an $\alpha > 0$ and choose in Theorem 1.5

$$k = \left\lceil \frac{\beta \ln r}{\ln \ln r} \right\rceil \quad \text{and} \quad m = \lceil \beta 5^k \ln r \rceil,$$

then for a suitable $\beta = \beta(\alpha) > 0$ we obtain a centrally symmetric polytope Q of dimension $r^{1+o(1)}$ and with an arbitrarily large number N of vertices such that r random vertices of Q span a face of Q with probability at least $1 - r^{-\alpha}$. As in Corollary 1.4, we have $f_{r-1}(Q) \geq (1 - r^{-\alpha}) \binom{N}{r}$.

In Section 2, we summarize the properties of the symmetric moment curve (1.1.1) and review several basic combinatorial facts needed for our proofs. We then prove Theorem 1.2 in Section 3 and Theorems 1.3 and 1.5 in Section 4. In Section 5, we sketch connections to error-correcting codes.

2. PRELIMINARIES

We utilize the following result of [B+11] concerning the symmetric moment curve (1.1.1).

(2.1) Theorem. Let $\mathcal{B}_k \subset \mathbb{R}^{2k}$,

$$\mathcal{B}_k = \text{conv} \left(U_k(t) : t \in \mathbb{S} \right),$$

be the convex hull of the symmetric moment curve. Then for every positive integer k there exists a number

$$\frac{\pi}{2} < \alpha_k < \pi$$

such that for an arbitrary open arc $\Gamma \subset \mathbb{S}$ of length α_k and arbitrary distinct $n \leq k$ points $t_1, \dots, t_n \in \Gamma$, the set

$$\text{conv} \left(U_k(t_1), \dots, U_k(t_n) \right)$$

is a face of \mathcal{B}_k .

For $k = 2$ with $\alpha_2 = 2\pi/3$ this result is due to Smilansky [Sm85].

We will also need the following technical lemma.

(2.2) Lemma. *Let $t_1, \dots, t_{2k} \in \mathbb{S}$ be distinct points no two of which are antipodal. Then the set of vectors*

$$\{U_k(t_1), \dots, U_k(t_{2k})\}$$

is linearly independent.

Proof. Seeking a contradiction, we assume that these $2k$ vectors are linearly dependent. Then they span a proper subspace in \mathbb{R}^{2k} , and hence there is a non-zero vector $C \in \mathbb{R}^{2k}$ that is orthogonal to all these vectors.

Consider the following trigonometric polynomial

$$f(t) = \langle C, U_k(t) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in \mathbb{R}^{2k} . Then $f(t) \not\equiv 0$ and t_1, \dots, t_{2k} are distinct roots of $f(t)$. Since $f(t + \pi) = -f(t)$, we conclude that $f(t)$ has at least $4k$ roots on the circle \mathbb{S} . On the other hand, substituting $z = e^{it}$, we can write

$$f(t) = \frac{p(z)}{z^{2k-1}},$$

where p is a polynomial with $\deg p \leq 4k - 2$, see [BN08] and [B+11]. Hence $p(z)$ has at least $4k$ distinct roots on the circle $|z| = 1$ and we must have $p(z) \equiv 0$, which is a contradiction. \square

We will also be using the following two well-known facts.

First, if P is a polytope and F is a face of P , then F is a polytope: it is the convex hull of the vertices of P that lie in F . Moreover, every face of F is also a face of P .

Second, if $T : \mathbb{R}^d \rightarrow \mathbb{R}^k$ is a linear transformation and $P \subset \mathbb{R}^d$ is a polytope, then $Q = T(P)$ is a polytope and for every face F of Q the inverse image of F ,

$$T^{-1}(F) = \left\{ x \in P : T(x) \in F \right\},$$

is a face of P . This face is the convex hull of the vertices of P mapped by T into vertices of F .

Finally, to estimate the dimension of the polytope $P_{k,n,m}$ in Theorem 1.3 we will rely on the following combinatorial lemma. For a set U of integers and a constant c , we define $cU := \{cu : u \in U\}$.

(2.3) Lemma. *Let K be the set of all odd integers in the closed interval $[1, 6k - 1]$, and let*

$$T = \bigcup_{j=0}^m 5^j K.$$

Then

$$|T| = 3k(m + 1) - m \lfloor (3k + 2)/5 \rfloor.$$

Proof. Denote by X the set of all elements of K that are not divisible by 5, and by S the complement of X in K . Then the sets $X, 5X, 5^2X, \dots, 5^mX$ are pairwise disjoint and their union consists of all elements of T that are not divisible by 5^{m+1} . On the other hand, every element of T that is divisible by 5^{m+1} is of the form $5^m s$ for some $s \in S$ and every element of the form $5^m s$ for $s \in S$ belongs to T and is divisible by 5^{m+1} . Thus

$$T = \left(\bigcup_{j=0}^m 5^j X \right) \cup 5^m S,$$

and the sets in the above union are pairwise disjoint. Hence

$$|T| = (m+1)|X| + |S| = (m+1)|K| - m|S|.$$

The statement now follows from the fact that there are $3k$ elements in K and that exactly $\lfloor (3k+2)/5 \rfloor$ of them are divisible by 5. \square

3. CENTRALLY SYMMETRIC 2-NEIGHBORLY POLYTOPES

(3.1) Proof of Theorem 1.2. The transformation

$$t \mapsto t + \pi \pmod{2\pi}$$

maps the set A_m onto itself. Since $\Psi_m(t + \pi) = -\Psi_m(t)$, the polytope P_m is centrally symmetric. Consider the projection $\mathbb{R}^{6(m+1)} \rightarrow \mathbb{R}^6$ that forgets all but the first 6 coordinates. Then the image of P_m is the polytope

$$(3.1.1) \quad Q_m = \text{conv} \left(U_3(t) : t \in A_m \right).$$

By Theorem 2.1, the polytope Q_m has $4 \cdot 3^{m+1}$ distinct vertices: $U_3(t)$ for $t \in A_m$. Furthermore, the inverse image of each vertex $U_3(t)$ of Q_m in P_m consists of a single vertex $\Psi_m(t)$ of P_m . Therefore, $\Psi_m(t)$ for $t \in A_m$ are all the vertices of P_m without duplicates.

To compute the dimension d of P_m , we observe that for all $t \in \mathbb{S}$, the third coordinate of $U_3(t)$ coincides with the first coordinate of $U_3(3t)$ while the fourth coordinate of $U_3(t)$ coincides with the second coordinate of $U_3(3t)$. Therefore, the polytope P_m lies in a subspace, denote it by \mathcal{L} , of codimension $2m$, and hence $\dim P_m \leq 4m + 6$. If the dimension of P_m is strictly smaller than $4m + 6$, then P_m lies in an affine hyperplane of \mathcal{L} . As in the proof of Lemma 2.2, such an affine hyperplane corresponds to a trigonometric polynomial $f(t)$ of degree $5 \cdot 3^m$ that has at least $4 \cdot 3^{m+1} = 12 \cdot 3^m$ roots (all points of A_m). This is however impossible, as no nonzero trigonometric polynomial of degree $5 \cdot 3^m$ has more than

$$2 \cdot 5 \cdot 3^m = 10 \cdot 3^m < 12 \cdot 3^m$$

roots (cf. the proof of Lemma 2.2). We conclude that $\dim P_m = 4m + 6$.

We prove that P_m is 2-neighborly by induction on m . It follows from Lemma 2.2 that P_0 is the convex hull of a set consisting of six linearly independent vectors and their opposite vectors. Combinatorially, P_0 is a 6-dimensional cross-polytope and hence the induction base is established.

Suppose now that $m \geq 1$. Let $t_1, t_2 \in A_m$ be such that

$$t_1 \neq t_2, t_2 + \pi \pmod{2\pi}.$$

Then there are two cases to consider.

$$\text{Case I: } t_1 - t_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \pmod{2\pi},$$

and

$$\text{Case II: } t_1 - t_2 \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right) \pmod{2\pi}.$$

In the first case, consider the polytope Q_m defined by (3.1.1) and the projection $P_m \rightarrow Q_m$ as above. By Theorem 2.1,

$$[U_3(t_1), U_3(t_2)]$$

is an edge of Q_m . Since the inverse image of a vertex $U_3(t)$ of Q_m in P_m consists of a single vertex $\Psi_m(t)$ of P_m , we conclude that

$$[\Psi_m(t_1), \Psi_m(t_2)]$$

is an edge of P_m .

In the second case, consider the map $\phi : A_m \rightarrow A_{m-1}$,

$$\phi(t) = 3t \pmod{2\pi}.$$

Then

$$\phi(A_m) = A_{m-1}$$

and for every t the inverse image of t , $\phi^{-1}(t)$, consists of 3 equally spaced points from A_m . In addition, we have

$$\phi(t_1) \neq \phi(t_2) + \pi \pmod{2\pi},$$

although we may have $\phi(t_1) = \phi(t_2)$. In any case, by the induction hypothesis, the interval (possibly contracting to a point)

$$(3.1.2) \quad [\Psi_{m-1}(3t_1), \Psi_{m-1}(3t_2)]$$

is a face of P_{m-1} .

Let us consider the projection $\mathbb{R}^{6(m+1)} \rightarrow \mathbb{R}^{6m}$ that forgets the first 6 coordinates. The image of P_m under this projection is P_{m-1} , and since (3.1.2) is a face of P_{m-1} , the set

$$(3.1.3) \quad \text{conv} \left(\Psi_m(x_{ij}) : \begin{array}{l} \phi(x_{ij}) = \phi(t_i) \quad \text{for } i = 1, 2 \\ \text{and } j = 1, 2, 3 \end{array} \right)$$

is a face of P_m (it is the inverse image of (3.1.2) under this projection). However, the face (3.1.3) is a convex hull of at most six distinct points no two of which are antipodal. Since by Lemma 2.2, any set of at most six distinct points $U_3(x_{ij})$ no two of which are antipodal is linearly independent, the face (3.1.3) is a simplex. Therefore,

$$[\Psi_m(t_1), \Psi_m(t_2)]$$

is a face of (3.1.3), and hence of P_m . \square

(3.2) *Remark.* Tweaking the construction of Theorem 1.2, allows us to produce d -dimensional centrally symmetric polytopes with an arbitrarily large number N of vertices that have at least $(1 - (\delta_2)^d) \binom{N}{2}$ edges, where one can choose any $\delta_2 > 3^{-1/4} \approx 0.77$ for all sufficiently large N and d .

To do so, fix an integer $s \geq 3$, and consider the curve Ψ_m as in Theorem 1.2. However, instead of working with the set A_m as in the proof Theorem 1.2, start with the set

$$W_0 = \left\{ \frac{\pi j}{2} : j = 0, 1, 2, 3 \right\}$$

of 4 equally spaced points on \mathbb{S} . Now replace each point t of W_0 by a cluster of s points on \mathbb{S} that lie very close to t . Moreover, do it in such a way, that the resulting subset of \mathbb{S} , which we denote by W_0^s , is centrally symmetric. For $m \geq 1$, define W_m^s recursively by

$$W_m^s := \phi^{-1}(W_{m-1}^s), \quad \text{where } \phi(x) = 3x \pmod{2\pi}.$$

Thus W_m^s consists of $4 \cdot 3^m$ clusters of s points each.

We claim that the polytope

$$P_m^s := \text{conv}(\Psi_m(t) : t \in W_m^s)$$

is a centrally symmetric polytope of dimension $d = 4m + 6$, with $N = N(s) = 4s \cdot 3^m$ vertices, and such that for every two distinct points $t_1, t_2 \in W_m^s$, the interval $[\Psi_m(t_1), \Psi_m(t_2)]$ is an edge of P_m^s , provided t_1 and t_2 are not from antipodal clusters. The proof of this claim is identical to the proof of Theorem 1.2, except that for the base case (the case of $m = 0$) we appeal to Theorem 2.1.

Thus each vertex of P_m^s is incident to all other vertices except itself and (possibly) the Ψ_m -images of the s points from the antipodal cluster. Therefore, the polytope P_m^s has at least

$$\frac{N(N-s-1)}{2} = \binom{N}{2} \left(1 - \frac{s}{N-1}\right) \approx \binom{N}{2} \left(1 - \frac{1}{4 \cdot 3^m}\right)$$

edges. Taking an arbitrarily large s yields the promised result on δ_2 . \square

4. CENTRALLY SYMMETRIC POLYTOPES WITH MANY FACES

(4.1) Proof of Theorem 1.3. We observe that the transformation

$$t \mapsto t + \pi \pmod{2\pi}$$

maps the set $A_{m,n}$ onto itself and that

$$\Psi_{k,m}(t + \pi) = -\Psi_{k,m}(t) \quad \text{for all } t \in \mathbb{S}.$$

Hence P is centrally symmetric. Consider the projection $\mathbb{R}^{6k(m+1)} \rightarrow \mathbb{R}^{6k}$ that forgets all but the first $6k$ coordinates. Then the image of $P_{k,m,n}$ is the polytope

$$(4.1.1) \quad Q_{k,m,n} = \text{conv}\left(U_{3k}(t) : t \in A_{m,n}\right).$$

By Theorem 2.1, the polytope $Q_{k,m,n}$ has $n5^m$ distinct vertices: $U_{3k}(t)$ for $t \in A_{m,n}$. Furthermore, the inverse image of each vertex $U_{3k}(t)$ of $Q_{k,m,n}$ in $P_{k,m,n}$ consists of a single vertex $\Psi_{k,m}(t)$ of $P_{k,m,n}$. Therefore, $\Psi_{k,m,n}(t)$ for $t \in A_{m,n}$ are all the vertices of $P_{k,m,n}$ without duplicates.

To estimate the dimension of $P = P_{k,m,n}$, we observe that for all $t \in \mathbb{S}$, the fifth coordinate of $U_{3k}(t)$ coincides with the first coordinate of $U_{3k}(5t)$ while the sixth coordinate of $U_{3k}(t)$ coincides with the second coordinate of $U_{3k}(5t)$, etc. Taking into account all coincidences of coordinates, we infer from Lemma 2.3 that the polytope P lies in a subspace of dimension $6k(m+1) - 2m\lfloor(3k+2)/5\rfloor$, and hence $\dim P \leq 6k(m+1) - 2m\lfloor(3k+2)/5\rfloor$. Moreover, if $n > 2(6k-1)$, then an argument identical to the one used in the proof of Theorem 1.2 (by counting roots of trigonometric polynomials) shows that $\dim P = 6k(m+1) - 2m\lfloor(3k+2)/5\rfloor$.

We prove Part (2) by induction on m . The statement trivially holds for $m = 0$. Let us assume that $m \geq 1$ and consider the map $\phi : A_{m,n} \rightarrow A_{m-1,n}$ defined by

$$\phi(t) = 5t \pmod{2\pi}.$$

Then

$$\phi(A_{m,n}) = A_{m-1,n}$$

and for every $t \in A_{m-1,n}$, the inverse image of t , $\phi^{-1}(t)$, consists of 5 equally spaced points from $A_{m,n}$. We note that if t is a random point uniformly distributed in

$A_{m,n}$, then $\phi(t)$ is uniformly distributed in $A_{m-1,n}$. The proof of the theorem will follow from the following two claims.

Claim I. Let $t_1, \dots, t_k \in A_{m,n}$ be arbitrary, not necessarily distinct, points. If

$$(4.1.2) \quad \text{conv}\left(\Psi_{k,m-1}(5t_i), \quad i = 1, \dots, k\right)$$

is a face of $P_{k,m-1,n}$ then

$$(4.1.3) \quad \text{conv}\left(\Psi_{k,m}(t_i), \quad i = 1, \dots, k\right)$$

is a face of $P_{k,m,n}$.

Claim II. Let $s_1, \dots, s_k \in A_{m-1,n}$ be arbitrary, not necessarily distinct, points. Then the conditional probability that

$$\text{conv}\left(\Psi_{k,m}(t_i) : \quad i = 1, \dots, k\right)$$

is not a face of $P_{k,m,n}$ given that

$$\phi(t_i) = s_i \quad \text{for} \quad i = 1, \dots, k$$

does not exceed $1 - 5^{-k+1}$.

To prove Claim I, we consider the projection $\mathbb{R}^{6k(m+1)} \rightarrow \mathbb{R}^{6km}$ that forgets the first $6k$ coordinates. The image of $P_{k,m,n}$ under this projection is $P_{k,m-1,n}$ and if (4.1.2) is a face of $P_{k,m-1,n}$ then

$$(4.1.4) \quad \text{conv}\left(\Psi_{k,m}(x_{ij}) : \quad \phi(x_{ij}) = \phi(t_i) \quad \text{for} \quad i = 1, \dots, k \right. \\ \left. \text{and} \quad j = 1, 2, 3, 4, 5\right)$$

is a face of $P_{k,m,n}$ as it is the inverse image of (4.1.2) under this projection. The face (4.1.4) is the convex hull of at most $5k$ distinct points and no two points x_{ij} in (4.1.4) are antipodal. Since by Lemma 2.2 a set of up to $6k$ distinct points $U_{3k}(x_{ij})$ no two of which are antipodal is linearly independent, the face (4.1.4) is a simplex. Therefore, the set (4.1.3) is a face of (4.1.4), and hence also a face of $P_{k,m,n}$. Claim I now follows.

To prove Claim II, we fix a sequence $s_1, \dots, s_k \in A_{m-1,n}$ of not necessarily distinct points. Then there are exactly 5^k sequences $t_1, \dots, t_k \in A_{m,n}$ of not necessarily distinct points such that $\phi(t_i) = s_i$ for $i = 1, \dots, k$. Choose an arbitrary t_1 subject to the condition $\phi(t_1) = s_1$. Let $\Gamma \subset \mathbb{S}$ be a closed arc of length $2\pi/5$

centered at t_1 . Then for $i = 2, \dots, k$ there is at least one $t_i \in \Gamma$ such that $\phi(t_i) = s_i$. By Theorem 2.1, for such a choice of t_2, \dots, t_k , the set

$$(4.1.5) \quad \text{conv}\left(U_{3k}(t_i) : i = 1, \dots, k\right)$$

is a face of the polytope $Q_{k,m,n}$ defined by (4.1.1). Considering the projection

$$P_{k,m,n} \longrightarrow Q_{k,m,n}$$

as above, we conclude that (4.1.3) is a face of $P_{k,m,n}$ as it is the inverse image of (4.1.5).

Hence the conditional probability that (4.1.3) is not a face is at most

$$\frac{5^{k-1} - 1}{5^{k-1}} = 1 - 5^{-k+1}.$$

□

(4.2) Proof of Corollary 1.4. Let us choose points t_1, \dots, t_k independently at random from the uniform distribution in $A_{m,n}$. Then the probability that the points are all distinct is

$$\frac{(N-1) \cdots (N-k+1)}{N^{k-1}}.$$

From Theorem 1.3, the conditional probability that

$$(4.3.1) \quad \text{conv}\left(\Psi_{k,m}(t_1), \dots, \Psi_{k,m}(t_k)\right)$$

is not a face, given that t_1, \dots, t_k are distinct, does not exceed

$$(1 - 5^{-k+1})^m \frac{N^{k-1}}{(N-1) \cdots (N-k+1)}.$$

Arguing as in the proof of Theorem 1.3 (Section 4.1), we conclude that if t_1, \dots, t_k are distinct and (4.3.1) is a face, then that face is a $(k-1)$ -dimensional simplex.

□

(4.3) Proof of Theorem 1.5. By construction, Q is a centrally symmetric polytope whose vertex set consists of the vertices of the r copies of P . Let us pick r vertices of Q independently at random from the uniform distribution and let k_i be the number of vertices picked from the i -th copy of P , $i = 1, \dots, r$. Then the probability that $k_i > k$ does not exceed

$$\binom{r}{k+1} r^{-k-1} < \frac{1}{(k+1)!}.$$

Therefore, the probability that $k_1, \dots, k_r \leq k$ is at least $1 - r/(k+1)!$. Now, the picked r vertices span a face of Q if and only if for all i with $k_i > 0$ the chosen k_i vertices from the i -th copy of P span a face of P . The result then follows by Theorem 1.3. □

5. CONNECTIONS TO ERROR-CORRECTING CODES

Here we briefly touch upon a well-known connection between centrally symmetric polytopes with many faces and the coding theory, see, for example, [RV05].

Let \mathbb{R}^N be N -dimensional Euclidean space with the standard basis e_1, \dots, e_N and the ℓ^1 -norm

$$\|x\|_1 = \sum_{i=1}^N |x_i| \quad \text{for } x = (x_1, \dots, x_N).$$

Let $L \subset \mathbb{R}^N$ be a subspace, let v_i be the orthogonal projection of e_i onto L , and let

$$P = \text{conv}\left(\pm v_i, \quad i = 1, \dots, N\right)$$

be the orthogonal projection of the standard cross-polytope (octahedron) in \mathbb{R}^N onto L .

Let $L^\perp \subset \mathbb{R}^N$ be the orthogonal complement of L . Suppose that we are given a point $a \in \mathbb{R}^N$, $a = (a_1, \dots, a_N)$, which is obtained by changing (corrupting) some (unknown) k coordinates of an unknown point $c \in L^\perp$, $c = (c_1, \dots, c_N)$, and that our goal is to find c . One, by now standard, way of attempting to do that is to try to find c as the solution to the linear programming problem of minimizing the function

$$(5.1) \quad x \mapsto \|x - a\|_1 \quad \text{for } x \in L^\perp.$$

Indeed, let

$$I_+ = \left\{ i : c_i > a_i \right\} \quad \text{and} \quad I_- = \left\{ i : c_i < a_i \right\}.$$

Then c is the unique minimum point of (5.1) if

$$\text{conv}\left(v_i \text{ for } i \in I_+ \quad \text{and} \quad -v_i \text{ for } i \in I_-\right)$$

is a face of P . By constructing polytopes P with many $(k - 1)$ -dimensional faces we produce subspaces L^\perp with the property that the points of L^\perp can be efficiently reconstructed from many of the different ways of corrupting some k of their coordinates.

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