MATH 669: COMBINATORICS OF POLYTOPES

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Abstract. These are rather condensed notes, not really proofread or edited, presenting key definitions and results of the course that I taught in Winter 2010 term. Problems marked by $^\circ$ are easy and basic, problems marked by * may be difficult.

Contents

1. Basic definitions

We work in a finite-dimensional real vector space V. Once we choose a basis of V, we may identify $V = \mathbb{R}^d$.

(1.1) Definitions. For two points $a, b \in V$, we define the *interval* $[a, b] \subset V$ by

$$[a,b] = \Big\{ x \in V : \ x = \lambda a + (1-\lambda) b : \ 0 \leq \lambda \leq 1 \Big\}.$$

A set $X \subset V$ is *convex* if for all $a, b \in X$ we have $[a, b] \subset X$. The empty set $\emptyset$ is convex. Given points, $a_1, \ldots, a_n \in V$, a point

$$a = \sum_{i=1}^{n} \lambda_i a_i \quad \text{where} \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n$$

is called a *convex combination* of $a_1, \ldots, a_n$. The *convex hull* $\operatorname{conv}(A)$ of a set $A \subset V$ is the set of all convex combinations of points from A. A *polytope* is the convex hull of a finite set of points.

Given a linear functional $\ell : V \longrightarrow \mathbb{R}$, not identically 0, and a real $\alpha \in \mathbb{R}$, the set

$$H_- = \Big\{ x \in V : \ \ell(x) \leq \alpha \Big\}$$

is called a *(closed) halfspace*. A *polyhedron* is the intersection of finitely many halfspaces.

(1.2) Problems.

1°. Prove that $\operatorname{conv}(A)$ is the minimal under inclusion convex set containing A.

2°. Prove that $\operatorname{conv}(\operatorname{conv}(A)) = \operatorname{conv}(A)$, that $\operatorname{conv}(A) \subset \operatorname{conv}(B)$ provided $A \subset B$ and that $\operatorname{conv}(A) \cup \operatorname{conv}(B) \subset \operatorname{conv}(A \cup B)$. Prove that if $u \notin \operatorname{conv}(A)$, $v \notin \operatorname{conv}(A)$, $u \in \operatorname{conv}\big(A \cup \{v\}\big)$ and $v \in \operatorname{conv}\big(A \cup \{u\}\big)$ then $u = v$.

3. Let us identify $\mathbb{C} = \mathbb{R}^2$. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial. Prove that the zeros of the derivative f' of f lie in the convex hull of the zeros of f (Gauss - Lucas Theorem).

4°. Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. Let us define:

$$\begin{aligned} &\Delta_{d-1} = \operatorname{conv}\left(e_1, \ldots, e_d\right), \quad O_d = \operatorname{conv}\left(\pm e_1, \ldots, \pm e_d\right) \quad \text{and} \\ &I_d = \operatorname{conv}\left(\pm e_1 \pm e_2 \ldots \pm e_d\right). \end{aligned}$$

Prove that Δ_{d-1}, O_d, and I_d are polyhedra and find the minimal set of linear inequalities defining each.

Polytope Δ_{d-1} is called the standard $(d-1)$-dimensional *simplex*, polytope O_d is called the standard d-dimensional *octahedron* or *cross-polytope* and polytope I_d is called the standard d-dimensional *cube*.

5^*. Prove that a polytope is a polyhedron and a bounded polyhedron is a polytope (Weyl - Minkowski Theorem, to be proven later).

6^*. Prove that the set of diagonals of $n \times n$ symmetric matrices with the eigenvalues $\lambda_1, \ldots, \lambda_n$ is the convex hull of the vectors in $\mathbb{R}^n$ whose coordinates are permutations of $\lambda_1, \ldots, \lambda_n$ (Schur-Horn Theorem, we will prove at least a part of it later).

2. Carathéodory's Theorem

(2.1) Theorem. *Let* $\dim V = d$ *and let* $A \subset V$ *be a set. Then every point* $x \in \operatorname{conv}(A)$ *is a convex combination of some* $d+1$ *points from* A.

Proof. Let us choose a point $x \in \operatorname{conv}(A)$. Then x is a convex combination of some points from A,

$$x = \sum_{i=1}^{n} \lambda_i a_i \quad \text{where} \quad \sum_{i=1}^{n} \lambda_i = 1$$

and where without loss of generality we may assume that

$$\lambda_i > 0 \quad \text{for} \quad i = 1, \ldots, n.$$

If $n \leq d+1$ we are done, since we can append the combination by 0s as needed. It suffices to prove that if $n > d+1$ then we can represent x as a convex combination of fewer a_is.

Let us consider a homogeneous system of linear equations in real variables $\alpha_1, \ldots, \alpha_n$:

$$\sum_{i=1}^{n} \alpha_i a_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 0.$$

The number of equations is $d+1$, so there is a non-trivial solution $\alpha_1, \ldots, \alpha_n$. In particular, for some i we must have $\alpha_i > 0$. For $t \geq 0$ let us define

$$\lambda_i(t) = \lambda_i - t\alpha_i \quad \text{for} \quad i = 1, \ldots, n.$$

Clearly,

$$x = \sum_{i=1}^{n} \lambda_i(t) a_i \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i(t) = 1.$$

Let us choose

$$t = \min_{i:\ \alpha_i > 0} \frac{\lambda_i}{\alpha_i}.$$

Then $\lambda_i(t) \geq 0$ and for some j we have $\lambda_j(t) = 0$. Hence we expressed x as a convex combination of fewer points. $\square$

(2.2) Problems.

1. Suppose that the set A is path-connected. Prove that every point $x \in \operatorname{conv}(A)$ is a convex combination of some d points from A.

2^*. Let $A_1, \ldots, A_{d+1} \subset V$ be sets, where $\dim V = d$. Suppose that $x \in \operatorname{conv}(A_i)$ for $i = 1, \ldots, d+1$. Prove that there exist points $a_i \in A_i$ such that $x \in \operatorname{conv}(a_1, \ldots, a_{d+1})$. This is the "colored Carathéodory Theorem" proved in I. Bárány, A generalization of Carathéodory's theorem, *Discrete Math.* **40** (1982), no. 2-3, 141–152.

3. Radon's Theorem

(3.1) Theorem. *Let* $\dim V = d$ *and let* $A \subset V$ *be a set with at least* $d+2$ *points,* $|A| \geq d+2$. *Then one can find subsets* $R, B \subset A$ *such that* $R \cap B = \emptyset$ *and* $\operatorname{conv}(R) \cap \operatorname{conv}(B) \neq \emptyset$.

Proof. Let $a_1, \ldots, a_n$, $n \geq d+2$ be some distinct points from A. Let us consider the following system of homogeneous linear equations in real variables $\beta_1, \ldots, \beta_n$:

$$\sum_{i=1}^{n} \beta_i a_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \beta_i = 0.$$

The number of equations is $d+1$ and since $n > d+1$ there exists a non-trivial solution to the system. In particular, for some i we have $\beta_i > 0$ and for some j we have $\beta_j < 0$. Let

$$\gamma = \sum_{i:\ \beta_i > 0} \beta_i = \sum_{i:\ \beta_i < 0} (-\beta_i) > 0.$$

Let

$$R = \Big\{ a_i :\ \beta_i > 0 \Big\} \quad \text{and} \quad B = \Big\{ a_i :\ \beta_i < 0 \Big\}.$$

Clearly, $R \cap B = \emptyset$ and for the point

$$p = \sum_{i:\ \beta_i > 0} \frac{\beta_i}{\gamma} a_i = \sum_{i:\ \beta_i < 0} \left(-\frac{\beta_i}{\gamma} \right) a_i$$

we have $p \in \operatorname{conv}(R) \cap \operatorname{conv}(B)$. □

(3.2) Problem.

1^*. Suppose that $\dim V = d$ and that $A \subset V$ is a set such that $|A| \geq (d+1)(k-1)+1$ for some integer k. Prove that one can find pairwise disjoint subsets $A_1, \ldots, A_k \subset A$ such that

$$\operatorname{conv}(A_1) \cap \ldots \cap \operatorname{conv}(A_k) \neq \emptyset.$$

This is Tverberg's Theorem, see H. Tverberg and S. Vrećica, On generalizations of Radon's theorem and the ham sandwich theorem, *European J. Combin.* **14** (1993), no. 3, 259–264 for a relatively intuitive proof.

4. Helly's Theorem

(4.1) Theorem. *Suppose that* $\dim V = d$ *and let* $S_1, \ldots, S_m \subset V$ *be convex sets such that*

$$S_{i_1} \cap \ldots \cap S_{i_{d+1}} \neq \emptyset \quad \textit{for all} \quad 1 \leq i_1 < i_2 < \ldots < i_{d+1} \leq m.$$

Then

$$\bigcap_{i=1}^{m} S_i \neq \emptyset.$$

Proof. The proof is by induction on m. The case of $m = d+1$ is tautological. Suppose that $m > d+1$. By the induction hypothesis, the intersection of every $(m-1)$ of sets S_i is not empty. Therefore, for $i = 1, \ldots, m$, there is a point p_i such that

$$p_i \in S_j \quad \text{provided} \quad j \neq i.$$

If some two points p_{i_1} and p_{i_2} coincide, we will have constructed a point $p = p_{i_1} = p_{i_2}$ which belongs to every set S_i. If the points $p_1, \ldots, p_m$ are distinct, we apply Theorem 3.1 to the set $A = \{p_1, \ldots, p_m\}$ and claim that there are disjoint subsets $R, B \subset A$ such that $\operatorname{conv}(R) \cap \operatorname{conv}(B) \neq \emptyset$. Let us choose a point $p \in \operatorname{conv}(R) \cap \operatorname{conv}(B)$. Let us consider a set S_i. Then either $p_i \notin R$ or $p_i \notin B$. If $p_i \notin R$, we have $R \subset S_i$ and hence $p \in S_i$ since S_i is convex. If $p_i \notin B$, we have $B \subset S_i$ and hence $p \in S_i$ since S_i is convex. In either case, $p \in S_i$ for $i = 1, \ldots, m$. □

(4.2) Problems.

1°. Let $\{S_i : i \in I\}$ be a possibly infinite family of compact convex subsets $S_i \subset V$, $\dim V = d$, such that the intersection of every $d+1$ of sets S_i is non-empty. Prove that the intersection of all sets S_i is non-empty.

2. Let us fix a $k \leq d+1$ and let $S_1, \ldots, S_m \subset V$ be convex sets such that the intersection of every k of sets S_i is non-empty. Let $L \subset V$ be a subspace such that $\dim L = d - k + 1$. Prove that there exists a translation $L + x$, $x \in V$, of L which intersects every set S_i.

3. Let $S_1, \ldots, S_m; C \subset V$ be convex sets, $\dim V = d$. Suppose that for every $d+1$ sets $S_{i_1}, \ldots, S_{i_{d+1}}$ there exists a translation $C + x, x \in V$, which *intersects* every $S_{i_1}, \ldots, S_{i_{d+1}}$. Prove that there exists a translation $C + x$ which *intersects* every set S_i for $i = 1, \ldots, m$.

4. In Problem 3, replace *intersects* by *contains.* Prove that the statement still holds.

5. In Problem 3, replace *intersects* by *is contained in.* Prove that the statement still holds.

6. Let $A \subset V$, $\dim V = d$, be a convex compact set. Prove that one can find a $u \in V$ such that $(-1/d)A + u \subset A$. Here and elsewhere

$$\alpha A = \Big\{ \alpha x : \ x \in A \Big\} \quad \text{and} \quad B + u = \Big\{ x + u : \quad x \in B \Big\}$$

for sets $A, B \subset V$, a vector $u \in V$ and a number $\alpha \in \mathbb{R}$.

7. Let $S \subset V$ be a finite set of points, $\dim V = d$. Prove that there is a point $p \in V$ such that every closed halfspace containing p contains at least $|S|/(d+1)$ of the points from S.

8*. Prove that for any $0 < \alpha \leq 1$ there exists a $\beta = \beta(\alpha, d) > 0$ with the following property: if $S_1, \ldots, S_m \subset V$ are convex sets in a d-dimensional space such that for at least $\alpha \binom{m}{d+1}$ of $(d+1)$-tuples $S_{i_1}, \ldots, S_{i_{d+1}}$ we have $S_{i_1} \cap \ldots \cap S_{i_{d+1}} \neq \emptyset$ then there is a point $p \in V$ which belongs to at least βm of the sets $S_1, \ldots, S_m$. This is the *Fractional Helly Theorem*, the optimal value is $\beta = 1 - (1-\alpha)^{1/(d+1)}$ whereas a weaker bound $\beta \geq \alpha/(d+1)$ is much easier to prove, see Section 8.1 of J. Matoušek, *Lectures on Discrete Geometry*, Graduate Texts in Mathematics, 212, Springer-Verlag, New York, 2002.

5. Euler characteristic

(5.1) Definitions. Let V be a finite-dimensional real vector space. For a set $A \subset V$ we define the *indicator* of A as the function $[A] : V \longrightarrow \mathbb{R}$ such that

$$[A](x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

We define the following real vector spaces

$$\mathcal{C}(V) = \operatorname{span}\Big\{ [A] : \ A \subset V \text{ is a closed convex set} \Big\},$$

$$\mathcal{C}_b(V) = \operatorname{span}\Big\{ [A] : \ A \subset V \text{ is a compact convex set} \Big\},$$

$$\mathcal{P}(V) = \operatorname{span}\Big\{ [A] : \ A \subset V \text{ is a polyhedron} \Big\} \quad \text{and}$$

$$\mathcal{P}_b(V) = \operatorname{span}\Big\{ [A] : \ A \subset V \text{ is a bounded polyhedron} \Big\}.$$

A linear functional, or more generally, a linear transformation defined on any of the spaces $\mathcal{C}(V), \mathcal{C}_b(V), \mathcal{P}(V)$ and $\mathcal{P}_b(V)$ is called a *valuation*.

(5.2) Theorem. *There exists a unique linear functional (valuation) $\chi : \mathcal{C}(V) \longrightarrow \mathbb{R}$, called the Euler characteristic, such that $\chi([A]) = 1$ for every non-empty closed convex set $A \subset V$.*

Proof. The uniqueness is immediate: for $f \in \mathcal{C}(V)$,

$$f = \sum_{i \in I} \alpha_i \, [A_i] \,, \tag{5.2.1}$$

where $A_i \subset V$ are closed convex sets and α_i are real numbers, we should have

$$\chi(f) = \sum_{i \in I:\ A_i \neq \emptyset} \alpha_i. \tag{5.2.2}$$

We need to prove the existence of χ.

First, we prove the existence of a linear functional $\chi : \mathcal{C}_b(V) \longrightarrow \mathbb{R}$ such that $\chi([A]) = 1$ for every non-empty compact convex set $A \subset V$. We identify $V = \mathbb{R}^d$ and proceed by induction on d.

For $d = 0$ we have $V = \{0\}$, so we define $\chi(f) = f(0)$ for all $f \in \mathcal{C}_b(V)$. Suppose we established the existence of χ for $V = \mathbb{R}^d$. Our goal is to show that χ exists in $\mathbb{R}^{d+1}$.

Let $H_\tau \subset \mathbb{R}^{d+1}$ be the affine hyperplane consisting of the points with the last coordinate τ. We have

$$\mathbb{R}^{d+1} = \bigcup_{\tau \in \mathbb{R}} H_\tau.$$

By the induction hypothesis there is a linear functional $\chi_\tau : \mathcal{C}_b\left(H_\tau\right) \longrightarrow \mathbb{R}$ such that $\chi_\tau([A]) = 1$ for all non-empty compact convex sets $A \subset H_\tau$. For a function $f \in \mathcal{C}_b\left(\mathbb{R}^{d+1}\right)$, let f_τ be its restriction onto H_τ. Then, if f is defined by (5.2.1), we have

$$f_\tau = \sum_{i \in I} \alpha_i \left[A_i \cap H_\tau\right]$$

and hence $f_\tau \in \mathcal{C}_b\left(H_\tau\right)$. Next, we claim that for all $f \in \mathcal{C}_b\left(\mathbb{R}^{d+1}\right)$ and $\tau \in \mathbb{R}$ the one-sided limit

$$\lim_{\epsilon \longrightarrow 0+} \chi_{\tau-\epsilon}\left(f_{\tau-\epsilon}\right) \tag{5.2.3}$$

exists and that for every $f \in \mathcal{C}_b\left(\mathbb{R}^{d+1}\right)$ there are at most finitely many values of τ where the limit (5.2.3) is not equal to $\chi_\tau\left(f_\tau\right)$. In view of (5.2.1) it suffices to check the claim if $f = [A]$, where $A \subset \mathbb{R}^{d+1}$ is a non-empty compact convex set. Let $\tau_{\min}$ be the minimum value of the last coordinate for a point in A and let $\tau_{\max}$ be the maximum value of the last coordinate for a point in A.

Using (5.2.2), we obtain

$$\lim_{\epsilon \longrightarrow 0+} \chi_{\tau-\epsilon}\left(f_{\tau-\epsilon}\right) = \begin{cases} 0 & \text{if } \tau > \tau_{\max} \\ 1 & \text{if } \tau_{\min} < \tau \leq \tau_{\max} \\ 0 & \text{if } \tau \leq \tau_{\min} \end{cases}$$

(5.2.4) and

$$\chi_\tau\left(f_\tau\right) = \begin{cases} 0 & \text{if } \tau > \tau_{\max} \\ 1 & \text{if } \tau_{\min} \leq \tau \leq \tau_{\max} \\ 0 & \text{if } \tau < \tau_{\min}, \end{cases}$$

which proves the claim. Now, for $f \in \mathcal{C}_b\left(\mathbb{R}^{d+1}\right)$, we define

$$\chi(f)=\sum_{\tau \in \mathbb{R}}\left(\chi_\tau\left(f_\tau\right)-\lim_{\epsilon \longrightarrow 0+} \chi_{\tau-\epsilon}\left(f_{\tau-\epsilon}\right)\right). \tag{5.2.5}$$

Since only finitely many terms of the sum (5.2.5) can be non-zero (namely, when τ is the minimum coordinate of a point in one of the sets A_i in (5.2.1)), the sum (5.2.5) is well-defined. By (5.2.4), it satisfies the requirements of the theorem.

Finally, we have to extend χ from $\mathcal{C}_b(V)$ to $\mathcal{C}(V)$. Let us identify $V=\mathbb{R}^d$ and let

$$B_r=\left\{\left(x_1, \ldots, x_d\right): \quad \sum_{i=1}^d x_i^2 \leq r^2\right\}$$

be the ball of radius r centered at the origin. Then for every $f \in \mathcal{C}(V)$ we have $f \cdot\left[B_r\right] \in \mathcal{C}_b(V)$ and hence $\chi\left(f \cdot\left[B_r\right]\right)$ is well-defined. We let

$$\chi(f)=\lim_{r \longrightarrow+\infty} \chi\left(f \cdot\left[B_r\right]\right).$$

It is straightforward to check the limit exists and satisfies the conditions of the theorem. □

In the course of the proof, we obtained the following important corollary.

(5.3) Corollary. *Let $f \in \mathcal{C}_b\left(\mathbb{R}^d\right)$. Then*

$$\chi(f)=\sum_{\tau \in \mathbb{R}}\left(\chi\left(f_\tau\right)-\lim_{\epsilon \longrightarrow 0+} \chi\left(f_{\tau-\epsilon}\right)\right),$$

where f_τ is the restriction of f onto the affine hyperplane in $\mathbb{R}^d$ consisting of the points with the last coordinate τ.

(5.4) Problems.

1°. Show that the indicators $[A]$, where $A \subset V$ are non-empty closed convex sets, do not form a basis of $\mathcal{C}(V)$ unless $\dim V=0$.

2°. Prove that the spaces $\mathcal{C}(V), \mathcal{C}_b(V), \mathcal{P}(V)$ and $\mathcal{P}_b(V)$ are closed under point-wise multiplication of functions.

3°. Prove the inclusion-exclusion formula for sets $A_1, \ldots, A_m \subset V$:

$$\left[\bigcup_{i=1}^m A_i\right]=\sum_{k=1}^m(-1)^{k-1} \sum_{1 \leq i_1<\ldots<i_k \leq m}\left[A_{i_1} \cap \ldots \cap A_{i_k}\right].$$

4. Let $A_1, \ldots, A_m$ be closed convex sets such that their union is convex and the intersection of any k of the sets is not empty. Prove that the intersection of some $k+1$ of the sets is not empty.

5. Let $\Delta_{d-1} \subset \mathbb{R}^d$ be the standard d-dimensional simplex defined by the equation $x_1+\ldots+x_d=1$ and inequalities $x_i \geq 0$ for $i=1,\ldots,d$. Let $F_i \subset \Delta_{d-1}$ be the i-th facet of Δ_{d-1} defined by the equation $x_i=0$. Suppose that $A_1,\ldots,A_d$ are closed convex sets such that $\Delta \subset A_1 \cup \ldots \cup A_d$ and such that $A_i \cap F_i = \emptyset$ for $i=1,\ldots,d$. Prove that $A_1 \cap \ldots \cap A_d \neq \emptyset$.

6. Let $A_1,\ldots,A_m$ be closed convex sets such that $A_1 \cap \ldots \cap A_m \neq \emptyset$. Prove that $\chi\left([A_1 \cup \ldots \cup A_m]\right)=1$.

7. Let $\operatorname{int} I_d$ be the interior of the standard d-dimensional cube in $\mathbb{R}^d$ defined by the inequalities $-1 < x_i < 1$ for $i=1,\ldots,d$. Prove that $[\operatorname{int} I_d] \in \mathcal{C}\left(\mathbb{R}^d\right)$ and that $\chi\left([\operatorname{int} I_d]\right)=(-1)^d$.

8*. Let θ be a function which associates with every polyhedron $P \subset V$ a real number $\theta(P)$. Suppose that $\theta(\emptyset)=0$ and that for every polyhedron $P \subset V$ and for every affine hyperplane $H \subset V$ bounding the closed halfspaces H_- and H_+ we have $\theta(P)=\theta\left(P \cap H_+\right)+\theta\left(P \cap H_-\right)-\theta\left(P \cap H\right)$. Prove that there exists a unique valuation $\Theta: \mathcal{P}(V) \longrightarrow \mathbb{R}$ such that $\Theta([P])=\theta(P)$ for every polyhedron $P \subset V$. See H. Groemer, On the extension of additive functionals on classes of convex sets, *Pacific J. Math.* **75** (1978), no. 2, 397–410.

6. Polyhedra and linear transformations

(6.1) Theorem. *Let V and W be finite-dimensional real vector spaces and let $T: V \longrightarrow W$ be a linear transformation. Then*

(1) *If $P \subset V$ is a polyhedron then $T(P) \subset W$ is a polyhedron;*
(2) *There is a unique linear transformation $\mathcal{T}: \mathcal{P}(V) \longrightarrow \mathcal{P}(W)$ such that $\mathcal{T}([P])=[T(P)]$ for any polyhedron $P \subset V$.*

Proof. To prove Part (1), let us consider first the following model case. Suppose that $V=\mathbb{R}^d$, $W=\mathbb{R}^{d-1}$ and $T:(x_1,\ldots,x_d) \longmapsto (x_1,\ldots,x_{d-1})$ is the map forgetting the last coordinate. Suppose that $P \subset \mathbb{R}^d$ is defined by a finite system of linear inequalities:

$$\sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for} \quad i \in I.$$

Let

$$I_+=\left\{i \in I:\ a_{id}>0\right\}, \quad I_-=\left\{i \in I:\ a_{id}<0\right\} \quad \text{and} \quad I_0=\left\{i \in I:\ a_{id}=0\right\}.$$

Then $(x_1,\ldots,x_{d-1}) \in T(P)$ if and only if

$$\sum_{j=1}^{d-1} a_{ij} x_j \leq b_i \quad \text{for} \quad i \in I_0 \tag{6.1.1}$$

and there exists $x_d \in \mathbb{R}$ such that

$$x_d \leq \frac{b_i}{a_{id}} - \sum_{j=1}^{d-1} \frac{a_{ij}}{a_{id}} x_j \quad \text{for} \quad i \in I_+ \quad \text{and}$$

$$x_d \geq \frac{b_i}{a_{id}} - \sum_{j=1}^{d-1} \frac{a_{ij}}{a_{id}} x_j \quad \text{for} \quad i \in I_-.$$

The system has a solution x_d if and only if

$$\text{(6.1.2)} \quad \frac{b_{i_1}}{a_{i_1 d}} - \sum_{j=1}^{d-1} \frac{a_{i_1 j}}{a_{i_1 d}} x_j \leq \frac{b_{i_2}}{a_{i_2 d}} - \sum_{j=1}^{d-1} \frac{a_{i_2 j}}{a_{i_2 d}} x_j \quad \text{for all} \quad i_1 \in I_- \quad \text{and} \quad i_2 \in I_+.$$

Hence the image $T(P)$ is defined by the finite set of linear inequalities (6.1.1)–(6.1.2) in $x_1, \ldots, x_{d-1}$, so $T(P)$ is a polyhedron. If $I_- = \emptyset$ or $I_2 = \emptyset$ there are no inequalities (6.1.2) and if $I_0 = \emptyset$ there are no inequalities (6.1.1). This procedure of obtaining the inequalities for $T(P)$ from those for P is called the *Fourier-Motzkin elimination.*

Next, we remark that if $T : V \longrightarrow W$ is an isomorphism then $T(P) \subset W$ is trivially a polyhedron and if $\ker T = \{0\}$ then $T : V \longrightarrow T(V)$ is an isomorphism and $T(P)$ is a polyhedron in $T(V)$ and hence in W. Finally, we can represent an arbitrary T as a composition of linear transformations

$$V \longrightarrow V \oplus W \longrightarrow W, \quad x \longmapsto (x, Tx) \longmapsto Tx.$$

The first transformation has zero kernel while the second is obtained by erasing the first $\dim V$ coordinates and hence can be dealt with by an iteration of the Fourier-Motzkin elimination procedure. This concludes the proof of Part (1).

For every $y \in W$ the inverse image $T^{-1}(y) \subset V$ is an affine subspace in W. For any $f \in \mathcal{P}(V)$, we can write

$$f = \sum_{i \in I} \alpha_i \left[P_i\right] \quad \text{and} \quad f\left[T^{-1}(y)\right] = \sum_{i \in I} \alpha_i \left[P_i \cap T^{-1}(y)\right],$$

for some polyhedra $P_i \subset V$ and some numbers $\alpha_i \in \mathbb{R}$. Hence for any $f \in \mathcal{P}(V)$ and any $y \in W$ the product $f\left[T^{-1}\right]$ lies in $\mathcal{P}(V)$. Let us define a function $g : W \longrightarrow \mathbb{R}$ by

$$g(y) = \chi\left(f\left[T^{-1}(y)\right]\right).$$

If $f = [P]$ where $P \subset V$ is a polyhedron then

$$g(y) = \begin{cases} 1 & \text{if} \quad P \cap T^{-1}(y) \neq \emptyset \\ 0 & \text{if} \quad P \cap T^{-1}(y) = \emptyset \end{cases}$$

and hence $g = [T(P)]$. Therefore, for every $f \in \mathcal{P}(V)$ we have $g \in \mathcal{P}(W)$ and the transformation $f \longmapsto g$ is the desired transformation $\mathcal{T}$. The uniqueness of $\mathcal{T}$ is obvious. $\square$

(6.2) Corollary. *Let $P \subset V$ be a polytope. Then P is a polyhedron.*

Proof. We can write $P = \operatorname{conv}(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in V$. Let

$$\Delta_{n-1} = \left\{ (\lambda_1, \ldots, \lambda_n) : \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n \right\}$$

be the standard $(n-1)$-dimensional simplex. Clearly, Δ_{n-1} is a polyhedron. We consider the linear transformation $T : \mathbb{R}^n \longrightarrow V$,

$$T(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i a_i.$$

Clearly, $T(\Delta_{n-1}) = P$. By Theorem 6.1 P is a polyhedron. $\square$

(6.3) Problems.

1°. Construct an example of a closed convex set $A \subset \mathbb{R}^2$ and a linear transformation $T : \mathbb{R}^2 \longrightarrow R$ such that $T(A)$ is not closed.

2°. It follows from the example of Problem 1 above that if $A \subset V$ is defined by a system of infinitely many linear inequalities and if $T : V \longrightarrow W$ is a linear transformation then $T(A)$ doesn't have to be defined by a system of infinitely many linear inequalities. Where does the proof of Part (1) of Theorem 6.1 break down for infinite sets of linear inequalities?

3°. Prove a version of Theorem 6.1 replacing polyhedra by compact convex sets and $\mathcal{P}(V)$ by $\mathcal{C}_b(V)$.

7. Minkowski sum

(7.1) Definitions. Let $A \subset V$ be a set and let $\alpha \in \mathbb{R}$ be a number. We define the *scaling*

$$\alpha A = \{\alpha x : \ x \in A\}, \quad \alpha A \subset V.$$

Let $A, B \subset V$ be sets. We define their *Minkowski sum*

$$A + B = \{a + b : \ a \in A, b \in B\}, \quad A + B \subset V.$$

(7.2) Theorem. *Let V be a finite-dimensional space. Then*

(1) *If $P_1, P_2 \subset V$ are polyhedra then $P_1 + P_2 \subset V$ is a polyhedron;*
(2) *There exists a unique bilinear operation $* : \mathcal{P}(V) \times \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$, called convolution, such that $[P_1] * [P_2] = [P_1 + P_2]$ for any two polyhedra $P_1, P_2 \subset V$.*

Proof. For polyhedra $P_1, P_2 \subset V$, let us define

$$P_1 \times P_2 = \{(x, y) : \ x \in P_1, y \in P_2\}, \quad P_1 \times P_2 \subset V \oplus V.$$

Clearly, $P_1 \times P_2$ is a polyhedron. Let us define a linear transformation

$$T : V \oplus V \longrightarrow V \quad \text{where} \quad T(x, y) = x + y.$$

Then

$$P_1 + P_2 = T\left(P_1 \times P_2\right)$$

and hence $P_1 + P_2$ is a polyhedron by Part (1) of Theorem 6.1, which proves Part (1).

By Part (2) of Theorem 6.1, there is a map $\mathcal{T} : \mathcal{P}(V \oplus V) \longrightarrow \mathcal{P}(V)$ such that

$$\mathcal{T}\left([P_1] \times [P_2]\right) = [P_1 + P_2].$$

For $f, g \in \mathcal{P}(V)$ let us define $f \times g : V \oplus V \longrightarrow V$ by

$$(f \times g)(x, y) = f(x)g(y).$$

Clearly $[P_1] \times [P_2] = [P_1 \times P_2]$ for polyhedra $P_1, P_2 \subset V$. Therefore, $f \times g \in \mathcal{P}(V \oplus V)$ for $f, g \in \mathcal{P}(V)$. Now we define

$$f * g = \mathcal{T}(f \times g).$$

The proof of Part (2) now follows. □

(7.3) Problems.

1°. Let A be a convex set and suppose that $\alpha, \beta \geq 0$. Prove that

$$(\alpha + \beta)A = \alpha A + \beta A.$$

2°. Prove that

$$f * [0] = [0] * f = f$$

for all $f \in \mathcal{P}(V)$.

3*. Let $P \subset \mathbb{R}^d$ be a d-dimensional bounded polyhedron. Prove that $[\operatorname{int} P] \in \mathcal{P}(V)$ and that

$$[P] * [-\operatorname{int} P] = (-1)^d [0],$$

where $\operatorname{int} P$ is the interior of P.

4°. Prove a version of Theorem 7.2 replacing polyhedra by convex compact sets and $\mathcal{P}(V)$ by $\mathcal{C}_b(V)$.

5*. Let $K_i \subset V$, $i \in I$, be a finite family of convex compact sets and let $\alpha_i \in \mathbb{R}$ be numbers such that

$$\sum_{i \in I} \alpha_i \left[K_i\right] = 0.$$

Prove that

$$\sum_{i:\ \alpha_i > 0} \alpha_i K_i = \sum_{i:\ \alpha_i < 0} (-\alpha_i) K_i,$$

where in the latter identity the sum is understood as the Minkowski sum and the products are understood as scalings.

8. Examples of valuations

First, we consider valuations on $\mathcal{C}_b(V)$. We consider a Euclidean structure on V.

(8.1) Intrinsic volumes. Clearly, volume of a convex compact set gives rise to a valuation: if

$$\sum_{i \in I} \alpha_i \, [A_i] = 0$$

for some convex compact sets $A_i \subset V$ and some real $\alpha_i \in \mathbb{R}$ then

$$\sum_{i \in I} \alpha_i \operatorname{vol}(A_i) = 0.$$

Let $L \subset V$ be a subspace, $\dim L = k$, and let vol_k be the k-dimensional volume in L. Denoting by $A_i|L$ the orthogonal projection of A_i onto L, we deduce from Problem 3° of Section 6.3 that

$$\sum_{i \in I} \alpha_i \, [A_i|L] = 0$$

and hence

$$\sum_{i \in I} \alpha_i \operatorname{vol}_k (A_i|L) = 0.$$

Hence the volume of the orthogonal projection of a convex compact set onto a subspace gives rise to a valuation. Let us define $\omega_k(A)$ as the average k-dimensional volume of the orthogonal projection of A onto a random k-dimensional subspace (where the average is taken with respect to the Haar measure on the Grassmannian $G_k(\mathbb{R}^n)$ of all k-dimensional subspaces in $\mathbb{R}^n$). Then ω_k, called the *k-th intrinsic volume* of A, gives rise to a valuation: if

$$\sum_{i \in I} \alpha_i \, [A_i] = 0$$

for convex compact sets $A_i \subset V$ and reals $\alpha_i \in \mathbb{R}$ then

$$\sum_{i \in I} \alpha_i \omega_k (A_i) = 0.$$

(8.2) Problems.

1°. Show that $\omega_k(A) \leq \omega_k(B)$ if $A \subset B$ are convex compact sets, that $\omega_k(\alpha A) = \alpha^k \omega_k(A)$ for a convex compact set A and a number $\alpha \geq 0$ and that $\omega_k(U(A)) = \omega_k(A)$, where U is an isometry of V.

2° Show that ω_0 is the Euler characteristic χ.

3. Suppose that $\dim V = d$. Show that there is a constant $c(d)$ such that

$$\omega_{d-1}(A) = c(d) \Big(\text{the surface area of } A\Big)$$

for all convex compact sets $A \subset V$ with a non-empty interior. In particular, $c(2) = 1/\pi$.

(8.3) Support function. Let us choose a linear functional $\ell : V \longrightarrow \mathbb{R}$. For a convex compact set $A \subset V$ let us define

$$h(A;\ell) = \max_{x \in A} \ell(x).$$

The value of $h(A;\ell)$ is called the *support function* of A in the direction of ℓ.

(8.4) Problems. 1. Show that $h(A;\ell)$ gives rise to a valuation: if $A_i \subset V$ are compact convex sets such that

$$\sum_{i \in I} \alpha_i \left[A_i\right] = 0$$

then

$$\sum_{i \in I} \alpha_i h\left(A_i;\ell\right) = 0$$

for any linear $\ell : V \longrightarrow \mathbb{R}$.

2°. Let $A, B \subset V$ be convex compact sets. Show that

$$h(A+B;\ell) = h(A;\ell) + h(B;\ell)$$

for any linear $\ell : V \longrightarrow \mathbb{R}$.

Finally, we consider an interesting valuation on $\mathcal{P}_b\left(\mathbb{R}^3\right)$.

(8.5) Dehn invariant. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called *additive* if

$$f(a+b) = f(a) + f(b) \quad \text{for all} \quad a, b \in \mathbb{R}$$

(it does not have to be continuous or even measurable). Let us choose an additive function f such that $f(\pi) = 0$. For a polytope $P \subset \mathbb{R}^3$, let us define a real number

$$D_f(P) = \sum_{\text{the edges of } P} (\text{length of the edge}) f(\text{the dihedral angle of } P \text{ at the edge}).$$

The number $D_f(P)$ is called the *Dehn invariant* of P. Strictly speaking, the Dehn invariant $D(P)$ is defined not as a real number but as an element of the module $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$, where both $\mathbb{R}$ and $\mathbb{R}/\pi\mathbb{Z}$ are considered as additive groups and modules over $\mathbb{Z}$ with the natural action, so

$$D(P) = \sum_{\text{the edges of } P} (\text{length of the edge}) \otimes_{\mathbb{Z}} (\text{the dihedral angle of } P \text{ at the edge} \mod \pi),$$

but we will work with $D_f(P)$ instead.

(8.6) Problems.

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an additive function such that $f(\pi) = 0$. Show that $f(\pi x) = 0$ for any $x \in \mathbb{Q}$.

2. Let $a \in \mathbb{R}$ be a number such that $a/\pi \notin \mathbb{Q}$. Deduce from the axiom of choice that there is an additive function f such that $f(\pi) = 0$ and $f(a) \neq 0$.

3*. Show that $D_f(P)$ gives rise to valuation: if $P_i \subset \mathbb{R}^3$ are polytopes and $\alpha_i \in \mathbb{R}$ are real numbers such that

$$\sum_{i \in I} \alpha_i \left[P_i\right] = 0$$

then

$$\sum_{i \in I} \alpha_i D_f \left(P_i\right) = 0$$

for every additive f such that $f(\pi) = 0$.

4°. Let $I \subset \mathbb{R}^3$ be the standard 3-dimensional cube. Show that $D_f\left(I\right) = 0$ for any additive f such that $f(\pi) = 0$.

5°. Let $\Delta \subset \mathbb{R}^3$ be a regular tetrahedron. Show that all dihedral angles of Δ are equal to $\arccos \dfrac{1}{3}$.

6*. Show that $\dfrac{1}{\pi} \arccos \dfrac{1}{3}$ is an irrational number. See, for example, Chapter I of M. Aigner and G.M. Ziegler, *Proofs from The Book*, Springer-Verlag, Berlin, 2004.

7. Deduce from Problems 2–6 above that one cannot cut a regular cube into finitely many polyhedral pieces to reassemble them into a regular tetrahedron (the solution to Hilbert's 3rd Poblem). Show that the axiom of choice (used in Problem 2) is not really needed to draw that conclusion.

8. Let $P, Q \subset \mathbb{R}^2$ be two polygons of equal area. Show that P can be cut into finitely many polygonal pieces that can be reassembled into Q (Bolyai-Gerwien Theorem).

9. The structure of polyhedra

(9.1) Definitions. Let V be a vector space and let $A \subset V$ be a set. Let $\ell : V \longrightarrow \mathbb{R}$ be a linear function and let $\alpha \in \mathbb{R}$ be a number. Suppose that $\ell(x) \leq \alpha$ for all $x \in A$. The set

$$F = \Big\{ x \in A : \quad \ell(x) = \alpha \Big\}$$

is called a *face* of A. We often treat $\emptyset$ and A as faces of A. Faces other than A and $\emptyset$ are called *proper faces*.

A point $v \in A$ is called *extreme* if whenever $v = (v_1 + v_2)/2$ for some $v_1, v_2 \in A$, we must have $v_1 = v_2 = v$. An extreme point of a polyhedron is called a *vertex*.

Let

$$P = \Big\{ x \in V : \quad \ell_i(x) \leq \alpha_i \quad \text{for} \quad i \in I \Big\} \tag{9.1.1}$$

be a polyhedron. We assume that $\ell_i \neq 0$ for all $i \in I$.

We say that the inequality $\ell_i(x) \leq \alpha_i$ is *active* on x if we have in fact $\ell_i(x) = \alpha_i$.

Let $u \in V \setminus \{0\}$ be a non-zero vector and let $a \in V$ be a point. The set

$$\Big\{ a + \tau u : \quad \tau \in \mathbb{R} \Big\}$$

is called a *line* through a in the direction of u and the set

$$\Big\{ a + \tau u : \quad \tau \geq 0 \Big\}$$

is called a *ray* emanating from a in the direction of u.

The *relative interior* of a convex set $A \subset V$ is the interior of A relative to the smallest affine subspace containing A. The *dimension* $\dim A$ of a convex set $A \subset V$ is the dimension of the smallest affine subspace containing A.

(9.2) Lemma. *Suppose that F is a face of A and that v is an extreme point of F. Then v is an extreme point of A.*

Proof. Let $\ell : V \longrightarrow \mathbb{R}$ be a linear function and let $\alpha \in \mathbb{R}$ be a number such that $\ell(x) \leq \alpha$ for all $x \in A$ and $F = \big\{ x \in A : \ \ell(x) = \alpha \big\}$. Suppose that

$$v = \frac{v_1 + v_2}{2}$$

for $v_1, v_2 \in A$. Then $\ell(v_1), \ell(v_2) \leq \alpha$ and

$$\frac{\ell(v_1) + \ell(v_2)}{2} = \ell(v) = \alpha.$$

Hence we should have $\ell(v_1) = \ell(v_2) = \alpha$ and $v_1, v_2 \in F$. Since v is an extreme point of F we must have $v_1 = v_2 = v$, which proves that v is an extreme point of A. □

(9.3) Lemma. *Let $P \subset V$ be a polyhedron defined by (9.1.1) and let $v \in P$ be a point. Let*

$$I_v = \Big\{ i \in I : \quad \ell_i(x) = \alpha_i \Big\}$$

be the set of indices of the inequalities active on v. Then v is a vertex of P if and only if

$$\operatorname{span}\Big\{ \ell_i : \quad i \in I_v \Big\} = V^*.$$

Proof. Suppose that

$$\operatorname{span}\Big\{\ell_i : \quad i \in I_v\Big\} = V^*.$$

Let us write

$$v = \frac{v_1 + v_2}{2} \quad \text{for some} \quad v_1, v_2 \in P.$$

Then

$$\ell_i(v_1), \ell_i(v_2) \leq \alpha_i \quad \text{and} \quad \frac{\ell_i(v_1) + \ell_i(v_2)}{2} = \ell_i(v) = \alpha_i \quad \text{for all} \quad i \in I_v.$$

Therefore,

$$\ell_i(v_1) = \ell_i(v_2) = \ell_i(v) = \alpha_i \quad \text{for all} \quad i \in I_v.$$

Since the functions $\ell_i : i \in I_v$ span the dual space V^*, the system of linear equations

$$\ell_i(x) = \alpha_i \quad \text{for} \quad i \in I_v$$

has at most one solution in V, so we must have $v_1 = v_2 = v$ and v is an extreme point.

Suppose that

$$\operatorname{span}\Big\{\ell_i : \quad i \in I_v\Big\} \neq V^*.$$

Then there exists $u \in V$, $u \neq 0$, such that

$$\ell_i(u) = 0 \quad \text{for all} \quad i \in I_v.$$

Let

$$v_1 = v - \epsilon u \quad \text{and} \quad v_2 = v + \epsilon u \quad \text{so} \quad v = \frac{v_1 + v_2}{2}.$$

Since for $i \notin I_v$ we have $\ell_i(v) < \alpha_i$, for all sufficiently small $\epsilon > 0$ we have $v_1, v_2 \in P$, so v is not a vertex of P. $\square$

In particular, a polyhedron defined by n linear inequalities in a d-dimensional space does not have more than $\binom{n}{d}$ vertices.

(9.4) Lemma. *Let $P \subset V$ be a polyhedron. Then either P contains an interior point or P lies in a proper affine subspace of V.*

Proof. Suppose that P is defined by the system (9.1.1). If for every $i \in I$ there is a point $x_i \in P$ such that $\ell_i(x_i) < \alpha_i$ then the point

$$x = \frac{1}{|I|} \sum_{i \in I} x_i$$

is an interior point of P. Otherwise, there is an $i \in I$ such that $\ell_i(x) = \alpha_i$ for all $x \in P$. $\square$

(9.5) Lemma. *Let $P \subset \mathbb{R}^d$ be a polyhedron. Then P is bounded if and only if P does not contain a ray.*

Proof. Clearly, if P contains a ray then P is unbounded. Suppose that P is unbounded, so there is a sequence $x_n \in P$ such that $\|x_n\| \geq n$, where $\| \cdot \|$ is the Euclidean norm. Suppose that P is defined by the system (9.1.1). Let

$$u_n = \frac{x_n}{\|x_n\|}, \quad \text{so} \quad \|u_n\| = 1.$$

We observe that

$$\ell_i(u_n) \leq \frac{\alpha_i}{n} \quad \text{for all} \quad i \in I.$$

Let u be a limit point of u_n, so $\|u\| = 1$ and

$$\ell_i(u) \leq 0 \quad \text{for all} \quad i \in I.$$

Then for every $x \in P$ the ray emanating from x in the direction of u is contained in P.

Now we can prove the second part of the Weyl-Minkowski Theorem, cf. Corollary 6.2.

(9.6) Theorem. *Let $P \subset V$ be a bounded polyhedron. Then P is a polytope that is the convex hull of the set of its vertices.*

Proof. We need to prove that every point $x \in P$ is a convex combination of vertices of P. We proceed by induction on $\dim V$.

The case of $\dim V = 0$ is trivial.

Suppose that P is defined by the system (9.1.1). If $\ell_i(x) = \alpha_i$ for some $i \in I$ then x lies in a proper face F of P and the result follows by the induction hypothesis and Lemma 9.2. If $\ell_i(x) < \alpha_i$ for all $i \in I$, let us consider a line L through x. Since P is bounded, the intersection $L \cap P$ is a closed interval, so

$$L \cap P = [a, b],$$

where $a, b \in P$ necessarily lie in proper faces of P. Arguing as above, we conclude that a and b are convex combinations of some vertices of P. Since $x \in [a, b]$, point x is also a convex combination of some vertices of P. $\square$

(9.7) Theorem. *Let $P \subset V$ be a non-empty polyhedron. Then P contains a vertex if and only if P does not contain a line.*

Proof. Let P be defined by system (9.1.1). If P contains a line in the direction $u \neq 0$ then, necessarily, $\ell_i(u) = 0$ for all $i \in I$. Then for any $v \in P$ we can write

$$v = \frac{v_1 + v_2}{2} \quad \text{where} \quad v_1 = v + u \quad \text{and} \quad v_2 = v - u,$$

so P has no vertices.

Suppose that P contains no lines. We proceed by induction on $\dim V$ with the trivial base of $\dim V = 0$. Without loss of generality, by Lemma 9.4 we may assume that P has a non-empty interior. Let us pick a point x in the interior of P and let us consider a line L through x in any direction u. Since P contains no lines, the intersection $P \cap L$ is either a non-empty closed interval or a ray. In either case, it contains a point a which necessarily lies in a proper face F of P. Hence there is a proper face F of P and the result follows by the induction hypothesis and Lemma 9.2. $\square$

(9.8) Definitions. Let $K \subset V$ be a polyhedron. Then K is called a *cone* if $0 \in K$ and for every $x \in K$ and every $\lambda \geq 0$ we have $\lambda x \in K$. Equivalently, K is a polyhedral cone if K is defined by finitely many homogeneous linear inequalities:

$$K = \Big\{ x \in V : \quad \ell_i(x) \leq 0 \quad \text{for} \quad i \in I \Big\}. \tag{9.8.1}$$

Let $P \subset V$ be a polyhedron. The set

$$K_P = \Big\{ u \in V : \quad x + \lambda u \in P \quad \text{for all} \quad x \in P \quad \text{and all} \quad \lambda \geq 0 \Big\}$$

is called the *recession cone* of P. Equivalently, if P is defined by (9.1.1), cone K_P is defined by (9.8.1). A cone without lines (equivalently, for which 0 is the vertex) is called *pointed*.

We say that a point $u \in K$, $u \neq 0$, spans an *extreme ray* of K if whenever $u = (u_1 + u_2)/2$ with $u_1, u_2 \in K$, we must have $u_1 = \lambda_1 u$ and $u_2 = \lambda_2 u$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

A vector

$$x = \sum_{i=1}^n \lambda_i u_i \quad \text{where} \quad \lambda_i \geq 0 \quad \text{for all} \quad i = 1, \ldots, n$$

is called a *conic combination* of vectors $u_1, \ldots, u_n$. The set of all conic combinations of vectors from a set A is called the *conic hull* of A and denoted $\operatorname{co}(A)$.

(9.9) Theorem. *Let $K \subset V$ be a pointed polyhedral cone such that $K \neq \{0\}$. Then there exists an affine hyperplane $H \subset V$ such that $0 \notin H$, $P = K \cap H$ is a polytope and $K = \operatorname{co}(P)$.*

Proof. Suppose that K is defined by system (9.8.1). Let

$$\ell = \sum_{i \in I} \ell_i.$$

Let us pick any $u \in K \setminus \{0\}$. We claim that $\ell(u) < 0$. Indeed, if $\ell(u) = 0$ we necessarily have $\ell_i(u) = 0$ for all $i \in I$ and then the line through the origin in the direction of u is contained in K, which is a contradiction.

In particular, we conclude that $\ell \neq 0$. Let us define

$$H = \Big\{ x \in V : \quad \ell(x) = -1 \Big\}.$$

Then H is an affine hyperplane not containing the origin and $P = K \cap H$ is a polyhedron. Since for every $x \in K \setminus \{0\}$ we have $\ell(x) < 0$, there exists a $\lambda > 0$ such that $\ell(\lambda x) = -1$. It follows that $K = \mathrm{co}(P)$. By Theorem 9.6, it remains to prove that P is bounded and by Lemma 9.5 it remains to prove that P contains no rays. Suppose that P contains a ray in the direction $u \neq 0$. Then, necessarily $\ell_i(u) \leq 0$ for all $i \in I$, so $u \in K \setminus \{0\}$. Additionally, since u is parallel to H, we must have $\ell(u) = 0$, which as we proved above is impossible. $\square$

(9.10) Theorem. *Let $P \subset V$ be a non-empty polyhedron without lines. Then*

$$P = M + K_P,$$

where M is the convex hull of the vertices of P and K_P is the recession cone of P.

Proof. Clearly,

$$M + K_P \subset P.$$

It remains to show the reverse inclusion. We prove it by induction on $\dim V$.

The case of $\dim V = 0$ is trivial. Let us pick a point $x \in P$. If x lies in a proper face F of P then by the induction hypothesis x can be written as a sum of a vector from K_F and a convex combination of vertices of F. The result follows by Lemma 9.2 and the inclusion $K_F \subset K_P$.

Suppose, therefore, that x lies in the interior of P. If $K_P = \{0\}$, the result follows by Theorem 9.6. If $K_P \neq \{0\}$, let us pick a vector $u \in K_P \setminus \{0\}$ and consider a line L through x in the direction of u. The intersection $L \cap P$ is a ray $\big\{ a + \tau u : \quad \tau \geq 0 \big\}$. Point a lies in a proper face of P and as we argued above belongs to $M + K_P$. Hence $x \in M + K_P$ as well. $\square$

(9.11) Theorem. *Let $P \subset V$ be a polyhedron. Then P can be represented as the Minkowski sum*

$$P = L + M + K,$$

where $L \subset V$ is a subspace, $M \subset V$ is a polytope and $K \subset V$ is a polyhedral cone without lines.

Proof. Suppose that P is defined by system (9.1.1). We note that if a line $\big\{ \tau u : \ \tau \in \mathbb{R} \big\}$ lies in P then $\ell_i(u) = 0$ for all $i \in I$. Hence

$$L = \Big\{ u \in V : \quad \ell_i(u) = 0 \quad \text{for} \quad i \in I \Big\}$$

is the largest under inclusion subspace contained in P.

Let us consider the projection $pr : V \longrightarrow V/L$ and let $Q = pr(P)$. By Theorem 6.1, $Q \subset V/L$ is a polyhedron. Furthermore, Q does not contain lines (since L is the largest subspace contained in P). Therefore, by Theorem 9.10, we can write $Q = M + K$, where M is a polytope and K is a cone without lines. By introducing a scalar product in V, we may identify V/L with a subspace of V. Since for all $x \in V/L$, we have $pr^{-1}(x) = x + L$, we obtain that $P = L + M + K$. $\square$

(9.12) Theorem. *Let $P \subset V$ be a polyhedron defined by (9.1.1). For a point $v \in P$, let $I_v \subset I$ be the set of inequalities active on v. Let*

$$F_v = \Big\{ x \in P : \quad \ell_i(x) = \alpha_i \quad \textit{for} \quad i \in I_v \Big\}.$$

Then

(1) *F_v is a face of P;*

(2) *If $F \subset P$ is a non-empty face of P then $F = F_v$ for any v in the relative interior of F.*

Proof. Let

$$\ell = \sum_{i \in I_v} \ell_i \quad \text{and} \quad \alpha = \sum_{i \in I_v} \alpha_i.$$

Clearly, $\ell(x) \leq \alpha$ for all $x \in P$ and

$$F_v = \Big\{ x \in P : \quad \ell(x) = \alpha \Big\}.$$

Hence F_v is a face, which proves Part (1).

To prove Part (2), suppose that a face $F \subset P$ is defined by the equation

$$F = \Big\{ x \in P : \quad \ell(x) = \alpha \Big\},$$

where $\ell : V \longrightarrow \mathbb{R}$ is a linear function and α is a real number such that $\ell(x) \leq \alpha$ for all $x \in P$. Let $v \in F$ be a point in the relative interior and let $I_v \subset I$ be the set of inequalities active on v. We claim that

(9.12.1)
$$\ell \ \in \ \operatorname{span}\Big\{ \ell_i : \quad i \in I_v \Big\}.$$

Indeed, if (9.12.1) if violated, one can find a point $u \in V$ such that $\ell_i(u) = 0$ for all $i \in I_v$ and $\ell(u) \neq 0$. Choosing $-u$, if necessary, we may assume that $\ell(u) > 0$. Then for all sufficiently small $\epsilon > 0$ we have $v' = v + \epsilon u \in P$ and $\ell(v') > \ell(v) = \alpha$, which is a contradiction. Hence (9.12.1) holds and

$$\ell = \sum_{i \in I_v} \lambda_i \ell_i \quad \text{and hence} \quad \alpha = \sum_{i \in I_v} \lambda_i \alpha_i$$

for some real λ_i. Therefore,

$$F_v \ \subset \ F.$$

Suppose now that there is a point $w \in F \setminus F_v$. Then there exists a $j \in I_v$ such that $\ell_j(w) < \alpha_j$. Therefore,

$$G = \Big\{ x \in F : \quad \ell_j(x) = \alpha_j \Big\}$$

is a proper face of F containing v, which is a contradiction, since v was chosen in the relative interior of F. $\square$

(9.13) Problems.

1. Let us fix a linear functional $\ell : V \longrightarrow \mathbb{R}$. For a compact convex set $A \subset V$, let $F_\ell(A)$ be the face of A in the direction of ℓ:

$$F_\ell(A) = \Big\{ y \in A : \quad \ell(y) = \max_{x \in A} \ell(x) \Big\}.$$

Prove that the correspondence $A \longmapsto F_\ell(A)$ gives rise to a valuation on $\mathcal{C}_b(V)$: if

$$\sum_{i \in I} \alpha_i \,[A_i] = 0 \quad \text{then} \quad \sum_{i \in I} \alpha_i \,[F_\ell\,(A_i)] = 0.$$

2. Construct examples of convex compact sets $C \subset B \subset A$ such that B is a face of A, C is a face of B but C is not a face of A.

3°. Show that the intersection of every two faces of a set is a face of the set.

4°. Prove that a vertex of a polyhedron is a face of the polyhedron.

5°. Let $P_1, P_2 \subset V$ be polyhedra and let $Q = P_1 + P_2$. Prove that every face of Q is the Minkowski sum of a face of P_1 and a face of P_2.

6. Let $P_1, P_2 \subset V$ be non-empty polyhedra and let $Q = P_1 \cap P_2$. Prove that every vertex v of Q can be written as $v = F_1 \cap F_2$, where F_1 is a face of P_1, F_2 is a face of P_2 and $\dim F_1 + \dim F_2 \leq \dim V$.

7. Let $A \subset V$ be a convex set, where V is a finite-dimensional space. Prove that either A contains an interior point or lies in a proper affine subspace of V.

8. Let $A \subset V$ be a closed convex set. Suppose that A has finitely many faces. Prove that A is a polyhedron.

10. The Euler–Poincaré formula

(10.1) Lemma. *Let $P \subset \mathbb{R}^k$ be a polytope with a non-empty interior* $\operatorname{int} P$. *Then* $[\operatorname{int} P] \subset \mathcal{C}_b\left(\mathbb{R}^k\right)$ *and*

$$\chi\left([\operatorname{int} P]\right) = (-1)^k.$$

Proof. By Corollary 6.2, P is a bounded polyhedron and by Theorem 9.12 every point of P lies in the relative interior of a face of P. Since faces of P are polyhedra and the intersection of any number of faces of P is a face (possibly empty) of P, we have $[\operatorname{int} P] \subset \mathcal{C}_b\left(\mathbb{R}^k\right)$.

Now we use the induction on k. The case of $k = 0$ is obvious and for $k > 0$ we use Corollary 5.3. Namely, let $H_\tau \subset \mathbb{R}^k$ be the affine hyperplane consisting of the points with the last coordinate τ. Then

$$\chi\left([\operatorname{int} P]\right) = \sum_{\tau \in \mathbb{R}} \Big(\chi\left([\operatorname{int} P \cap H_\tau]\right) - \lim_{\epsilon \longrightarrow 0+} \chi\left([\operatorname{int} P \cap H_{\tau - \epsilon}]\right) \Big).$$

By the induction hypothesis, for τ equal to the maximum value of the last coordinate on P the corresponding summand is $0 - (-1)^{k-1} = (-1)^k$, while for all other τ the corresponding summand is 0. $\square$

Now we can establish the Euler-Poincaré formula.

(10.2) Theorem. *Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope and let $f_i(P)$ denotes the number of i-dimensional faces of P. Then*

$$\sum_{i=0}^{d-1} (-1)^i f_i(P) = 1 + (-1)^{d-1}.$$

Proof. Using Theorem 9.12, we can write

$$[P] = [\operatorname{int} P] + \sum_F [\operatorname{int} F],$$

where the sum is taken over all proper faces F of P and $\operatorname{int} F$ is the relative interior of F. Applying the Euler characteristic to the both parts of the identity and using Lemma 10.1, we conclude

$$1 = (-1)^d + \sum_F (-1)^{\dim F},$$

from which the proof follows. □

(10.3) Problems.

1. Let $P \subset V$ be an unbounded polyhedron without lines. Prove that $\chi([\operatorname{int} P]) = 0$.

2. Let $P \subset \mathbb{R}^d$ be a non-empty unbounded polyhedron without lines, let $f_i^0(P)$ be the number of bounded i-dimensional faces of P and let $f_i^\infty(P)$ be the number of unbounded i-dimensional faces of P. Prove that

$$\sum_{i=0}^{d-1} (-1)^i f_i^0(P) = 1 \quad \text{and} \quad \sum_{i=1}^{d} (-1)^{i+1} f_i^\infty(P) = 1.$$

3. Let $P \subset \mathbb{R}^3$ be a 3-dimensional polytope. For a vertex v of P let us define the *curvature* $\kappa(v)$ at v as 2π minus the sum of the angles at v of the facets of P containing v. Prove the *Gauss-Bonnet formula*

$$\sum_v \kappa(v) = 4\pi,$$

where the sum is taken over all vertices v of P.

11. The Birkhoff polytope

(11.1) Definitions. A *doubly stochastic* matrix is an $n \times n$ non-negative matrix with all row sums and all column sums equal to 1. A *permutation* matrix $\pi(\sigma)$ corresponding to a permutation σ of the set $\{1, \dots, n\}$ is the $n \times n$ matrix such that

$$\pi(\sigma)_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise.} \end{cases}$$

The set of all permutations σ of $\{1, \dots, n\}$ is the *symmetric group* S_n. Let $\mathbb{R}^{n \times n}$ be the vector space of all $n \times n$ matrices. The *Birkhoff polytope* B_n is defined as the bounded polyhedron in $\mathbb{R}^{n \times n}$ that consists of all doubly stochastic matrices.

(11.2) Theorem. *The vertices of B_n are the $n \times n$ are the $n \times n$ permutation matrices. Consequently,*

$$B_n = \operatorname{conv}\Big(\pi(\sigma): \quad \sigma \in S_n\Big).$$

Proof. It is immediate that a permutation matrix is necessarily a vertex of B_n. Let us prove that there are no other vertices.

We proceed by induction on n. The case of $n = 1$ is trivial.

Let $A \subset \mathbb{R}^{n \times n}$ be the affine subspace of the $n \times n$ matrices with row and column sums equal to 1. We claim that $\dim A = (n-1)^2$. Indeed, every matrix $x \in A$, $x = (x_{ij})$, is uniquely specified by its $(n-1) \times (n-1)$ upper left corner, since for other entries we should have

$$\begin{aligned} x_{in} =& 1 - \sum_{j=1}^{n-1} x_{ij} \quad \text{for} \quad i = 1, \ldots, n-1 \\ x_{nj} =& 1 - \sum_{i=1}^{n-1} x_{ij} \quad \text{for} \quad j = 1, \ldots, n-1 \quad \text{and} \\ x_{nn} =& (2-n) + \sum_{\substack{1 \leq i \leq n-1 \\ 1 \leq j \leq n-1}} x_{ij}. \end{aligned}$$

The polyhedron $B_n \subset A$ is defined by n^2 inequalities $x_{ij} \geq 0$. Let us consider a vertex $v = (v_{ij})$ of B_n. By Lemma 9.3, at least $(n-1)^2$ of the inequalities should be active on v and hence at least $(n-1)^2 = n^2 - 2n + 1$ entries v_{ij} should be 0. We observe that no row can have all zeros and that there must be a row with $(n-1)$ zero, since otherwise the total number of zeros would have been at most $n(n-2) < (n-1)^2$. The remaining entry in a row with $(n-1)$ zero must be 1 and hence all other entries in the corresponding column with 1 must be 0. If we cross out the row and the column containing 1, we obtain an $(n-1) \times (n-1)$ doubly stochastic matrix w which should be a vertex of B_{n-1}. By the induction hypothesis, w is a permutation matrix and hence v must also be a permutation matrix. $\square$

(11.3) Problems.

1. Consider a polytope of $n \times n$ symmetric doubly stochastic matrices. Prove that every vertex of the polytope is either a permutation matrix or the average of two permutation matrices.

2. Let G be a finite group acting in a finite-dimensional vector space V, let $A \subset V$ be a convex compact set such that $g(A) = A$ for all $g \in G$ and let $L \subset V$ be the subspace $L = \{x \in V : \ g(x) = x \text{ for all } g \in G\}$. Prove that every extreme point of $A \cap L$ is a convex combination of at most $|G|$ extreme points of A.

3. Prove that the interval $[\pi(\sigma), \pi(\tau)]$ is an edge (1-dimensional face) of B_n if and only if $\sigma^{-1}\tau$ is a cycle (that is, a permutation consisting of a single cycle and fixed points).

We prove the Schur part of the Schur–Horn Theorem.

(12.1) Theorem. *Let A be an $n \times n$ real symmetric matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $l = (\lambda_1, \ldots, \lambda_n)$ and let $a = (a_{11}, \ldots, a_{nn})$ be the vector of the diagonal entries of A. Then a lies in the convex hull of the vectors obtained from l by a permutation of the coordinates:*

$$a \ \in \ \operatorname{conv}\Big(\pi(\sigma) l : \quad \sigma \in S_n\Big).$$

Proof. Matrix A can be written in the form $A = U \Lambda U^T$, where $U = (u_{ij})$ is an orthogonal matrix and Λ is the diagonal matrix having $\lambda_1, \ldots, \lambda_n$ on the diagonal. Then

$$a_{ii} = \sum_{j=1}^{n} \lambda_j u_{ij}^2 \quad \text{for} \quad i = 1, \ldots, n.$$

Let $C = (c_{ij})$ be the matrix defined by $c_{ij} = u_{ij}^2$. Since U is an orthogonal matrix, C is doubly stochastic and we have $a = Cl$. By Theorem 11.2, matrix C can be written as a convex combination of permutation matrices $\pi(\sigma)$ and the proof follows. $\square$

(12.2) Problems.

1*. Prove the Horn part of the Schur-Horn Theorem: for every vector $a \in \operatorname{conv}\Big(\pi(\sigma) l : \quad \sigma \in S_n\Big)$ there is a real symmetric $n \times n$ matrix A with diagonal a and the eigenvalues $l = (\lambda_1, \ldots, \lambda_n)$.

2. Complete the following alternative proof of Theorem 12.1. Its advantage is that it can be generalized to other situations, in particular, to the orbits of other Lie groups.

Let V be the space of $n \times n$ symmetric matrices with the scalar product $\langle A, B \rangle = \operatorname{trace}(AB)$. Let $W \subset V$ be the subspace consisting of the diagonal matrices. Consider the orthogonal projection $pr : V \longrightarrow W$ that replaces the non-diagonal entries of a matrix with zeros. Consider the orbit

$$O = \Big\{ U \Lambda U^T : \quad U \quad \text{is orthogonal} \Big\}$$

of a diagonal matrix Λ. Notice that $O \cap W$ consists of the diagonal matrices obtained from Λ by permuting the entries on the diagonal. Our goal is to show that

$$pr(O) \ \subset \ \operatorname{conv}\big(O \cap W\big).$$

Argue that it suffices to show that for any $B \in W$ the maximum of the linear function $\ell(X) = \operatorname{trace}(XB)$ is attained at a point of $O \cap W$. Moreover, it is enough to show that for B with distinct diagonal entries. Note that O is a smooth manifold and that the tangent space at $X \in O$ consists of all matrices of the type $XY - YX$ for a skew-symmetric Y: $Y^T = -Y$. Then X is a critical point of ℓ if and only if

$$0 = \operatorname{trace}\bigl(XYB - YXB\bigr) = \operatorname{trace}\bigl(YBX - YXB\bigr) = \operatorname{trace}\bigl(Y(BX - XB)\bigr)$$
$$\text{for all skew-symmetric} \quad Y.$$

The last condition implies that $BX - XB = 0$ and since the diagonal entries of B are distinct, X must be a diagonal matrix.

13. Transportation polyhedra

(13.1) Definition. Let G be a directed graph with a finite set V of vertices and a finite set E of directed edges $v \to w$ for certain pairs $v, w \in V$. We forbid loops $v \to v$ and multiple edges. Let $b = \bigl\{b_v : \ v \in V\bigr\}$ be real numbers assigned to the vertices. If $b_v > 0$, we call b_v the *demand* at v, if $b_v < 0$, we call b_v the *supply* at v. The pair (G, b) is called a *transportation network*. An assignment $x_e : e \in E$ of non-negative numbers to the edges of G is called a *feasible flow* if at every vertex v the balance condition is satisfied:

$$\sum_{\substack{e \in E: \\ e \text{ ends in } v}} x_e - \sum_{\substack{e \in E: \\ e \text{ begins in } \ v}} x_e = b_v \quad \text{for all} \quad v \in V.$$

We define the polyhedron $T(G; b) \subset \mathbb{R}^E$ as the set of all feasible flows.

(13.2) Lemma. *Suppose that $x = (x_e : \ e \in E)$ is a vertex of $T(G; b)$. Let $E_x \subset E$ be the set of edges e where $x_e > 0$. Then E_x contains no cycles, that is, configurations of the type $v_1 - v_2 - \ldots - v_m - v_1$ where "$-$" stands for an edge in either direction.*

Proof. Suppose that there is a cycle $v_1 - v_2 - \ldots - v_m - v_1$ with positive flows on the edges. We choose a sufficiently small $\epsilon > 0$ and construct two feasible flows y and z, where

flow y is obtained from x by adding ϵ to the flow on each edge of the type $v_i \to v_{i+1}$, $v_m \to v_1$ and subtracting ϵ from the flow on each edge of the type $v_i \leftarrow v_{i+1}$ and $v_m \leftarrow v_1$;

flow z is obtained from x by adding ϵ to the flow on each edge of the type $v_i \leftarrow v_{i+1}$, $v_m \leftarrow v_1$ and subtracting ϵ from the flow on each edge of the type $v_i \to v_{i+1}$ and $v_m \to v_1$.

Since $x = (y + z)/2$, flow x cannot be a vertex. □

(13.3) Theorem. *Suppose that $b_v : \ v \in V$ are integer. Let $x \in T(G; b)$ be a vertex. Then x_e is integer for all $e \in E$.*

Proof. By Lemma 13.2 it suffices to prove the following statement:

suppose that $\{x_e\}$ is a feasible flow in a transportation network with integer demand/supplies and such that the set of edges e with $x_e > 0$ has no cycles. Then x_e are integers.

We proceed by induction on the number of non-zero x_e. If all $x_e = 0$ then clearly x_e are all integers. If not all $x_e = 0$ then there is a vertex v such that for only one edge e incident to v we have $x_e > 0$. Let w be the other vertex incident to e. We modify the network by setting

$$b_w := b_w + b_v \quad \text{and} \quad b_v := 0,$$

modify the feasible flow by setting $x_e := 0$ and apply the induction hypothesis. $\square$

(13.4) Problems.

1°. Show that the Birkhoff polytope of Section 11 is a transportation polyhedron.

2°. Consider the polytope of $m \times n$ non-negative matrices with positive integer row sums $r_1, \ldots, r_m$ and positive integer column sums $c_1, \ldots, c_n$. Show that each vertex of the polytope is an integer matrix.

3°. Show that if $T(G;b)$ is non-empty then $\sum_{v \in V} b_v = 0$.

4. Suppose that $T(G;b)$ is non-empty. Show that $T(G;b)$ is bounded if and only if G does not contain a directed cycle $v_1 \to v_2 \to \ldots \to v_m \to v_1$.

5*. Consider the polytope P_n of all $n \times n \times n$ non-negative arrays $\{x_{ijk}\}$ with all sectional sums equal to 1:

$$\sum_{j,k} x_{ijk} = 1 \quad \text{for} \quad i = 1, \ldots, n;$$

$$\sum_{i,j} x_{ijk} = 1 \quad \text{for} \quad k = 1, \ldots, n;$$

$$\sum_{i,k} x_{ijk} = 1 \quad \text{for} \quad j = 1, \ldots, n.$$

Prove that for any sequence of rational numbers $1 > \sigma_1 > \ldots > \sigma_p > 0$ one can find a positive integer b such that the numbers $(b-1)/b > \sigma_1 > \ldots > \sigma_p > 1/b$ is the set of values (without multiplicities) of the non-zero coordinates of a vertex of P_n for some n.

See M.B. Gromova, The Birkhoff-von Neumann theorem for polystochastic matrices [translation of Operations research and statistical simulation, No. 2 (Russian), 3–15, 149, Izdat. Leningrad. Univ., Leningrad, 1974]. Selected translations. *Selecta Math. Soviet.* **11** (1992), no. 2, 145–158.

14. The permutation polytope

(14.1) Definitions. Let us fix a vector $a = (a_1, \ldots, a_n)$. The convex hull of all vectors in $\mathbb{R}^n$ obtained by permutations of the coordinates of a is called the *permutation polytope* $P(a)$:

$$P(a) = \operatorname{conv}\Big(\big(a_{\sigma(1)}, \ldots, a_{\sigma(n)}\big) : \quad \sigma \in S_n \Big) = \operatorname{conv}\Big(\pi(\sigma) a : \quad \sigma \in S_n \Big),$$

see Definitions 11.1. By $\langle x, y\rangle$ we denote the standard scalar product of vectors $x, y \in \mathbb{R}^n$:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \quad \text{for} \quad x = (x_1, \ldots, x_n) \quad \text{and} \quad y = (y_1, \ldots, y_n)\,.$$

(14.2) Lemma. *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two real n-vectors. Suppose that $a_i > a_j$ and $b_i < b_j$ for some $i \neq j$. Let $\overline{b}$ be the vector obtained from b by swapping coordinates b_i and b_j. Then*

$$\langle \overline{b}, a\rangle \;>\; \langle b, a\rangle.$$

Proof. We have

$$\langle \overline{b}, a\rangle - \langle b, a\rangle = b_i a_j + b_j a_i - b_i a_i - b_j a_j = (b_j - b_i)(a_i - a_j) > 0.$$

□

(14.3) Theorem. *Let us fix a vector $a = (a_1, \ldots, a_n)$ where $a_1 > \ldots > a_n$ and let us consider the permutation polytope $P(a)$. Then the $(n-k)$-dimensional faces F of $P(a)$ containing a are as follows:*

we choose a partition of the integer interval $1, \ldots, n$ into k non-empty consecutive integer intervals $I_1, \ldots, I_k$ and let

$$F_{I_1, \ldots, I_k} = \operatorname{conv}\Big\{\pi(\sigma)a : \quad \sigma(I_j) = I_j \quad \textit{for} \quad j = 1, \ldots, k\Big\}.$$

Proof. Let $b \in \mathbb{R}^n$ be a vector defining a face F of $P(a)$ containing a, so

$$\langle b, a\rangle = \max_{x \in P(a)} \langle b, x\rangle \quad \text{and} \quad F = \Big\{x \in P(a) : \quad \langle b, x\rangle = \langle b, a\rangle\Big\}.$$

By Lemma 14.2 we must have $b_1 \geq b_2 \geq \ldots \geq b_n$. Suppose that

$$b_1 = \ldots = b_{m_1} > b_{m_1+1} = \ldots = b_{m_1+m_2} > b_{m_1+m_2+1} \ldots$$

and let us define the partition

$$I_1 = \{1, \ldots, m_1\}, I_2 = \{m_1 + 1, \ldots, m_1 + m_2\} \ldots$$

In other words, the coordinate b_i does not change as long as i stays within a subinterval I_j of the partition and gets smaller when i moves from I_j to I_{j+1}. It is now clear that the corresponding face F is the convex hull of all vectors $\pi(\sigma)a$, where σ ranges over all permutations $\sigma \in S_n$ that map each of the intervals I_j onto itself. Geometrically, the face $F = F_{I_1, \ldots, I_k}$ is the direct product of permutation polytopes in spaces $\mathbb{R}^{I_1}, \ldots, \mathbb{R}^{I_k}$ and hence $\dim F = n - k$. □

(14.4) Corollary. *Let $a=(a_1, \ldots, a_n)$ be a vector with distinct coordinates. Let us choose a partition*

$$\{1, \ldots, n\} = S_1 \cup \ldots \cup S_k,$$

where S_j are non-empty pairwise disjoint subsets.

Let $s_j = |S_j|$ for $j=1, \ldots, k$ and let us define consecutive subintervals

$$\begin{gathered}I_1 = \{1, \ldots, s_1\}, \ I_2 = \{s_1+1, \ldots, s_1+s_2\}, \ldots, \\ I_k = \{s_1 + \ldots + s_{k-1}+1, \ldots, n\},\end{gathered}$$

so that $|I_j| = |S_j|$ for $j=1, \ldots, k$.

Let $F = F_{S_1, \ldots, S_k}$ be the convex hull of the vectors $x=(x_1, \ldots, x_n)$ such that for all $j=1, \ldots, k$, the set of the coordinates $\{x_i : \ i \in S_j\}$ coincides with the set of the coordinates $\{a_i : \ i \in I_j\}$. Then F is a face of $P(a)$ of dimension $n-k$ and all $(n-k)$-faces of $P(a)$ appear this way.

□

(14.5) Problems.

1°. Let P be a polytope and let $F \subset P$ be a face of P. Prove that F is the convex hull of the vertices of P that belong to F.

2°. Suppose that the coordinates $a_1, \ldots, a_n$ of a are not all equal. Prove that $\dim P(a) = n-1$.

3°. Suppose that $a_1 > \ldots > a_n$. Prove that two vertices of $P(a)$ are the endpoints of an edge (a face of dimension 1) if and only if one vertex is obtained from the other by swapping the coordinates with two adjacent values a_i and a_{i+1} for $i=1, \ldots, n-1$.

4. Let $a=(a_1, \ldots, a_n)$ where $a_1 \geq a_2 \geq \ldots \geq a_n$. Prove that $P(a)$ is defined in $\mathbb{R}^n$ by the inequalities

$$\sum_{i \in S} x_i \ \leq \ \sum_{i=1}^{|S|} a_i \quad \text{for all} \quad S \subset \{1, \ldots, n\}, \quad 0 < |S| < n$$

and the equation

$$\sum_{i=1}^n x_i = \sum_{i=1}^n a_i.$$

5. Let $a=(a_1, \ldots, a_n)$ where $a_1 \geq a_2 \geq \ldots \geq a_n$ and let $x=(x_1, \ldots, x_n)$ where $x_1 \geq x_2 \geq \ldots \geq x_n$. Prove that $x \in P(a)$ if and only if

$$\sum_{i=1}^k x_i \leq \sum_{i=1}^k a_i \quad \text{for} \quad k=1, \ldots, n-1 \quad \text{and} \quad \sum_{i=1}^n x_i = \sum_{i=1}^n a_i$$

(Rado's Theorem).

6. Let $a = (1, \ldots, n)$ and let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Prove that $P(a)$ is a translation of the Minkowski sum of $\binom{n}{2}$ intervals

$$\left[\frac{e_i - e_j}{2}, \ \frac{e_j - e_i}{2}\right] \quad \text{for} \quad i > j.$$

15. Cyclic polytopes

(15.1) Definitions. Let t be a real parameter. The curve

$$\gamma(t) = \left(t, t^2, \ldots, t^d\right) \in \mathbb{R}^d$$

is called the *moment curve.*

Let us pick numbers

$$0 < t_1 < \ldots < t_n$$

and let

$$v_i = \gamma(t_i) \quad \text{for} \quad i = 1, \ldots, n.$$

The polytope

$$P = \operatorname{conv}\Big(v_1, \ldots, v_n\Big)$$

is called the *cyclic polytope.* Often, we suppress the dependence on the choice of parameters $t_1, \ldots, t_n$ in the notation for P and denote it just by $C(d, n)$.

(15.2) Theorem. *Let $k \leq d/2$ and let $v_{i_1}, \ldots, v_{i_k}$ be distinct vertices of $C(d, n)$. Then*

$$F = \operatorname{conv}\left(v_{i_1}, \ldots, v_{i_k}\right)$$

is a face of $C(d, n)$.

Proof. Let us consider a polynomial

$$p(t) = t^{d-2k}\left(t - t_{i_1}\right)^2 \cdots \left(t - t_{i_k}\right)^2.$$

Thus

$$p\left(t_{i_1}\right) = \ldots = p\left(t_{i_k}\right) = 0 \quad \text{and} \quad p(t) > 0 \quad \text{for} \quad t \neq t_{i_1}, \ldots, t_{i_k}; \quad t > 0.$$

Since $\deg p = d$, we can write

$$p(t) = \alpha_0 - \alpha_1 t - \ldots - \alpha_d t^d.$$

Let $a = (\alpha_1, \ldots, \alpha_d)$. Then

$$p(t) = \alpha_0 - \langle a, \gamma(t) \rangle.$$

Hence

$$\langle a, v_{i_1} \rangle = \ldots = \langle a, v_{i_k} \rangle = \alpha_0 \quad \text{and} \quad \langle a, v_i \rangle < \alpha_0 \quad \text{for} \quad i \neq i_1, \ldots, i_k,$$

which concludes the proof. $\square$

(15.3) Problems.

1. Prove that any affine hyperplane in $\mathbb{R}^d$ intersects the moment curve $\gamma(t)$ in not more than d points.

2. Prove the *Gale's eveness condition*: the convex hull of $\{v_i : \ i \in I\}$ is a *facet* of $C(d,n)$ (that is, a face of dimension $d-1$) if and only if $|I| = d$ and any two indices not in I are separated by an even number of indices in I.

3. Describe the faces of $C(4,n)$.

4. Suppose that $d = 2m$ is even. Consider the *trigonometric moment curve*

$$\omega(t) = (\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos mt, \sin mt), \quad 0 \leq t < 2\pi$$

in $\mathbb{R}^d$. Prove that any affine hyperplane in $\mathbb{R}^d$ intersects $\omega(t)$ in not more than d points.

5. Let $\omega(t)$ be the trigonometric moment curve of Problem 2, let $0 \leq t_1 < t_2 < \ldots < t_n < 2\pi$ and let $v_i = \omega(t_i)$ for $i = 1, \ldots, n$. Consider $P = \operatorname{conv}\Big(v_1, \ldots, v_n\Big)$. Prove that the convex hull of every $k \leq d/2$ vertices of P is a face of P.

6. Consider letter "Y", that is, three intervals joined at an endpoint. Prove that one cannot embed "Y" in $\mathbb{R}^d$ by a continuous embedding so that every affine hyperplane intersects the image in at most d points.

7. Let $P \subset \mathbb{R}^d$ be a polytope with n vertices and let $k > d/2$. Suppose that every k vertices of P are the vertices of some proper face of P. Prove that $n \leq d+1$.

16. Polarity

(16.1) Definition. Let V be Euclidean space with the scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ and let $A \subset V$ be a non-empty set. The *polar* A° of A is defined by

$$A^\circ = \Big\{ c \in V : \quad \langle c, x \rangle \leq 1 \quad \text{for all} \quad x \in A \Big\}.$$

(16.2) Lemma. *Let $A \subset V$ be a non-empty closed convex set and let $b \notin A$ be a point. Then there exists a vector $c \in A$ and a number $\alpha \in \mathbb{R}$ such that $\langle c, x \rangle < \alpha$ for all $x \in A$ and $\langle c, b \rangle > \alpha$.*

Proof. First, we prove that there exists a point $a \in A$ such that $\|a - b\| \leq \|x - b\|$ for all $x \in A$. Indeed, let us choose a ball centered at b of a sufficiently large radius r,

$$B_r = \Big\{ x \in V : \quad \|x - b\| \leq r \Big\},$$

which intersects A. Since A is closed, the intersection $A \cap B_r$ is compact and hence the function $x \longmapsto \|x - b\|$ attains its minimum on $A \cap B_r$, which is necessarily the minimum on A. Let

$$c = b - a \neq 0 \quad \text{and} \quad \alpha = \frac{1}{2} \langle b - a, \ b + a \rangle.$$

Since A is convex, for every $x \in A$ and every $0 \leq t \leq 1$, we have $tx + (1-t)a \in A$ and therefore

$$\|tx + (1-t)a - b\|^2 \;\geq\; \|a - b\|^2.$$

Since

$$\|tx+(1-t)a-b\|^2 = \|t(x-a)+(a-b)\|^2 = \|a-b\|^2 + 2t\langle x-a,\ a-b\rangle + t^2\|x-a\|^2,$$

we conclude that

$$\langle c,\ x-a\rangle \;\leq\; 0 \quad \text{for all} \quad x \in A$$

and hence

$$\langle c,\ x\rangle \;\leq\; \langle c,\ a\rangle \quad \text{for all} \quad x \in A. \tag{16.2.1}$$

On the other hand,

$$\langle c,\ b\rangle - \langle c,\ a\rangle = \|c\|^2 \; > 0,$$

and hence

$$\langle c,\ b\rangle \;>\; \langle c,\ a\rangle. \tag{16.2.2}$$

Since α is the average of $\langle c, a\rangle$ and $\langle c, b\rangle$, the proof follows by (16.2.1) and (16.2.2).
□

(16.3) Theorem.

(1) *Let $A \subset V$ be a closed convex set containing 0. Then*

$$(A^\circ)^\circ = A;$$

(2) *There exists a linear transformation $\mathcal{D} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ such that*

$$\mathcal{D}\left([A]\right) = [A^\circ]$$

for all non-empty closed convex sets $A \subset V$;

(3) *Let $P \subset V$ be a non-empty polyhedron. Then P° is a polyhedron.*

Proof. Let us choose an $x \in A$. Then $\langle c, x\rangle \leq 1$ for all $c \in A^\circ$ and hence $x \in (A^\circ)^\circ$. Suppose that there exists a $b \in (A^\circ)^\circ$ such that $b \notin A$. Since A is closed and convex, by Lemma 16.2 there is a vector $c \in V$ and a number $\alpha \in \mathbb{R}$ such that $\langle c, x\rangle < \alpha$ for all $x \in A$ and $\langle c, b\rangle > \alpha$. Since $0 \in A$ we conclude that $\alpha > 0$. Scaling, if necessary, $c \longmapsto \alpha^{-1}c$, we may assume that $\alpha = 1$. Then $c \in A^\circ$ and $\langle c, b\rangle > 1$, which contradicts that $b \in (A^\circ)^\circ$. Hence Part (1) is proven.

For $\epsilon > 0$ let us define a function $G_\epsilon : V \times V \longrightarrow \mathbb{R}$ by

$$G_\epsilon(x, y) = \begin{cases} 1 & \text{if } \langle x, y\rangle < 1 + \epsilon \\ 0 & \text{otherwise.} \end{cases}$$

We claim that for all $f \in \mathcal{C}(V)$ and all $y \in V$

the function $g_{y,\epsilon}(x) = f(x)G_\epsilon(x,y)$ lies in $\mathcal{C}(V)$ and
the limit $h(y) = \lim_{\epsilon \longrightarrow 0+} \chi\left(g_{y,\epsilon}\right)$ exists.

By linearity, it suffices to check the above statements for $f = [A]$, where A is a non-empty closed convex set. Let

$$H_{y,\epsilon} = \Big\{x \in V : \quad \langle x, y \rangle \geq 1 + \epsilon\Big\}.$$

Thus $H_{y,\epsilon}$ is a closed halfspace and

$$g_{y,\epsilon} = [A] - \left[A \cap H_{y,\epsilon}\right],$$

so $g_{y,\epsilon} \in \mathcal{C}(V)$. Therefore,

$$\chi\left(g_{y,\epsilon}\right) = \begin{cases} 1 & \text{if } \ \langle x, y \rangle < 1 + \epsilon \quad \text{for all} \quad x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$h(y) = \begin{cases} 1 & \text{if } \ \langle x, y \rangle \leq 1 \quad \text{for all} \quad x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we claim that the linear transformation $\mathcal{D} : f \longmapsto h$ maps $\mathcal{C}(V)$ into $\mathcal{C}(V)$ and maps the indicator of a non-empty closed convex set to the indicator of its polar. Indeed, if $f = [A]$ then $h = [A^\circ]$ and the proof of Part (2) follows.

By Theorem 9.11 there is a decomposition

$$P = L + K + M,$$

where $L \subset V$ is a subspace, $K \subset V$ is a pointed polyhedral cone and M is a polytope.

We observe that

$$L^\circ = \Big\{c \in V : \quad \langle c, x \rangle = 0 \quad \text{for all} \quad x \in L\Big\},$$

so L° is the orthogonal complement of L. Indeed, if $\langle c, x \rangle \neq 0$ for some $x \in L$ then by scaling $x \longmapsto \lambda x$ for an appropriate $\lambda \in \mathbb{R}$ we find $x \in L$ such that $\langle c, x \rangle > 1$, so $c \notin L^\circ$.

Next, we claim that

$$K^\circ = \Big\{c \in V : \quad \langle c, x \rangle \leq 0 \quad \text{for all} \quad x \in K\Big\}.$$

Indeed, if $\langle c, x \rangle > 0$ for some $x \in K$ then by scaling $x \longmapsto \lambda x$ for an appropriate $\lambda > 0$ we find $x \in K$ such that $\langle c, x \rangle > 1$, so $c \notin K^\circ$.

By Theorem 9.9, we can write

$$K = \operatorname{co}\Big(u_i: \quad i = 1, \ldots, m\Big)$$

for some vectors $u_1, \ldots, u_m \in V$. Then

$$K^\circ = \Big\{c \in V: \quad \langle c, u_i\rangle \leq 0 \quad \text{for} \quad i = 1, \ldots, m\Big\}.$$

Assuming that

$$M = \operatorname{conv}\Big(v_i: \quad i = 1, \ldots, n\Big),$$

we conclude that

$$M^\circ = \Big\{c \in V: \quad \langle c, v_i\rangle \leq 1 \quad \text{for} \quad i = 1, \ldots, m\Big\}.$$

In particular, L°, K° and M° are polyhedra. We claim that

$$P^\circ = L^\circ \cap K^\circ \cap M^\circ.$$

Indeed, any $x \in P$ can be represented as a sum $x = u + w + y$, where $u \in L$, $w \in K$ and $y \in M$. Then, for every $c \in L^\circ \cap K^\circ \cap M^\circ$, we have

$$\begin{aligned}
\langle c, x\rangle =& \langle c, u\rangle + \langle c, w\rangle + \langle c, y\rangle \\
=& \qquad\quad\;\; \langle c, w\rangle + \langle c, y\rangle \\
\leq& \qquad\qquad\qquad\quad\;\; \langle c, y\rangle \\
\leq& \qquad\qquad\qquad\qquad 1
\end{aligned}$$

and hence $c \in P^\circ$. Suppose that $c \in P^\circ$. Let us choose a point $x \in P$. Then, for any $u \in L$ and any $\lambda \in \mathbb{R}$ we have $x + \lambda u \in P$ and so we must have $\langle c, u\rangle = 0$, so $c \in L^\circ$. Similarly, for any $u \in K$ and any $\lambda \geq 0$ we have $x + \lambda u \in P$, so we must have $\langle c, u\rangle \leq 0$ and $c \in K^\circ$. Finally, $M \subset P$, so we must have $c \in M^\circ$. $\square$

(16.4) Problems.

1°. Find the polars of the origin, the whole space, the unit ball, the standard cube and octahedron, see Problem 4 of Section 1.2.

2°. Show that $A \subset B$ implies $B^\circ \subset A^\circ$, that $\left(\bigcup_{i \in I} A_i\right)^\circ = \bigcap_{i \in I} A_i^\circ$ and that $(\alpha A)^\circ = \alpha^{-1} A^\circ$ for $\alpha > 0$.

3. Let $K_1, K_2 \subset V$ be two polyhedral cones. Show that $(K_1 + K_2)^\circ = K_1^\circ \cap K_2^\circ$ and that $(K_1 \cap K_2)^\circ = K_1^\circ + K_2^\circ$.

4. Let $I \cup J = \{1, \ldots, d\}$ be a partition, $I \cap J = \emptyset$. Let $K_1, K_2 \subset \mathbb{R}^d$ be two sets defined by

$$\begin{aligned}
K_1 =& \Big\{(x_1, \ldots, x_d): \quad x_i \geq 0 \quad \text{for} \quad i \in I \quad \text{and} \quad x_j > 0 \quad \text{for} \quad j \in J\Big\}, \\
K_2 =& \Big\{(x_1, \ldots, x_d): \quad x_i \leq 0 \quad \text{for} \quad i \in I \quad \text{and} \quad x_j > 0 \quad \text{for} \quad j \in J\Big\}.
\end{aligned}$$

Prove that $\mathcal{D}([K_1]) = (-1)^{|J|}[K_2]$.

(17.1) Theorem. *Let V be Euclidean space, $\dim V = d$, and let $P \subset V$ be a polytope containing the origin in its interior. Let $Q = P^\circ \subset V$. For a proper face F of P let us define $\hat{F} \subset Q$ by*

$$\hat{F} = \Big\{ c \in Q : \quad \langle c, x \rangle = 1 \quad \text{for all} \quad x \in F \Big\}.$$

Then

(1) *Q is a polytope containing the origin in its interior;*
(2) *$\hat{F}$ is a face of Q;*
(3) *$\dim F + \dim \hat{F} = d - 1$;*
(4) *If G is a proper face of P such that $F \subset G$ then $\hat{G} \subset \hat{F}$;*
(5) *Let G be a proper face of Q. Let*

$$F = \Big\{ x \in P : \quad \langle x, c \rangle = 1 \quad \text{for all} \quad c \in G \Big\}.$$

Then F is a proper face of P and $\hat{F} = G$.

Proof. Let

$$P = \operatorname{conv}\Big(v_i, \ i = 1, \ldots, n \Big),$$

where $v_1, \ldots, v_n$ are the vertices of P. Then

$$Q = \Big\{ c \in V : \quad \langle c, v_i \rangle \leq 1 \quad \text{for} \quad i = 1, \ldots, n \Big\}. \tag{17.1.1}$$

In particular, Q is a polyhedron. For $\rho > 0$ let

$$B_\rho = \Big\{ x \in V : \quad \|x\| \leq \rho \Big\}$$

be the ball of radius ρ centered at the origin. Since for some $\epsilon > 0$ we have

$$B_\epsilon \ \subset \ P \ \subset \ B_{1/\epsilon},$$

we have

$$B_\epsilon \ \subset \ Q \ \subset B_{1/\epsilon}$$

and so Q is a bounded polyhedron containing the origin in its interior. Part (1) now follows.

Suppose that

$$F = \operatorname{conv}\Big(v_i : \quad i \in I \Big). \tag{17.1.2}$$

Then

$$\hat{F} = \Big\{ c \in Q : \quad \langle c, v_i \rangle = 1 \quad \text{for all} \quad i \in I \Big\}.$$

Let

$$u = \frac{1}{|I|} \sum_{i \in I} v_i$$

be the average of the vertices v_i on the face F. Because of (17.1.1), we can write

$$\hat{F} = \Big\{ c \in Q : \quad \langle c, u \rangle = 1 \Big\}.$$

Therefore, $\hat{F}$ is a face of Q, so Part (2) follows.

Since F is a proper face of P there exists a $c \in V$ and a number α such that

$$\langle c, x \rangle \leq \alpha \quad \text{for all} \quad x \in P \quad \text{and} \quad F = \Big\{ x \in P : \quad \langle c, x \rangle = \alpha \Big\}.$$

Since P contains the origin in its interior, $\alpha > 0$ and by scaling $c \longmapsto \alpha^{-1} c$ we can assume that $\alpha = 1$. Hence by (17.1.2),

$$\langle c, v_i \rangle = 1 \quad \text{for} \quad i \in I \quad \text{and} \quad \langle c, v_i \rangle < 1 \quad \text{for} \quad i \in I. \tag{17.1.3}$$

In particular, $c \in \hat{F}$. Let $L = \operatorname{span}(F)$ and let $L^\perp$ be the orthogonal complement to L. Thus $\dim L = \dim F + 1$ and $\dim L^\perp = d - \dim F - 1$. We observe that for any $w \in L^\perp$ and a sufficiently small $\epsilon > 0$ the perturbation $c \longmapsto c + \epsilon w$ satisfies (17.1.3) and hence lies in $\hat{F}$. Therefore, $\dim \hat{F} \geq d - \dim F - 1$. Moreover, for any $c' \in \hat{F}$ and any $x \in F$ we have $\langle c - c', x \rangle = 0$ and hence $\dim \hat{F} \leq d - \dim L = d - \dim F - 1$. This concludes the proof of Part (3).

Part (4) is obvious.

By Part (1) of Theorem 16.3, we have $P = Q^\circ$. We conclude that F is a proper face of P by exchanging the roles of P and Q and by Part (2) of the Theorem, so $F = \hat{G}$. Moreover, by Part (3) of the Theorem, we have $\dim F = d - \dim G - 1$ and hence $\dim \hat{F} = \dim G$. Clearly, $G \subset \hat{F}$. Let us pick a point x in the relative interior of G and hence in the relative interior of $\hat{F}$. By Theorem 9.12, both $\hat{F}$ and G are defined by turning the inequalities of Q active on x into equations, and hence $G = \hat{F}$, which proves Part (5). □

(17.2) Face figure. Let $P \subset V$ be a d-dimensional polytope containing the origin in its interior and let $F \subset P$ be a k-dimensional face of P, $0 \leq k \leq d-1$. Let $Q = P^\circ$, let $\hat{F}$ be the face of Q constructed in Theorem 17.1 and let H be the polar of $\hat{F}$ computed in the affine span of $\hat{F}$ with respect to the origin chosen in the relative interior of $\hat{F}$. Hence H is a polytope and $\dim H = d - 1 - k$. Theorem 17.1 implies that there is an inclusion-preserving bijection between the faces G of P containing F and the faces of H, where F itself corresponds to the empty face of H and P corresponds to H. The partially ordered (by inclusion) set of all faces G of P containing F is called the *face figure* of F in P and denoted P/F. Hence the face figure P/F is isomorphic to the partially ordered (by inclusion) set of all faces of a $(d-k-1)$-dimensional polytope H (the empty face and H itself included).

(17.3) Problems.

1°. Let P be a d-dimensional polytope and let $F \subset G$ be two faces of P such that $\dim G - \dim F = 2$. Prove that there are precisely two faces H_1, H_2 of P such that $F \subset H_1, H_2 \subset G$ and the inclusions are proper.

2. Let P be a d-dimensional polytope and let $\operatorname{int} P$ be its interior. Prove that

$$(-1)^d[\operatorname{int} P] = \sum_F (-1)^{\dim F}[F],$$

where the sum is taken over all faces F of P, including P but not including the empty face.

3. Let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{R}^4$ and let

$$P = \operatorname{conv}\Big(e_i + e_j,\ -e_i - e_j,\ e_i - e_j \quad \text{for} \quad 1 \leq i \neq j \leq 4\Big).$$

Prove that P° can be obtained from P by an invertible linear transformation (such polytopes are called *self-dual*).

18. Regular triangulations and subdivisions

(18.1) Definitions. Points $v_1, \ldots, v_m \in V$ are called *affinely independent* if whenever $\alpha_1 v_1 + \ldots + \alpha_m v_m = 0$ for some real α's such that $\alpha_1 + \ldots + \alpha_m = 0$, one must have $\alpha_1 = \ldots = \alpha_m = 0$. Equivalently, $v_1, \ldots, v_m$ are affinely independent if and only if the vectors $w_i = (v_i, 1) \in V \oplus \mathbb{R}$, $i = 1, \ldots, m$, are linearly independent. The convex hull of a set of affinely independent points is called a *simplex*. Equivalently, a simplex is the image $T(\Delta_{m-1})$ of the standard simplex Δ_{m-1} (see Problem 4 of Section 1.2), where T is a composition of a translation and a linear transformation with zero kernel.

Let $P \subset \mathbb{R}^{d-1}$ be a $(d-1)$-dimensional polytope. A *triangulation* of P is a representation of P as a finite union of $(d-1)$-dimensional simplices such that every two simplices are either disjoint or intersect by a common proper face. We are interested in triangulations *without new vertices*, that is, when the vertices of the simplices of the triangulation are vertices of the polytope P. More generally, a *polytopal subdivision* is a representation of P as a finite union of $(d-1)$-dimensional polytopes such that every two polytopes are either disjoint or intersect by a common proper face (again, we are interested in subdivisions without new vertices). We say that subdivision S_1 *refines* subdivision S_2 if every polytope of subdivision S_2 is a union of polytopes of subdivision S_1.

(18.2) Lemma. *Let $v_1, \ldots, v_{d+1} \in \mathbb{R}^{d-1}$ be points. Suppose that there exist real $\tau_1, \ldots, \tau_{d+1}$ such that the points $w_i = (v_i, \tau_i) \in \mathbb{R}^d$, $i = 1, \ldots, d+1$, are affinely independent. Then the set of such vectors $t = (\tau_1, \ldots, \tau_{d+1})$ is open and dense in $\mathbb{R}^{d+1}$.*

Proof. Points $w_1, \ldots, w_{d+1}$ are affinely independent if and only if the determinant of the $(d+1) \times (d+1)$ matrix with the columns $(w_1, 1), \ldots, (w_{d+1}, 1)$ is not 0. The last condition can be written as $p(t) \neq 0$, $t = (\tau_1, \dots, \tau_{d+1})$, for some polynomial $p : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}$. We have either $p(t) \equiv 0$ for $t \in \mathbb{R}^{d+1}$ or the set $\Big\{t \in \mathbb{R}^d : \quad p(t) \neq 0\Big\}$ is open and dense in $\mathbb{R}^{d+1}$. $\square$

(18.3) Regular subdivisions. Let $P \subset \mathbb{R}^{d-1}$ be a polytope with the vertices $v_1, \dots, v_n$. We assume that $\dim P = d-1$. For $t = (\tau_1, \dots, \tau_n)$ we construct a polyhedron $Q(t) \subset \mathbb{R}^d = \mathbb{R}^{d-1} \oplus \mathbb{R}$ as follows.

Let $w_i = (v_i, \tau_i)$ for $i = 1, \dots, n$ and let

$$Q(t) = \Big\{(x, \sigma) : \quad (x, \tau) \in \operatorname{conv}(w_1, \dots, w_n) \quad \text{for some} \quad \tau \geq \sigma\Big\}.$$

Alternatively,

$$Q(t) = \operatorname{conv}(w_1, \dots, w_n) + R,$$

where R is the ray $\big\{(0, \tau) : \quad \tau \leq 0\big\} \subset \mathbb{R}^{d-1} \oplus \mathbb{R}$. We call $Q(t)$ a *lifting* of P.

For any $x \in P$, the line $\big\{(x, \tau) : \quad -\infty < \tau < +\infty\big\}$ intersects the boundary of $Q(t)$ at a single point which lies in some bounded face of $Q(t)$. Hence the projections of the bounded facets (faces of dimension $(d-1)$) of $Q(t)$ form a subdivision of P, which is called a *regular subdivision.* If each bounded facet of $Q(t)$ is a simplex, we get a triangulation of P, called a *regular triangulation.*

(18.4) Problems.

1°. Prove that the set of all vectors $t = (\tau_1, \dots, \tau_n)$, $t \in \mathbb{R}^n$, such that all bounded facets of $Q(t)$ are simplices is open and dense in $\mathbb{R}^n$.

2. Prove that every subdivision of a convex polygon $P \subset \mathbb{R}^2$ by its non-intersecting diagonals is a regular subdivision.

19. The secondary polytope

(19.1) Definitions. Let $P \subset \mathbb{R}^{d-1}$ be a polytope with the vertices $v_1, \dots, v_n$. We assume that $\dim P = d-1$. Following I.M. Gelfand, M. Kapranov and A. Zelevinsky, we define its *secondary polytope* $\Sigma(P) \subset \mathbb{R}^n$. First, we interpret $\mathbb{R}^n$ as the space of all real-valued functions ψ on the vertices of the polytope with the scalar product

$$\langle \phi, \ \psi \rangle = \sum_{i=1}^n \phi(v_i) \psi(v_i).$$

Then, for each triangulation T of P we define $\phi_T \in \mathbb{R}^n$ by

$$\phi_T(v) = \sum_{\substack{\Delta \in T: \\ v \in \Delta}} \operatorname{vol} \Delta.$$

In words: the value of ϕ_T on a vertex v is the sum of the volumes of the simplices of the triangulation T that contain v. We define

$$\Sigma(P) = \operatorname{conv}\Big(\phi_T : \quad T \text{ is a triangulation of } P\Big).$$

We consider only triangulations without new vertices.

A function $g : P \longrightarrow \mathbb{R}$ is called *concave* if

$$g\left(\sum_{i=1}^{n} \alpha_i v_i\right) \geq \sum_{i=1}^{n} \alpha_i g(v_i)$$
$$\text{whenever} \quad \sum_{i=1}^{n} \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n.$$

Gelfand, Kapranov and Zelevinsky defined the secondary polytope $\Sigma(A)$ more generally, for any set $A \subset \mathbb{R}^{d-1}$ of points spanning $\mathbb{R}^{d-1}$ affinely, not necessarily the set of vertices of a convex polytope.

(19.2) Lemma. *Let $\Delta \subset \mathbb{R}^{d-1}$ be a $(d-1)$-dimensional simplex with the vertices $v_1, \ldots, v_d$ and let $f : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ be an affine function (a linear function plus a constant). Then*

$$\int_{\Delta} f(x)\ dx = \frac{\operatorname{vol} \Delta}{d} \sum_{i=1}^{d} f(v_i).$$

Proof. Applying an invertible affine transformation, we may assume that

$$\Delta = \operatorname{conv}\Big(0, e_1, \ldots, e_{d-1}\Big),$$

where $e_1, \ldots, e_{d-1}$ is the standard basis of $\mathbb{R}^{d-1}$. If $f(x)$ is a constant, the formula obviously holds. If $f(x) = x_i$, the left hand side of the formula evaluates to

$$\begin{aligned}\frac{1}{(d-2)!} \int_0^1 x(1-x)^{d-2}\ dx =& \frac{1}{(d-2)!} \int_0^1 (1-x)x^{d-2}\ dx \\ =& \frac{1}{(d-2)!} \left(\frac{1}{d-1} - \frac{1}{d}\right) = \frac{1}{d!}\end{aligned}$$

and so does the right hand side. $\square$

(19.3) Lemma. *For a function $\psi \in \mathbb{R}^n$ and a triangulation T of P, let us define a function $g_{\psi,T} : P \longrightarrow \mathbb{R}$ as follows: if $\Delta = \operatorname{conv}(v_i :\ i \in I)$ is a simplex of T containing x and*

$$x = \sum_{i \in I} \alpha_i v_i \quad \textit{where} \quad \sum_{i \in I} \alpha_i = 1 \quad \textit{and} \quad \alpha_i \geq 0 \quad \textit{for} \quad i \in I,$$

we define

$$g_{\psi,T}(x) = \sum_{i \in I} \alpha_i \psi(v_i).$$

Then

$$\langle \psi,\ \phi_T \rangle = d \int_P g_{\psi,T}(x)\ dx.$$

Proof. Since the restriction of $g_{\psi,T}$ on every simplex Δ of the triangulation is an affine function which coincides with ψ on the vertices of Δ, by Lemma 19.2 we obtain:

$$\begin{aligned} d\int_P g_{\psi,T}(x)\ dx =& d\sum_{\Delta\in T}\int_\Delta g_{\psi,T}(x)\ dx = \sum_{\Delta \in T} (\operatorname{vol}\Delta)\cdot \left(\sum_{v \text{ is a vertex of } \Delta} \psi(v) \right) \\ =& \sum_{i=1}^n \psi(v_i) \left(\sum_{\substack{\Delta\in T: \\ v_i \in \Delta}} \operatorname{vol}\Delta \right) = \langle \psi,\ \phi_T\rangle. \end{aligned}$$

□

(19.4) Lemma. *Let us choose a function $\psi \in \mathbb{R}^n$, let $t = (\psi(v_1), \ldots, \psi(v_n))$, let $Q(t)$ be a lifting of P and let S be the corresponding regular polytopal subdivision of P as in Section 18.3.*

(1) *Let T be a triangulation of P such that $g_{\psi,T}$ is concave. Then for any triangulation T' of P we have*

$$g_{\psi,T}(x)\ \geq\ g_{\psi,T'}(x) \quad \textit{for all} \quad x \in P;$$

(2) *Let T be a triangulation which refines S. Then $g_{\psi,T}$ is concave;*
(3) *Let T' be a triangulation which does not refine S. Then for some $x \in P$ we have $g_{\psi,T'}(x) < g_{\psi,T}(x)$, where T is a triangulation which refines S.*

Proof. Let $x \in P$ be a point and let $\Delta = \operatorname{conv}(v_i :\ i \in I)$ be a simplex of triangulation T' such that $x \in \Delta$. Then

$$x = \sum_{i\in I} \alpha_i v_i \quad \text{for some} \quad \alpha_i \geq 0 \quad \text{such that} \quad \sum_{i \in I} \alpha_i = 1. \tag{19.4.1}$$

Since $g_{\psi,T}$ is concave, we have

$$g_{\psi,T}(x) \geq \sum_{i\in I} \alpha_i g_{\psi,T}(v_i) = \sum_{i \in I} \alpha_i \psi(v_i) = g_{\psi,T'}(x)$$

and Part (1) follows.

Let

$$x = \sum_{i=1}^n \alpha_i v_i \quad \text{where} \quad \sum_{i=1}^n \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n.$$

Then for

$$\tau = \sum_{i=1}^{n} \alpha_i \psi(v_i)$$

we have

$$(x, \tau) \ \in \ \operatorname{conv}(w_1, \ldots, w_n) \quad w_i = (v_i, \psi(v_i)) \quad \text{for} \quad i = 1, \ldots, n.$$

Therefore, for some $\sigma \geq \tau$ the point (x, σ) lies in a bounded facet of $Q(t)$. Let $\Delta = \operatorname{conv}(v_i : \ i \in I)$ be a simplex of triangulation T containing x. If T refines S then the point (x, σ) lies in the convex hull $\operatorname{conv}(w_i : \ i \in I)$ and

$$g_{\sigma,T}(x) = \sigma \geq \tau = \sum_{i=1}^{n} \alpha_i \psi(v_i) = \sum_{i=1}^{n} \alpha_i g_{\psi,T}(v_i),$$

and Part (2) follows.

Suppose that T' does not refine S. Then there exists a simplex Δ of T', $\Delta = (v_i : \ i \in I)$ such that $\operatorname{conv}(w_i : \ w_i \in I)$ does not lie in a bounded facet of $Q(t)$. Then, for some $x \in \Delta$ as in (19.4.1) and

$$\tau = \sum_{i \in I} \alpha_i \psi(v_i),$$

the point $(x, \tau) \in Q(t)$ does not lie in a bounded facet of $Q(t)$. Therefore, for some $\sigma > \tau$ the point (x, σ) lies in a bounded facet of $Q(t)$ and hence

$$g_{\psi,T'}(x) = \tau < \sigma = g_{\psi,T}(x)$$

and Part (3) follows. □

(19.5) Theorem. *Let $P \subset \mathbb{R}^{d-1}$ be a $(d-1)$-dimensional polyhedron with n vertices $v_1, \ldots, v_n$ and let $\Sigma(P) \subset \mathbb{R}^n$ be its secondary polytope. Then the vertices of $\Sigma(P) \subset \mathbb{R}^n$ are the vectors ϕ_T, where T is a regular triangulation of P. Moreover, the faces of $\Sigma(P)$ are the sets*

$$\operatorname{conv}\left(\phi_T : \quad T \quad \textit{refines} \quad S\right),$$

where S is a regular polytopal subdivision of P.

Proof. Let us choose a vector $\psi \in \mathbb{R}^n$. By Lemma 19.3,

$$\langle \psi, \ \phi_T \rangle = d \int_P g_{\psi,T} \ dx.$$

By Lemma 19.4, the maximum of the integral is attained when T refines the regular polyhedral subdivision S of P obtained from the lifting $Q(t)$ for $t = (\psi(v_1), \ldots, \psi(v_n))$. By Lemma 18.2, for ψ from an open dense set the subdivision S is a triangulation, and hence there is a unique, necessarily regular, triangulation T maximizing $\langle \psi, \ \phi_T \rangle$. □

(19.6) Problems.

1. Prove that $\dim \Sigma(P) = n - d$, where n is the number of vertices of P and $d-1$ is the dimension of P.

2. Let $P \subset \mathbb{R}^2$ be a polygon. The polytope $\Sigma(P)$ is called an *associahedron*. Describe the faces of $\Sigma(P)$ and compute their dimensions.

20. Fiber polytopes

We discuss an alternative construction of the secondary polytope due to L. Billera and B. Sturmfels, see L.J. Billera and B. Sturmfels, Fiber polytopes, *Ann. of Math. (2)* **135** (1992), no. 3, 527–549.

Let $P \subset \mathbb{R}^{d-1}$ be a $(d-1)$-dimensional polytope with the vertices $v_1, \ldots, v_n$ and let $\Delta_{n-1} \subset \mathbb{R}^n$ be the standard simplex, $\Delta_{n-1} = \operatorname{conv}(e_1, \ldots, e_n)$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Let us consider the linear transformation $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{d-1}$ defined by

$$\pi\left(\sum_{i=1}^n \alpha_i e_i\right) = \sum_{i=1}^n \alpha_i v_i.$$

Hence

$$\pi\left(\Delta_{n-1}\right) = P.$$

A map $\gamma : P \longrightarrow \Delta_{n-1}$ is called a *section* of π if

$$\pi\left(\gamma(x)\right) = x \quad \text{for all} \quad x \in P.$$

We consider only Lebesgue measurable sections, so that we can define the integral

$$\int_P \gamma(x) \ dx$$

as a point in $\mathbb{R}^n$.

(20.1) Theorem. *We have*

$$\Sigma(P) = \left\{ d \int_P \gamma(x) \ dx : \quad \gamma \quad \textit{is a section of} \quad \pi \right\}.$$

Proof. It is clear that the set

$$\left\{ d \int_P \gamma(x) \ dx : \quad \gamma \quad \text{is a section of} \quad \pi \right\} \tag{20.1.1}$$

is convex.

For a triangulation T of P we construct a particular section γ_T as follows. Let Δ be a simplex of the triangulation, $\Delta = \operatorname{conv}(v_i : \quad i \in I)$. Then for every $x \in \Delta$ there is a unique representation

$$x = \sum_{i \in I} \alpha_i v_i \quad \text{where} \quad \sum_{i \in I} \alpha_i = 1 \quad \text{and} \quad \alpha_i \geq 0 \quad \text{for} \quad i \in I.$$

We define

$$\gamma_T(x) = \sum_{i \in I} \alpha_i e_i.$$

Since γ_T is a linear function on every simplex Δ of the triangulation T, applying Lemma 19.2, we get

$$\begin{aligned} d \int_P \gamma_T(x)\, dx = &\sum_{\Delta \in T} d \int_\Delta \gamma_T(x)\, dx = \sum_{\Delta \in T} (\operatorname{vol} \Delta) \left(\sum_{i:\ v_i \in \Delta} e_i \right) \\ = &\sum_{i=1}^n \left(\sum_{\Delta:\ v_i \in \Delta} \operatorname{vol} \Delta \right) e_i = \phi_T, \end{aligned}$$

as defined by Definition 19.1. Hence the set (20.1.1) contains vectors ϕ_T.

Let us choose a vector $\psi \in \mathbb{R}^n$ (which we interpret as a function on the vertices of P) and find a section $\gamma(x)$ maximizing

$$\left\langle \psi,\ \int_P \gamma(x)\, dx \right\rangle = \int_P \big\langle \psi,\ \gamma(x) \big\rangle\, dx.$$

Clearly, it suffices to maximize $\big\langle \psi,\ \gamma(x) \big\rangle$ for each $x \in P$. Let

$$\gamma(x) = \sum_{i=1}^n \alpha_i e_i \quad \text{where} \quad \sum_{i=1}^n \alpha_i = 1 \quad \text{and} \quad \alpha_1, \ldots, \alpha_n \geq 0. \tag{20.1.2}$$

Since $\pi(\gamma(x)) = x$, we must have

$$x = \sum_{i=1}^n \alpha_i v_i. \tag{20.1.3}$$

Besides,

$$\big\langle \psi,\ \gamma(x) \big\rangle = \sum_{i=1}^n \alpha_i \psi\left(v_i\right). \tag{20.1.4}$$

Hence our goal is to choose $\alpha_1, \ldots, \alpha_n$ in (20.1.2) in such a way that (20.1.3) holds and (20.1.4) is maximized.

For $i = 1, \dots, n$ let $w_i = (v_i, \psi(v_i)) \in \mathbb{R}^d$ be the lifting of the vertices v_i. Thus the maximum value of (20.1.4) is the largest $\tau \in \mathbb{R}$ such that

$$(x, \tau) \in \operatorname{conv}(w_1, \dots, w_n).$$

Now it is clear how to construct $\gamma(x)$: we let $t = (\psi(v_1), \dots, \psi(v_n))$, consider the lifting $Q(t)$ of P as in Section 18.3, pick a traingulation T refining the regular subdivision produced by $Q(t)$ and let $\gamma = \gamma_T$, in which case as we have shown

$$d \int_P \gamma_T(x)\ dx = \phi_T.$$

Hence we proved that the convex set (20.1.1) contains all vectors ϕ_T, where T is a triangulation of P, and that for every $\psi \in \mathbb{R}^n$ the linear function $\theta \longmapsto \langle \psi, \theta \rangle$ attains its maximum on (20.1.1) at some vector ϕ_T, possibly among other points. It follows that (20.1.1) contains $\Sigma(P)$.

Suppose that (20.1.1) contains a point $\phi \notin \Sigma(P)$. Since $\Sigma(P)$ is a polyhedron, there exists $\psi \in \mathbb{R}^n$ such that

$$\langle \psi, \phi \rangle \ > \ \langle \psi, \phi_T \rangle \quad \text{for all} \quad \phi_T.$$

But as we proved, the maximum of the linear function $\theta \longmapsto \langle \psi, \theta \rangle$ is attained, in particular, at some ϕ_T, which is a contradiction. $\square$

Theorem 20.1 suggests the following more general construction of the *fiber polytope*. Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope, let $Q \subset \mathbb{R}^n$ be an n-dimensional polytope and let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ be a linear transformation such that $\pi(Q) = P$. A function $\gamma : P \longrightarrow Q$ is called a *section* if $\pi(\gamma(x)) = x$ for all $x \in P$. We consider Lebesgue measurable sections only. The set

$$\Sigma(Q, P) = \left\{ \int_P \gamma(x)\ dx : \quad \gamma \quad \text{is a section} \right\}$$

is called the *fiber polytope* associated with the map $\pi : Q \longrightarrow P$.

(20.2) Problems.

1. Prove that $\Sigma(Q, P)$ is indeed a polytope of dimension $n - d$.

2^*. Let $\Delta_{n-1} \subset \mathbb{R}^n$ be the standard simplex, let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a map,

$$\pi(x_1, \dots, x_n) = a_1 x_1 + \ldots + a_n x_n,$$

where $a_1, \dots, a_n$ are distinct numbers, and let $P = \pi(\Delta_{n-1})$.

Prove that $\Sigma(\Delta_{n-1}, P)$ is an $(n-2)$-dimensional parallelepiped.

3^*. Let $Q \subset \mathbb{R}^n$ be the n-dimensional cube,

$$Q = \Big\{ (x_1, \dots, x_n) : \quad 0 \leq x_i \leq 1 \quad \text{for} \quad i = 1, \dots, n \Big\},$$

let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the map

$$\pi(x_1, \dots, x_n) = x_1 + \ldots + x_n$$

and let $P = \pi(Q)$. Prove that $\Sigma(Q, P)$ is a permutation polytope.

(21.1) Definitions. Let $P \subset V$ be a polyhedron and let $v \in P$ be a point. The set

$$\operatorname{fcone}(P, v) = \Big\{ u \in V : \quad v + \epsilon u \in P \quad \text{for some} \quad \epsilon > 0 \Big\}$$

is called the *cone of feasible directions* of P at v. Alternatively, if P is defined by a set of linear inequalities,

$$P = \Big\{ x \in V : \quad \ell_i(x) \ \leq \ \alpha_i, \quad i \in I \Big\}$$

and

$$I_v = \Big\{ i \in I : \quad \ell_i(v) = \alpha_i \Big\}$$

are the inequalities active on v, then

$$\operatorname{fcone}(P, v) = \Big\{ u \in V : \quad \ell_i(u) \leq 0 \quad \text{for all} \quad i \in I_v \Big\}.$$

The set

$$\operatorname{tcone}(P, v) = v + \operatorname{fcone}(P, v)$$

is called the *tangent cone* of P at v. Alternatively,

$$\operatorname{tcone}(P, v) = \Big\{ x \in V : \quad \ell_i(x) \leq \alpha_i \quad \text{for all} \quad i \in I_v \Big\}.$$

Note that the tangent cone is not really a cone since its vertex does not have to be at the origin.

Let $\mathcal{P}(V)$ be the algebra of polyhedra and let $\mathcal{P}_l(V) \subset \mathcal{P}(V)$ be the subspace spanned by the indicators of polyhedra with lines. For two functions $f, g \in \mathcal{P}(V)$, we say that

$$f \equiv g \quad \text{modulo polyhedra with lines}$$

provided

$$f - g \ \in \ \mathcal{P}_l(V).$$

(21.2) Theorem. *Let $P \subset V$ be a polyhedron. Then*

$$[P] \equiv \sum_v [\operatorname{tcone}(P, v)] \quad \textit{modulo polyhedra with lines},$$

where the sum is taken over all vertices v of P.

Proof. First, we prove the theorem assuming that P is a polytope. In this case, without loss of generality, we assume that V is Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and P contains the origin in its interior. Let $v_1, \ldots, v_n$ be the vertices of P.

Let $Q = P^\circ$ be the polar of P, so Q is a polytope (see Theorem 17.1). For a vertex v_i, let us compute $\operatorname{tcone}(P, v_i)^\circ$. We have

$$\operatorname{tcone}(P, v_i) = v_i + \operatorname{co}\Big(v_j - v_i : \quad \text{for all} \quad j \neq i\Big).$$

Then

$$\operatorname{tcone}(P, v_i)^\circ = \Big\{c \in V : \quad \langle c, v_i \rangle \leq 1 \quad \text{and} \quad \langle c, v_i \rangle \geq \langle c, v_j \rangle \quad \text{for all} \quad j \neq i\Big\}.$$

In other words,

$$\operatorname{tcone}(P, v_i)^\circ = \Big\{c \in V : \quad 1 \geq \langle c, v_i \rangle = \max_{x \in P} \langle c, x \rangle\Big\}.$$

Let $\hat{v}_i \subset Q$ be the facet of Q dual to v_i, see Theorem 17.1. Then $\operatorname{tcone}(P, v_i)^\circ$ is the pyramid over $\hat{v}_i$ with the vertex at 0,

$$\operatorname{tcone}(P, v_i)^\circ = \operatorname{conv}\Big(0, \hat{v}_i\Big).$$

Now, we have

$$[Q] - \sum_{i=1}^n [\operatorname{conv}(0, \hat{v}_i)] = \sum_j \alpha_j \, [K_j]\,,$$

where $K_j = \operatorname{conv}(0, F_j)$ for a face F_j of Q with $\dim F_j \leq \dim Q - 2$ and $\alpha_j \in \mathbb{R}$. Hence K_j are polyhedra lying in proper subspaces of V. Applying Theorem 16.3, we conclude that

$$[P] - \sum_{i=1}^n [\operatorname{tcone}(P, v_i)] = \sum_j \alpha_j \left[K_j^\circ\right].$$

Since K_j° contains a line orthogonal to $\operatorname{span}(K_j)$, the proof follows.

Suppose now that $P \subset V$ is an unbounded polyhedron. If P contains a line then by Theorem 9.7 polyhedron P has no vertices and the result holds trivially. Suppose that P does not contain a line. Then by Theorem 9.10 we can write $P = M + K_P$, where M is the convex hull of the vertices of P and K_P is the recession cone of P. As we proved,

$$[M] \equiv \sum_v [\operatorname{tcone}(M, v)] \quad \text{modulo polyhedra with lines},$$

where the sum is taken over all vertices v of P. By Theorem 7.2, we have

$$[P] = [M + K_P] \equiv \sum_v [\operatorname{tcone}(M, v) + K_P] \quad \text{modulo polyhedra with lines.}$$

It remains to notice that

$$[\operatorname{tcone}(M, v) + K_P] = [\operatorname{tcone}(P, v)]\,.$$

□

(21.3) Problems.

1°. Suppose that $v=0$. Prove that

$$\operatorname{tcone}(P, v) = \operatorname{fcone}(P, v) = \bigcup_{t \geq 0} tP.$$

2°. Let $T: V \longrightarrow W$ be a linear transformation, let $P \subset V$ be a polyhedron, and let $v \in P$ be a point. Prove that

$$T\big(\operatorname{tcone}(P, v)\big) = \operatorname{tcone}\big(T(P), T(v)\big).$$

3°. Let P be a polyhedron with the recession cone K_P and let M be the convex hull of the set of vertices of P. Prove that for a vertex v of P we have

$$\operatorname{tcone}(P, v) = \operatorname{tcone}(M, v) + K_P.$$

4. Let P be a polytope. Prove that

$$\sum_v [\operatorname{fcone}(P, v)] \equiv [0] \quad \text{modulo polyhedra with lines},$$

where the sum is taken over all vertices v of P.

5. Let P be a polyhedron. Prove that

$$\sum_v [\operatorname{fcone}(P, v)] \equiv [K_P] \quad \text{modulo polyhedra with lines},$$

where K_P is the recession cone of P and the sum is taken over all vertices v of P.

6°. Let P be a polytope with the vertices $v_1, \ldots, v_n$. Prove that

$$\operatorname{fcone}(P, v_i) = \operatorname{co}\Big(v_j - v_i : \quad j \neq i\Big).$$

7. Let P be a polytope with the vertices $v_1, \ldots, v_n$. Prove that

$$\operatorname{fcone}(P, v_i) = \operatorname{co}\Big(v_j - v_i : \quad \operatorname{conv}(v_j, v_i) \quad \text{is an edge of } P\Big).$$

8°. Check that $\mathcal{P}_l(V) \subset \mathcal{P}(V)$ is an ideal with respect to the convolution $*$: if $f \in \mathcal{P}_l(V)$ then $f * g \in \mathcal{P}_l(V)$ for all $g \in \mathcal{P}(V)$, cf. Theorem 7.2.

9. Let P be a polytope. For a non-empty face F of P let us define the tangent cone $\operatorname{tcone}(P, F)$ as follows: we pick a point v in the relative interior of F and let $\operatorname{tcone}(P, v) = \operatorname{tcone}(P, F)$. Prove that the tangent cone so defined does not depend on the choice of v and that

$$[P] = \sum_F (-1)^{\dim F} [\operatorname{tcone}(P, F)],$$

where the sum is taken over all faces F of P, including $F = P$ (the *Brianchon-Gram* Theorem).

(22.1) Definition. A cone $K \subset V$ is called *simple* if $K = \operatorname{co}(u_1, \ldots, u_n)$ where $u_1, \ldots, u_n$ is a basis of V.

For a polyhedron $P \subset V$, where V is Euclidean space with the scalar product $\langle \cdot, \cdot \rangle$, we consider the integral

$$\int_P e^{\langle c,x \rangle} \, dx$$

as a function of $c \in V$.

(22.2) Examples.
a) Let $V = \mathbb{R}^n$ and let

$$K = \mathbb{R}^n_+ = \Big\{ (x_1, \ldots, x_n) : \quad x_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n \Big\}.$$

Let $c = (c_1, \ldots, c_n)$. Then

$$\int_{\mathbb{R}^n_+} e^{\langle c,x \rangle} \, dx = \prod_{i=1}^n \int_0^{+\infty} e^{c_i x_i} \, dx_i = \prod_{i=1}^n \frac{1}{-c_i},$$

provided $c_i < 0$ for $i = 1, \ldots, n$, in which case the integral converges absolutely and uniformly on compact subsets of $- \operatorname{int} \mathbb{R}^n_+$

b) Let $K = \operatorname{co}(u_1, \ldots, u_n)$, where $u_1, \ldots, u_n$ is a basis of V. This example reduces to a) by a change of variables, and we obtain

$$\int_K e^{\langle c,x \rangle} \, dx = |u_1 \wedge \ldots \wedge u_n| \prod_{i=1}^n \frac{1}{\langle -c, u_i \rangle},$$

where $|u_1 \wedge \ldots \wedge u_n|$ denotes the volume of the parallelepiped

$$\left\{ \sum_{i=1}^n \alpha_i u_i : \quad 0 \leq \alpha_i \leq 1 \quad \text{for} \quad i = 1, \ldots, n \right\}$$

spanned by $u_1, \ldots, u_n$. The integral converges absolutely for all $c \in \operatorname{int} K^\circ$ uniformly on compact subsets of $\operatorname{int} K^\circ$.

c) Let $K \subset V$ be a pointed polyhedral cone. By Theorem 9.9, we may write $K = \operatorname{co}(P)$, where P is a polytope in an affine hyperplane not containing 0. Triangulating P (see Section 18), we may write

$$[K] \equiv \sum_{i \in I} [K_i] \quad \text{modulo lower-dimensional cones,}$$

where $K_i \subset V$ are simple cones as in b). Hence

$$\int_K e^{\langle c,x\rangle}\ dx = \sum_{i\in I}\int_{K_i} e^{\langle c,x\rangle}\ dx.$$

Therefore,

$$\int_K e^{\langle c,x\rangle}\ dx = \sum_{i\in I}\alpha_i\prod_{j=1}^{n}\frac{1}{\langle -c,u_{ij}\rangle},$$

where $\alpha_i > 0$ and $u_{i1},\ldots,u_{in}$ is a basis of V for all $i \in I$. The integral converges absolutely for $c \in \operatorname{int} K^\circ$ uniformly on all compact subsets of $\operatorname{int} K^\circ$. It follows form the proof of Theorem 9.9 that $\operatorname{int} K^\circ \neq \emptyset$.

The following result was proved by J. Lawrence and independently by A. Khovanskii and A. Pukhlikov at about the same time, c. 1990.

(22.3) Theorem. *Let V be Euclidean space and let $\mathcal{M}(V)$ be the space of functions on V that are finite linear combinations of functions*

$$c \longmapsto \frac{e^{\langle c,v\rangle}}{\langle c,u_1\rangle\cdots\langle c,u_n\rangle},$$

where $v \in V$ and $u_1,\ldots,u_n$ is a basis of V. There exists a unique linear transformation

$$\Phi : \mathcal{P}(V) \longrightarrow \mathcal{M}(V)$$

such that

(1) *If $P \subset V$ is a polyhedron without lines and K_P its recession cone then for all $c \in \operatorname{int} K_P^\circ$ the integral*

$$\int_P e^{\langle c,x\rangle}\ dx$$

converges absolutely and uniformly on compact subsets of $\operatorname{int} K_P^\circ$ to a function $\phi(P;c) \in \mathcal{M}(V)$ such that $\Phi([P]) = \phi(P;c)$.

(2) *If P contains a line then $\Phi([P]) = 0$.*

Proof. We proceed by induction on $n = \dim V$. For $n = 0$ the result is trivial.

Suppose that $n > 0$. The proof consists of three steps.

Step 1. We prove that if P is a polyhedron without lines then for all $c \in \operatorname{int} K_P^\circ$ the integral

$$\int_P e^{\langle c,x\rangle}\ dx$$

converges absolutely and uniformly for compact subsets of $\operatorname{int} K_P^\circ$ to a function $\phi(P;c) \in \mathcal{M}(V)$.

Let us pick a sufficiently generic unit vector $u \in \operatorname{int} K_P^\circ$ and let us slice V by affine hyperplanes

$$H_t = \Big\{x : \quad \langle -u, x\rangle = t\Big\}, \quad t \in \mathbb{R}.$$

By Theorem 9.10 it follows that the function $x \longrightarrow \langle -u, x\rangle$ attains its finite minimum t_0 on P, so we can write

$$\int_P e^{\langle c,x\rangle}\, dx = \int_{t_0}^{+\infty} \left(\int_{P_t} e^{\langle c,x\rangle}\, dx\right) dt, \tag{22.3.1}$$

where $P_t = P \cap H_t$. By Lemma 9.5, P_t is a bounded and hence is a polytope. We choose u in such a way that u is not orthogonal to any face of P of positive dimension. In this case, every k-dimensional face of P_t is the intersection of H_t and a $(k+1)$-dimensional face of P. In particular, every vertex of P_t is the intersection of H_t and an edge of P. Let $t_1 < \ldots < t_k$ be the values of t for which H_t passes through a vertex of P. Then for every open interval (t_j, t_{j+1}), and, if P is unbounded, for the open ray $(t_k, +\infty)$, the vertices $v_1(t), \ldots, v_m(t)$ of P_t change linearly with t:

$$v_i(t) = a_i + tw_i \quad \text{for} \quad i = 1, \ldots, m$$

while the cone of feasible directions $K_i = \operatorname{fcone}(P_t, v_i(t))$ doesn't change. For a given c we can always choose u in such a way that the function $x \longrightarrow \langle c, x\rangle$ is not constant on the edges of P_t.

By Theorem 21.2,

$$[P_t] \equiv \sum_{i=1}^m [v_i(t) + K_i] \quad \text{modulo polyhedra with lines}$$

and hence by the induction hypothesis

$$\int_{P_t} e^{\langle c,x\rangle}\, dx = \sum_{i=1}^m e^{\langle c, v_i(t)\rangle} f_i(c), \tag{22.3.2}$$

where $f_i(c)$ is a linear combination of functions of the type

$$\frac{1}{\langle c, w_1\rangle \cdots \langle c, w_{n-1}\rangle},$$

cf. Example 22.2.

It remains to notice that for $v(t) = a + tw$ we have

$$\begin{gathered}\int_{t_j}^{t_{j+1}} e^{\langle c,\ v(t)\rangle}\, dt = \frac{e^{\langle c, v(t_{j+1})\rangle} - e^{\langle c, v(t_j)\rangle}}{\langle c, w\rangle}\\ \text{and that}\\ \int_{t_k}^{+\infty} e^{\langle c,\ v(t)}\, dt = \frac{e^{\langle c, v(t_k)\rangle}}{\langle -c, w\rangle}\end{gathered} \tag{22.3.3}$$

since in the latter case w is the direction of an unbounded edge of P, so $w \in K_P$ and since $c \in \operatorname{int} K_P^\circ$, we have $\langle c, w \rangle < 0$, so the integral converges.

Using (22.3.1)–(22.3.3), we conclude Step 1.

Step 2. We prove that there is a unique valuation $\Phi : \mathcal{P}(V) \longrightarrow \mathcal{M}(V)$ such that $\Phi([P]) = \phi(P; c)$ if P is a polyhedron without lines and $\phi(P; c)$ is the function constructed at Step 1.

Let us write

$$[V] = \sum_{j \in J} \beta_j \, [Q_j] \, ,$$

where $\beta_j \in \mathbb{R}$ and Q_j are some polyhedra without lines. Then, for any polyhedron P we have

$$[P] = [P][V] = \sum_{j \in J} \beta_j \, [P \cap Q_j] \, .$$

It follows that $\mathcal{P}(V)$ is spanned by the indicators of polyhedra without lines.

Suppose now that

$$\sum_{i \in I} \alpha_i \, [P_i] = 0, \tag{22.3.4}$$

where $\alpha_i \in \mathbb{R}$ and P_i are polyhedra without lines. Multiplying (22.3.4) by $[Q_j]$ we obtain

$$\sum_{i \in I} \alpha_i \, [P_i \cap Q_j] = 0.$$

For the recession cones we have $K_{P_i \cap Q_j} \subset K_{Q_j}$ and hence $K_{Q_j}^\circ \subset K_{P_i \cap Q_j}^\circ$. Therefore, for all $c \in \operatorname{int} K_{Q_j}^\circ$ we have

$$\sum_{i \in I} \alpha_i \int_{P_i \cap Q_j} e^{\langle c, x \rangle} \, dx = 0$$

and all the integrals converge absolutely and uniformly on compact subsets of $\operatorname{int} K_{Q_j}^\circ$. Hence we have

$$\sum_{i \in I} \alpha_i \phi(P_i \cap Q_j; c) = 0 \quad \text{for all} \quad c \in \operatorname{int} K_{Q_j}^\circ .$$

Since the above identity holds for a non-empty open set of c and functions ϕ are meromorphic, we get

$$\sum_{i \in I} \alpha_i \phi(P_i \cap Q_j; c) = 0 \quad \text{for all} \quad j \in J. \tag{22.3.5}$$

Similarly, we have

$$[P_i] = \sum_{j \in J} \beta_j \, [P_i \cap Q_j]$$

and hence

$$\int_{P_i} e^{\langle c,x \rangle} \, dx = \sum_{j \in J} \beta_j \int_{P_i \cap Q_j} e^{\langle c,x \rangle} \, dx \quad \text{for all} \quad c \in \operatorname{int} K_{P_i}^{\circ},$$

which implies that

$$\phi(P_i; c) = \sum_{j \in J} \beta_j \phi(P_i \cap Q_j; c). \tag{22.3.6}$$

Then, using (22.3.5) and (22.3.6) we deduce from (22.3.4) that

$$\sum_{i \in I} \alpha_i \phi(P_i; c) = \sum_{i \in I} \alpha_i \left(\sum_{j \in J} \beta_j \phi(P_i \cap Q_j; c) \right) = \sum_{j \in J} \beta_j \left(\sum_{i \in I} \alpha_i \phi(P_i \cap Q_j; c) \right) = 0.$$

Hence the correspondence $[P] \longmapsto \phi(P; c)$ preserves linear relations among the indicators of polyhedra without lines and therefore can be extended uniquely to a valuation $\Phi : \mathcal{P}(V) \longmapsto \mathcal{M}(V)$, which concludes Step 2.

Step 3. We prove that $\Phi([P]) = 0$ if P is a polyhedron with a line.

Let $P \subset V$ be a polyhedron without lines, let $u \in V$ be a vector and let $P + u$ be the translation of P. Then $K_{P+u} = K_P$ and for all $c \in \operatorname{int} K_P^{\circ}$ we have

$$\int_{P+u} e^{\langle c,x \rangle} \, dx = e^{\langle c,u \rangle} \int_P e^{\langle c,x \rangle} \, dx.$$

Therefore,

$$\phi(P + u; c) = e^{\langle c,u \rangle} \phi(P; c) \quad \text{and} \quad \Phi([P + u]) = e^{\langle c,u \rangle} \Phi([P]).$$

By linearity, the same identity holds if P is any polyhedron. If P contains a line then there is a vector $u \neq 0$ such that $P + u = P$. Then we have

$$\Phi([P]) = \Phi([P + u]) = e^{\langle c,u \rangle} \Phi([P]),$$

from which $\Phi([P]) = 0$. $\square$

(22.4) Problems.

1. Let $P \subset \mathbb{R}^n$ be a polytope. Prove that

$$\sum_v \Phi[\operatorname{fcone}(P, v)] = 0,$$

where the sum is taken over all vertices v of P.

2. Let $P \subset \mathbb{R}^n$ be a polytope with integer vertices (that is, the coordinates of each vertex are integer). Let $c \in \mathbb{R}^n$ be an integer vector which is not orthogonal to any edge of P. Prove that

$$\int_P e^{2\pi i \langle c,x \rangle} \, dx = 0.$$

23. A formula for the volume of a polytope

Let V be Euclidean n-dimensional space and let $P \subset V$ be a polytope. Using Theorem 22.3, we may write

$$\int_P e^{\langle c,x\rangle}\,dx = \sum_v \phi(\mathrm{tcone}(P,v);c) = \sum_v e^{\langle c,v\rangle}\phi(\mathrm{fcone}(P,v);c), \tag{23.1}$$

where sum is taken over all vertices v of P and $\phi(\mathrm{fcone}(P,v);c)$ is a rational function of c of degree $-n$ obtained by extending the integral

$$\int_{\mathrm{fcone}(P,v)} e^{\langle c,x\rangle}\,dx$$

by an analytic continuation in c from its domain of convergence, which is the interior of $\mathrm{fcone}(P,v)^\circ$.

Identity (23.1) is known as *Brion's Theorem.*

Substituting $c = 0$ in the left hand side of (23.1) we obtain the volume of P. However, $c = 0$ is the pole of every term in the right hand side of (23.1). This difficulty can be handled as follows.

Let t be a small parameter and let us replace c by tc in (23.1). We get

$$\int_P e^{t\langle c,x\rangle}\,dx = \sum_v e^{t\langle c,v\rangle} t^{-n}\phi(\mathrm{fcone}(P,v);c).$$

Using the standard expansion

$$e^z = 1 + z + \frac{z^2}{2} + \ldots + \frac{z^n}{n!} + \ldots$$

we observe that the left hand side of (23.1) is an analytic function of t and $\mathrm{vol}\,P$ is its constant term. Each summand is a meromorphic function of t and its constant term is

$$\frac{\langle c,v\rangle^n}{n!}\phi(\mathrm{fcone}(P,v);c).$$

Hence we obtain the formula

$$\mathrm{vol}\,P = \sum_v \frac{\langle c,v\rangle^n}{n!}\phi(\mathrm{fcone}(P,v);c). \tag{23.2}$$

Curiously, the right hand side is a function of c while the left hand side is just a constant. When the cone of feasible directions at each vertex v of P is simple,

$$\mathrm{fcone}(P,v) = \mathrm{co}\Big(u_{1v},\ldots,u_{nv}\Big)$$

for some basis $u_{1v},\ldots,u_{nv}$ of V, we obtain

$$\mathrm{vol}\,P = \sum_v \frac{\langle c,v\rangle^n}{n!}\,|u_{1v}\wedge\ldots\wedge u_{nv}|\,\frac{1}{\langle -c,u_{1v}\rangle\cdots\langle -c,u_{nv}\rangle}, \tag{23.3}$$

where the sum is taken over all vertices v on P and c is any vector not orthogonal to any of the edges of P. Formula (23.3) is due to J. Lawrence.

Problem.

1°. Prove that

$$\sum_v \frac{\langle c, v\rangle^k}{k!}\phi(\operatorname{fcone}(P, v); c) = 0 \quad \text{for} \quad k = 0, \ldots, n-1.$$

24. Simple polytopes and their h-vectors

(24.1) Definitions. Let P be a polytope. Vertices v and u of P are called *neighbors* if $[v, u]$ is an edge of P. A polytope P is called *simple* if for every vertex v of P the cone of feasible directions $\operatorname{fcone}(P, v)$ is simple, that is, the conic hull of linearly independent vectors. For a d-dimensional polytope P we define $f_k(P)$ as the number of k-dimensional faces of P, where we agree that $f_d(P) = 1$. The vector $(f_0(P), \ldots, f_d(P))$ is called the *f-vector* of P. For a simple polytope P we define

$$h_k(P) = \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} f_i(P) \quad \text{for} \quad k = 0, \ldots, d.$$

The vector $(h_0(P), \ldots, h_d(P))$ is called the *h-vector* of P.

(24.2) Lemma. *Let P be a simple d-dimensional polytope. Then*

(1) *Every vertex v has exactly d neighbors.*
(2) *For every vertex v and for every $0 \leq k \leq d$ of its neighbors there is a unique k-dimensional face of P containing v and the k neighbors of v.*
(3) *Every k-dimensional face F of P containing a vertex v of P contains exactly k neighbors of v and $\operatorname{fcone}(F, v)$ is a simple k-dimensional cone.*
(4) *The intersection of any $0 \leq k \leq d$ facets of P containing v is a $(d-k)$-dimensional face of P.*
(5) *Let v be a vertex of P, let ℓ be a linear function such that $\ell(u) < \ell(v)$ for every neighbor u of v. Then the maximum of ℓ on P is attained at v and only at v.*
(6) *Every face of P is a simple polytope.*

Proof. Parts (1)–(5) deal with a particular vertex v of P. Without loss of generality, we may assume that $P \subset \mathbb{R}^d$, that $v = 0$ and that

$$\operatorname{tcone}(P, v) = \operatorname{fcone}(P, v) = \mathbb{R}^d_+. \tag{24.2.1}$$

Then the neighbors $u_1, \ldots, u_d$ of v are points $u_i = \alpha_i e_i$, where e_i is the i-th standard basis vector and $\alpha_i > 0$. The k-dimensional face containing v and $u_{i_1}, \ldots, u_{i_k}$ is defined by the inequality $\sum_{i \neq i_1, \ldots, i_k} x_i \geq 0$ which is active on v and $u_{i_1}, \ldots, u_{i_k}$. Let F is a k-dimensional face of P containing v. Then $F = P \cap H$ where H is a hyperplane $H = \{x : \ \ell(x) = 0\}$ such that $\ell(x) \leq 0$ for all $x \in P$. That is, $\ell(x) = c_1 x_1 + \ldots + c_d x_d$, where $c_i \leq 0$ for $i = 1, \ldots, d$. Then $\operatorname{fcone}(F, v) = \operatorname{tcone}(F, v)$

is the cone consisting of the points $(x_1, \ldots, x_d)$ with $x_i = 0$ whenever $c_i < 0$ and $x_i \geq 0$ whenever $c_i = 0$. This is a simple cone of dimension $k = \dim F$ and hence the number of coordinates i with $c_i = 0$ is k. Then F contains u_i with $c_i = 0$. The i-th facet containing v is defined by the coordinate inequality $x_i \geq 0$ and hence the intersection of the facets defined by the inequalities $x_{i_1} \geq 0, \ldots, x_{i_k} \geq 0$ is the face defined by the inequality $\sum_{i \in \{i_1, \ldots, i_k\}} x_i \geq 0$. If $\ell(u_i) < \ell(v)$ for $i = 1, \ldots, d$ then $\ell(x) = c_1 x_1 + \ldots + c_d x_d$, where $c_i < 0$ for $i = 1, \ldots, d$ and since $P \subset \mathbb{R}^d_+$, the maximum of ℓ on P is attained at $v = 0$. This proves Parts (1)–(5).

Part (6) follows from Part (3). $\square$

(24.3) Lemma. *Let $a_0, \ldots, a_d$ and $b_0, \ldots, b_d$ be numbers. Then the two systems of linear equations*

$$b_k = \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} a_i \quad \textit{for} \quad k = 0, \ldots, d$$

and

$$a_i = \sum_{k=i}^{d} \binom{k}{i} b_k \quad \textit{for} \quad i = 0, \ldots, d$$

are equivalent.

In particular,

$$f_i(P) = \sum_{k=i}^{d} \binom{k}{i} h_k(P) \quad \textit{for} \quad i = 0, \ldots, d.$$

Proof. Let us introduce two polynomials

$$a(t) = \sum_{i=0}^{d} a_i t^i \quad \text{and} \quad b(t) = \sum_{k=0}^{d} b_k t^k.$$

Then the first system is equivalent to the identity $b(t) = a(t-1)$ whereas the second system is equivalent to the identity $a(t) = b(t+1)$. $\square$

(24.4) Theorem. *Let P be a simple d-dimensional polytope and let ℓ be a linear function which is not constant on any edge of P. For a vertex v we define the index of v with respect to ℓ as the number of neighbors u of v such that $\ell(u) < \ell(v)$. Then for $k = 0, \ldots, d$ the number of vertices of P of index k is equal to $h_k(P)$ and, in particular, does not depend on ℓ.*

Proof. Let $h_k(P; \ell)$ be the number of vertices of index k with respect to ℓ. Since ℓ is not constant on the edges of P, on every face F of P the function ℓ attains its maximum at a unique vertex v of F which is necessarily a vertex of P.

Let us consider the correspondence

$$\psi_i : \quad i\text{-dimensional faces of } P \longmapsto \text{vertices of } P,$$

which with every i-dimensional face F of P associates the vertex v of P where the maximum of ℓ on F is attained. Let us compute the number of i-dimensional faces F mapped to the same vertex v of index k. Thus at exactly k of the d neighbors of v the function ℓ attains a smaller value than it does at v. By Parts (2), (3) and (5) of Lemma 24.2, we uniquely select an i-dimensional face F of P such that $\psi_i(F) = v$ by selecting i neighbors from the set of k neighbors of v with the smaller value of ℓ. Therefore,

$$\left|\psi_i^{-1}(v)\right| = \binom{k}{i} \quad \text{provided the index of } v \text{ is } k.$$

This gives us the equation

$$f_i(P) = \sum_{k=i}^{d} \binom{k}{i} h_k(P;\ell) \quad \text{for} \quad i = 0, \ldots, d.$$

By Lemma 24.3, the equations are equivalent to

$$h_k(P;\ell) = \sum_{i=k}^{d} (-1)^{i-k} \binom{i}{k} f_i(P) = h_k(P) \quad \text{for} \quad k = 0, \ldots, d.$$

□

(24.5) Corollary. *Let P be a d-dimensional simple polytope. Then*

$$h_k(P) = h_{d-k}(P) \quad \textit{for} \quad k = 0, \ldots, d.$$

Proof. Let us pick a linear function ℓ not constant on any edge of P. Then, by Theorem 24.4, the number of vertices of P of index k with respect to ℓ is $h_k(P)$. On the other hand, every vertex of index k with respect to ℓ has index $d-k$ with respect to $-\ell$. Since the number of vertices having index $d-k$ with respect to $-\ell$ is $h_{d-k}(P)$, the proof follows. □

The formulas of Corollary 24.5 are called the *Dehn-Sommerville equations.*

(24.6) Problems.

1°. Prove that $h_k(P) \geq 1$ provided $d \geq 1$ and that $\sum_{k=0}^{d} h_k(P) = f_0(P)$.

2°. Check that $h_0(P) = h_d(P)$ is the Euler-Poincaré formula.

3°. Check that for $d = 3$ the Dehn-Sommerville equations are equivalent to $f_0(P) - f_1(P) + f_2(P) = 2$ and $3f_0(P) - 2f_1(P) = 0$.

4°. Check that for $d = 4$ the Dehn-Sommerville equations are equivalent to $f_0(P) - f_1(P) + f_2(P) - f_3(P) = 0$ and $f_1(P) = 2f_0(P)$.

5°. Let P be a d-dimensional simplex. Prove that $h_k(P) = 1$ for $0 \leq k \leq d$.

6°. Let P be a d-dimensional cube. Prove that P is simple and that $h_k(P) = \binom{d}{k}$.

7. Let P be a 3-dimensional simple polytope and let p_k be the number of k-gons among its facets. Prove that

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k \geq 7} (k-6)p_k.$$

8. For a permutation σ of the set $\{1, \ldots, n\}$, let us define a *descent* as a number $i = 2, \ldots, n$ such that $\sigma(i) < \sigma(i-1)$. Let $E(n,k)$ be the number of permutations σ having precisely $k-1$ descents, $k = 1, \ldots, n$. Let $a = (a_1, \ldots, a_n)$ be a point with distinct coordinates and let $P = P(a)$ be the corresponding permutation polytope. Prove that P is simple, that $h_k(P) = E(n, k+1)$ and that $E(n,k) = E(n-k+1)$ for $k = 1, \ldots, n$.

25. The upper bound theorem

(25.1) Lemma. *Let P be a d-dimensional simple polytope and let F be a facet of P. Then*

$$h_k(F) \ \leq \ h_k(P) \quad \textit{for} \quad k = 0, \ldots, d.$$

Moreover, if the intersection of every $k+1$ of facets of P is non-empty then

$$h_k(F) = h_k(P).$$

Proof. Since F is a facet there exists a linear function ℓ on the ambient space V and a number α such that

$$\ell(x) \ \geq \ \alpha \quad \text{for all} \quad x \in P \quad \text{and} \quad F = \Big\{ x \in P : \quad \ell(x) = \alpha \Big\}.$$

In particular, if u and v are vertices of P such that $v \in F$ and $u \notin F$ then $\ell(v) < \ell(u)$. Let $\tilde{\ell}$ be a sufficiently generic linear function sufficiently close to ℓ, so that $\tilde{\ell}$ is not constant on edges of P and $\tilde{\ell}(v) < \tilde{\ell}(u)$ for any two vertices u and v of P with $v \in F$ and $u \notin F$. Then the index of every vertex v of F with respect to $\tilde{\ell}$ is equal to the index of v as a vertex of P with respect to $\tilde{\ell}$, so $h_k(F) \leq h_k(P)$.

Suppose that the intersection of every $k+1$ facets of P is non-empty. Let v be a vertex of P of index k with respect to $\tilde{\ell}$. Numbering the neighbors $u_1, \ldots, u_d$ of v in the increasing order of $\tilde{\ell}(u_i)$, we obtain

$$\tilde{\ell}(u_i) > \tilde{\ell}(v) \quad \text{for} \quad i = k+1, \ldots, d.$$

By Part (2) of Lemma 24.2, there exists a $(d-k)$-dimensional face G of P containing v and $u_{k+1}, \ldots, u_d$. By Part (5) of Lemma 24.2, we have $\tilde{\ell}(v) \leq \tilde{\ell}(x)$ for all $x \in G$. On the other hand, G can be represented as the intersection of k facets of P, $G = F_1 \cap \ldots \cap F_k$ (we obtain F_i as the unique facet containing v and all its neighbors except u_i). Since

$$G \cap F = F \cap F_1 \cap \ldots \cap F_k \neq \emptyset,$$

for a vertex $u \in G \cap F$ we must have $\tilde{\ell}(u) \geq \tilde{\ell}(v)$. This proves that $v \in F$ and hence $h_k(F) = h_k(P)$. □

(25.2) Lemma. *Let P be a d-dimensional simple polytope. Then*

$$\sum_F h_k(F) = (d-k)h_k(P) + (k+1)h_{k+1}(P) \quad \textit{for} \quad k = 0, \ldots, d-1,$$

where the sum is taken over all facets F of P.

Proof. Let us choose a generic function ℓ not constant on edges of P and let v be a vertex of P. If the index of v is smaller than k then the index of v on any facet of P will be smaller than k. If the index of v is k then there are precisely $(d-k)$ facets containing v for which the index of v is k (such a facet contains v and all but one neighbor u of v for which $\ell(u) > \ell(v)$). If the index of v is $k+1$ then there are precisely $k+1$ facets of P containing v for which the index of v is k (such a facet contains v and all but one neighbor u of v for which $\ell(u) < \ell(v)$). If the index of v is greater than $k+1$ then the index of v on every facet of P containing v is greater than k. □

(25.3) Corollary. *Let P be a d-dimensional simple polytope with n facets. Then*

$$h_{k+1}(P) \ \leq \ \frac{n-d+k}{k+1} h_k(P) \quad \textit{for} \quad k = 0, \ldots, d-1.$$

Moreover, if every $k+1$ facets of P have a non-empty intersection then

$$h_{k+1}(P) = \frac{n-d+k}{k+1} h_k(P).$$

Proof. Follows by Lemma 25.1 and Lemma 25.2. □

(25.4) Corollary. *Let P be a d-dimensional simple polytope with n facets. Then*

$$h_k(P) \ \leq \ \binom{n-d+k-1}{k} \quad \textit{for} \quad k = 0, \ldots, d.$$

Moreover, if every k facets of P have a non-empty intersection then

$$h_k(P) = \binom{n-d+k-1}{k}.$$

Proof. Follows by Corollary 25.3 since $h_0(P) = 1$. □

(25.5) Proposition. *Let $C(d,n)$ be the d-dimensional cyclic polytope and let $C(d,n)^\circ$ be its polar with the origin chosen in the interior of $C(d,n)$. Then $C(d,n)^\circ$ is a simple d-dimensional polytope with n facets and for every d-dimensional simple polytope P with n facets, we have*

$$h_k(P) \ \leq \ h_k\left(C(d,n)^\circ\right) \quad \textit{for} \quad k=0,\ldots,d.$$

Proof. It is easy to show (see Problem 1 of Section 15.3, for example), that every facet of $C(d,n)$ is a $(d-1)$-dimensional simplex. By Theorem 17.1, $C(d,n)^\circ$ is a simple polytope with n facets. By Theorem 15.2, every $k \leq d/2$ vertices of $C(d,n)$ are the vertices of a proper face of $C(d,n)$. Therefore, by Theorem 17.1, every $k \leq d/2$ facets of $C(d,n)^\circ$ have a non-empty intersection. Therefore, by Corollary 25.4,

$$h_k\left(C(d,n)^\circ\right) = \binom{n-d+k-1}{k} \quad \text{for} \quad 0 \leq k \leq d/2.$$

Since by Corollary 25.4

$$h_k(P) \leq \binom{n-d+k-1}{k} \quad \text{for} \quad k=0,\ldots,d,$$

we obtain

$$h_k(P) \leq h_k\left(C(d,n)^\circ\right) \quad \text{for} \quad 0 \leq k \leq d/2.$$

Since by Corollary 24.5,

$$h_k(P) = h_{d-k}(P) \quad \text{and} \quad h_k\left(C(d,n)^\circ\right) = h_{d-k}\left(C(d,n)^\circ\right),$$

the proof follows. □

(25.6) Theorem. *Let $C(d,n)$ be the d-dimensional cyclic polytope and let $C(d,n)^\circ$ be its polar with the origin chosen in the interior of $C(d,n)$. Then for every d-dimensional polytope P with n facets, we have*

$$f_i(P) \ \leq \ f_i\left(C(d,n)^\circ\right) \quad \textit{for} \quad i=0,\ldots,d.$$

Proof. We prove the theorem assuming, additionally, that P is simple (see Problem 2 of Section 25.8 below). In this case, the theorem follows by Proposition 25.5 since

$$f_i(P) = \sum_{k=i}^{d} \binom{k}{i} h_k(P) \quad \text{and} \quad f_i\left(C(d,n)^\circ\right) = \sum_{k=i}^{d} \binom{k}{i} h_k\left(C(d,n)^\circ\right).$$

□

(25.7) Theorem. *Let $C(d,n)$ be the d-dimensional cyclic polytope with n vertices. Then for any d-dimensional polytope P with n vertices we have*

$$f_i(P) \ \leq \ f_i\big(C(d,n)\big) \quad \textit{for} \quad i = 0, \dots, d.$$

Proof. Follows from Theorem 25.6 and Theorem 17.1. $\square$

Theorem 25.7 is known as the *Upper Bound Theorem.* It was proved by P. McMullen and the proof of this section follows his approach.

(25.8) Problems.

1. Let P be a simple d-dimensional polytope. Prove that for d even,

$$f_0(P) \ \leq \ f_{d/2}(P) + 2 \sum_{i=d/2}^{d} f_i(P)$$

while for d odd,

$$f_i(P) \ \leq \ 2 \sum_{i=(d+1)/2}^{d} f_i(P).$$

2*. Let P be a d-dimensional polytope with n facets. Prove that there is a simple d-dimensional polytope $\tilde{P}$ with n facets, such that

$$f_i(P) \ \leq \ f_i(\tilde{P}) \quad \text{for} \quad i = 0, \dots, d.$$

See, for example, Section 5.2 of B. Grünbaum, *Convex Polytopes. Second edition. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler*, Graduate Texts in Mathematics, **221**, Springer-Verlag, New York, 2003.

26. Balinski's Theorem

(26.1) Definitions. Given a polyhedron P, we define its *graph* $G(P)$ as an undirected graph whose vertices are the vertices of P and whose edges are the edges (1-dimensional faces) of P.

(26.2) Lemma. *Let $P \subset V$ be a polytope, let $\ell : V \longrightarrow \mathbb{R}$ be a linear function and let $v \in P$ be a vertex. Suppose that*

$$\ell(v) \ < \ \max_{x \in P} \ell(x).$$

Then there exists a vertex u of P such that $\ell(u) > \ell(v)$ and $[u, v]$ is an edge of P.

Proof. Let us consider $\operatorname{fcone}(P, v)$. By Theorem 9.9 there exists an affine hyperplane H, $0 \notin H$, such that $Q = \operatorname{fcone}(P, v) \cap H$ is a polytope and

$$\operatorname{fcone}(P, v) = \operatorname{co}(Q).$$

Clearly, we must have $\ell(w) > 0$ for some vertex w of Q. Next, we note that the vertices of Q are the intersections of the edges of $\operatorname{fcone}(P, v)$ with H. Therefore, $w + v$ lies on an edge of P, one endpoint of which is v while the other is u. $\square$

The following theorem was proved by M.L. Balinski in 1961.

(26.3) Theorem. *Let P be a d-dimensional polytope. Then the graph $G(P)$ is d-connected, that, is, the graph obtained from $G(P)$ by removing any $d-1$ vertices and adjacent edges is connected. Equivalently, for any two vertices of $G(P)$ there exist d pairwise vertex-disjoint paths connected them.*

Proof. The equivalence of two definitions of d-connectedness is a Graph Theory result, which we don't discuss here.

We proceed by induction on d. Clearly, the result holds for $d=1$.

Suppose that $P \subset V$, where $\dim V = d$. Let us choose any $d-1$ vertices $v_1, \ldots, v_{d-1}$ which we remove from $G(P)$ with the adjacent edges, as well as yet another vertex v_0. Then there exists a non-zero linear functional $\ell : V \longrightarrow \mathbb{R}$ and a number $\alpha \in \mathbb{R}$ such that

$$\ell\left(v_i\right) = \alpha \quad \text{for} \quad i = 0, \ldots, d.$$

Let

$$\alpha_+ = \max_{x \in P} \ell(x) \quad \text{and} \quad \alpha_- = \min_{x \in P} \ell(x).$$

Let us define the following two faces of P:

$$F_+ = \Big\{x \in P : \quad \ell(x) = \alpha_+\Big\} \quad \text{and} \quad F_- = \Big\{x \in P : \quad \ell(x) = \alpha_-\Big\}.$$

Since $\dim P = d$, we have that F_+ and F_- are proper faces of P and that $F_+ \cap F_- = \emptyset$.

By Lemma 26.2, for every vertex v of $P \setminus F_+$ there exists a path in $G(P)$ connecting v with a vertex of F_+ such that the values of ℓ strictly increase along the path and for every vertex $v \in P \setminus F_-$ there exists a path in $G(P)$ connecting v with a vertex of F_- such that the values of ℓ strictly decrease along the path. We have the following three cases:

Case 1. We have $\alpha_- < \alpha < \alpha_+$. Then for every vertex $v \neq v_1, \ldots, v_d$ such that $\ell(v) \geq \alpha$ there exists a path (possibly consisting of a single point) which does not contain $v_1, \ldots, v_d$ and which connects v with a vertex of F_+ and for every vertex $v \neq v_1, \ldots, v_d$ such that $\ell(v) \leq \alpha$ there exists a path (possibly consisting of a single point) which does not contain $v_1, \ldots, v_d$ and which connects v with a vertex of F_-. Note that for v_0 both paths exist.

Case 2. We have $\alpha = \alpha_-$. Then for every vertex $v \neq v_1, \ldots, v_d$ there exists a path which does not contain $v_1, \ldots, v_d$ and which connects v with a vertex of F_+.

Case 3. We have $\alpha = \alpha_+$. Then for every vertex $v \neq v_1, \ldots, v_d$ there exists a path which does not contain $v_1, \ldots, v_d$ and which connects v with a vertex of F_-.

In any case, by the induction hypothesis, we conclude that the graph obtained from $G(P)$ by removing $v_1, \ldots, v_{d-1}$ and the adjacent edges is connected. □

(26.4) Problems.

1°. Let P be an unbounded polyhedron without lines. Prove that $G(P)$ is connected.

2°. Let I_d be the d-dimensional cube. Prove that graph $G\left(I_d\right)$ is not $(d+1)$-connected.

(27.1) Definitions. Let P be a polytope and let $G(P)$ be its graph. An *acyclic orientation* O of $G(P)$ assigns to every edge $[u, v]$ a direction $u \longrightarrow v$ or $v \longrightarrow u$ in such a way that there are no directed cycles $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_n \longrightarrow v_1$. Let S be a set of vertices of P and let O be an acyclic orientation of $G(P)$. A vertex v is called a *sink* of S if all edges $[u, v]$ with $u, v \in S$ are directed $u \longrightarrow v$. The set S is called an *initial set* if all edges $[u, v]$ with $u \notin S$ and $v \in S$ are directed $v \longrightarrow u$. The *index* of a vertex v is the number of edges $u \longrightarrow v$. We identify a face F of P with the set of its vertices in $G(P)$.

We present an algorithm, due to G. Kalai, to recover the facial structure of a simple polytope P from its graph $G(P)$.

(27.2) Lemma. *Let P be a d-dimensional simple polytope, let $G(P)$ be its graph and let O be an acyclic orientation of $G(P)$. For $k = 0, \ldots, d$ let $h_k^O(P)$ be the number of vertices of index k. Then*

$$\sum_{k=0}^{d} 2^k h_k^O(P) \ \geq \ \sum_{i=0}^{d} f_i(P)$$

with the equality if and only if every face F of P (we count P as its own face) has precisely one sink.

Proof. Since the orientation O is acyclic, every set S of vertices has at least one sink. By Parts (2) and (3) of Lemma 24.2, every vertex of index k is a sink of exactly 2^k faces of P: for every set of vertices $u_i : i \in I$ such that $u_i \longrightarrow v$ there is a unique face F_I of P of dimension $|I|$ that contains v and $u_i : i \in I$. Since F_I contains no other neighbors of v, vertex v is a sink of F. Therefore,

$$\begin{aligned} \sum_{k=0}^{d} 2^k h_k^O(P) &= \sum_{v \text{ is a vertex of } P} 2^{\text{index of } v} \\ &= \sum_{F \text{ is a face of} P} \text{the number of sinks in } F. \end{aligned}$$

□

(27.3) Lemma. *Let P be a d-dimensional simple polytope and let F be its face. Then there exists an acyclic orientation O of $G(P)$ such that*

(1) *Face F is an initial set.*
(2) *Every face of P has a unique sink.*

Proof. Since F is a face there exists a linear function ℓ and a number α such that $\ell(x) \geq \alpha$ for all $x \in P$ and

$$F = \Big\{ x \in P : \quad \ell(x) = \alpha \Big\}.$$

Let $\tilde{\ell}$ be a sufficiently generic linear function sufficiently close to ℓ so that $\tilde{\ell}$ is not constant on edges of P and $\tilde{\ell}(v) < \tilde{\ell}(u)$ for any two vertices v and u of P such that $v \in F$ and $u \notin F$, cf. proof of Lemma 25.1. Let us define O by directing $u \longrightarrow v$ if $\tilde{\ell}(v) > \tilde{\ell}(u)$. □

(27.4) Reconstructing the faces of P from $G(P)$.

Input: Graph $G(P)$ of a d-dimensional simple polytope P.

Output: The list of faces of P.

The algorithm: For every acyclic orientation O of $G(P)$ we compute the number $h_k^O(P)$ of vertices of index k and the quantity

$$f^O = \sum_{k=0}^{d} 2^k h_k^O(P).$$

We compute

$$f = \min_O f^O,$$

where the minimum is taken over all acyclic orientations. We call an acyclic orientation *good* if $f^O = f$.

We output F as the set of vertices of a k-dimensional face of P if the subgraph induced by F is connected and k-regular and F occurs as an initial set in some good acyclic orientation F.

(27.5) Theorem. *The algorithm is correct.*

Proof. By Lemmas 27.2 and 27.3 for every good orientation O every face F of P has a unique sink. If F is a k-dimensional face of P then by Theorem 26.3 the subgraph induced by F is connected while by Part (6) of Lemma 24.2 it is k-regular. By Lemma 27.3, every k-dimensional face will be included in the output of the algorithm.

Let O be a good acyclic orientation and let H be an initial set such that the subgraph induced by H is connected and k-regular. Let us prove that H is the set of vertices of a k-dimensional face of P.

Since O is acyclic, H contains a sink v. Since H is k-regular, v has k neighbors $u_1, \ldots, u_k$ in H and we must have $u_i \longrightarrow v$ for $i = 1, \ldots, k$. By Part (2) of Lemma 24.2 there is a k-dimensional face F of P which contains v and $u_1, \ldots, u_k$. Then v must be a sink of F and since orientation O is good, v must be the only sink of F. Let w be a vertex of F. Since v is the unique sink of F, there is a directed path $w = w_0 \longrightarrow w_1 \longrightarrow \ldots \longrightarrow w_n = v$ in $G(P)$, where $w_1, \ldots, w_n$ are vertices of F. If $w \notin H$ then for some $i = 0, \ldots, n-2$ we must have $w_i \notin H$ and $w_{i+1} \in H$, which is a contradiction since H is an initial set. Therefore $F \subset H$. Since the graphs induced by F and H are both k-regular and connected, we must have $F = H$. □

(28.1) Definitions. Let G be a connected graph. The *length* of a path in G is the number of edges in the path. The *distance* between any two vertices of the graph is the smallest length of a path connecting the vertices. The *diameter* of the graph G is the largest distance between two vertices of the graph. Let $\Delta(d, n)$ be the maximum diameter of $G(P)$ for a polyhedron P of dimension at most d and with at most n facets. It follows by Problem 1 of Section 26.4 that $\Delta(d, n) < +\infty$. We say that a path in $G(P)$ *visits* a facet F of P if the path contains a vertex of F.

The famous *Hirsch Conjecture* states that the diameter of the graph $G(P)$ of a d-dimensional polytope P with n facets does not exceed $n - d$, see E.D. Kim and F. Santos, An update on the Hirsch conjecture, *preprint* `arXiv:0907.1186`, 2009 and E.D. Kim and F. Santos, Companion to "An update on the Hirsch conjecture", *preprint* `arXiv:0912.4235`, 2009. Below we reproduce an upper bound on the the diameter of $G(P)$, where P is a (possibly unbounded) polyhedron, due to G. Kalai and D. Kleitman.

(28.2) Lemma. *Let P be a d-dimensional polyhedron with n facets and without lines. For a vertex w of P let us define k_w as the largest integer such that the set of facets of P that can be visited from w by a path of length at most k_w contains not more than $n/2$ facets. We let $k_w = -1$ if w belongs to more than $n/2$ facets of P. Then*

(1) *The distance in $G(P)$ between any two vertices u and v does not exceed*
$$2 + k_u + k_v + \Delta(d - 1, n - 1);$$
(2) *Suppose that $k_w > 0$. Let Q_w be the polyhedron defined by the inequalities that define the facets of P that can be visited by a path of length at most k_w from w. Then w is a vertex of Q_w and every path in $G(Q_w)$ of length at most k_w is a path in $G(P)$.*
(3) *We have*
$$k_w \leq \Delta(d, n/2).$$

Proof. Since the set of facets visited by a path from u of length at most $k_u + 1$ contains more than $n/2$ facets and the set of facets visited by a path from v of length at most $k_v + 1$ contains more than $n/2$ facets, there is a facet F of P visited by a path Π_u from u of length at most $k_u + 1$ and by a path Π_v from v of length at most $k_v + 1$. Moreover, F is a $(d-1)$- dimensional polyhedron with at most $(n-1)$ facets. Connecting the endpoints of Π_u and Π_v in F, we obtain a path connecting u and v of length at most $2 + k_u + k_v + \Delta(d - 1, n - 1)$, which proves Part (1).

The inequalities that define the facets of P containing w are also inequalities defining Q_w. Therefore, $\operatorname{tcone}(P, w) = \operatorname{tcone}(Q_w, w)$, so w is a vertex of Q_w. Let Π be a path from w in $G(Q_w)$ of length $k \leq k_w$. We prove by induction on k that Π is a path in $G(P)$. The case of $k = 0$ has been dealt with. Suppose that $k > 1$ and let us consider the path $\hat{\Pi} \subset \Pi$ consisting of the first $k-1$ edges of Π. Suppose that the other endpoint of $\hat{\Pi}$ is u. Then the inequalities defining the facets of P containing u are also inequalities defining Q_w and hence $\operatorname{tcone}(P, u) = \operatorname{tcone}(Q_w, u)$. Therefore the remaining edge of Π follows an edge e of P with one endpoint at u. Since this edge is bounded in Q_w, it is bounded in P and let v be the other endpoint of e in $G(P)$. Then v is the intersection of e with a facet F of P and since $k \leq k_w$, the inequality defining the facet F is also an inequality defining a facet of Q_w. Then v is a vertex of Q_w and the remaining edge of Π is $[u, v]$, which proves Part (2).

Let u be a vertex of P such that the distance between w and u in $G(P)$ is k_w. Such a vertex exists since otherwise all vertices and hence all facets of P can be visited by a path from w of length at most k_w, which is a contradiction. As follows by Part (2), the distance between w and u in $G(Q_w)$ is also k_w. Since Q_w is defined by at most $n/2$ inequalities, the proof of Part (3) follows. □

(28.3) Theorem. *We have*

$$\Delta(d, n) \;\leq\; 2d \log_2 n + n^{1+\log_2 d}.$$

Proof. By Parts (1) and (3) of Lemma 28.2, we obtain

$$\Delta(d, n) \;\leq\; 2 + 2\Delta(d, n/2) + \Delta(d-1, n-1).$$

Iterating ($d \longmapsto d-1$), we obtain

$$\Delta(d, n) \;\leq\; 2d + 2d\Delta(d, n/2).$$

Iterating ($n \longmapsto n/2$), we obtain

$$\Delta(d, n) \;\leq\; 2d \log_2 n + (2d)^{\log_2 n} \;\leq\; 2d \log_2 n + n^{1+\log_2 d}.$$

□

(28.4) Problem.

1. Let P be a d-dimensional polytope with n facets. Suppose that the vertices of P are 0-1 vectors. Prove that the diameter of $G(P)$ does not exceed $n - d$.

29. Edges of a centrally symmetric polytope

(29.1) Definition. A polytope P is called *centrally symmetric* if $P = -P$.

The following result was proved by A. Barvinok and I. Novik.

(29.2) Theorem. *Let P be a d-dimensional centrally symmetric polytope. Then*

$$f_1(P) \;\leq\; \frac{f_0^2(P)}{2}\left(1 - 2^{-d}\right).$$

Proof. Let us define

$$h = \sum_v [P+v],$$

where the sum is taken over all vertices v of P. Clearly, $P+v \subset 2P$. Let us normalize the Lebesgue measure on $2P$ in such a way that $\operatorname{vol}(2P)=1$. Then $\operatorname{vol}(P+v) = 2^{-d}$ and

$$\int_{2P} h\ dx = \sum_v \int_{2P} [P+v]\ dx = f_0(P)2^{-d}.$$

Therefore, by the Hölder inequality, we have

$$\int_{2P} h^2\ dx \ \geq \ f_0^2(P)2^{-2d}.$$

On the other hand,

$$\int_{2P} h^2\ dx = \sum_{(v,u)} \operatorname{vol}\Big((P+u)\cap(P+v)\Big),$$

where the sum is taken over all unordered pairs u and v of vertices.

Suppose that $\operatorname{vol}\Big((P+u)\cap(P+v)\Big) > 0$. Then there exist $x,y \in \operatorname{int} P$ such that $x+u = y+v$. Thus $(u-v)/2 = (y-x)/2$. Since $P=-P$, we have $y, -x \in \operatorname{int} P$ and so $(y-x)/2 \in \operatorname{int} P$. Therefore, the midpoint of the interval $[u,-v]$ lies in the interior of P and so $[u,-v]$ is not an edge of P. Consequently,

$$\sum_{(v,u)} \operatorname{vol}\Big((P+u)\cap(P+v)\Big) \ \leq \ 2^{-d}f_0(P) + 2^{-d+1}\left(\binom{f_0(P)}{2} - f_1(P)\right).$$

Summarizing,

$$2^{-d}f_0(P) + 2^{-d+1}\left(\binom{f_0(P)}{2} - f_1(P)\right) \ \geq \ f_0^2(P)2^{-2d}$$

and the proof follows. □

(29.3) Problems.

1°. Recall that the standard octahedron $O_n \subset \mathbb{R}^n$ is defined as

$$O_n = \operatorname{conv}\Big(e_i, -e_i : \quad i=1,\ldots,n\Big),$$

where $e_1,\ldots,e_n$ is the standard basis in $\mathbb{R}^n$. Prove that the $(r-1)$-dimensional faces of O_n are as follows: let $S_-, S_+ \subset \{1,\ldots,n\}$ be disjoint subsets such that $|S_-|+|S_+| = r$. Then

$$F_{S_+,S_-} = \left\{ (x_1,\ldots,x_n) \in O_n : \quad \begin{array}{ccc} x_i \geq 0 & \text{for} & i \in S_+, \\ x_i \leq 0 & \text{for} & i \in S_-, \\ x_i = 0 & \text{for} & i \notin S_+ \cup S_- \end{array} \right\}.$$

2. Let $L \subset \mathbb{R}^n$ be a subspace. Let us consider a projection $pr : \mathbb{R}^n \longrightarrow \mathbb{R}^n/L$ with the kernel L and let $Q = pr(O_n)$, where O_n is the standard octahedron. Let $y \in L$ be a unique vector minimizing $\|u - x\|_1$ for $u \in L$, where

$$\|a\|_1 = \sum_{i=1}^n |a_i| \quad \text{for} \quad a = (a_1, \ldots, a_n)\,.$$

Let

$$S_+ = \Big\{i: \quad y_i > x_i\Big\} \quad \text{and} \quad S_- = \Big\{i: \quad y_i < x_i\Big\}$$

so that $|S_+| + |S_-| = r$. Prove that $pr\left(F_{S_+,S_-}\right)$ is an $(r-1)$-dimensional face of Q.

3. Let $L \subset \mathbb{R}^n$ be a subspace and let $pr : \mathbb{R}^n \longrightarrow \mathbb{R}^n/L$ be a projection with the kernel L. Let $Q = pr(O_n)$ and suppose that the projection of every $(r-1)$-dimensional face of O_n is an $(r-1)$-dimensional face of Q. Suppose further, that there is a point $y \in L$ such that y differ from x in r coordinates. Prove that y is the unique minimum of $\|u - x\|_1$ for $u \in L$.

30. Approximating a convex body by an ellipsoid

(30.1) Definitions. Let V be Euclidean space, let $B \subset V$,

$$B = \Big\{x \in V: \quad \|x\|^2 \leq 1\Big\},$$

be the unit ball, let $T : V \longrightarrow V$ be an invertible linear transformation and let $a \in V$ be a point. The set $E = T(B) + a$ is called an *ellipsoid* and a is called its *center*.

(30.2) Theorem. *Let $K \subset V$ be a centrally symmetric compact convex set with a non-empty interior. Then there exists an ellipsoid $E \subset V$ centered at 0 such that*

$$E \ \subset \ K \ \subset \ \left(\sqrt{\dim V}\right) E.$$

Proof. We choose E to be the ellipsoid of the maximum volume among those centered at 0 and contained in K (that such an ellipsoid exists is proven by a standard compactness argument). Hence $E \subset K$ and we have to prove that $K \subset \left(\sqrt{\dim V}\right) E$. Without loss of generality we assume that $K = B$, the unit ball. Suppose that there is a point $x \in K$ such that $\|x\| > \sqrt{\dim V}$. Let us introduce a coordinate system in V, thus identifying $V = \mathbb{R}^d$ and $x = (r, 0, \ldots, 0)$, where $r > \sqrt{d}$. Since K is symmetric and convex, we have $\operatorname{conv}(B, x, -x) \subset K$. Our goal is to inscribe an ellipsoid $E \subset \operatorname{conv}(B, x, -x)$ such that $\operatorname{vol} E > \operatorname{vol} B$ thus obtaining a contradiction with the existence of x.

We look for E in the form

$$E = \left\{ (x_1, \ldots, x_d) : \quad \frac{x_1^2}{\alpha^2} + \frac{1}{\beta^2} \sum_{i=2}^{d} x_i^2 \ \leq \ 1 \right\},$$

where $\alpha > 1$ and $0 < \beta < 1$. Then

$$\operatorname{vol} E = \alpha \beta^{d-1} \operatorname{vol} B.$$

To make sure that $E \subset \operatorname{conv}(B, x, -x)$ for some choice of α and β, by symmetry it suffices to show the inclusion for $d = 2$, in which case we write x for x_1 and y for x_2.

Suppose that $d = 2$ and let $(a, b) \in \partial B$ be a point on the unit circle, so $a^2 + b^2 = 1$. The equation of the tangent line to B through (a, b) is $ax + by = 1$ and if we insist that the line passes through the point $(r, 0)$, we obtain the equation

$$\frac{x}{r} + \frac{y\sqrt{r^2 - 1}}{r} = 1. \tag{30.2.1}$$

Similarly, the equation of the tangent line to E through a point $(a, b) \in E$ such that $a^2/\alpha^2 + b^2/\beta^2 = 1$ is

$$\frac{a}{\alpha^2} x + \frac{b}{\beta^2} y = 1. \tag{30.2.2}$$

Hence we obtain a common tangent to B and E passing through $(r, 0)$ when

$$\frac{a}{\alpha^2} = \frac{1}{r} \quad \text{and} \quad \frac{b}{\beta^2} = \frac{\sqrt{r^2 - 1}}{r}.$$

Therefore $a = \alpha^2/r$ and $b = \beta^2 \sqrt{r^2 - 1}/r$. Substituting a and b into the equation $a^2/\alpha^2 + b^2/\beta^2 = 1$, we obtain

$$\alpha^2 = r^2 - (r^2 - 1)\beta^2. \tag{30.2.3}$$

If $\beta = 1 - \epsilon$ for a sufficiently small $\epsilon > 0$ and (α, β) satisfy (30.2.3) then $E \subset K$. Letting $\beta = 1 - \epsilon$ we obtain from (30.2.3) that $\alpha = 1 + (r^2 - 1)\epsilon + O(\epsilon^2)$. Therefore,

$$\alpha \beta^{d-1} = \exp\bigl\{ \ln \alpha + (d - 1) \ln \beta \bigr\} = \exp\left\{ (r^2 - 1)\epsilon - (d - 1)\epsilon + O(\epsilon^2) \right\} > 1$$

provided $r > \sqrt{d}$ and $\epsilon > 0$ is sufficiently small. The obtained contradiction shows that in fact $r \leq \sqrt{d}$ and $K \subset \sqrt{d} B$. $\square$

(30.3) Problems.

1. Prove that for any centrally symmetric convex compact set K with a non-empty interior there exists a unique ellipsoid E which has the maximum volume among all ellipsoids inscribed in K and centered at the origin.

2. Let K be a convex compact (not necessarily symmetric) set with a non-empty interior. Prove that there exists a unique ellipsoid E which has the maximum volume among all ellipsoids inscribed in K.

3. Let $E \subset K$ be the maximum volume ellipsoid of Problem 2 and suppose that its center is at the origin. Prove that $K \subset (\dim K)E$.

4. Let K be a centrally symmetric convex compact set with a non-empty interior. Prove that there exists a unique ellipsoid E of the minimum volume among all ellipsoids containing K and centered at the origin. Prove that $(\dim K)^{-1/2}E \subset K$.

5. Let K be a convex compact (not necessarily symmetric) set with a non-empty interior. Prove that there exists a unique ellipsoid E of the minimum volume among all ellipsoids containing K. Suppose that the center of E is at the origin. Prove that $(\dim K)^{-1}E \subset K$.

31. Spherical caps

(31.1) Lemma. *Let*

$$\mathbb{S}^{n-1} = \left\{ (x_1, \ldots, x_n) : \quad \sum_{i=1}^n x_i^2 = 1 \right\}$$

be the unit sphere in $\mathbb{R}^n$ and let μ be the (unique) rotation invariant Borel probability measure on $\mathbb{S}^{n-1}$. For $0 \leq \epsilon \leq 1$ let us define the spherical cap

$$C_\epsilon = \left\{ (x_1, \ldots, x_n) \in \mathbb{S}^{n-1} : \quad x_1 \geq \epsilon \right\}.$$

Then

$$\mu\left(C_\epsilon\right) \ \leq \ \frac{1}{\sqrt{2}} \exp\left\{ -\frac{(n-1)\epsilon^2}{6} \right\}.$$

Proof. Let ν be the Gaussian probability measure on $\mathbb{R}^n$ with the density

$$\frac{1}{\pi^{n/2}} e^{-\|x\|^2}.$$

Since ν is rotation invariant, the push-forward of ν under the radial projection $\mathbb{R}^n \setminus \{0\}$, $x \longmapsto x/\|x\|$ is the measure μ on $\mathbb{S}^{n-1}$.

Let

$$A_\epsilon = \left\{ (x_1, \ldots, x_n) : \quad x_1^2 \geq \epsilon^2 \|x\|^2 \right\}.$$

Then

$$\mu\left(C_\epsilon\right)=\frac{1}{2}\nu\left(A_\epsilon\right)$$

and hence our goal is bound $\nu\left(A_\epsilon\right)$. We have

$$\begin{aligned}
\nu\left(A_\epsilon\right)=&\frac{1}{\pi^{n/2}}\int_{A_\epsilon} e^{-\|x\|^2}\ dx \ \leq\ \frac{1}{\pi^{n/2}}\int_{A_\epsilon} e^{\left(x_1^2-\epsilon^2\|x\|^2\right)/2}e^{-\|x\|^2}\ dx\\
\leq&\frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{\left(x_1^2-\epsilon^2\|x\|^2\right)/2}e^{-\|x\|^2}\ dx\\
=&\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\|x_1\|^2/2}\ dx_1\right)\prod_{i=1}^{n-1}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(1+\epsilon^2/2)x_i^2}\ dx_i\right)\\
=&\sqrt{2}\left(\frac{1}{\sqrt{1+\epsilon^2/2}}\right)^{n-1} = \sqrt{2}\exp\left\{-\frac{(n-1)}{2}\ln\left(1+\frac{\epsilon^2}{2}\right)\right\}\\
\leq&\sqrt{2}\exp\left\{-\frac{(n-1)\epsilon^2}{6}\right\}
\end{aligned}$$

and the proof follows. □

We will also need a lower bound on the measure of a (small) spherical cap.

(31.2) Lemma. *Let*

$$\mathbb{S}^{n-1}=\left\{(x_1,\dots,x_n):\quad \sum_{i=1}^n x_i^2=1\right\}$$

be the unit sphere in $\mathbb{R}^n$ and let μ be the (unique) rotation invariant Borel probability measure on $\mathbb{S}^{n-1}$. For $0\leq\delta\leq 2$ and $y\in\mathbb{S}^{n-1}$, let us define the spherical cap

$$A_\delta(y)=\left\{x\in\mathbb{S}^{n-1}:\quad \|x-y\|\leq\delta\right\}.$$

Then

$$\mu\left(A_\delta\right)\ \geq\ \frac{\delta^n}{(2+\delta)^n}.$$

Proof. Suppose that $\Sigma\subset\mathbb{S}^{n-1}$ is a δ-net, that is, a finite set of points such that every point $x\in\mathbb{S}^{n-1}$ is within distance δ from some point $y\in\Sigma$. Then the caps $A_\delta(y)$ for $y\in\Sigma$ cover $\mathbb{S}^{n-1}$ and hence

$$\mu\left(A_\delta(y)\right)\ \geq\ \frac{1}{|\Sigma|}.$$

Let us now construct a δ-net as follows. Let Σ be the maximal (under inclusion) set $\Sigma \subset \mathbb{S}^{n-1}$ such that the distance between any two points of Σ is at least δ. Clearly, Σ is a δ-net. For $y \in \Sigma$, let

$$B_{\delta/2}(y) = \left\{ x \in \mathbb{R}^n : \quad \|x - y\| \leq \frac{\delta}{2} \right\}$$

be the ball centered at y of radius $\delta/2$. Hence the balls $B_{\delta/2}(y)$ are pairwise disjoint and contained in the ball $B_{1+\delta/2}(0)$. Therefore,

$$|\Sigma| \cdot \operatorname{vol} B_{\delta/2}(y) \ \leq \ \operatorname{vol} B_{1+\delta/2}(0)$$

and hence

$$|\Sigma| \ \leq \ \frac{(2+\delta)^n}{\delta^n}.$$

The proof now follows. □

(31.3) Problem.

1. Let $B = \{x \in \mathbb{R}^n : \ \|x\| \leq 1\}$ be the unit ball. Prove that

$$\operatorname{vol} B = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

32. An inequality for the number of faces of a centrally symmetric polytope

The following result is due to V.D. Milman, T. Figiel and J. Lindenstrauss.

(32.1) Theorem. *There exists a constant $\gamma > 0$ such that*

$$\ln f_0(P) \cdot \ln f_{d-1}(P) \ \geq \ \gamma d$$

for any $d > 1$ and any d-dimensional centrally symmetric polytope P.

Proof. By Theorem 30.2 there is an ellipsoid E centered at the origin such that $E \subset P \subset \sqrt{d}E$. Applying a linear transformation, if necessary, we assume that $E = B$, the unit ball defined in the ambient space V by the inequality

$$B = \left\{ x \in V : \quad \|x\| \leq 1 \right\}.$$

Let v_i, $i = 1, \ldots, f_{d-1}(P)$ be the vertices of P and let $\overline{v}_i = v_i/\|v_i\|$ be their radial projections onto the unit sphere $\mathbb{S}^{d-1}$. Let us choose an $0 < \epsilon < 1$ and let us consider the spherical cap

$$C_\epsilon\left(\overline{v}_i\right) = \left\{ c \in \mathbb{S}^{d-1} : \quad \langle c, \overline{v}_i \rangle \ \geq \ \epsilon \right\} \quad \text{for} \quad i = 1, \ldots, f_{d-1}(P).$$

Let μ the the rotation invariant Borel probability measure on $\mathbb{S}^{d-1}$. By Lemma 31.1, we have

$$\mu\left(C_\epsilon\left(\overline{v}_i\right)\right) \ \leq \ \exp\left\{-\frac{(d-1)\epsilon^2}{6}\right\}.$$

We can choose

$$\epsilon \ \leq \ \gamma_0\sqrt{\frac{\ln f_0(P)}{d}}$$

for some absolute constant $\gamma_0 > 0$ such that

$$f_0(P)\exp\left\{-\frac{(d-1)\epsilon^2}{6}\right\} \ < \ \frac{1}{2}.$$

Letting

$$X_\epsilon = \bigcup_i C_\epsilon\left(\overline{v}_i\right),$$

we conclude that

$$\mu\left(X_\epsilon\right) \ < \ \frac{1}{2} \tag{32.1.1}$$

and that for any $c \in \mathbb{S}^{d-1} \setminus X_\epsilon$, we have

$$\max_{x\in P}\langle c, x\rangle \ = \ \max_i\langle c, v_i\rangle \ \leq \ \sqrt{d}\max_i\langle c, \overline{v}_i\rangle \ \leq \ \epsilon\sqrt{d} \ \leq \ \gamma_0\sqrt{\ln f_0(P)}. \tag{32.1.2}$$

Suppose that P is defined by the inequalities

$$P = \Big\{x \in V: \quad \langle u_j, x\rangle \leq \alpha_j, \ j = 1, \ldots, f_{d-1}(P)\Big\}.$$

Since P contains the origin in its interior, we must have $\alpha_j > 0$ and, rescaling if necessary, we assume that $\alpha_j = 1$ for $j = 1, \ldots, f_{d-1}(P)$.

Let $\overline{u}_j = u_j/\|u_j\|$ be the radial projections of u_j onto the unit sphere $\mathbb{S}^{d-1}$. Since $\overline{u}_j \in P$, we conclude that $\|u_j\| \leq 1$ for $j = 1, \ldots, f_{d-1}(P)$. Let us choose a $0 < \delta < 1$ and let us consider the spherical cap

$$C_\delta\left(\overline{u}_j\right) = \Big\{c \in \mathbb{S}^{d-1}: \quad \langle c, \overline{u}_j\rangle \ \geq \delta\Big\} \quad \text{for} \quad j = 1, \ldots, f_{d-1}(P).$$

We can choose

$$\delta \ \leq \ \gamma_1\sqrt{\frac{\ln f_{d-1}(P)}{d}}$$

for some absolute constant $\gamma_1 > 0$ such that

$$f_{d-1}(P)\exp\left\{-\frac{(d-1)\delta^2}{6}\right\} \ < \ \frac{1}{2}.$$

Letting

$$Y_\delta = \bigcup_j C_\delta\left(\overline{u}_j\right),$$

we conclude that

$$\mu\left(Y_\delta\right) \ < \ \frac{1}{2}. \tag{32.1.3}$$

Moreover, for every $c \in \mathbb{S}^{d-1} \setminus Y_\delta$ we have

$$|\langle c, u_j\rangle| \ \le \ |\langle c, \overline{u}_j\rangle| \ \le \ \delta \quad \text{for} \quad j=1,\ldots,f_{d-1}(P)$$

and hence $\delta^{-1}c \in P$. Therefore, for every $c \in \mathbb{S}^{d-1} \setminus Y_\delta$ we have

$$\max_{x \in P}\langle c, x\rangle \ \ge \ \langle c, \delta^{-1}c\rangle \ = \ \delta^{-1} \ \ge \ \gamma_1^{-1}\sqrt{\frac{d}{\ln f_{d-1}(P)}}. \tag{32.1.4}$$

By (32.1.1) and (32.1.3), there exists a $c \in \mathbb{S}^{d-1}$ such that $c \notin X_\epsilon$ and $c \notin Y_\delta$. For such a c, we have by (32.1.2) and (32.1.4)

$$\gamma_1^{-1}\sqrt{\frac{d}{\ln f_{d-1}(P)}} \ \le \ \max_{x \in P}\langle c, x\rangle \ \le \ \gamma_0\sqrt{\ln f_0(P)}.$$

Therefore,

$$\ln f_0(P) \cdot \ln f_{d-1}(P) \ \ge \ \gamma d \quad \text{for} \quad \gamma = (\gamma_0\gamma_1)^{-2},$$

as desired. □

(32.2) Problems.

1. Let P be a d-dimensional polytope such that $B \subset P \subset \rho B$ for the unit ball B and some $\rho > 1$. Prove that

$$\ln f_{d-1}(P) \cdot \ln f_0(P) \ \ge \ \gamma d^2/\rho^2,$$

for some absolute constant $\gamma > 0$.

2. Let P be a d-dimensional centrally symmetric polytope. Prove that

$$\ln f_l(P) \cdot \ln f_k(P) \ \ge \ \gamma(l-k) \quad \text{for all} \quad 0 \le k \le l \le d-1$$

and some absolute constant $\gamma > 0$.

33. Gale transforms and symmetric Gale transforms

Let $P \subset \mathbb{R}^d$ be a d-dimensional polytope with the vertices $v_1, \ldots, v_n$. By translating P, we can assume that $v_1 + \ldots + v_n = 0$. Moreover, by applying an invertible linear transformation, we can represent P as the orthogonal projection of the standard simplex $\Delta_{n-1} \subset \mathbb{R}^n$ onto a d-dimensional subspace L. We define L by its basis consisting of the rows of the $d \times n$ matrix whose columns are $v_1, \ldots, v_n$.

(33.1) Theorem. *Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, let $u = e_1 + \ldots + e_n$ and let $L \subset \mathbb{R}^n$ be a d-dimensional subspace orthogonal to u. Let v_i be the orthogonal projection of e_i onto L and let $P = \operatorname{conv}(v_1, \ldots, v_n)$. Let $\hat{L} = (L \oplus \mathbb{R}u)^{\perp}$ be the $(n-d-1)$-dimensional subspace orthogonal to L and to u and let $\hat{v}_i$ be the orthogonal projection of e_i onto L. Then $\{v_i : \ i \in I\}$ is the set of vertices of a proper face of P if and only if*

$$0 = \sum_{i \notin I} \lambda_i \hat{v}_i \quad \textit{for some} \quad \lambda_i > 0 \quad \textit{for} \quad i \notin I$$

or, equivalently, if

$$0 \in \operatorname{int} \operatorname{conv}\big(\hat{v}_i : \quad i \notin I\big),$$

where by " int " we understand the relative interior (that is, the interior relative to the affine span of the set $\hat{v}_i : \ i \notin I$).

Proof. Suppose that $\{v_i : \ i \in I\}$ is the set of vertices of a proper face of P. Then there exists a vector $c \in L$ and a number α such that

$$\begin{aligned} &\langle c, v_i \rangle = \alpha \quad \text{for} \quad i \in I \quad \text{and} \\ &\langle c, v_i \rangle > \alpha \quad \text{for} \quad i \notin I. \end{aligned} \tag{33.1.1}$$

Since $\langle c, v_i \rangle = \langle c, e_i \rangle$ for all i, we obtain

$$\begin{aligned} &\langle c - \alpha u, e_i \rangle = 0 \quad \text{for} \quad i \in I \quad \text{and} \\ &\langle c - \alpha u, e_i \rangle > 0 \quad \text{for} \quad i \notin I. \end{aligned} \tag{33.1.2}$$

Thus we can write

$$c - \alpha u = \sum_{i \notin I} \lambda_i e_i \quad \text{for} \quad \lambda_i = \langle c - \alpha u, e_i \rangle > 0. \tag{33.1.3}$$

Projecting the above identity onto $\hat{L}$, we get

$$0 = \sum_{i \notin I} \lambda_i \hat{v}_i \quad \text{where} \quad \lambda_i > 0 \quad \text{for all} \quad i \notin I. \tag{33.1.4}$$

Conversely, suppose that (33.1.4) holds. Then the vector $\sum_{i \notin I} \lambda_i e_i$ lies in $\left(\hat{L}\right)^{\perp} = L \oplus \mathbb{R}u$ and hence for some $\alpha \in \mathbb{R}$ we have (33.1.3) where

$$c \ \in \ \left(\hat{L} \oplus \mathbb{R}u\right)^{\perp} = L.$$

Then (33.1.2) and hence (33.1.1) hold and so $\{v_i : \ i \in I\}$ are the vertices of a proper face of P. $\square$

The correspondence

$$\{v_i : \ i = 1, \ldots, n\} \ \leftrightarrow \ \{\hat{v}_i : \ i = 1, \ldots, n\}$$

is called the *Gale transform* after D. Gale.

The *symmetric Gale transform* was introduced by P. McMullen and G.C. Shephard. It is based on representing a centrally symmetric polytope as the projection of the octahedron O_n.

(33.2) Theorem. *Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and let $L \subset \mathbb{R}^n$ be a d-dimensional subspace. Let v_i be the orthogonal projection of e_i onto L and let $P = \operatorname{conv}(v_1, \ldots, v_n; -v_1, \ldots, -v_n)$. Let $L^{\perp} \subset \mathbb{R}^n$ be the orthogonal complement to L and let $\overline{v}_i$ be the orthogonal projection of e_i onto $L^{\perp}$. Then the set $\{\epsilon_i v_i : \ i \in I\}$ where $\epsilon_i \in \{-1, 1\}$ is the set of vertices of a proper face of P if and only if*

$$\sum_{i \in I} \epsilon_i \overline{v}_i = \sum_{i \notin I} \delta_i \overline{v}_i \quad \textit{where} \quad |\delta_i| < 1 \quad \textit{for} \quad i \notin I$$

or, equivalently, if

$$\sum_{i \in I} \epsilon_i \overline{v}_i \ \in \ \operatorname{int} \operatorname{conv} \left(\sum_{i \notin I} \delta_i \overline{v}_i : \quad \textit{for} \quad \delta_i \in \{-1, 1\} \right),$$

where by " int *" we understand the relative interior.*

Proof. Suppose that $\{\epsilon_i v_i : \ i \in I\}$ is the set of vertices of a proper face of P. Then there exists a vector $c \in L$ and a number $\alpha > 0$ such that

$$\begin{aligned} &\langle c, v_i \rangle = \epsilon_i \alpha \quad \text{for} \quad i \in I \quad \text{and} \\ &|\langle c, v_i \rangle| < \alpha \quad \text{for} \quad i \notin I. \end{aligned} \tag{33.2.1}$$

Since $\langle c, v_i \rangle = \langle c, e_i \rangle$ for all i, we obtain

$$\begin{aligned} &\langle c, e_i \rangle = \epsilon_i \alpha \quad \text{for} \quad i \in I \quad \text{and} \\ &|\langle c, e_i \rangle| < \alpha \quad \text{for} \quad i \notin I. \end{aligned} \tag{33.2.2}$$

Thus we can write

$$c = \sum_{i \in I} \epsilon_i \alpha e_i + \sum_{i \notin I} \lambda_i e_i \quad \text{where} \quad |\lambda_i| < \alpha \quad \text{for} \quad i \notin I. \tag{33.2.3}$$

Projecting the above identity onto $L^\perp$ and dividing by α, we obtain

$$\sum_{i \in I} \epsilon_i \overline{v}_i = \sum_{i \notin I} \delta_i \overline{v}_i \quad \text{where} \quad |\delta_i| < 1 \quad \text{for all} \quad i \notin I. \tag{33.2.4}$$

Conversely, suppose that (33.2.4) holds. Then the vector $\sum_{i \in I} \epsilon_i e_i - \sum_{i \notin I} \delta_i e_i$ lies in L and hence we have (32.2.3) for some $c \in L$ and $\alpha = 1$. Therefore, both (33.2.2) and (33.2.1) hold and hence $\{\epsilon_i v_i : i \in I\}$ are the vertices of a proper face of P. □

(33.3) Problems.

1. Describe a possible facial structure of a d-dimensional polytope with $d+2$ vertices.

2. Construct an example of a $(d-1)$-dimensional centrally symmetric polytope P with $2d$ vertices such that every subset of $k < d/2$ vertices, not containing a pair of antipodal vertices, is the set of vertices of some face of P.

34. Almost Euclidean subspaces of ℓ^1 and centrally symmetric polytopes with many faces

(34.1) Definitions. Let us consider the following two norms in $\mathbb{R}^n$: the usual Euclidean norm

$$\|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \quad \text{for} \quad x = (x_1, \ldots, x_n)$$

and the ℓ^1 norm

$$\|x\|_1 = \sum_{i=1}^n |x_i| \quad \text{for} \quad x = (x_1, \ldots, x_n).$$

It is not hard to show that

$$\|x\|_2 \ \leq \ \|x\|_1 \ \leq \ \sqrt{n}\|x\|_2 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

For a $0 < \rho < 1$ we say that a subspace $L \subset \mathbb{R}^n$ is *ρ-Euclidean* if

$$\rho\sqrt{n}\|x\|_2 \ \leq \ \|x\|_1 \ \leq \ \sqrt{n}\|x\|_2 \quad \text{for all} \quad x \in L.$$

The following is a rephrasing of a result due to N. Linial and I. Novik.

(34.2) Theorem. *Let $L \subset \mathbb{R}^n$ be a subspace, let v_i be the orthogonal projection of e_i onto L, and let $P = \operatorname{conv}\big(v_1, \ldots, v_n; -v_1, \ldots, -v_n\big)$. Suppose that the complementary subspace $L^\perp \subset \mathbb{R}^n$ is ρ-Euclidean and let $k < \rho^2 n/4$ be a positive integer. Then every set of k vertices of P not containing a pair of antipodal vertices is the set of vertices of some proper face of P.*

Proof. Let $\overline{v}_i$ be the orthogonal projection of e_i onto $L^\perp$ for $i = 1, \ldots, n$.

Let $I \subset \{1, \ldots, n\}$ be a set, $|I| = k$, and let $\epsilon_i \in \{-1, 1\}$ for $i \in I$ be signs such that the set $\big\{\epsilon_i v_i : \ i \in I\big\}$ is *not* the set of vertices of a face of P. By Theorem 33.2, the point $\sum_{i \in I} \epsilon_i \overline{v}_i$ does not lie in the relative interior of the convex hull of the points $\sum_{i \notin I} \delta_i \overline{v}_i$ for $\delta_i \in \{-1, 1\}$ and hence there exists a vector $c \in L^\perp$, $c \neq 0$, such that

$$\left\langle c, \ \sum_{i \in I} \epsilon_i \overline{v}_i \right\rangle \ \geq \ \max_{\substack{\delta_i = \pm 1 \\ i \notin I}} \left\langle c, \ \sum_{i \notin I} \delta_i \overline{v}_i \right\rangle = \sum_{i \notin I} \left| \langle c, \overline{v}_i \rangle \right|.$$

In particular,

$$\sum_{i \in I} \left| \langle c, \overline{v}_i \rangle \right| \ \geq \ \sum_{i \notin I} \left| \langle c, \overline{v}_i \rangle \right|.$$

On the other hand, $\langle c, \overline{v}_i \rangle = \langle c, e_i \rangle$ for $i = 1, \ldots, n$ and hence

$$\sum_{i \in I} \left| \langle c, e_i \rangle \right| \ \geq \ \sum_{i \notin I} \left| \langle c, e_i \rangle \right|.$$

Therefore,

$$\sum_{i \in I} \left| \langle c, e_i \rangle \right| \ \geq \ \frac{1}{2} \sum_{i=1}^{n} \left| \langle c, e_i \rangle \right| \ = \ \frac{1}{2} \|c\|_1.$$

On the other hand, by the Cauchy- Schwarz inequality

$$\sum_{i \in I} \left| \langle c, e_i \rangle \right| \ \leq \ \sqrt{k} \left(\sum_{i \in I} \langle c, e_i \rangle^2 \right)^{1/2} \ \leq \ \sqrt{k} \|c\|_2 \ < \ \frac{\rho}{2} \sqrt{n} \|c\|_2$$

and

$$\|c\|_1 \ > \ \rho \sqrt{n} \|c\|_2,$$

which is a contradiction. $\square$

(34.3) Problems.

1°. Construct an example of a 1-dimensional subspace of $\mathbb{R}^n$ which is 1-Euclidean.

2. For $0 < \rho < 1$ and a positive integer k construct an example of a k-dimensional subspace of $\mathbb{R}^n$ for which is ρ-Euclidean for a sufficiently large n.

(35.1) Definitions. Let V be an n-dimensional vector space with Euclidean norm $\|\cdot\|$ and let $p : V \longrightarrow \mathbb{R}$ be some other norm on V. Let

$$B = \Big\{ x \in V : \quad \|x\| \leq 1 \Big\}$$

be the Euclidean unit ball, and let

$$K = \Big\{ x \in V : \quad p(x) \leq 1 \Big\}$$

be the unit ball in norm p.

Let $G_k(V)$ be Grassmannian manifold of all k-dimensional subspaces $L \subset V$ with the unique rotation invariant Borel probability measure $\nu_{k,n}$.

The following result is due to S. Szarek who was building on the work of B. Kashin.

(35.2) Theorem. *Suppose that $B \subset K$ and that*

$$\left(\frac{\operatorname{vol} K}{\operatorname{vol} B} \right)^{1/n} \leq \gamma$$

for some $\gamma > 1$. Then for any $1 \leq k \leq n-1$ a random subspace $L \in G_k(V)$ satisfies

$$p(x) \leq \|x\| \leq (12\gamma)^{\frac{n}{n-k}} p(x) \quad \text{for all} \quad x \in L$$

with probability at least $1 - 2^{-n}$.

Proof. Since $B \subset K$ we have

$$p(x) \leq \|x\| \quad \text{for all} \quad x \in V.$$

We note that

$$\frac{\operatorname{vol} K}{\operatorname{vol} B} = \int_{\mathbb{S}^{n-1}} p^{-n}(x) \ d\mu_{n-1}(x),$$

where μ_{n-1} is the rotation invariant Borel probability measure on the unit Euclidean sphere $\mathbb{S}^{n-1}$. Indeed, for a point $x \in V \setminus 0$ we have $p^{-1}(x)x \in \partial K$ and hence we obtain K by stretching B in the direction of $x \in \mathbb{S}^{n-1}$ by the factor of $p^{-1}(x)$.

Next, we note that for any continuous function $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ we have

$$\int_{\mathbb{S}^{n-1}} f(x) \ d\mu_{n-1}(x) = \int_{G_k(V)} \left(\int_{\mathbb{S}^{n-1} \cap L} f(x) \ d\mu_{k-1,L}(x) \right) \ d\nu_{k,n}(L),$$

where $\mu_{k-1,L}$ is the rotation invariant Borel probability measure on the $(k-1)$-dimensional unit sphere $\mathbb{S}^{n-1} \cap L$ in the k-dimensional subspace L. This follows from the uniqueness of the rotation invariant Borel probability measure on $\mathbb{S}^{n-1}$.

Summarizing,

$$\int_{G_k(V)} \left(\int_{\mathbb{S}^{n-1} \cap L} p^{-n}(x) \; d\mu_{k-1,L}(x) \right) \; d\nu_{k,n}(L) \; \leq \; \gamma^n$$

and hence with probability at least $1-2^{-n}$ a random subspace $L \in G_k(V)$ satisfies

$$\int_{\mathbb{S}^{n-1} \cap L} p^{-n}(x) \; d\mu_{k-1,L}(x) \; \leq \; (2\gamma)^n. \tag{35.2.1}$$

Suppose that a subspace $L \in G_k(V)$ satisfies (35.2.1). Then, for any $0 < \alpha < 1$ we have

$$\mu_{k-1,L} \left\{ x \in \mathbb{S}^{n-1} \cap L : \quad p(x) \; \leq \; \frac{\alpha}{2\gamma} \right\} \; \leq \; \alpha^n. \tag{35.2.2}$$

Suppose that for some $y \in \mathbb{S}^{n-1} \cap L$ we have

$$p(y) \; \leq \; \frac{\alpha}{4\gamma}.$$

Let us consider the $(k-1)$-dimensional spherical cap

$$A(y) = \left\{ x \in \mathbb{S}^{n-1} \cap L : \quad \|x-y\| \leq \frac{\alpha}{4\gamma} \right\}.$$

Then for every $x \in A(y)$ we have

$$p(x) \; \leq \; p(y) + p(x-y) \; \leq \; p(y) + \|x-y\| \; \leq \; \frac{\alpha}{2\gamma}.$$

Using the estimate of Lemma 31.2 on the measure of a spherical cap, we conclude that

$$\mu_{k-1,L} \left\{ x \in \mathbb{S}^{n-1} \cap L : \quad p(x) \; \leq \; \frac{\alpha}{2\gamma} \right\} \; \geq \; \left(\frac{\alpha}{12\gamma} \right)^k. \tag{35.2.3}$$

However, (35.2.3) contradicts (35.2.2) if

$$\alpha \; < \; (12\gamma)^{-\frac{k}{n-k}}.$$

Therefore, for all $y \in \mathbb{S}^{n-1} \cap L$ we have

$$p(y) \; \geq \; (2\gamma)^{-1}(12\gamma)^{-\frac{k}{n-k}} \; \geq \; (12\gamma)^{-\frac{n}{n-k}}$$

and the proof follows. $\square$

(35.3) Example: almost Euclidean subspaces of ℓ^1. Let $V = \mathbb{R}^n$ with the usual Euclidean norm $\| \cdot \|$. Let us define

$$p(x) = n^{-1/2} \|x\|_1 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Then the unit ball of $p(x)$ is the dilated octahedron $K = \sqrt{n} O_n$ and $B \subset \sqrt{n} O(n)$, where B is the Euclidean unit ball.

We have (cf. Problem 31.3)

$$\left(\frac{\operatorname{vol} K}{\operatorname{vol} B} \right)^{1/n} = \left(\frac{n^{n/2} \Gamma(n/2+1) 2^n}{\pi^{n/2} n!} \right)^{1/n} \leq \sqrt{\frac{2e}{\pi}}.$$

It follows by Theorem 35.2 that for any $0 < \epsilon < 1$ there is a constant $\rho(\epsilon) > 0$ and an integer $k \geq (1-\epsilon)n$ such that a random subspace $L \in G_k(\mathbb{R}^n)$ is $\rho(\epsilon)$-Euclidean with probability at least $1 - 2^{-n}$.

(35.4) Problem.

1. Prove that the probability that $m \geq n$ independent random points $x_1, \ldots,$ $x_m \in \mathbb{S}^{n-1}$ lie in a halfspace is

$$2^{-m+1} \sum_{k=0}^{n-1} \binom{m-1}{k}.$$

See J.G. Wendel, A problem in geometric probability, *Math. Scand.* **11** (1962), 109–111.