Math 669 Problems, Set no. I

Due Thursday, February 18

1. Problem. Let V be a real vector space, let $S \subset V$ be a set and let $a, b \in V$ be points such that $a \notin \operatorname{conv}(S)$ and $b \notin \operatorname{conv}(S)$. Prove that if $a \in \operatorname{conv}(S \cup \{b\})$ and $b \in \operatorname{conv}(S \cup \{a\})$ then $a = b$.

2. Problem. Let $f(z)$ be a non-constant polynomial in one complex variable z and let $z_1, \ldots, z_m$ be the roots of f. Identify the set of complex numbers $\mathbb{C}$ with the plane $\mathbb{R}^2$ by identifying $z = x + iy$ with (x, y). Let $w \in \mathbb{C}$ be a root of the derivative f' of f. Prove that $w \in \operatorname{conv}(z_1, \ldots, z_m)$ (Gauss-Lucas Theorem).

3. Problem. Let $A_1, \ldots, A_{d+1} \subset \mathbb{R}^d$ be sets and let $a \in \mathbb{R}^d$ be a point such that $a \in \operatorname{conv}(A_i)$ for $i = 1, \ldots, d+1$. Prove that there are points $a_i \in A_i$ for $i = 1, \ldots, d+1$ such that $a \in \operatorname{conv}(a_1, \ldots, a_{d+1})$ (Bárány Theorem).

Hint: Argue that without loss of generality one may assume that all the sets A_i are finite. Choose $a_i \in A_i$ so that the Euclidean distance from a to $\operatorname{conv}(a_1, \ldots, a_{d+1})$ is the smallest possible.

4. Problem. Construct an example of a closed set $A \subset \mathbb{R}^2$ such that $\operatorname{conv}(A)$ is not closed.

5. Problem. Let $S \subset \mathbb{R}^d$ be a set of $(k-1)(d+1)+1$ points in $\mathbb{R}^d$, where $k > 1$ is an integer. Prove that one can find k pairwise disjoint subsets $S_i \subset S$ for $i = 1, \ldots, k$ such that

$$\operatorname{conv}(S_1) \cap \operatorname{conv}(S_2) \cap \ldots \cap \operatorname{conv}(S_k) \neq \emptyset$$

(Tverberg Theorem).

Hint: Identify $\mathbb{R}^d$ with the affine hyperplane $x_1 + \ldots + x_{d+1} = 1$ in $\mathbb{R}^{d+1}$. Let W be a $(k-1)$-dimensional real vector space and let $w_1, \ldots, w_k \in W$ be vectors such that $w_1 + \ldots + w_k = 0$ is their only linear dependence, up to a factor. In the space $\mathbb{R}^{d+1} \otimes W$ consider the $(k-1)(d+1)+1$ subsets

$$A_s = \{s \otimes w_1,\ s \otimes w_2,\ \ldots,\ s \otimes w_k\} \quad \text{where} \quad s \in S.$$

Apply the result of Problem 3.

6. Problem. Let $A_1, \ldots, A_m$ be convex sets in a d-dimensional real vector space V. Let $k \leq d+1$ and suppose that every k of the sets have a point in common. Prove that for any subspace $L \subset V$ such that $\dim L = d-k+1$ there is a translation $L + u = \{x + u : x \in L\}$ that intersects every set A_i, $i = 1, \ldots, m$.

7. Problem. Let $A_1, \ldots, A_m; C$ be convex sets in a d-dimensional real vector space V. Suppose that for any $d+1$ sets $A_{i_1}, \ldots, A_{i_{d+1}}$ there is a vector $u \in V$ such that the translate $C+u = \{x+u : \; x \in C\}$ intersects every set $A_{i_1}, \ldots, A_{i_{d+1}}$:

$$A_{i_1} \cap (C+u) \neq \emptyset, \; A_{i_2} \cap (C+u) \neq \emptyset, \; \ldots, \; A_{i_{d+1}} \cap (C+u) \neq \emptyset$$

(vector u may depend on $A_{i_1}, \ldots, A_{i_{d+1}}$). Prove that there is a vector $w \in V$ such that

$$A_i \cap (C+w) \neq \emptyset \quad \text{for} \quad i = 1, \ldots, m.$$

8. Problem. Let $A \subset \mathbb{R}^d$ be a compact convex set. Prove that there is a point $u \in \mathbb{R}^d$ such that

$$-\frac{1}{d}A + u \subset A \quad \text{where} \quad -\frac{1}{d}A + u = \left\{ -\frac{1}{d}x + u : \; x \in A \right\}.$$

9. Problem. Let $A_1, A_2, A_3, A_4 \subset \mathbb{R}^2$ be closed convex sets such that the set $A_1 \cup A_2 \cup A_3 \cup A_4$ is convex. Suppose that all pairwise intersections $A_1 \cap A_2$, $A_1 \cap A_3$, $A_1 \cap A_4$, $A_2 \cap A_3$, $A_2 \cap A_4$ and $A_3 \cap A_4$ are non-empty. Prove that at least three of the four intersections $A_1 \cap A_2 \cap A_3$, $A_1 \cap A_2 \cap A_4$, $A_1 \cap A_3 \cap A_4$ and $A_2 \cap A_3 \cap A_4$ are non-empty and if all the four intersections are non-empty then $A_1 \cap A_2 \cap A_3 \cap A_4 \neq \emptyset$.

10. Problem. Let $\Delta \subset \mathbb{R}^d$ be the standard $(d-1)$-dimensional simplex defined by

$$\Delta = \left\{ (x_1, \ldots, x_d) : \quad \sum_{i=1}^{d} x_i = 1 \quad \text{and} \quad x_i \geq 0 \quad \text{for} \quad i = 1, \ldots d \right\}.$$

For $i = 1, \ldots, d$, let $\Gamma_i \subset \Delta$ be the facet of Δ defined by the equation $x_i = 0$. Let $K_1, \ldots, K_d \subset \mathbb{R}^d$ be closed convex sets such that

$$\Delta \subset \bigcup_{i=1}^{d} K_i \quad \text{and} \quad K_i \cap \Gamma_i = \emptyset \quad \text{for} \quad i = 1, \ldots, d.$$

Prove that

$$\bigcap_{i=1}^{d} K_i \neq \emptyset$$

(Klee Theorem).