Math 669 Problems, Set no. II

Due Thursday, March 18

1. Problem. For sets $A, B \subset \mathbb{R}^d$, we define their Minkowski sum as

$$A + B = \{x + y : \ x \in A, y \in B\}.$$

Prove that the Minkowski sum of convex sets is convex and that the Minkowski sum of compact sets is compact.

2. Problem. Let $\mathcal{K}(\mathbb{R}^d)$ be the algebra of convex compact sets in $\mathbb{R}^d$. Show that there is a bilinear operation

$$\star : \ \mathcal{K}(\mathbb{R}^d) \times \mathcal{K}(\mathbb{R}^d) \longrightarrow \mathcal{K}(\mathbb{R}^d)$$

such that

$$[A] \star [B] = [A + B]$$

for any non-empty convex compact sets $A, B \subset \mathbb{R}^d$. Here $A + B$ is the Minkowski sum. The operation $\star$ is bilinear if

$$(\alpha f + \beta g) \star h = \alpha(f \star h) + \beta(g \star h) \quad \text{and} \quad h \star (\alpha f + \beta g) = \alpha(h \star f) + \beta(h \star g)$$

for any $f, g, h \in \mathcal{K}(\mathbb{R}^d)$ and $\alpha, \beta \in \mathbb{R}$.

3. Problem. Let $A, B \subset \mathbb{R}^d$ be non-empty convex compact sets such that $A \cup B$ is convex. Prove the following identity for Minkowski sums:

$$(A \cap B) + (A \cup B) = A + B.$$

Hint: One way (which extends to more general identities) to prove it is to argue that for any linear function $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$, we have

$$\max_{x \in A \cap B} \ell(x) + \max_{x \in A \cup B} \ell(x) = \max_{x \in A} \ell(x) + \max_{x \in B} \ell(x).$$

4. Problem. Let $A \subset \mathbb{R}^2$ be a convex polygon (convex hull of finitely many points with a non-empty interior). Show that

$$\omega_1(A) = \frac{1}{\pi} \text{perimeter of } A,$$

where ω_1 is the first intrinsic volume.

5. Problem. Let $A \subset \mathbb{R}^d$ be a convex body, let $B \subset \mathbb{R}^d$ be the unit ball. Show that for any $\epsilon > 0$, we have

$$\mathrm{vol}(A + \epsilon B) = \sum_{k=0}^{d} \binom{d}{k} \frac{\nu_d}{\nu_k} \omega_k(A) \epsilon^{d-k},$$

where ν_k is the volume of the unit ball in $\mathbb{R}^k$ and $\omega_k(A)$ is the k-th intrinsic volume. *Hint:* Using Problem 2, prove that the correspondence

$$A \longmapsto \mathrm{vol}(A + \epsilon B)$$

for convex compact sets $A \subset \mathbb{R}^d$ extends to a valuation on $\mathcal{K}(\mathbb{R}^d)$. Use Hadwiger's Theorem characterizing valuations invariant under rigid motions of $\mathbb{R}^d$ and continuous with respect to the Hausdorff distance.

6. Problem. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ where $m < n$ be the projection

$$(\xi_1, \ldots, \xi_n) \longmapsto (\xi_1, \ldots, \xi_m).$$

Let $C \subset \mathbb{R}^n$ be a convex body, that is, a convex compact set with non-empty interior. For $x \in T(C)$, let

$$f(x) = \ln \mathrm{vol}\left(T^{-1}(x) \cap C\right),$$

where we measure volume in the $n - m$ dimensional affine subspace $T^{-1}(x)$. Show that

$$f(\alpha x + \beta y) \ \geq \alpha f(x) + \beta f(y)$$

for any $x, y \in T(C)$ and any $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Deduce that if C is symmetric about the origin, that is, $C = -C$, then

$$\max_{x \in T(C)} f(x) = f(0).$$

7. Problem. Let $V = \mathbb{R}_\infty$ be the real vector space of all infinite sequences $x = (\xi_1, \xi_2, \ldots)$ for which all but finitely many terms ξ_i are zero. Let $A \subset V$ be the set of all such sequences $x \neq 0$ such that the last non-zero term ξ_n is strictly positive. Prove that A is convex and that there is no hyperplane $H \subset V$ such that $0 \in H$ and H isolates A (strictly or not).

8. Problem. Let $A \subset \mathbb{R}^d$ be a non-empty closed convex set. Prove that A contains an extreme point if and only if A does not contain lines.

9. Problem. Construct an example of a real vector space V and a non-empty convex set $A \subset V$ such that the intersection of A with every line in V is a closed bounded interval, possibly empty or a point, and A has no extreme points.

10. Problem. Construct an example of a convex compact set $A \subset \mathbb{R}^3$ such that the set of extreme points of A is not closed.